Linear coupling of electromagnetic waves in inhomogeneous weakly-ionized media

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An account is given of the current state of the problem of linear coupling between electromagnetic waves. Conversion effects are discussed for waves in the radio, optical, and x-ray frequency ranges in different inhomogeneous, weakly-anisotropic media, such as magnetoactive plasma (in the laboratory and in space), "plasma + magnetized vacuum" system, liquid crystals, ferroelectrics, lightguides, and so on. The linear coupling phenomenon arises when waves traverse an inhomogeneous portion of a medium in which dispersion curves converge and the wave polarization structure undergoes a change. A qualitative analysis is presented of linear wave conversion, which can be used as a basis for determining in a relatively simple manner the possibility and efficacy of wave coupling, and to exhibit the characteristic dependence of the conversion effect on the properties of the inhomogeneous medium. The qualitative analysis is accompanied by a discussion of the solution of a number of standard problems that are of interest in connection with specific applications of the linear wave-conversion effect.

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CONTENTS

1. INTRODUCTION

The problem of linear wave coupling (in its rigorous formulation) has been with us for more than a quarter of a century. Systematic research in this branch of physics began in the 1950s in connection with the study of wave propagation in ionospheric plasma and in broad-range irregular microwave and acoustic waveguides (see, for example, Refs. 1–10). It was found that the propagation of waves through an inhomogeneous portion of a medium was accompanied by a nonadiabatic change in the geometrical optics amplitude. This phenomenon was called linear wave coupling or linear mode conversion. At present, the linear coupling phenomenon is being studied in a variety of branches of physics, including optics, quantum radiophysics, hydrodynamics, acoustics, electrodynamics, plasma physics, and so on. The total number of papers devoted to this topic can now be counted in thousands, and continues to increase rapidly. Several reviews and review-type papers have been published (for the current state of the relevant mathematical theory—see Refs. 26–28 and the references cited therein). It is no exaggeration to say that the problem of linear wave coupling is among the most important problems in the linear theory of oscillations and waves.

However, until recently, the theory was unable to provide a qualitative and, at the same time, sufficiently detailed description of linear conversion effects in a

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more or less general physical situation, i.e., for waves of arbitrary nature in an arbitrary, anisotropic, inhomogeneous medium. The point is that linear coupling problems for which solutions are available are largely confined to special cases of wave propagation. The consequence of this is that it is very difficult to envision linear wave conversion not only in some newly emerging extensive range of applications, for example, in seismology,[29] or the optics of inhomogeneous media,[55] but even in the classical case of a plasma in a newly created or newly discovered situation. It may actually appear that the theory of waves and oscillations is powerless in the face of this problem, and that linear coupling proceeds quite differently in each particular case.[30]

The aim of this review is not only to familiarize the reader with new results and new areas of application of the linear coupling effect, but also to provide a quantitative analysis of linear conversion and to demonstrate its possibilities. The qualitative approach will enable us to perform a unified analysis of linear wave coupling in very different situations, for example, in magnetoactive plasmas, liquid crystals, ferrodielectrics, elastic bodies, lightguides, planar waveguides, and so on. This is particularly important because the phenomenon is only just beginning to be studied in these areas, and the number of interested parties is much greater than the number of "specialists in linear coupling."

Specifically, our review is devoted to the coupling of electromagnetic waves in weakly anisotropic, smoothly inhomogeneous media. This area includes at present the greatest number of experimental investigations, and important results have already been reported.[1)]

It is well known[31-33] that, in a linear homogeneous stationary medium, an arbitrary radiation field can be looked upon as a superposition of normal waves of the form $\mathscr{E}_i A_i(\omega, \mathbf{k}_i) \exp[i(\omega t - \mathbf{k}_i \mathbf{r})]$, where the frequency and the wave vector $\mathbf{k}_i$ are connected by the dispersion relation, and $\mathscr{E}_i$ is a vector defining the polarization of the ith normal wave. This decomposition can be used in an inhomogeneous medium as well, except that the parameters of the medium (refractive indices $n_i = |\mathbf{k}_i| c/\omega$ and polarization coefficients) are then local properties of the medium and, consequently, are functions of position. When the properties of the medium vary slowly, the complex wave amplitudes $f_i(\mathbf{r}) = A_i(\mathbf{r}) \exp[i\varphi_i(\mathbf{r})]$ in the above expansion are determined by the eikonal equation for the phase $\varphi_i(\mathbf{r})$ (which is not then equal to $-\mathbf{k}_i \mathbf{r}$) and the law of conservation of energy for the amplitude $A_i(\mathbf{r})$ of each of the waves. This approach is called the adiabatic approximation, the WKB approximation, or the geometrical optics approximation.[31-36]

However, under certain definite conditions, the wave field may not obey the geometrical optics approximation in an inhomogeneous portion of the medium (even when the medium has slowly-varying properties), and one then speaks of the linear wave coupling phenomenon. The essence of this phenomenon is that, when the inhomogeneous portion of the medium is traversed by the radiation, the amplitude ratio and the phase difference between the waves comprising the radiation are found to vary in a manner that is different from that predicted by the geometrical optics approximation. Moreover, the waves are no longer independent (we are discussing waves whose polarizations $\mathscr{E}_i(\mathbf{r})$ at each point in the medium are determined by its local properties; see below). There is a mutual conversion both of waves traveling in opposite directions (incident and reflected waves) and waves propagating (transporting energy) in the same direction. In particular, when a geometrical optics wave of a particular type (for example, ordinary waves) is incident on the region in which coupling occurs, two coherent geometrical optics waves (ordinary and extraordinary) are found to leave the region. This process is linear, since it is not connected with a violation of the principle of superposition of wave fields. The adjective, "linear," will be omitted henceforth, since we shall not consider nonlinear interactions.

Let us consider a simple example of wave coupling. Suppose the radiation propagates in a plane layered medium in the direction of the z axis (anisotropy axis $\mathbf{L}$) and undergoes a smooth azimuthal rotation through a finite angle ψ_0 (see, for example, Fig. 1 with $\alpha = \pi/2$; in a magnetoactive plasma, the anisotropy axis $\mathbf{L}$ is parallel to the magnetic field $\mathbf{B}_0$). Next, suppose that, at entry to the above inhomogeneous region, the radiation is a linearly polarized wave with $\mathbf{E} \parallel \mathbf{L}$, which is an extraordinary wave in geometrical optics. In the geometrical optics approximation, the field $\mathbf{E}$ should then rotate as the anisotropy axis $\mathbf{L}$ rotates, remaining parallel to it throughout the process. This adiabatic regime corresponds to the absence of coupling, since the ordinary wave does not appear. However, when the anisotropy axis rotates at a sufficient rate, and the degree of anisotropy is very low, the electric vector will necessarily lag behind the anisotropy axis, and will cease to be parallel to it. It is well known[34] that, in the limit of an isotropic medium or of vacuum, the field $\mathbf{E}$ will cease to rotate with $\mathbf{L}$ altogether. This means that, in general, radiation leaving the inhomogeneous region will be a superposition of ordinary and extraordinary waves that will be polarized along and at right-angles to the anisotropy axis, respectively. Consequently, a conversion of the geometrical optics waves will have taken place.[42]

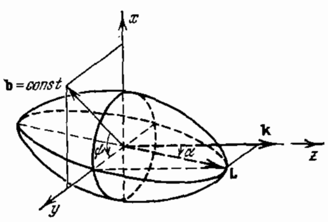

FIG. 1. "Rotating" set of coordinates and the optical ellipsoid $\mathrm{Re}\, \varepsilon_{kl}^{-1} x_k x_l = 1 (x_k = x, y, z)$. The y axis follows the anisotropy axis $\mathbf{L}$ and the vector $\mathbf{b}$ has a fixed direction in space: $\mathbf{b} \perp \mathbf{z}$.

[1)]In this review, we shall mostly refer to papers published in recent years. As far as earlier work is concerned, we shall refer to monographs and review papers in which these results are summarized.

Wave conversion is determined by the nature and scale of the inhomogeneity of the medium in the region of coupling, and therefore carries information about the structure of the inhomogeneity. By varying the degree of inhomogeneity, we can control the wave conversion efficiency and, consequently, the intensity and polarization of transmitted and reflected waves.

Consider the one-dimensional propagation of monochromatic waves in a stationary medium without sources (three-dimensional propagation will be treated in Sec. 2D). If we remove the time factor $\exp(i\omega t)$, the wave equations for the N components X_α of the field will take the form of the following first-order differential equations[2,14,32,36]

$$\mathbf{e}' = -iT\mathbf{e}, \tag{1.1}$$

where $\mathbf{e}$ represents the N-component column vector constructed from the complex field variables X_α ($\alpha = 1, \ldots, N$), and the square matrix $T = T(z)$ is determined by the local properties of the medium, i.e., it has the same form both in homogeneous and in inhomogeneous media, and does not contain differential operators. Frequency dispersion is represented by the function $T(\omega)$. The prime indicates differentiation with respect to the dimensionless position coordinate $\zeta = k_0 z$, measured along the direction of propagation (the z direction), and $k_0 = \omega/c$, and c = const is the characteristic wave velocity (in electrodynamics, the velocity of light in vacuum).

For the matrix T, we define at each point in the medium a complete set of eigenvectors $\mathbf{e}_i$ for which $\mathbf{e}_i\mathbf{e}_i^* = 1$ ($i = 1, \ldots, N$), and the refractive indices n_i are the eigenvalues in $T\mathbf{e}_i = n_i\mathbf{e}_i$. Next, we use the canonical transformation

$$\mathbf{e} = \sum_{i=1}^{N} f_i \vec{\mathscr{E}}_i, \quad \vec{\mathscr{E}}_i = \Phi_i \mathbf{e}_i, \tag{1.2}$$

and replace (1.1) with the equations for the complex amplitudes of the coupled waves, assuming[2)] that $n_i = n_i(\zeta)$, $\mathbf{e}_i = \mathbf{e}_i(\zeta)$, $f_i = f_i(\zeta)$:

$$f_i' + i n_i f_i = \sum_{j=1}^{N} a_{ij} f_j, \quad a_{ij} = -\vec{\mathscr{E}}_j' \cdot \vec{\mathscr{E}}^{i*}. \tag{1.3}$$

The dependence of f_i on the f_j ($i \neq j$) indicates that there is a linear relationship between waves in the inhomogeneous medium, where $a_{ij} \neq 0$. The factors $\Phi_i(\zeta)$ will be determined from the condition $a_{ii} \equiv -\vec{\mathscr{E}}_i' \cdot \vec{\mathscr{E}}^{i*} = 0$ or, equivalently, from

$$\frac{\Phi_i'}{\Phi_i} + \mathbf{e}_i' \cdot \mathbf{e}^{i*} = 0. \tag{1.4}$$

This condition signifies that the local values of the refractive indices for the coupled waves are independent of the inhomogeneity of the medium [see (1.3)]. It is readily verified that the geometrical optics asymptotic behavior of (1.1) leads to independent solutions of the form $\Phi_i \mathbf{e}_i \exp(-i\int n_i d\zeta)$, in which the factors Φ_i satisfy (1.4). Therefore, according to (1.2), coupling of the geometrical optics waves is described by a change in the amplitudes f_i.

For the electromagnetic field, we have $N = 4$. The derivation of equations describing the coupling of electromagnetic waves is given in Sec. 2, where the explicit form of the coupling coefficients a_{ij} is also given.

In the limit of weakly inhomogeneous media ($a_{ij} \to 0$), Eq. (1.3) yields the solution in the WKB approximation:

$$f_i = f_i(\zeta_{\text{in}}) \exp\left[-i \int_{\zeta_{\text{in}}}^{\zeta} n_i(\zeta)\, d\zeta\right]. \tag{1.5}$$

Linear coupling manifests itself as the difference between the solutions of (1.3) for an inhomogeneous medium ($a_{ij} \neq 0$) and the WKB solutions given by (1.5). The difference is that the polarization of the geometrical optics wave, which is set by the ratio X_α/X_β of the wave field components, is not adiabatically conserved as it should be locally for the given geometrical optics wave. In other words, different components of the field change in an uncoordinated manner (from the point of view of geometrical optics), altering the local structure of the particular normal wave $\mathbf{e}_i$ and thus generating other waves. Hence, it is clear that the final result of the analysis of wave coupling should be the prediction of the polarization structure of the total wave field. Generally speaking, this problem can be solved directly on the basis of the original set of equations given by (1.1) for the field components X_α, which is, indeed, occasionally done (Refs. 1, 3, 11, 12, 30, 36–39, and others). However, it will become clear later that a more effective and convenient approach is to use the above transformation to the coupled equations for the complex wave amplitudes, given by (1.3).

Since, in a nonabsorbing medium with $n_i^2 > 0$, the total stationary energy flux S_z carried by the coupled waves does not vary along the z direction, the wave-conversion process can be characterized by the redistribution of energy among the waves. It will become clear in Sec. 2 that the condition given by (1.4), or, more precisely, the condition $\mathrm{Re}\, a_{ii} = 0$, ensures that the energy flux density $\vec{\mathscr{E}}_i = \Phi_i \mathbf{e}_i$ will be constant. When this density is normalized to unity, the law of conservation of energy assumes the simple form[3)]

$$S_z = \sum_i f_i f_i^* - \sum_j f_j f_j^* = \text{const}, \tag{1.6}$$

where the summation indices i and j label geometrical optics waves propagating in the direction of the $+z$ and $-z$ axes, respectively. The foregoing refers to the case where $|\mathrm{Im}\, n_i| \ll |n_i|$ and the waves are weakly absorbed in the region of coupling. Henceforth, absorption will be neglected unless stated to the contrary.

At present, the approximation of two coupled waves (propagating in the same or in opposite directions) is usually employed in coupling theory and is given by

2)In the transformation leading to (1.3) we used the related (to $\vec{\mathscr{E}}_j$) set of vectors $\vec{\mathscr{E}}^{i*}$ ($\vec{\mathscr{E}}_j \cdot \vec{\mathscr{E}}^{i*} = \delta_{ij}$). It defines the so-called "transfer" waves that are the eigenvectors of the transposed matrix: $T^{\mathrm{T}} \vec{\mathscr{E}}^{i*} = n_i \vec{\mathscr{E}}^{i*}$. Hence $T_{\alpha\beta} = \sum_{i=1}^{N} n_i \mathscr{E}_{i\alpha} \mathscr{E}_\beta^{i*}$.

3)It is important to emphasize that this method is not valid in regions that are opaque to the ith wave ($n_i^2 < 0$) in which its energy flux vanishes and the original geometrical optics substitution given by (1.2) and (1.4) cannot be made (see Ref. 14).

$$f_1' + in_1 f_1 = a_{12} f_2,$$
$$f_2' + in_2 f_2 = a_{21} f_1. \tag{1.7}$$

This is valid in a smoothly inhomogeneous medium in which, in a given region, there are two coupled waves with converging (for waves propagating in the same direction) dispersion curves $n_{1,2}(\zeta)$, or with dispersion curves approaching zero (for waves propagating in opposite directions). Two-wave coupling is discussed qualitatively in Sec. 3. When the scale of the inhomogeneity in space is of the order of one wavelength, all the waves are usually coupled; the most interesting situation then is that of random inhomogeneities.[12,25,40] However, we shall not be concerned with such questions here.

2. BASIC EQUATIONS AND APPROXIMATIONS OF THE THEORY OF COUPLED ELECTROMAGNETIC WAVES

A. Normal waves in an anisotropic medium. Geometrical optics

The equations for a monochromatic electromagnetic field in a stationary anisotropic medium without sources have the following form:

$$\mathrm{rot}\,\mathbf{B} = ik_0\mathbf{D}, \quad \mathrm{rot}\,\mathbf{E} = -ik_0\mathbf{B}, \tag{2.1}$$

$$D_i(\omega, \mathbf{r}) = \varepsilon_{ij}(\omega, \mathbf{r})\,E_j(\omega, \mathbf{r}), \tag{2.2}$$

where $\mathbf{D}$ is the electric displacement, and $\mathbf{B}$ is the magnetic induction, $\mathbf{E}$ is the electric field, ε_{ij} is the complex permittivity tensor, $k_0 = \omega/c$, and spatial dispersion is ignored. For a plane electromagnetic wave propagating in the direction of the z axis, Eq. (2.1) reduces to $\mathbf{e}' = -iT\mathbf{e}$ (1.1) where

$$\mathbf{e} = \begin{pmatrix} E_x \\ -E_y \\ B_x \\ B_y \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \varepsilon^2\varepsilon_{yx}^{-1} & \varepsilon^2\varepsilon_{xx}^{-1} & 0 & 0 \\ \varepsilon^2\varepsilon_{yy}^{-1} & \varepsilon^2\varepsilon_{xy}^{-1} & 0 & 0 \end{pmatrix} \tag{2.3}$$

and $\varepsilon^2 = (\varepsilon^{-1}\varepsilon_{yy}^{-1} - \varepsilon_{xy}^{-1}\varepsilon_y^{-1})^{-1}$ and $\varepsilon_{ij}^{-1}(\omega, \mathbf{r})$ is the reciprocal permittivity tensor ($E_i = \varepsilon_{ij}^{-1}D_j$). The latter is convenient for a medium without sources,[31,41] in which the longitudinal component of the electric displacement is zero ($D_z = 0$).

The four normal waves in (2.3) and (1.1) consist of extraordinary (n_e) and ordinary (n_0) waves propagating in the direction of the z axis ($i = 1, 2$; $n_{1,2} = n_{e,0}$, $\mathbf{K}_{1,2} = \mathbf{K}_{e,0}$) and in the opposite direction ($i = 3, 4$; $n_{3,4} = -n_{e,0}$, $\mathbf{K}_{3,4} = \mathbf{K}_{e,0}$). The corresponding eigenvectors and eigenvalues are

$$\vec{\mathscr{E}}_i = \begin{pmatrix} E_x \\ -E_y \\ B_x \\ B_y \end{pmatrix}_i = \frac{\sqrt{8\pi/c}}{\sqrt{n_i(1+K_i^2)}} \begin{pmatrix} 1 \\ -iK_i \\ -in_iK_i \\ n_i \end{pmatrix}, \tag{2.4}$$

$$n_{e,0}^2 = \varepsilon^2\left[\frac{1}{2}(\varepsilon_{xx}^{-1} + \varepsilon_{yy}^{-1}) \pm i\varepsilon_{xy}^{-1}\sqrt{q^2 - \frac{\varepsilon_{yx}^{-1}}{\varepsilon_{xy}^{-1}}}\right], \tag{2.5}$$

$$K_{e,0} = q \mp \sqrt{q^2 - \frac{\varepsilon_{yx}^{-1}}{\varepsilon_{xy}^{-1}}}, \quad q = \frac{\varepsilon_{yy}^{-1} - \varepsilon_{xx}^{-1}}{2i\varepsilon_{xy}^{-1}}, \tag{2.6}$$

where $K_i \equiv -iE_{iy}/E_{ix}$ is the polarization coefficient of the ith wave. According to (2.6), the ordinary and extraordinary waves are polarized elliptically, i.e., the real part of the electric field $\mathrm{Re}[\mathbf{E}_i \exp(i\omega t)]$ of each wave describes an ellipse in space in the course of time. When there is no absorption, so that $\varepsilon_{ij} = \varepsilon_{ji}^*$, the x, y axes can be arranged so that $\mathrm{Re}\,\varepsilon_{xy}^{-1} = 0$ and, consequently, $\varepsilon_{yx}^{-1} = -\varepsilon_{xy}^{-1}$. It then follows from (2.6) that the polarization ellipses of the ordinary and extraordinary waves are mutually orthogonal: $K_e \cdot K_0 \equiv -1$. The ratio of the principal axes of these ellipses is determined by q. When $|q| \gg 1$, the ellipses degenerate into mutually perpendicular straight lines (along the x and y axes, respectively) and, when $|q| \ll 1$, they degenerate into circles (with opposite directions of rotation of the electric field vector).

The geometrical optics approximation is valid when the properties of the medium vary sufficiently slowly in the direction of wave propagation: the waves (2.4) then propagate independently of one another, their phases and amplitudes are determined by the refractive indices, in accordance with (1.5), and the polarizations follows the varying properties of the medium, in accordance with (2.4) and (2.6). The factor $1/\sqrt{n_i(1+K_i^2)}$ in the amplitude of each wave $\vec{\mathscr{E}}_i$ (2.4) corresponds to the law of conservation of energy in the particular wave, i.e., to a constant Poynting vector $|(c/8\pi)\,\mathrm{Re}[\mathbf{E}_i\mathbf{B}_i^*]_z| = \mathrm{const}$ [the fields in (2.4) are chosen so that this constant is equal to unity]. In a three-dimensionally inhomogeneous medium, each of the geometrical optics waves will propagate along the ray defined by the eikonal equation[31–34] $(\nabla\varphi i)^2 = n_i^2$.

B. Helical waves

In a homogeneous medium, the x, y axes assume fixed positions and are conveniently oriented to lie along the optical axes of the medium in the plane perpendicular to the direction of propagation, i.e., the z axis (or, more precisely, along the axes of the ellipse, $\mathrm{Re}[\varepsilon_{xx}^{-1}x^2 + (\varepsilon_{xy}^{-1} + \varepsilon_{yx}^{-1})\,xy + \varepsilon_{yy}^{-1}y^2] = 1$). In a uniaxial anisotropic medium, this means that the anisotropy axis determined, for example, by the magnetic field B_0 in a plasma, or the director $\mathbf{L}$ in a liquid crystal, lies in the y, z plane (Fig. 1). The optical axes in an inhomogeneous medium can, in general, rotate as they move along the z axis (their azimuthal rotation is characterized by the variation of the angle ψ in Fig. 1). If we consider the projections of $\mathbf{E}$ and $\mathbf{B}$ along the x and y axes of the coordinate frame that "rotates" together with the rotation of the optical axes (the z axis is fixed), the matrix T in the propagation equation (1.1) assumes the form

$$\tilde{T} = \begin{pmatrix} 0 & -i\psi' & 0 & 1 \\ i\psi' & 0 & 1 & 0 \\ \varepsilon^2\varepsilon_{yx}^{-1} & \varepsilon^2\varepsilon_{xx}^{-1} & 0 & i\psi' \\ \varepsilon^2\varepsilon_{yy}^{-1} & \varepsilon^2\varepsilon_{xy}^{-1} & -i\psi' & 0 \end{pmatrix}; \tag{2.7}$$

where $\psi' = k_0 d\psi/dz$ is the local rate of rotation of the optical axes in space. The tensor ε_{ij}^{-1} in the matrix $\tilde{T}$ (2.7) has a fixed form and does not explicitly depend on the angle ψ, in contrast to the matrix T (2.3) in the nonrotating coordinate frame.

When $\psi' = \mathrm{const}$, and the medium is homogeneous in all other respects, the equation $\mathbf{e}' = -i\tilde{T}\mathbf{e}$ (1.1) yields the normal waves in a medium with uniformly rotating optical axes. We then have

$$\vec{\tilde{\mathscr{E}}}_i = \begin{pmatrix} E_x \\ -E_y \\ B_x \\ B_y \end{pmatrix}_i = \frac{\sqrt{8\pi/c}}{\sqrt{\tilde{n}_i(1+\tilde{K}_i^2) + 2\psi'\tilde{K}_i}} \begin{pmatrix} 1 \\ -i\tilde{K}_i \\ -i(\tilde{n}_i\tilde{K}_i + \psi') \\ \tilde{n}_i + \psi'\tilde{K}_i \end{pmatrix}, \tag{2.8}$$

$$\tilde{n}^4-[\varepsilon^2(\varepsilon_{xx}^{-1}+\varepsilon_{yy}^{-1})+2\psi'^2]\tilde{n}^2-2i\varepsilon^2(\varepsilon_{xy}^{-1}-\varepsilon_{yx}^{-1})\psi'\tilde{n} + (\varepsilon^2\varepsilon_{xx}^{-1}-\psi'^2)(\varepsilon^2\varepsilon_{yy}^{-1}-\psi'^2)-\varepsilon^4\varepsilon_{xy}^{-1}\varepsilon_{yx}^{-1}=0, \tag{2.9}$$

$$\tilde{K}_i \equiv -iE_{iy}E_{ix}^{-1}=(\varepsilon^2\varepsilon_{yy}^{-1}-\psi'^2-\tilde{n}_i^2)(i\varepsilon^2\varepsilon_{xy}^{-1}+2\psi'\tilde{n}_i)^{-1}. \tag{2.10}$$

These waves will be called helical waves.[42–47] All quantities referring to these waves will be indicated by a tilde. It is clear from (2.9) that there are four helical waves with different refractive indices $\tilde{n}_i$ that are determined by a fourth-degree equation (in general, not a biquadratic equation, as was the case for the ordinary and extraordinary waves in a homogeneous medium)[4)] and its different polarization coefficients $\tilde{K}_i$. Equations (2.7)–(2.10) include absorption. However, even when there is low absorption, the polarization ellipses of the helical waves will not, in general, be orthogonal: $\tilde{K}_1\tilde{K}_2 \neq -1$. The description of the propagation of electromagnetic waves in terms of the helical waves is very convenient in cholesteric liquid crystals, in twisted waveguides, in helicoidal ferrodielectrics, and in plasmas with a sheared magnetic field.

The normal-wave structure in a medium with a periodic helical inhomogeneity is relatively simple when the optical axes rotate smoothly and the medium is weakly anisotropic [see (2.5)–(2.9)]:

$$|\psi'/\sqrt{\varepsilon}| \ll 1, \quad \frac{1}{2}|n_e^2-n_0^2| \ll \frac{1}{2}(n_e^2+n_0^2) \approx \varepsilon. \tag{2.11}$$

Since we shall frequently return to this important special case, let us write out the corresponding approximate formulas for two helical waves propagating in the same direction [they follow from (2.9) and (2.10) with $\varepsilon_{xy}^{-1}=-\varepsilon_{yx}^{-1})$]:[5)]

$$\tilde{n}_{1,2}=\sqrt{\varepsilon}\left[1 \pm \left(\frac{i}{2}\varepsilon\varepsilon_{xy}^{-1}+\psi'/\sqrt{\varepsilon}\right)\sqrt{\tilde{q}^2+1}\right], \tag{2.12}$$

$$\tilde{K}_{1,2}=\tilde{q}\mp\sqrt{\tilde{q}^2+1}, \quad \tilde{q}=\frac{\varepsilon(\varepsilon_{yy}^{-1}-\varepsilon_{xx}^{-1})}{4\left(\frac{i}{2}\varepsilon\varepsilon_{xy}^{-1}+\psi'/\sqrt{\varepsilon}\right)}. \tag{2.13}$$

It is clear from these two expressions that the gyrotropy ($i\varepsilon\varepsilon_{xy}^{-1}/2$) and the rotation of the optical axes of the medium ($\psi'/\sqrt{\varepsilon}$) lead to effects of the same kind and are, therefore, additive. In all other respects, the helical waves behave in this case in the same way as the ordinary and extraordinary waves. In the absence of rotation of the optical axes, the helical waves are identical with the ordinary and extraordinary waves. We note that, as in the case of (2.4), the energy flux density associated with the helical waves (2.8) is normalized to unity.

The geometrical optics of helical waves can be constructed by analogy with the geometrical optics of ordinary and extraordinary waves. The only difference is that the parameter ψ' must be regarded as a local characteristic of the medium, just as ε_{ij}^{-1} is (see Refs. 43 and 47).[6)]

C. Wave coupling for normal incidence on a plane-layered medium

Consider the propagation of waves in a plane-layered medium in the direction of the inhomogeneity axis (z axis). We shall use helical waves, i.e., we shall not ascribe a uniform rotation of the optical axes to the inhomogeneity of the medium. According to (1.2)–(1.4), we introduce the geometrical optics replacement of the form $\mathbf{e}=\sum_{i=1}^4 f_i\tilde{\vec{\mathscr{E}}}_i$, which ensures the transition to geometrical optics asymptotic behavior:

$$\begin{pmatrix} E_x \\ -E_y \\ B_x \\ B_y \end{pmatrix} = \sum_{i=1}^{4} f_i \frac{(8\pi/c)^{1/2}}{\sqrt{\tilde{n}_i(1+\tilde{K}_i^2)+2\psi'\tilde{K}_i}} \begin{pmatrix} 1 \\ -i\tilde{K}_i \\ -i(\tilde{n}_i\tilde{K}_i+\psi') \\ \tilde{n}_i+\psi'\tilde{K}_i \end{pmatrix}. \tag{2.14}$$

This leads us to the following equations for the interaction between helical waves:

$$f_i'+i\tilde{n}_i f_i = \sum_{j=1}^{4} \frac{(\delta_{ij}-1)\{\tilde{K}_j'[2\psi'+\tilde{K}_i(\tilde{n}_i+\tilde{n}_j)]+\tilde{n}_j'(1+\tilde{K}_i\tilde{K}_j)+\psi''(\tilde{K}_i+\tilde{K}_j)\}}{2\sqrt{\tilde{n}_i(1+\tilde{K}_i^2)+2\psi'\tilde{K}_i}\sqrt{\tilde{n}_j(1+\tilde{K}_j^2)+2\psi'\tilde{K}_j}} f_j. \tag{2.15}$$

We have assumed in the transformation to (2.15) that the polarizations of the helical waves are determined by the gyrotropy and rotation of the optical axes, but not by absorption or, more precisely, we have used equation $\varepsilon_{xy}^{-1}=-\varepsilon_{yx}^{-1}$. For this common case, the coupling coefficients in (2.15) are antisymmetric, i.e., $a_{ij}=-a_{ji}$. This can be verified by differentiating the identity

$$(\tilde{n}_i+\tilde{n}_j)(1+\tilde{K}_i\tilde{K}_j) \equiv -2\psi'(\tilde{K}_i+\tilde{K}_j), \tag{2.16}$$

which follows from (2.9) and (2.10).[7)] If there is no rotation of the optical axes ($\psi'=0$), the set of equations given by (2.15) yields the following equations for the coupling of ordinary and extraordinary waves:[2,48]

[4)]The difference between the refractive indices $\tilde{n}_1$ and $\tilde{n}_3$ (or $\tilde{n}_2$ and $\tilde{n}_4$) for two helical waves propagating in opposite directions occurs in magnetoactive (gyrotropic) media with rotating optical axes when $(\varepsilon_{xy}^{-1}-\varepsilon_{yx}^{-1})\psi' \neq 0$. This is connected with the fact that the direction of rotation of the optical axes of the medium in one case is the same as the direction of Faraday rotation, whereas, in the other, it is in the opposite direction.[43,122] The result is that, alongside the well-known nonreciprocity of Faraday rotation of the plane of polarization, we now have the additional polarizational nonreciprocity, namely, the difference between the polarization ellipses and phase velocities of normal waves propagating in opposite directions.

[5)]The expression for $\tilde{K}_{1,2}$ given by (2.13) is valid when the denominator in the expression for $\tilde{q}$ does not vanish: $|(i/2)\varepsilon\varepsilon_{xy}^{-1}+\psi'/\sqrt{\varepsilon}| \gg |(1/2)\varepsilon\varepsilon_{xy}^{-1}\psi'/\sqrt{\varepsilon}|$.

[6)]It is appropriate to emphasize here that a large number of periods of the helical structure is not essential to enable us to introduce and use helical modes even though, for any other periodic structure, this condition is necessary for the definition of normal waves. The point is that, for a given direction of propagation, it is only helical (azimuthal) rotation that will ensure that the constitutive relation (2.2) and Eqs. (2.1) will remain unaltered, and this enables us to perform a local analysis over an arbitrarily small segment of the structure. In particular, helical waves can be used to describe the propagation of radiation when the optical axes of the medium rotate through a finite angle $\psi_0 \lesssim 1$.

[7)]It is clear from (2.16) that the nonorthogonality of the polarizations of the helical waves that is characterized by the departure of $\tilde{K}_i\tilde{K}_j$ from -1 is proportional to the rate ψ' of rotation of the optical axes in space. We note that, at the degeneracy point for the two waves $\tilde{n}_i=\tilde{n}_j$ $(i \neq j)$, the expression under the square root in (2.14) will not be zero, since $\tilde{K}_i \neq \tilde{K}_j$ because of the particular choice of eigenvectors in (2.8). We note that, when $\tilde{K}_i=\tilde{K}_j$ and $\tilde{n}_i=\tilde{n}_j$, the expression under the square root will vanish by virtue of (2.16).

$$
\begin{aligned}
f_1' + in_e f_1 &= -\Psi \frac{n_e+n_0}{2\sqrt{n_e n_0}} f_2 - \frac{in_e'}{2n_e} f_3 + i\Psi \frac{n_e-n_0}{2\sqrt{n_e n_0}} f_4 \\
f_2' + in_0 f_2 &= \Psi \frac{n_e+n_0}{2\sqrt{n_e n_0}} f_1 + i\Psi \frac{n_e-n_0}{2\sqrt{n_e n_0}} f_3 - \frac{in_0'}{2n_0} f_4 \\
f_3' - in_e f_3 &= \frac{in_e'}{2n_e} f_1 - i\Psi \frac{n_e-n_0}{2\sqrt{n_e n_0}} f_2 - \Psi \frac{n_e+n_0}{2\sqrt{n_e n_0}} f_4 \\
f_4' - in_0 f_4 &= -i\Psi \frac{n_e-n_0}{2\sqrt{n_e n_0}} f_1 + \frac{in_0'}{2n_0} f_2 + \Psi \frac{n_e+n_0}{2\sqrt{n_e n_0}} f_3,
\end{aligned}
\tag{2.17}
$$

where $\Psi = -q'[2(q^2+1)]^{-1}$. Equations (2.15) and (2.17) include absorption provided it does not violate the condition $\varepsilon_{xy}^{-1} = -\varepsilon_{yx}^{-1}$. If there is no absorption, we have the law of conservation of energy in the form given by (1.6). This follows directly from (2.15) and (2.17), which have the form of (1.3). This can be verified if we recall that the coupling coefficients for these equations satisfy the following symmetry relations: $a_{ij} = -a_{ji}^*$ for waves i, j propagating in the same direction, and $a_{ij} = a_{ji}^*$ for waves i, j propagating in opposite directions.

It is clear from (2.15) and (2.17) that, when the medium is highly inhomogeneous (large values of the derivatives $\tilde{n}_i'$, $\tilde{K}_i'$, ψ''), all four geometrical optics waves are closely coupled with one another. The theory of wave coupling has not as yet been developed for this case. In the homogeneous medium, for which $a_{ij} = 0$ for all i, j, the normal waves propagate independently of one another. For a smooth inhomogeneity of scale $\Lambda \gg \lambda$ (λ is the wavelength in the medium) for which the coupling coefficients a_{ij} in (2.15) and (2.17) are small, the coupling between waves propagating in opposite directions and the coupling between waves propagating in the same direction can be considered separately under certain additional conditions. In the case of above-barrier reflection, the conversion of the former waves corresponds to reflection with a refractive index approaching (but different from) zero[7,49-51] (see Sec. 3D). This reflection (coupling of two waves of the same type) occurs when waves of the other type can be neglected [see (2.17)]:

$$
\begin{aligned}
f_1' + in_1 f_1 &= -\Omega f_3 \\
f_3' - in f_3 &= \Omega f_1
\end{aligned}, \quad \Omega = \frac{in'}{2n}. \tag{2.18}
$$

We note, by the way, that analogous equations for two helical waves propagating in opposite directions can be obtained from (2.15). Reflection from $n = 0$ points and from regions with negative n^2 will not be considered because, for regions for which $n^2 \leqslant 0$, the waves are nonpropagating, and the original geometrical optics substitution (1.2)–(1.4) is no longer meaningful (see Ref. 14).

The coupling of waves propagating in the same direction is described by equations of the form

$$
\begin{aligned}
f_1' + i\tilde{n}_1 f_1 &= -\tilde{\Psi} f_2 \\
f_2' + i\tilde{n}_2 f_2 &= \tilde{\Psi} f_1
\end{aligned}, \quad \tilde{\Psi} = -\frac{\tilde{q}'}{2(\tilde{q}^2+1)} \tag{2.19}
$$

[see (2.11)–(2.15) and below]. In this case, wave conversion occurs in the region where the refractive indices approach one another $|\tilde{n}_1 - \tilde{n}_2| \ll \tilde{n}_1 + \tilde{n}_2$. The absence of reflected waves under the condition $\Lambda \gg \lambda$ will be ensured when the two refractive indices $\tilde{n}_1$ and $\tilde{n}_2$ are sufficiently different from zero.

The law of conservation of energy follows from (2.18) and (2.19), and is written in the form

$$
|f_1|^2 - |f_3|^2 = \text{const} \tag{2.20}
$$

for waves propagating in opposite directions and

$$
|f_1|^2 + |f_2|^2 = \text{const} \tag{2.21}
$$

for the waves propagating in the same direction. This corresponds to purely imaginary and purely real coupling coefficients Ω and $\tilde{\Psi}$, respectively. Comparison of (2.18) and (2.19) will show that the coupling coefficient for the accompanying waves is determined by the change in $\tilde{q}$, i.e., the wave polarization $\tilde{K}_{1,2}$ (2.13) is independent of $\tilde{n}_{1,2}$, whereas the coupling coefficient of the colliding waves is determined exclusively by the change in their refractive indices. We emphasize that (2.18) and (2.19) are suitable for the description of arbitrary (including strong) wave conversion, despite the assumed continuity of the inhomogeneous variation in $\tilde{n}_{1,2}$ and $\tilde{q}$ on the scale of the wavelengths in the medium. The conditions for an effective coupling between the waves will be obtained in Sec. 3.

D. Quasi-isotropic approximation of geometrical optics (three-dimensional case). The Budden-Kravtsov equations

Under the conditions defined above, the interaction of waves in a three-dimensionally inhomogeneous medium can be reduced to the one-dimensional case by using the "quasi-isotropic" approximation of geometrical optics.[34,37,38,47,52] This approximation is valid in a weakly anisotropic, smoothly inhomogeneous medium.[8)] According to this method, the electric displacement and magnetic induction $\mathbf{D}(\mathbf{r})$, $\mathbf{B}(\mathbf{r})$ are sought in the form of an asymptotic series in powers of k_0^{-1}, as is usually done in the geometrical optics of an isotropic medium. We now use in (2.1) and (2.2) the smooth inhomogeneity and weak anisotropy ($|\varepsilon_{ij} - \varepsilon\delta_{ij}| \ll \varepsilon$) as small parameters, and introduce the geometrical optics replacement of the form of (2.14). This gives us the Budden-Kravtsov equations:[2,34]

$$
\begin{aligned}
\frac{df_1}{dl} + ik_0\tilde{n}_1 f_1 &= f_2 [2(\tilde{q}^2+1)]^{-1} \frac{d\tilde{q}}{dl}, \\
\frac{df_2}{dl} + ik_0\tilde{n}_2 f_2 &= -f_1 [2(\tilde{q}^2+1)]^{-1} \frac{d\tilde{q}}{dl}.
\end{aligned}
\tag{2.22}
$$

These two equations have the same form as (2.19) which was obtained in the one-dimensional case. However, the role of the dimensionless coordinate ξ is now played by $k_0 l$, where l is the path length along the quasi-isotropic ray. The shape of the ray is determined by the eikonal equation in the isotropic medium with refractive index $\sqrt{\varepsilon} \approx (n_0 + n_e)/2$. The introduction of the quasi-isotropic ray is connected with the fact that, approximately, we need not distinguish between ordinary and extraordinary rays under the conditions of weak anisotropy in the wave coupling region.[21,38] The amplitude factor $1/\sqrt{\tilde{n}_i} \approx \varepsilon^{-1/4}$ in (2.14) is now replaced by the new factor A, which satisfies the equation

---

8) Wave coupling has not been extensively investigated[12,20,21,24] in three-dimensionally inhomogeneous media with strong anisotropy and arbitrary inhomogeneity. Several special cases of oblique incidence on an anisotropic plane-layered medium have been considered in plasma physics,[13-16,48,53] in the optics of cholesteric liquid crystals,[54,55] in magneto-optics,[56] and in integrated optics.[23,57]

$\mathrm{div}(\sqrt{\varepsilon}A^2\mathbf{l}) = 0$, where $\mathbf{l}$ is the unit vector along the quasi-isotropic ray. This last equation is obtained from Maxwell's equations together with (2.22) as a condition for the consistency of the first order approximation equations in the above small parameters. The factor A, which takes into account the variation in the refractive index $\sqrt{\varepsilon}$ and refraction, ensures that energy conservation (2.21) is satisfied along the ray.

In the three-dimensional case, we use the orthonormal set of coordinates x, y, z transported along the quasi-isotropic ray and following the rotation of the optical axes (see Fig. 1, where the z axis is parallel to $\mathbf{l}$). The angle ψ is the angle between the y axis and the direction in the x, y plane specified by Rytov's law[31,34] $d\chi/dl = T^{-1}$ (T is the radius of curvature of the ray and the angle χ is measured from the ray normal in the x, y plane). In other words, the angle ψ is the angle between the y axis that follows the optical axis and the electric field vector $\mathbf{b}$ of the linearly polarized radiation propagating along the quasi-isotropic ray on the assumption that the medium is isotropic and its refractive index is $\sqrt{\varepsilon}$. With this definition, the derivative ψ' represents both the rotation of the optical axes and the twisting of the ray itself.[47]

Thus, in a weakly-anisotropic medium, the coupling of waves propagating in the same direction is described in a practically identical manner in both the one-dimensional and three-dimensional cases.

The coupling with reflected waves in the three-dimensionally inhomogeneous medium in the quasi-isotropic approximation is evidently impossible to take into account because reflections occurring in this type of medium violate the locally plane structure of the wave field, which is essential for the validity of the quasi-isotropic approximation (Refs. 21, 34, 38).

E. Equations for the transfer of wave polarization in inhomogeneous media

In the above discussion, we considered monochromatic waves of given polarization at each point in space. However, in practice, we frequently have to deal with waves whose frequencies are concentrated within a narrow interval $\Delta\omega \ll \omega$.[33,59-62] Their polarizations are slowly-varying and, generally speaking, random functions of time. The theory of transfer of this type of radiation therefore employs quantities of the form $\overline{E_\alpha E_\beta^*}$ ($\alpha, \beta = x, y$), in which a time average has been taken within the interval $\Delta t \gg 1/\Delta\omega$ (or a statistical average over the ensemble). The corresponding transfer equations, including coupling between the waves in a three-dimensionally inhomogeneous weakly anisotropic medium, can be deduced from the Budden-Kravtsov equations (2.22). Neglecting reflected waves, we find from these equations that the quadratic combinations $J_{ij} = \overline{f_i f_j^*}$ of the complex amplitudes of the two waves propagating in the same direction satisfy the following transfer equations:[9)]

[9)]Equations (2.23) are referred to as "polarization transfer equations,"[67,68] since $f_i = \mathbf{e}\cdot\vec{\mathscr{E}}^{i*}$, where $\mathbf{e}$ is the resultant field and $\vec{\mathscr{E}}^{i*}$ represents the "transfer" waves that are orthogonal to the normal waves $\vec{\mathscr{E}}_j^{i*}$ [see (1.2) and (1.3)].

$$\begin{aligned} J'_{11} &= 2(\mathrm{Im}\,\tilde{n}_1)J_{11} - \tilde{\Psi}(J_{12}+J_{12}^*),\\ J'_{22} &= 2(\mathrm{Im}\,\tilde{n}_2)J_{22} + \tilde{\Psi}(J_{12}+J_{12}^*),\\ J'_{12} &= i(\tilde{n}_2^* - \tilde{n}_1)J_{12} + \tilde{\Psi}(J_{11}-J_{22}). \end{aligned} \tag{2.23}$$

Since the properties of the medium are assumed to be determined, we can transform to the quantities $\overline{E_\alpha E_\beta^*}$ with the aid of the geometrical optics substitution (2.14) and, using the quasi-isotropic approximation, we obtain

$$\varepsilon(\varepsilon^{-1}I_{\alpha\beta})' = \tilde{T}_{\alpha\beta\gamma\delta}I_{\gamma\delta}, \quad I_{\alpha\beta} = \frac{c}{8\pi}\varepsilon A^{-2}\overline{E_\alpha E_\beta^*} \approx \frac{c}{8\pi}\varepsilon^{-1}A^{-2}\overline{D_\alpha D_\beta^*}, \tag{2.24}$$

where the polarization tensor $I_{\alpha\beta}$ refers to a unit frequency interval and unit solid angle along the ray.

The coordinate system used in the above expressions is defined in Sec. 2D. The polarization transfer tensor $\tilde{T}_{\alpha\beta\gamma\delta}$ will be shown to be expressible in terms of the components of the normal waves $\vec{\mathscr{E}}_i$ in the same way as in the homogeneous medium;[64-66] however, it now depends on the coordinate $\zeta = k_0 l$ along the ray. Thus, in the quasi-isotropic approximation, the transfer equations given by (2.24) in an inhomogeneous medium differ from the corresponding equations for the homogeneous medium only by the dependence of the coefficients on the space coordinates. The transfer equations thus take into account the coupling between the waves (this was previously demonstrated by numerical calculations[63] in special cases). This result was expected because the original Maxwell equations did not contain derivatives of the medium parameters. This can be readily exploited to generalize the transfer equations given by (2.24) for the inhomogeneous medium to the case where the coupled normal waves are nonorthogonal (nonorthogonality was not taken into account above). To show this, it is sufficient, as in the homogeneous medium, to take into account the nonorthogonality of the helical waves $\vec{\mathscr{E}}_i$ in the expression for the tensor $\tilde{T}_{\alpha\beta\gamma\delta}$ (see Refs. 33 and 67).

It follows from (2.23) [or (2.24)] that, in the absence of dichroism, with $\mathrm{Im}\,\tilde{n}_1 = \mathrm{Im}\,\tilde{n}_2$, the ratio of the intensities of the polarized (coherent) and unpolarized (noncoherent) components of the radiation remains the same. In this case, effects associated with coupling between the waves in the theory of transfer of partially polarized radiation obviously apply only to the polarized (coherent) component, and can be investigated on the basis of the Budden-Kravtsov equations given by (2.22) without introducing the transfer equations. However, analysis of the transfer equations is essential in the presence of dichroism, and of depolarization, scattering, and emission by the medium itself. The latter factors are usually statistical in character. They can be taken into account relatively simply by replacing the stochastic field equations with the transfer equations, having included in their right-hand sides the corresponding root-mean-square parameters of depolarization, scattering, and emission by the medium. The transfer of radiation under the conditions of linear wave coupling has not been examined with allowance for these factors although they are of undoubted interest in the theory of generation and propagation of waves, for example, in magnetoactive plasmas or liquid crystals.[33,59-62,69,70]

3. QUALITATIVE ANALYSIS OF LINEAR WAVE CONVERSION. THE COUPLING PARAMETER

Let us now consider the qualitative picture of the coupling of two waves propagating in the same direction, which is described by the set of coupled equations given by (2.19) with real coupling coefficients $\tilde{\Psi} = -a_{12} = a_{21}$ and refractive indices $\tilde{n}_{1,2}$. The coupling between two colliding waves (2.18) with purely imaginary coupling coefficients $\Omega = in'/(2n)$, and the more general case (1.7) with complex coupling coefficients and refractive indices (i.e., with allowance for absorption), can be analyzed analogously.

A. Conversion matrix and conversion coefficient. Limiting cases (geometrical optics, isotropic medium, jump in anisotropy)

The set of equations given by (2.19) enables us to find the values of the functions $f_1^{\rm out}$, $f_2^{\rm out}$ and the corresponding radiation field (2.14) at exit from an inhomogeneous layer from given values $f_1^{\rm in}$, $f_2^{\rm in}$ at entry. Since the set is linear, we have

$$f_1^{\rm out} = F_{11}f_1^{\rm in} + F_{12}f_2^{\rm in}, \quad f_2^{\rm out} = F_{21}f_1^{\rm in} + F_{22}f_2^{\rm in}, \tag{3.1}$$

where the four quantities F_{ij} form the conversion matrix[52]

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} = \begin{pmatrix} \sqrt{1-Q}\,e^{-i\gamma} & \sqrt{Q}\,e^{i(\varphi+\gamma)} \\ -\sqrt{Q}\,e^{-i(\varphi+\gamma)} & \sqrt{1-Q}\,e^{i\gamma} \end{pmatrix} \exp\left(-\int_{\zeta_{\rm in}}^{\zeta_{\rm out}} \tilde{n}\,d\zeta\right). \tag{3.2}$$

This is readily verified by substituting

$$f_{1,2} = \delta_{1,2}\exp\left(-i\int \tilde{n}\,d\zeta\right), \quad \tilde{n} = \frac{1}{2}(\tilde{n}_1 + \tilde{n}_2), \tag{3.3}$$

which reduces (2.19) to the symmetric form

$$\begin{aligned} \delta_1' + \frac{i}{2}(\tilde{n}_1 - \tilde{n}_2)\delta_1 &= -\tilde{\Psi}\delta_2 \\ \delta_2' - \frac{i}{2}(\tilde{n}_1 - \tilde{n}_2)\delta_2 &= \tilde{\Psi}\delta_1 \end{aligned}, \quad \tilde{\Psi} = -\frac{\tilde{q}'}{2(\tilde{q}^2+1)}. \tag{3.4}$$

The first column of the conversion matrix is obtained from the solution (δ_1, δ_2) under the boundary conditions $\delta_1^{\rm in} = 1$, $\delta_2^{\rm in} = 0$. Since energy conservation (2.21) has already set the ratio of the moduli of $\delta_1^{\rm out}$ and $\delta_2^{\rm out}$, we may introduce the notation

$$\delta_1^{\rm out} = \sqrt{1-Q}\,e^{-i\gamma}, \quad \delta_2^{\rm out} = -\sqrt{Q}\,e^{-i(\varphi+\gamma)}, \tag{3.5}$$

where possible phase differences are represented by the factors $\exp(-i\gamma)$ and $\exp[-i(\varphi+\gamma)]$. The second column of the conversion matrix is obtained under the boundary condition $\delta_1^{\rm in} = 0$, $\delta_2^{\rm in} = 1$. Simple substitution in (3.4) shows that $(-\delta_2^*, \delta_1^*)$ can be taken as the solution, where δ_1 and δ_2 are the previous solutions used in determining the first column. This enables us to write the conversion matrix in the form given by (3.2) if we also take into account the relationship between $\delta_{1,2}$ and $f_{1,2}$ given by (3.3).

The quantity Q in (3.4) is the so-called conversion coefficient that determines the relative intensity of waves of a given type (at exit) when a wave of another type is incident on the layer:

$$\begin{aligned} Q &= \frac{|f_1^{\rm out}|^2}{|f_1|^2+|f_2|^2} = \frac{|\delta_1^{\rm out}|^2}{|\delta_2^{\rm in}|^2} \quad \text{for} \quad f_1^{\rm in} = 0, \\ Q &= \frac{|f_2^{\rm out}|^2}{|f_1|^2+|f_2|^2} = \frac{|\delta_2^{\rm out}|^2}{|\delta_1^{\rm in}|^2} \quad \text{for} \quad f_2^{\rm in} = 0. \end{aligned} \tag{3.6}$$

It is clear that $0 \leq Q \leq 1$. The coefficient of conversion into the same wave is $1-Q$. This is clear from the law of conservation of energy given by (2.21), which was also taken into account in deriving (3.6). It follows from the symmetry of (3.4) that the two expressions in (3.6) lead to identical values of Q.

The quantity ϕ in (3.2) obviously determines the phase difference between waves of the first and second type at exit from the layer when a wave of the second type is incident upon it. In the geometrical optics approximation, the quantity γ is identical with the "Faraday" phase difference

$$\gamma_0 = \frac{1}{2}\int_{\zeta_{\rm in}}^{\zeta_{\rm out}} (\tilde{n}_1 - \tilde{n}_2)\,d\zeta. \tag{3.7}$$

The onset of coupling signifies that the conversion coefficient Q is nonzero, and the phase difference γ is different from the value given by (3.7). This is clear if we compare the conversion matrix F given by (3.2) with the matrix F_0 obtained by solving (3.4) in the absence of coupling when $\tilde{\Psi} = 0$:

$$F_0 = \begin{bmatrix} \exp\left(-i\int_{\zeta_{\rm in}}^{\zeta_{\rm out}} \tilde{n}_1\,d\zeta\right) & 0 \\ 0 & \exp\left(-i\int_{\zeta_{\rm in}}^{\zeta_{\rm out}} \tilde{n}_2\,d\zeta\right) \end{bmatrix}. \tag{3.8}$$

The last matrix corresponds to the geometrical optics approximation.

In the limiting case of an isotropic medium ($\tilde{n}_1 \approx \tilde{n}_2$), the equations given by (3.4) can also be readily solved, and the conversion matrix assumes the form

$$F = \begin{pmatrix} \cos\Delta\eta & \sin\Delta\eta \\ -\sin\Delta\eta & \cos\Delta\eta \end{pmatrix} \exp\left(-i\int_{\zeta_{\rm in}}^{\zeta_{\rm out}} \tilde{n}\,d\zeta\right), \tag{3.9}$$

where

$$\eta = -\int \tilde{\Psi}\,d\zeta = \frac{\pi}{4} + \frac{1}{2}\operatorname{arctg}\tilde{q}(\zeta), \tag{3.10}$$

$$\Delta\eta = \eta_{\rm out} - \eta_{\rm in} = \frac{1}{2}(\operatorname{arctg}\tilde{q}_{\rm out} - \operatorname{arctg}\tilde{q}_{\rm in}). \tag{3.11}$$

According to (3.10),

$$\tilde{q} = -\operatorname{ctg}2\eta, \quad \tilde{K}_2 \equiv -\tilde{K}_1^{-1} = \operatorname{tg}\eta, \tag{3.12}$$

i.e., the increment $\Delta\eta$ on η represents the wave polarization (2.13) along the propagation route $\zeta_{\rm in}$, $\zeta_{\rm out}$. This change determines the magnitude of the conversion coefficient in the limit of the isotropic medium: $Q = \sin^2\Delta\eta$ [see (3.9)]. The phase shift is then $\gamma = 0$ and the phase is $\varphi = 0$ or $\pm\pi$. Transforming to the field components E_x and E_y with the aid of (2.40), we can readily show that there is no change in the polarization of the resultant radiation in the layers $\zeta_{\rm in}$, $\zeta_{\rm out}$ in this case, and the matrix F given by (3.9) produces a simple conversion between two decompositions of this polarization, namely, decomposition into two orthogonal polarizations of normal waves at entry to the layer and decomposition into two other orthogonal polarizations at exit from the layer. We now adopt the convention that by normal waves in the isotropic medium we shall understand waves that are obtained by a limiting transition from the anisotropic medium and, consequently, the waves have a definite polarization [see (2.6) and (2.13)] for $\varepsilon_{ij}\to\varepsilon\delta_{ij}$.

In an anisotropic medium, $\tilde{n}_1\neq\tilde{n}_2$. However, for layers with a small geometrical optics phase difference (3.7), ($|\gamma_0|\ll 1$), the terms $\pm i/2(\tilde{n}_1-\tilde{n}_2)\delta_{1,2}$ in (3.4) can again be neglected in the limit of the isotropic medium. The conversion matrix F for such inhomogeneous layers (anisotropy "jumps") is identical with that given by (3.9).

B. Riccati equation. Strong and weak conversion

It is convenient to perform a qualitative analysis of the coupling process by using (3.4) [or (2.19)] together with the Riccati equation for the complex wave amplitudes $P=-if_1/f_2$ that follows from it:

$$\frac{dP}{d\eta}=i(P^2-1)-2iG(\eta)P. \tag{3.13}$$

The variable η is given by (3.10), and the characteristic function $G(\eta)$ has the form

$$G(\eta)=\frac{\widetilde{n}_2-\widetilde{n}_1}{2\widetilde{\Psi}}\equiv\frac{\widetilde{n}_1-\widetilde{n}_2}{2\eta'}\equiv(\widetilde{n}_1-\widetilde{n}_2)\frac{\widetilde{q}^2+1}{\widetilde{q}'}. \tag{3.14}$$

To obtain the Riccati equation given by (3.13), it is convenient to write (3.4) in the canonical form

$$\begin{aligned} d\delta_1/d\eta + iG(\eta)\,\delta_1 &= \delta_2,\\ d\delta_2/d\eta - iG(\eta)\,\delta_2 &= -\delta_1. \end{aligned} \tag{3.4a}$$

According to (2.14), the ratio P directly determines the polarization coefficient at any point along the ray:

$$K\equiv -i\frac{E_y}{E_x}=\frac{\widetilde{K}_1(\widetilde{K}_2-iP)}{\widetilde{K}_1-iP}. \tag{3.15}$$

The elements of the conversion matrix (3.2) can be expressed in terms of P in the following way. The conversion coefficient Q and phase ϕ are found by solving (3.13) under the boundary condition $P_0(\eta_{in})=0$:

$$P_0(\eta_{out})=-\frac{i\sqrt{Q}}{\sqrt{1-Q}}\exp(i\varphi) \tag{3.16}$$

(a wave of type 2 alone is incident on the layer). The validity of (3.16) becomes clear if we consider (3.2) and (3.6). In its turn, the phase shift γ is given by

$$\gamma-\int_{\eta_{in}}^{\eta_{out}}G(\eta)\,d\eta=-\operatorname{Re}\int_{\eta_{in}}^{\eta_{out}}P_0(\eta)\,d\eta, \tag{3.17}$$

where $P_0(\eta)$ is the solution of (3.13) under the same boundary condition. The formula given by (3.17) is obtained by integrating the standard substitution $P(\eta)=G(\eta)+i\delta_2^{-1}d\delta_2/d\eta$ that relates the Riccati equation (3.13) to (3.4a).

According to (3.13) and (3.4a), the entire process of variation of polarization along the ray (including the coupling region) is determined by the behavior of the characteristic function $G(\eta)$ within the range $\Delta\eta$ of the variable $\eta(\zeta)$ in the medium. For example, if, in the inhomogeneous medium, we can linearize (2.13) by taking $\tilde{q}=q_0\circ\zeta$, and the refractive index difference can be approximately represented by the quadratic expression $\tilde{n}_1-\tilde{n}_2=(\Delta n)_0\times\sqrt{1+q_0^2\zeta^2}$, then $\Delta\eta=\pi/2$, and the characteristic function is

$$G(\eta)=\frac{(\Delta n)_0}{q_0\sin^3 2\eta}. \tag{3.18}$$

It is clear that the solution of (3.13) with $G(\eta)$ given by (3.18) will describe not only wave coupling in the particular layer (see Secs. 4A and 7C), but the solution will be also valid in all other layers with the same characteristic function $G(\eta)$, independently of the specific refractive index distribution $\tilde{n}_{1,2}(\zeta)$ and the quantity $\tilde{q}(\zeta)$.

It is clear from the form of the function $\eta(\tilde{q})$ in (3.10) that, within the $\tilde{q}$ interval between $-\infty$ and $+\infty$, the variation in η will largely be confined to the region where $\tilde{q}^2\sim 1$, and the value of $\tilde{K}_{1,2}$ given by (2.13) will correspond to the elliptic polarization of the geometrical optics waves. On the contrary, for $\tilde{q}^2\gg 1$ (linear polarization), the variable η will be practically constant, remaining close to zero or $\pi/2$. For $\tilde{q}^2\ll 1$ (circular polarization), the values of η will be close to $\pi/4$.

In the limiting case of high values of the characteristic function $|G(\eta)|\gg 1$ between η_{in} and η_{out} [i.e., when $\tilde{n}_1\neq\tilde{n}_2$ and, according to (3.14), the variable η varies slowly along the ray], the approximate solution of (3.13) can be found in the form of an expansion in the small parameter $G_{min}^{-1}\ll 1$, where G_{min} is the smallest value of $|G(\eta)|$ within the interval between η_{in} and η_{out}. We then have

$$P(\eta_{out})=\left\{P(\eta_{in})-i\int_{\eta_{in}}^{\eta_{out}}\exp\left[2i\int_{\eta_{in}}^{\eta}G(\eta)\,d\eta\right]d\eta\right\}\times\exp\left[-2i\int_{\eta_{in}}^{\eta_{out}}G(\eta)\,d\eta\right]. \tag{3.19}$$

This result corresponds to the inclusion of the terms $-i$ and $-2iG(\eta)P$ (but not iP^2) in (3.13) and differs by the presence of the small corrections of the order of G_{min}^{-1} as compared with the geometrical optics solution in which only the term $-2iG(\eta)P$ is included.

The conversion coefficient is now not equal to zero, but remains small in comparison with unity (weak coupling):

$$Q\approx\left|\int_{\eta_{in}}^{\eta_{out}}\exp\left[2i\int_{\eta_{in}}^{\eta}G(\eta)\,d\eta\right]d\eta\right|^2\leqslant G_{min}^{-2}\ll 1. \tag{3.20}$$

The phase shift γ is not very different from the geometrical optics value γ_0 given by (3.7). The behavior of the phase ϕ of the created wave is more complicated. In simple cases, the new wave arises almost in quadrature with the incident wave: $\varphi\approx\pm\pi/2$ or $\varphi\approx -2\gamma_0\pm\pi/2$ [see (3.19)].

According to (3.16), the coupling is weak when $|P_0^2|\ll 1$ throughout. In this case, for any function $G(\eta)$ we can neglect the term iP^2 in (3.13), and obtain an expression identical with (3.19). Consequently, it describes weak conversion even for $|G(\eta)|\lesssim 1$. Moreover, if we assume approximately that, even for $|G(\eta)|\leq 1$, the phase of the function P is determined by an expression of the form given by (3.19), the solution of (3.13) for $|P|$ leads to a conversion coefficient whose form is similar to that of (3.20):

$$Q=\sin^2\left|\int_{\eta_{in}}^{\eta_{out}}\exp\left[2i\int_{\eta_{in}}^{\eta}G(\eta)\,dn\right]d\eta\right|. \tag{3.21}$$

Judging by special cases,[72] the formula given by (3.21) will also describe approximately both effective and strong conversion. This is characteristic for cases where phase discrepancies are small as compared with geometrical optics. The situation is typical for transitional layers because, for such layers, $|\gamma-\gamma_0| \lesssim (\Delta\eta/2)^2$ [see (3.7), (3.17), and (3.26)].

In the limiting case where $|G(\eta)| \ll 1$ ($\tilde{n}_1$ and $\tilde{n}_2$ are then very close to one another and η continues to vary along the ray), it is also possible to find explicit expressions for Q, ϕ, and γ. This can be done by substituting Q, which converts (3.13) with the function $G(\eta)$ into the analogous Riccati equation with $\tilde{G}(\eta) = 1/G(\eta)$, and then again using a solution such as (3.19). It is readily verified that this solution yields small corrections to Q, ϕ, and γ, obtained earlier in the limit of an isotropic medium (3.9) (when $G(\eta) = 0$). In particular,

$$Q \approx \sin^2 \Delta\eta \cdot [1 - O(G^2)], \tag{3.22}$$

$|\gamma-\gamma_0| \ll 1$, and the created wave is almost in phase ($\varphi \approx 0$) or in antiphase ($\varphi \approx \pm\pi$) with the incident wave. When $|\Delta\eta| \sim 1$ in the layer, this case corresponds to strong coupling.

According to (3.22), a small change in η along the ray (as compared with unity) leads to the appearance of only weak coupling: $Q \ll 1$. The foregoing is valid not only for $|G(\eta)| \ll 1$, but also for inhomogeneous layers with $|G(\eta)| \lesssim 1$.[10)] When $|G(\eta)| \gg 1$, we have $Q \ll 1$ for any increments $\Delta\eta$ along the ray [see (3.20)].

Summarizing the above discussion, we may conclude that effective coupling with conversion coefficient $Q \sim 1/2$ can occur only in inhomogeneous layers in which the characteristic function $G(\eta)$ falls to values $|G(\eta)| \lesssim 1$ within the interval $|\Delta\eta| \sim 1$. This conclusion can be confirmed by analyzing the phase structure of the Riccati equation given by (3.13).

Figure 2 shows a few characteristic examples of the behavior of the function $G(\eta)$. Curve 1 refers to the layer described by (3.18) and curve 2 represents the limiting polarization effect, when the radiation leaves the anisotropic medium ($\tilde{n}_1 \neq \tilde{n}_2$) and enters the isotropic medium ($\tilde{n}_1 = \tilde{n}_2$). Here the coupling is strong with conversion coefficient $Q_{max} = \sin^2\Delta\eta$ when $|\Delta\eta| \sim 1$, and the function $G(\eta)$ falls sufficiently rapidly from the values $|G(\eta)| \gg 1$ to values $|G(\eta)| \ll 1$. The result of this is that a large proportion of the increment $\Delta\eta$ and, consequently, the main change in the polarization coefficients of the geometrical optics waves occur for $|G(\eta)| \ll 1$. The same function $G(\eta)$ obtains over the transition from the homogeneous anisotropic medium to the highly inhomogeneous anisotropic medium for which $\tilde{n}_1 \neq \tilde{n}_2$ and $G(\eta)$ tends to zero with increasing η because of the high derivative $\tilde{q}'$ [see (3.14)].

10)The solution of (3.13) [or (3.4a)] is conveniently sought over a small range of values of η in terms of a series in $\Delta\eta$. This yields $Q \approx (\Delta\eta)^2 \ll 1$. In particular, the coupling effect turns out to be weak when the polarization of the geometrical optics waves remains either linear or circular along the beam: it is then clear from (3.10)–(3.12) that the increment $\Delta\eta$ is small along the beam.

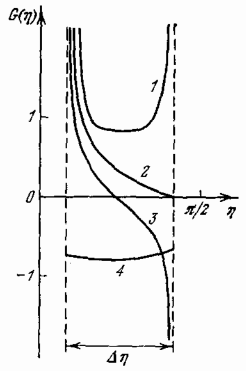

FIG. 2. Simple forms of the function $G(\eta)$ in transitional layers: 1—layer with constant sign of $\tilde{n}_1 - \tilde{n}_2$; 2—transitional layer between anisotropic and isotropic media; 3—layer with a change of sign of $\tilde{n}_1 - \tilde{n}_2$; 4—layer with $G(\eta) \approx \text{const} \sim 1$.

At this point, it is useful to note the following symmetry properties: as the curve representing $G(\eta)$ changes sides, so does the conversion matrix, and when the curve is symmetric (for example, curve 1 in Fig. 2), the conversion matrices for the two halves differ by the complex conjugation of the off-diagonal elements. All these conclusions can be readily verified by considering the set of coupling equations given by (3.4a).

The foregoing results can also be deduced from the Riccati equation for the polarization coefficient of the resultant radiation:

$$K' = i\,\frac{\tilde{n}_2 - \tilde{n}_1}{\tilde{K}_2 - \tilde{K}_1}(K - \tilde{K}_1)(K - \tilde{K}_2). \tag{3.23}$$

This follows from (3.13) after we substitute (3.15), but can also be written at once, as in a homogeneous medium, because the derivatives of the parameters of the medium do not appear in this equation (cf. the derivation of the transfer equations in Sec. 2E). The last feature enables us to extend this equation by including absorption, dichroism, nonorthogonality of normal waves, and strong rotation of the optical axes of the medium, because these factors are included in the coefficients $\tilde{n}_{1,2}$ and $\tilde{K}_{1,2}$. The corresponding generalization of the equation given by (3.13) for the amplitudes of the coupled waves can be obtained by employing (3.15) in the reverse direction.

C. Coupling parameter in transitional layers

The propagation path of electromagnetic waves in a smoothly inhomogeneous medium usually passes through regions with linear and circular polarizations of the geometrical optics waves and transitional regions between them. It is shown above that wave coupling occurs only in these transitional regions. For one transitional region, $\Delta\eta = \pm\pi/4$, and, for two, $\Delta\eta = \pm\pi/2$. The coupling is weaker when $|G(\eta)| \gg 1$ along the entire ray. It remains weak when the function $G(\eta)$ falls to values $|G(\eta)| \lesssim 1$ on a small interval $|\Delta\eta| \ll 1$, but retains high absolute values everywhere else along the ray. Conversely, the coupling becomes effective ($Q \sim 1/2$) when $|G(\eta)| \lesssim 1$ over a wide interval $|\Delta\eta| \sim 1$ (Fig. 2). Finally, strong coupling occurs when $|G(\eta)| \ll 1$ over an interval $|\Delta\eta|\sim1$. When $\Delta\eta=\pm\pi/4$ and $\Delta\eta=\pm\pi/2$, the conversion coefficient approaches its maximum values (1/2 and 1, respectively).

Consider a transitional layer characterized by a monotonic variation of η (and of $\tilde q$) and lying between two neighboring regions in which the geometrical optics approximation is valid. It is clear from the foregoing that, when the variation of $G(\eta)$ over most of the interval $\Delta\eta$ does not exceed the magnitude of the function itself, the effectiveness of the conversion in the transitional layer can be characterized by the parameter $G=G(\eta_0)$, where η_0 is a fixed point (for example, the midpoint) in the interval $\Delta\eta$. In the case of a transitional layer with $|\Delta\eta|\sim1$, the interval $\Delta\eta$ must include the region where $\tilde q^2\sim1$, and the coupling parameter can be defined by[11]

$$G=|G(\eta)|_{\tilde q^2\sim1}=\left|\frac{\tilde n_2-\tilde n_1}{2\tilde\Psi}\right|_{\tilde q^2\sim1}. \tag{3.24}$$

Depending on the sign of the ellipticity parameter $\tilde q$, the value $\tilde q^2=1$ corresponds to $\eta=3\pi/8$ and $\eta=\pi/8$, respectively [see (3.10)].

When the medium includes a transitional region in which there is an appreciable change in the polarization $\tilde K_{1,2}$ (2.13) of the geometrical optics waves (i.e. $|\Delta\eta|\sim1$ along the ray) and the relative change in the characteristic function $G(\eta)$ is less than or comparable with unity within this region, the values of the coupling parameter enable us to judge the degree of wave conversion. In particular,

$$\begin{aligned} &Q\ll1 && \text{for } G\gg1,\\ &Q\sim\tfrac{1}{2}Q_{\max} && \text{for } G\sim1,\\ &Q\approx Q_{\max} && \text{for } G\ll1, \end{aligned} \tag{3.25}$$

where $Q_{\max}=\sin^2\Delta\eta$. The function $Q(G)$ is shown schematically in Fig. 3. When $G\gg1$, the formula given by (3.20) can be employed and shows that $Q(G)$ can exhibit oscillations because, in this case of weak conversion, the coupling region may not be localized and phase effects are found to appear. However, the most interesting and nontrivial case is that of effective coupling ($Q\sim1/2$ for $G\sim1$, $|\Delta\eta|\sim1$). More sophisticated methods (see Sec. 3E) must then be used to obtain the analytic form of $Q(G)$.

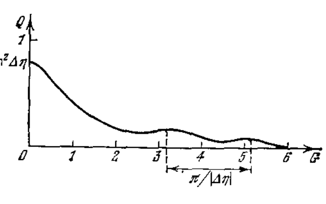

FIG. 3. Characteristic dependence of conversion coefficient Q on the coupling parameter G. A similar curve has been obtained numerically for $\Delta\eta=\pi/2$ in the case of a neutral current layer in plasma.[74]

[11]We note that, in Cohen's paper,[73] the "coupling parameter" was defined to be the function $G(\eta)$ and not the quantity G given by (3.24) and introduced previously in Refs. 3, 33, and 52.

We emphasize once again that the relations given by (3.25) with the conversion parameter (3.24) are valid only for a small increment $|\Delta\eta|\sim1$ in the transitional layer. The magnitude of Q is then 0.5–1. The quantity Q decreases when the increment $\Delta\eta$ is smaller. However, in any case, Eq. (3.22) shows that the maximum conversion coefficient is $Q_{\max}=\sin^2\Delta\eta$ and is reached when $|G(\eta)|\ll1$ and, in particular, $G\ll1$, in the transitional layer.

Next, we note that, according to (3.14) and (3.10),

$$2\int_{\eta_{\rm in}}^{\eta_{\rm out}}G(\eta)\,d\eta=\int_{\zeta_{\rm in}}^{\zeta_{\rm out}}(\tilde n_1-\tilde n_2)\,d\zeta. \tag{3.26}$$

Hence, it follows that, when in the transitional layer we have along the ray segment $\zeta_{\rm in}$, $\zeta_{\rm out}$ coincident with the region of effective coupling ($|\Delta\eta|\sim1$, $|G(\eta)|\lesssim1$), the order of magnitude of the absolute value of $\int_{\zeta_{\rm in}}^{\zeta_{\rm out}}(\tilde n_1-\tilde n_2)d\zeta$ does not exceed unity. Since the condition $\int(\tilde n_1-\tilde n_2)d\zeta\sim1$ determines the spatial period of beats between the geometrical optics waves, it is clear that effective coupling occurs when an appreciable change in the polarization of these waves occurs within this period (i.e., when the period of the beats occupies the interval $|\Delta\eta|\sim1$). It is, therefore, not surprising that, as can readily be verified, the coupling parameter (3.24) is of the order of the ratio of the inhomogeneity scale $\Lambda_{\tilde q}$ of $\tilde q$ at the point $\tilde q^2\sim1$ to the period of beats between the waves $2\pi/k_0|\tilde n_1-\tilde n_2|$ at the same point:

$$G=|2(\tilde n_1-\tilde n_2)k_0\Lambda_{\tilde q}|_{\tilde q^2\sim1}. \tag{3.27}$$

In other words, coupling occurs when the polarization of the resultant radiation does not succeed in following the variation in the polarizations $\tilde K_{1,2}$ of the geometrical optics waves. This conclusion is most readily understood if we turn to (3.23), which gives the polarization K, and if we consider the "Faraday" integral $\int(\tilde n_1-\tilde n_2)d\zeta$ as a new independent variable. Over the interval within which $\int(\tilde n_1-\tilde n_2)d\zeta\ll1$, the increment on the unknown function K in (3.23) is also small and, consequently, the polarization of the resultant radiation remains practically constant, whereas the polarizations of the geometrical optics waves $\tilde K_{1,2}$, which are determined by the local properties of the medium, can undergo a substantial change (in an inhomogeneous medium). However, the change in the polarization of the resultant radiation occurs over an interval on which $\int(\tilde n_1-\tilde n_2)d\zeta\gtrsim1$ (of course, subject to the condition $K\neq\tilde K_{1,2}$ which, by the way, is unimportant in the presence of coupling). This follows from (3.23) which shows that the characteristic scale of the change in the polarization, determined by the ratio $|K/K'|$ for $|K|\sim1$, is equal to the period of beats between the waves $2\pi/k_0|\tilde n_1-\tilde n_2|$, but is independent of the derivatives $\tilde K'_{1,2}$, $\tilde n'_{1,2}$. The result is that, as soon as the inhomogeneity scale $\Lambda_{\tilde q}=|\tilde q/\tilde q'|_{\tilde q^2\sim1}$ for the geometrical optics waves (2.13) becomes less than the period of the beats, the situation cannot be described by the geometrical optics approximation, and we have wave coupling.

The significance of the transformation of the original field equations given by (1.1) and (2.3) or (3.23) into the canonical coupling equations (3.4a) or (3.13) should now be clear. Information on the coupling phenomenon is implicit in the original equations and appears in the form of different functional characteristics of the inhomogeneous layer, for example, $\tilde{n}_{1,2}(\zeta)$ and $\tilde{q}(\zeta)$. On the contrary, in the canonical coupling equations, all the information about the conversion effect is borne by the single function $G(\eta)$, given by (3.14), which is thus the differential characteristic of the inhomogeneous layer. The function $G(\eta)$ explicitly contains the derivative $\tilde{q}'$ (it determines the spatial rate of change in the polarization of the geometrical optics waves). This enables us to introduce the local spatial "rate" of the coupling process and, in particular, to introduce the coupling parameter as the average of this "rate" in the transitional region in which coupling occurs.

When there are two or more transitional layers in which coupling occurs, the phase characteristics of wave conversion become of fundamental importance. In fact, multiplying together the conversion matrices (3.3) for two such layers (with parameters Q_I, φ_I, γ_I and Q_{II}, φ_{II}, γ_{II}, respectively), and bearing in mind the phase difference $\delta = 1/2k_0 \int(\tilde{n}_1 - \tilde{n}_2)d\zeta$ acquired by the waves in the gap between the layers, we obtain

$$Q=[\sqrt{Q_I(1-Q_{II})}+\sqrt{Q_{II}(1-Q_I)}]^2 -4\sqrt{Q_IQ_{II}(1-Q_I)(1-Q_{II})}\sin^2\left[\delta+\gamma_{II}+\frac{1}{2}(\varphi_{II}-\varphi_I)\right]. \quad (3.28)$$

In the case of two identical transitional regions forming an inhomogeneous layer that is symmetric with respect to the point $\zeta = 0$, we can rewrite (3.28) in the form (see Secs. 5B, 6A, and Refs. 50 and 169):

$$Q=\sin^2 s'\cdot \mathrm{ch}^{-2} s'', \quad (3.28a)$$

where

$$2s'=2\delta+2\gamma_{II}+\varphi_{II}-\varphi_I+\pi,\ s''=\mathrm{archch}\,[4Q_I(1-Q_I)]^{-1/2},\ Q_I=Q_{II}.$$

It is clear that complete conversion depends on wave interference, and conversion effects in the two neighboring layers can cancel out. On the other hand, when we have a periodic sequence of transitional layers, with weak coupling in each of them, complete conversion may be substantially amplified as a result of resonance interference between the waves.

D. Qualititative analysis of coupling between waves propagating in opposite directions

The coupling equations for the waves given by (2.18) can be reduced by the substitution $F=(f_1-if_3)/\sqrt{n}$ to the canonical wave equation $F''+n^2F=0$. The latter is investigated in a large number of papers (see Refs. 7–10, 26–28, 51, 75, 92, etc.). Here, we merely note the analogy between coupling phenomena for waves propagating in the same and in opposite directions. In accordance with (2.18), we confine our attention to above-barrier reflection ($n^2>0$). It follows from (2.18) that the reflection coefficient $\mathscr{P}=-if_3/f_1$ is a solution of the Riccati equation

$$\frac{d\mathscr{P}}{d\theta}=\mathscr{G}^2-1-2i\mathscr{G}(\theta)\mathscr{P}, \quad (3.29)$$

where $\theta=-\int[n'/(2n)]d\zeta=-\ln\sqrt{n}$, $\mathscr{G}(\theta)=2n^2/n'$. It differs formally from the Riccati equation for waves propagating in the same direction (3.13) merely by the replacements $\theta\to i\eta$, $\mathscr{G}(\theta)\to -iG(\eta)$ and the fact that, to determine the conversion coefficient we must seek the solution of (3.29) subject to the boundary condition $\mathscr{P}(\zeta_{out})=0$, and the solution of (3.13) subject to the boundary condition $P(\zeta_{in})=0$. It is therefore a relatively simple matter to extend the analysis given above to the case of coupling between two waves propagating in opposite directions. We need only allow for the fact that, when the interval of variation of the variable η, given by (3.10) in the case of waves propagating in the same direction, is restricted to the segment $[0,\pi/2]$, the interval of variation of η in the case of waves propagating in opposite directions is unlimited, i.e., $(-\infty;+\infty)$. (We note, by the way, that in the case of waves propagating in the same direction, the presence of absorption will also ensure that the variable η and the function $G(\eta)$ become complex, and the interval $\Delta\eta$ may become unlimited.) In particular, the maximum conversion coefficient is given by

$$Q_{max}=|\sin^2\Delta\eta|\,(|\sin^2\Delta\eta|+|\cos^2\Delta\eta|)^{-1} \quad (3.30)$$

with $\Delta\eta$ or $i\Delta\theta$ for waves propagating in the same and in opposite directions, respectively.

For a transitional layer in which $n(\zeta)$ is a monotonic function, effective reflection with conversion (reflection) coefficient $Q=|\mathscr{P}|^2/(|\mathscr{P}|^2+1)\sim 1/2$ can occur only in a layer in which the characteristic function $\mathscr{G}(\theta)$ falls to values $|\mathscr{G}(\theta)|\lesssim 1$ on $|\Delta\theta|\gtrsim 1$. Proceeding by analogy, we may introduce a coupling parameter $\mathscr{G}$ governing the conversion efficiency in a transitional layer n_{in}, n_{out}, and consider the dependence of the reflection coefficient and the phases of the waves propagating in opposite directions on the coupling parameter. When phase effects are relatively unimportant, we may again[5] use the approximate formula given by (3.21) with the replacement $\eta\to -i\theta$, $G(\eta)\to i\mathscr{G}(\theta)$ and $\sin\to\mathrm{sh}$. Phase effects and interference between the coupled waves propagating in the same direction (see Sec. 3C) become important in the case of reflection from a sequence of transitional layers.

E. Methods of solving linear coupling equations

Qualitative analysis of wave coupling enables us to identify regions of effective conversion and to estimate the size of the effect directly from the form of the inhomogeneity in the medium without resorting to the solution of the equations. Various approximate methods of solving the coupling equations are being used to obtain more detailed information. They include different variants of perturbation theory,[1,7,28,34,49,72,77,164] modifications of the WKB method,[26,78,79] the method of phase integrals, the method of undetermined Tsvan multipliers),[3,9,25,50] the method of the comparison equation (linear associated equation),[28,50] numerical methods,[64,74,80-87] the method of standard problems,[7,30,36-39,47,52,71,75,76,88-91,93] and so on.

The advances achieved in recent years in the theory of wave coupling are, in many respects, due to the application of the method of standard problems, i.e., the derivation of exact solutions for a sufficiently broad class of inhomogeneous layers and the comparative analysis of their properties. The principle of this method can be conveniently illustrated by considering the equation[52]

$$\frac{d^2\delta_2}{d\eta^2}+I(\eta)\,\delta_2=0,\quad I(\eta)=1-\frac{i\,dG(\eta)}{d\eta}+G^2(\eta), \tag{3.31}$$

which follows from (3.4a). Let us choose the characteristic function $G(\eta)$ so that its form corresponds to the particular type of coupling (see Sec. 3B and Fig. 2), and so that Eq. (3.31) reduces to one of the equations that we have investigated, whose solutions can be expressed in terms of known functions. According to (3.14), this choice imposes one differential relationship on all the functions defining the inhomogeneous profiles of the parameters of the medium. By integrating this relationship we obtain a class of functions corresponding to different specific realizations of the homogeneous layer (dependence of the parameters of the medium on position along the ray). Wave coupling in each of these realizations is described by the conversion matrix which we can find from the known asymptotic behavior of the special functions of the chosen standard problem. Thus, by working "outwards," i.e., by starting with the chosen standard function $G(\eta)$, we reduce the initial problem to other problems which, generally speaking, are simpler. These are: determination of the conversion matrix with the aid of (3.13), and determination of the set of realizations of inhomogeneous layers in which conversion is described by the chosen standard problem, i.e., with the aid of (3.14). Both these problems can be readily solved, and this will be demonstrated below. On the other hand, if we start by specifying a particular profile of the inhomogeneous layer, the wave coupling problem will often lead us to a complicated differential equation, whose solution is unknown. Nevertheless, the nature of the conversion effects can still then be identified by finding the function $G(\eta)$ and exploiting the qualitative analysis together with known standard problems with characteristic functions $G(\eta)$ of the same kind. Moreover, when the true function $G(\eta)$ is not very different from the standard function, we can readily find corrections to the standard conversion matrix by using perturbation theory.

4. WAVE CONVERSION IN MAGNETOACTIVE PLASMA

Plasma situated in an inhomogeneous magnetic field $\mathbf{B}_0(\mathbf{r})$ is an important and the most extensively studied example of a gyrotropic medium in which the conversion of electromagnetic waves is possible. In a smoothly inhomogeneous magnetoactive plasma, the coupling between ordinary and extraordinary waves[12)] occurs in regions where the refractive indices

[12)]We are concerned with high-frequency waves in cold plasmas. Analysis of coupling between low-frequency waves propagating, for example, in the Earth's ionosphere, can be performed in an analogous manner.[39,71,72] Moreover, we neglect spatial dispersion, the presence of which in "hot" plasma leads to coupling between plasma, Bernstein, cyclotron, and other waves,[1,3,11-13,17,18,94-97] and so on. This range of problems demands at least a separate review, and will not be examined here.

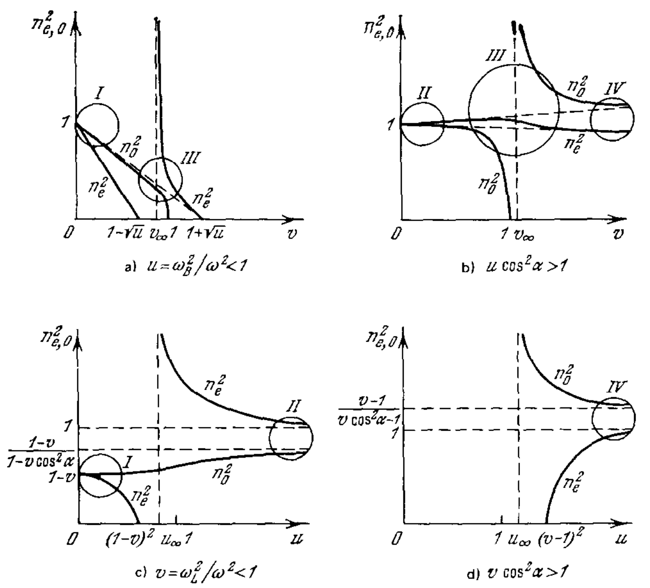

FIG. 4. Refractive indices for normal waves in magnetoactive plasma as functions of electron density v and magnetic field u (schematic). Circles show wave coupling regions.

$$n_{1,2}^2=1-v+v\sqrt{u}\cos\alpha\,(K_{1,2}+\sqrt{u}\cos\alpha)^{-1} \tag{4.1}$$

approach one another (see Fig. 4), and is connected with variation in polarization coefficients

$$K_{1,2}=q\mp\sqrt{q^2+1},\quad q=\frac{\sqrt{u}\sin^2\alpha}{2(1-v)\cos\alpha}, \tag{4.2}$$

where we have used (2.6) and the identity $\varepsilon_{xy}^{-1}\equiv-\varepsilon_{yx}^{-1}$ for a plasma with the magnetic field $\mathbf{B}_0$ lying in the y, z plane (Fig. 1); α is the angle between $\mathbf{B}_0$ and the direction of propagation (the z axis), $u=\omega_B^2/\omega_2$, $v=\omega_L^2/\omega^2$, $\omega_B=eB_0/mc$ is the electron gyrofrequency, $\omega_L=(4\pi e^2N/m)^{1/2}$ is the plasma frequency, and N is the electron density of the plasma (e and m are the electron charge and mass, respectively).

It is clear from (4.2) that the variation in the polarization of ordinary and extraordinary waves along the ray and their coupling occur because of the inhomogeneity of the magnetic field $\mathbf{B}_0(\mathbf{r})$, both in magnitude and direction, which results from the inhomogeneous distribution of electron density. Conversion is then most effective in a weak quasitransverse or a strong quasilongitudinal magnetic field:[33,52]

$$\sqrt{u}\ll 1-v,\quad \cos^2\alpha\ll 1,\quad v<1; \tag{4.3}$$

$$u\gg 1,\qquad \sin^2\alpha\ll 1,\qquad v\ll\sqrt{u}. \tag{4.4}$$

The coupling parameter (3.24) is substantially reduced in these regions as compared with the value $G\sim k_0 v\Lambda_q$ that is characteristic for plasmas with $\alpha\sim u\sim|1-u|\sim 1$ and $v\lesssim 1-v$. It follows that, when conditions (4.3) and (4.4) are satisfied, effective coupling is possible in denser and more homogeneous plasma. The inhomogeneity in plasma density turns out to be important when we consider propagation along a strong magnetic field in post-critical plasma ($1\ll v\ll\sqrt{u}$, region IV in Fig. 4), or in the neighborhood of plasma resonance ($v\sim 1$, region III in Fig. 4). The last case is complicated by the vanishing of the function $n^2(\zeta)$ at the point $v = 1$ and the singularity of this function at $v = v_\infty$. We shall not examine this in the present paper (see Refs. 1, 3, and 14).

A. Weak quasitransverse magnetic field

When the inhomogeneity scale Λ_q is largely determined by the change in the direction of the magnetic field along the ray, the coupling parameter G in (3.24) in the case of (4.3) is given by[3,13,52,73]

$$G_\perp = \frac{\sqrt{2}\, k_0 v u}{(1-v)^{3/2}} \left| \frac{\cos\alpha}{d\cos\alpha/dl} \right|_{q^2\sim 1}. \tag{4.5}$$

According to Sec. 3C, in a transitional layer from quasilongitudinal propagation ($q_{\rm in}^2 \to 0$) to quasiperiodic propagation ($q_{\rm out}^2 \to \infty$), or the other way round, the maximum conversion coefficient is 1/2 and this is reached for $G_\perp \ll 1$. In that case, an incident circularly polarized wave of a given type will traverse the inhomogeneous region Δ_q with practically no change in polarization. Consequently, the region of quasitransverse propagation in which normal waves are linearly polarized will be entered by coherent ordinary and extraordinary waves of equal intensity, and their superposition will ensure circular polarization of the resultant equation. As $G_\perp$ increases and the increment $\Delta\eta$ decreases for $q_{\rm in}^2 \neq 0$, $q_{\rm out}^2 \neq \infty$, the conversion efficiency will fall (see Sec. 3C and Fig. 3).

When the waves successively traverse two transitional layers with $q^2 \sim 1$, the conversion of ordinary and extraordinary waves may become complete: $Q \approx 1$ for $G_\perp \ll 1$. This occurs when the waves traverse a region of transverse magnetic field in which its longitudinal component in the direction of propagation changes sign (Fig. 5a). When $\cos\alpha = l/\Lambda$, and, u, ψ' = const, we then have [compare this with (3.18)]

$$G(\eta) = -\frac{v u^{3/2}}{4(1-v)^{5/2}} k_0 \Lambda \cos^{-3} 2\eta, \tag{4.6}$$

and Eq. (3.31) has an exact solution that can be expressed in terms of parabolic cylinder functions.[38,50,93,165] The conversion coefficient is given by[38]

$$Q = \exp\left(-\frac{\pi G_\perp}{4\sqrt{2}}\right) = \exp\left(-\frac{\omega_\perp^4}{\omega^4}\right), \tag{4.7}$$
$$\omega_\perp = (\pi \omega_L^2 \omega_B^3 \Lambda / 8c)^{1/4}$$

(this result was originally obtained by the method of phase integrals[3,99]). The expression for Q in terms of $\omega_\perp$ in (4.7) is written for $v \ll 1$, in which case, $G_\perp \propto \omega^{-4}$ [see (4.5)]. The critical frequency $\omega_\perp$ is introduced on the basis of the condition $\pi G_\perp/4\sqrt{2} = 1$.

At low frequencies for which $\omega \ll \omega_\perp$, where $G_\perp \gg 1$, the waves propagate independently, following their dispersion curves (Fig. 5a) and reversing the sign of rotation in accordance with the change in the sign of the longitudinal component of the magnetic field. As the frequency increases, the geometrical optics approximation breaks down and, when $\omega \gg \omega_\perp$ and $G_\perp \ll 1$, the radiation does not "feel" the change in the magnetic field, and retains the sign of its circular polarization just as for propagation in a vacuum. Finally, when $G_\perp \sim 1$, we have the intermediate case in which the incidence of a wave of a given type on the coupling region results in the emergence of two coherent waves from the region, namely, the ordinary and the extraordinary waves. The ratio of their intensities is $Q/(1-Q)$, and depends on frequency [see (4.7)]. When $\omega \sim \omega_\perp$, this ratio becomes equal to unity, which corresponds to linear polarization. The frequency $\omega_\perp$ of reversal of the sign of the polarization (or the value of the coupling parameter $G_\perp$ at a given frequency) is a convenient source of information about local plasma parameters [magnetic fields in the solar corona[3,84,85,100] (Fig. 6a), electron density in ionospheric or laboratory plasma,[38] and plasma in the inner magnetosphere of Jupiter[83,101]].

As an example, we recall observations of the reversal of the sign of polarization of the slowly-varying component and microwave bursts of solar radiofrequency emission at about 3 GHz (see Refs. 33 and 100, and the references therein). This effect can be explained by assuming that the emitted radiation passes a coronal region of a quasitransverse magnetic field in which $\omega_\perp/2\pi \sim 3$ GHz. It can be used in (4.7) to show that the magnetic field in the coupling region is $B_0 \sim 5$ Oe if we suppose that $N \sim 3 \cdot 10^8$ cm^{-3} and $\Lambda \sim 10^{10}$ cm; the defini-

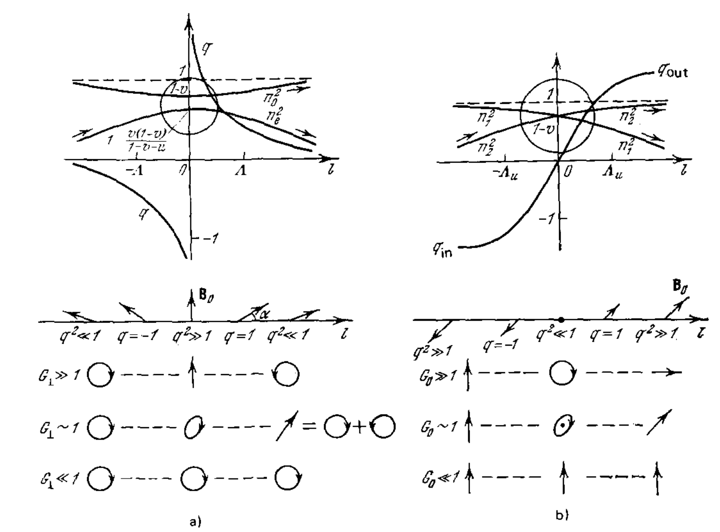

FIG. 5. Behavior of dispersion branches and wave polarizations in the case of propagation through a quasitransverse magnetic field (a) and neutral current layer (b).

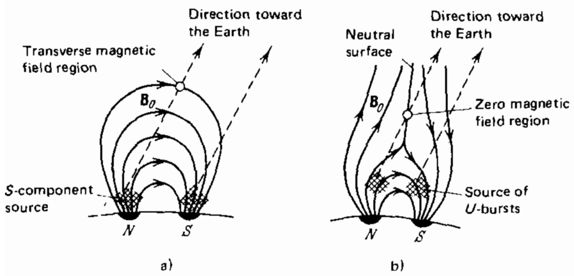

FIG. 6. Emergence of solar radiofrequency radiation from an active region above a bipolar sunspot group through a transverse magnetic field (a) and zero magnetic field (b).

tion of $\omega_\perp$ then shows that this estimate is relatively insensitive to the parameters N, Λ, but is much more sensitive to the frequency $\omega_\perp$.

Wave conversion is also found to occur when a region of zero magnetic field is traversed, in which the main contribution to the scale Λ_q is provided by the change in the magnitude of the field and the corresponding coupling parameter[33,52,74]

$$G_0 = 4\sqrt{2}\, k_0 v \sqrt{1-v}\cos^2\alpha \left| \frac{\sqrt{u}}{\mathrm{d}\sqrt{u}/\mathrm{d}l} \right|_{q^2\sim 1} \leqslant 1. \tag{4.8}$$

This situation occurs in neutral current layers in plasmas (see Fig. 5b), which play an important role in the explanation of the origin of magnetic substorms and solar flares. The phenomenon of wave coupling can be used as a basis for the diagnostics of current layers[33,74,100] (Fig. 6b): by measuring the conversion coefficient Q and knowing the function $Q(G)$ in a current layer (it is calculated in Ref. 74), we can determine the coupling parameter G and, consequently, establish the connection between the layer parameters ω_L, Λ_u and $\cos\alpha$.

B. Strong quasilongitudinal magnetic field

In a strong quasilongitudinal magnetic field, the coupling parameters associated with the variation in the magnetic field in both direction ($\alpha = \alpha(\zeta)$) and magnitude ($u = u(\zeta)$) are respectively given by

$$\begin{aligned} G_\parallel &= \frac{\sqrt{2}k_0 v}{\sqrt{u}} \left| \frac{\sin\alpha}{\mathrm{d}\sin\alpha/\mathrm{d}l} \right|_{q^2\sim 1}, \\ G_\infty &= \frac{\sqrt{2}\, k_0 v \sin^2\alpha}{|1-v|} \left| \frac{\sqrt{u}}{\mathrm{d}\sqrt{u}/\mathrm{d}l} \right|_{q^2\sim 1}, \end{aligned} \tag{4.9}$$

where, in accordance with (4.4), the subscripts "parallel" and "infinite" indicate that $\sin^2\alpha \ll 1$ and $u \gg 1$, respectively. Let us illustrate the situation by considering the Epstein-type transitional layer (see Ref. 7):

$$q(\sigma) = \frac{q_{\mathrm{in}} - q_{\mathrm{out}}}{1+\exp(4\sqrt{2}\sigma/G)} + q_{\mathrm{out}}, \quad \sigma = \int \frac{n_1 - n_2}{2\sqrt{q^2+1}}\,\mathrm{d}\zeta, \tag{4.10}$$

where $q_{\mathrm{in}} > q_{\mathrm{out}} > 0$ and $G = G_\parallel$ or $G = G_\infty$. The corresponding function (3.14) is

$$G(\eta) = -\frac{G}{2\sqrt{2}} \frac{(q_{\mathrm{in}} - q_{\mathrm{out}})\,(1+\mathrm{ctg}^2\, 2\eta)^{3/2}}{(q_{\mathrm{in}} + \mathrm{ctg}\, 2\eta)\,(q_{\mathrm{out}} + \mathrm{ctg}\, 2\eta)} \tag{4.11}$$

and is illustrated by curve 1 in Fig. 2. The solution of (3.31) in the case where (4.11) is valid takes the form of hypergeometric functions and is analyzed in Ref. 52. It turns out that the conversion efficiency increases monotonically with decreasing G and with increasing difference $q_{\mathrm{in}} - q_{\mathrm{out}}$ across the layer [i.e., with increasing increment $\Delta\eta$, given by (3.11)]. When the layer contains a transition from quasitransverse propagation ($q_{\mathrm{in}} \to \infty$) to quasilongitudinal propagation ($q_{\mathrm{out}} \to 0$), we have

$$Q = \left[1 + \exp\left(\frac{\pi G}{2\sqrt{2}}\right)\right]^{-1} \tag{4.12}$$

In complete correspondence with the results of the qualitative analysis, the conversion coefficient is $Q \ll 1$ for $G \gg 1$ and $Q \sim 1/2$ for $G \lesssim 1$. The coupling determined by the parameters $G_\parallel$ and G_∞ has not been previously considered although it is interesting, for example, in the investigation of semiconductor plasma in a strong magnetic field, or in the analysis of the polarization characteristics of the radioemission escaping from the ionosphere of Jupiter and from the magnetospheres of neutron stars.

C. Conversion effects in the "plasma + magnetized vacuum" system

In strong magnetic fields of up to 5×10^{12} Oe, the immediate neighborhood of neutron stars—x-ray pulsars—must be looked upon as a medium consisting of tenuous plasma ($v \ll 1$) and magnetized vacuum.[102-104] In a medium of this kind,

$$q = \frac{\sqrt{u}\sin^2\alpha}{2\cos\alpha}\left[1 - \frac{3a(u-1)}{vu}\right], \tag{4.13}$$

where

$$a = \left(\frac{1}{45\pi}\right)\left(\frac{e^2}{\hbar c}\right)\left(\frac{B_0}{B_{\mathrm{cr}}}\right)^2 \ll 1, \quad B_{\mathrm{cr}} = \frac{m^2c^3}{e\hbar} = 4{,}4\cdot10^{13}\ \mathrm{Oe}.$$

The region where

$$\left|1 - \frac{1}{3} vu/a(u-1)\right| \ll 1 \tag{4.14}$$

is the most interesting. In this region, the influence of plasma on the polarization of normal waves in the x-ray range is comparable with the influence of the magnetized vacuum, and the variation in q can be assured by the inhomogeneity in plasma density. Wave conversion is possible under these conditions.[102]

When the condition $\sqrt{u}\sin^2\alpha \gg 2|\cos\alpha|$ for quasitransverse propagation is satisfied in the plasma, the polarization of waves will be nearly linear. Suppose that the plasma density decreases in the direction of propagation of the waves. According to (4.13), the quantity q will then decrease from values $|q| \gg 1$ down to zero. It will then change sign, and its absolute magnitude will again become large. In accordance with the foregoing, the polarization of the wave will at first be determined by the plasma and will be nearly linear with the vector $\mathbf{E}$ lying along the x axis for the extraordinary wave, and the vector $\mathbf{E}$ lying along the y axis for the ordinary wave. In the "cancellation" region, which includes the layer where $1-(1/3)vu/a(u-1) = 0$, the polarization will be nearly circular. Finally, in tenuous plasma with low values of v, the polarization of the waves will be determined exclusively by the magnetized vacuum, and will again be nearly linear. However, the extraordinary wave will then be polarized along the y

axis, and the ordinary wave along the x axis. In the coupling region, where $q^2 \sim 1$, the corresponding coupling parameter is given by[104]

$$G_v = 12\sqrt{2}\, k_0 a u^{-1} \operatorname{ctg}^2\alpha \left| \frac{v}{\mathrm{d}v/\mathrm{d}l} \right|_{q^2 \sim 1}, \qquad (4.15)$$

where $u = \text{const} > 1$, $\alpha = \text{const}$.

The rotation of the plane of polarization that was seen above to accompany the transition from the plasma into the region where the properties of the waves are determined by the magnetized vacuum will obviously occur only in the geometrical optics approximation when $G_v \gg 1$ and there is no coupling. If, on the other hand, the coupling is strong, there will be no rotation of the plane of polarization during the propagation process, and the waves transform from extraordinary to ordinary (and vice versa). For a linear density profile, the conversion efficiency is described by the familiar formula (4.7) in which $G_\perp$ is replaced with G_v.[104] It follows from the condition $G_v \lesssim 1$ that, in a broad range of angles $\alpha \sim 1$, there is appreciable coupling if $\hbar\omega \lesssim 1$ keV (soft x rays). The density in the coupling region can be found from the condition $1 - 1/3vu/a(u-1) = 0$, according to which $N_0 \sim 10^{20}$ cm^{-3}. Since the wave coupling phenomenon occurs in a broad range of angles, it may be expected that it will be important in the evolution of the polarization of radiation emitted by x-ray pulsars.[13)]

D. Limiting polarization effect

Mutual wave conversion may have an influence on the polarization of radiation entering vacuum from magnetoactive plasma (for example, plasma produced in thermonuclear fusion installations or the magnetospheric plasma; see Refs. 1–3, 33, 58, 86, 98, and 101, where the respective experiments and observations are discussed). In such cases, there is an inhomogeneous region that connects the anisotropic medium to the vacuum (or an isotropic medium). When the polarization of radiation that has traversed this region differs from that calculated in the geometrical optics approximation, we say that the limiting polarization effect has taken place. It is clear physically that, when v or u is sufficiently small, i.e., when the degree of anisotropy is low, the influence of the magnetic field $\mathbf{B}_0(\mathbf{r})$ on the polarization should tend to zero. As $v \to 0$, the waves should propagate as in a vacuum, and, as $u \to 0$, they should propagate as in an isotropic plasma. This is why, beginning with a certain level, the polarization of radiation emerging into vacuum (or an isotropic plasma) will cease to vary and, in the course of subsequent propagation, the polarization ellipse will assume a fixed shape, which will be the same as it was at entry into the region of weak anisotropy ("limiting polarization"). Conversely, according to the geometrical optics approximation, the polarization ellipse should, even here, have a variable shape, depending on the variation in the parameters v, u, and α [see (4.2)]. This means that wave coupling occurs for small values of v and u in the transitional region between the anisotropic and the isotropic medium.

The foregoing qualitative analysis (see Sec. 3C) enables us to identify the following features of the limiting polarization effect. To begin with, it is clear that, if the polarization of normal waves (4.2) varies little [$|\Delta\eta| \ll 1$, see (3.11)] as they emerge into the isotropic medium, the limiting polarization will obey the geometrical optics approximation and, consequently, the limiting polarization effect will be practically absent.[14)] According to Sec. 3C, both here and in the general case of emergence from an anisotropic medium, corresponding to $\zeta \to +\infty$, the polarization of the radiation (which can be arbitrary) will cease to vary whenever the "Faraday" integral $\int_{-\infty}^{\zeta}(n_1 - n_2)\mathrm{d}\zeta$ ceases to increase. The interval of values of ζ in which $\int_{\zeta}^{\infty}(n_1 - n_2)\mathrm{d}\zeta \sim 1$ is referred to as the limiting polarization region.

If, on the other hand, the transition from anisotropic to isotropic medium is accompanied by an appreciable change in the polarization of normal waves ($|\Delta\eta| \sim 1$), i.e., quasitransverse propagation is replaced by quasi-longitudinal propagation (or vice versa), then wave coupling must be taken into account. It becomes appreciable and leads to the limiting polarization effect when the coupling parameter is $G \lesssim 1$ in the coupling region $q^2 \sim 1$. Depending on the position of the coupling region relative to the limiting polarization region, there are then three different possible situations. When the coupling region is located closer to the isotropic medium than the limiting polarization region, the increase in the "Faraday" integral $\int (n_1 - n_2)\mathrm{d}\zeta$ in the coupling region is small and $G \ll 1$, in which case the limiting polarization effect corresponds to strong coupling and, to an extent, can be regarded as trivial (see Sec. 3A). When the coupling region lies on the other side of the limiting polarization region, i.e., within the body of the anisotropic medium, the evolution of limiting polarization is largely determined by wave conversion in the coupling region because the polarization obeys the geometrical optics approximation during subsequent propagation toward the isotropic medium and, in particular, in the limiting polarization region. The third possible situation is, in our view, the most interesting. Here, the coupling region coincides with the limiting polarization region and $G \sim 1$ (in this case, the characteristic function $G(\eta)$ has the form of Curve 2 in Fig. 2). This situation has been analyzed in Ref. 106 by considering

[13)]Analysis of radiowave conversion effects in the magnetospheres of radio pulsars[105] must be performed with allowance for the fact that the plasma in the magnetospheres of neutron stars (radio pulsars) is probably relativistic.

[14)]It is precisely this type of situation that has been analyzed in the well-known paper by Budden,[107] who examined the atmospheric propagation of a wave reflected from the Earth's ionosphere, assuming that the inhomogeneity was connected with a variation in the effective collision frequency $s = \nu_{\rm eff}/\omega$. The quantity q in (4.2) is equal to[1] $\sqrt{u}\,\sin^2\alpha \times [2(1 - v - is)\cos\alpha]^{-1}$ and is a slowly-varying function of s for $s \ll 1$ and $v \neq 1$. The increment $|\Delta\eta| \ll 1$ and the evolution of the limiting polarization do not then, in fact, accompany wave coupling: $Q_{\max} \ll 1$, see (3.30). It is obvious that this conclusion is also valid for tenuous ($v \ll 1$) magnetoactive plasmas in which the inhomogeneity on the boundary with the vacuum is connected with the change in plasma concentration.[35,52]

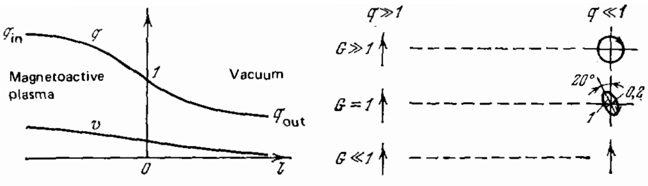

FIG. 7. Limiting polarization in the transitional layer (4.16) between magnetoactive plasma and vacuum (schematic). The ordinary wave is incident on the layer from the left.

the example of the standard problem[15)] for a transitional layer with an inhomogeneous magnetic field, described by the equation

$$\frac{dq}{d\zeta} = -v\sqrt{u}\cos\alpha\,(q^2+1)^{3/2}\frac{1}{G(\eta)}, \quad G(\eta) = G\operatorname{ctg}\left[\frac{(1/2)\pi(\eta_0-\eta)}{\Delta\eta}\right], \tag{4.16}$$

where $\Delta\eta$ is given by (3.11), $0 < \Delta\eta < (\eta_0 - \pi/4)$, and G = const (see Fig. 7). We then have[106]

$$Q = [1 + |\operatorname{ctg}(G_1\Delta\eta)\operatorname{ctg}(G_2\Delta\eta)|]^{-1}, \tag{4.17}$$

$$e^{i\varphi} = \frac{4\pi^2}{\Delta\eta}\sqrt{|\operatorname{ctg}(G_1\Delta\eta)\operatorname{ctg}(G_2\Delta\eta)|} \times \frac{\Gamma(-2G_1\Delta\eta/\pi)\,\Gamma(2G_2\Delta\eta/\pi)}{\Gamma^2(-G_1\Delta\eta/\pi)\,\Gamma^2(G_2\Delta\eta/\pi)}\,e^{i4(\ln 2)G\Delta\eta/\pi}, \tag{4.18}$$

$$\gamma = \frac{1}{2}\int_{\zeta_{in}}^{\zeta_{out}}(n_1-n_2)\,d\zeta + \arg\left[\frac{\Gamma((1/2)-2iG\Delta\eta/\pi)}{\Gamma((1/2)-(G_1\Delta\eta/\pi))\,\Gamma((1/2)+(G_2\Delta\eta/\pi))}\right], \tag{4.19}$$

where Γ is the gamma-function and $G_{1,2} = \sqrt{1-G^2} \pm iG$. The phases ϕ and γ enable us to determine the shape and orientation of the limiting polarization ellipse (see Fig. 7). The conversion coefficient Q characterizes the limiting polarization effect which, according to (4.17) is stronger for larger increments $\Delta\eta$ and smaller coupling parameters G.

The difference between the region of departure from geometrical optics and the region of limiting polarization is emphasized in Ref. 108. Denisov[30] considers more complicated and less frequently encountered models of limiting polarization regions containing many transitional layers, i.e., many local coupling regions with $q^2 \sim 1$ (or, in other words, a single distributed coupling region). For example, one model discussed by Denisov[30] is

$$\Psi \equiv -\frac{q'}{2(q^2+1)} = \text{const}, \quad n_2 - n_1 = \begin{cases} h(\zeta_1-\zeta) & \text{for} \quad \zeta < \zeta_1, \\ 0 & \text{for} \quad \zeta > \zeta_1 \end{cases} \tag{4.20}$$

(h = const), for which the system of transitional layers has the form

$$q \equiv -\operatorname{ctg} 2\eta = -\operatorname{ctg}[2\eta_1 + 2\Psi(\zeta_1-\zeta)] \tag{4.21}$$

(see Fig. 8). In each of the transitional layers in which q changes from $+\infty$ to $-\infty$, the conversion process is characterized by the coupling parameter G determined

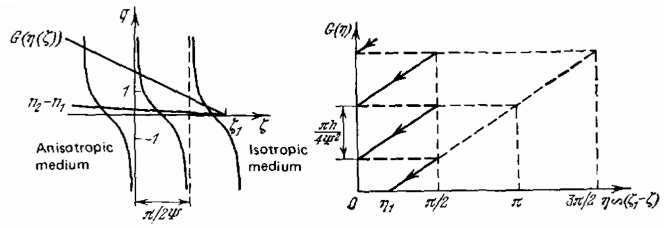

FIG. 8. A system of transitional layers and the characteristic function $G(\eta(\zeta)) = (h/2\Psi^2)\Psi(\zeta_1-\zeta)$ in the region between the anisotropic and isotropic media for the model defined by (4.20).

in accordance with the qualitative analysis given in Sec. 3. As the radiation emerges from the anisotropic medium and passes through the successive transitional layers, the coupling parameter undergoes a successive reduction because the function $G(\eta(\zeta))$ decreases (see Fig. 8). Complete conversion is determined by the interference of waves in the successive transitional layers (Sec. 3C).

This situation arises when $\frac{1}{2}h/\Psi^2 \ll 1$, and effective conversion with coupling parameter $G \lesssim 1$ occurs in a large number ($\sim 2\Psi^2/h \gg 1$) of transitional layers. The "Faraday" integral evaluated along the path crossing these layers is then large: $\int(n_2-n_1)d\zeta \equiv 2\int G(\eta)d\eta \sim 2\Psi^2/h \gg 1$. Accordingly, the qualitative analysis shows that the polarization of radiation emerging into the isotropic medium will vary even in the region where $G(\eta) \ll 1$. This can be seen from the exact solution given by (4.20), which is expressed in terms of the parabolic cylinder functions.

The picture is simplified in the other limiting cases, where $\frac{1}{2}h/\Psi^2 \gg 1$ and the function $G(\eta)$ decreases so rapidly that the transition from $G(\eta) \gg 1$ to $G(\eta) \ll 1$ occurs in a single (last) transitional layer. According to the exact solution given by Denisov,[30] the conversion process is weak: $Q \sim 2\Psi^2/h \ll 1$. This result is in agreement with the qualitative analysis and follows from (3.20). In this case, the remaining transition layers in the anisotropic medium do not affect wave coupling or the limiting polarization effect.

E. Plasma with a sheared magnetic field

In the foregoing discussion, we did not take into account the shear of the lines of force of the magnetic field B_0, i.e., the variation in the direction of the projection of the magnetic field on to the plane perpendicular to the direction of propagation (see Sec. 2B). This inhomogeneity is typically encountered in tokamaks,[43,58] in the magnetospheres of rotating accreting neutron stars,[105] in rotational discontinuities and shock waves, and in low-frequency helical and magnetohydrodynamic waves.[44] These situations are also known to involve wave coupling, which becomes possible even as a result of inhomogeneity in the density of tenous plasma ($v = v(\zeta) \ll 1$).

The wave-coupling theory is readily extended to the case of plasma with a sheared field. Instead of ordinary

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[15)]The method of phase integrals is ineffective in this case because it will not yield the phase of the waves and, consequently, will not solve the problem of limiting polarization (for example, it cannot be used to determine the orientation of the output polarization ellipse).

and extraordinary waves, we must then consider the helical waves introduced in Sec. 2B. The conversion of these waves is characterized by the coupling parameter G given by (3.24), and occurs in regions where $\tilde{q}^2 \sim 1$. Since q and $\tilde{q}$ are different [see (2.6) and (2.13)], the position of the coupling region $\tilde{q}^2 \sim 1$ in plasma with a magnetic field shear is different from that in the absence of shear, for which the coupling region is determined by the condition $q^2 \sim 1$. The presence of shear gives rise to the conversion of helical waves in new intervals of the plasma parameters, and for a broader range of inhomogeneities. In particular, inhomogeneous shear ($\psi'' \neq 0$) leads to helical-wave conversion. A detailed analysis of this is given in Ref. 47, but will not be reproduced here because, fundamentally, the coupling phenomenon in plasma with a magnetic-field shear is very similar to the analogous phenomenon in liquid crystals of the cholesteric type (see Sec. 5).

5. LINEAR COUPLING IN THE OPTICS OF LIQUID CRYSTALS

Whereas, in magnetoactive plasma, the study of magnetic-field shear and helical-wave conversion is merely an extension of wave-coupling theory, the investigation of the entire range of problems associated with the rotation of the optical axes of the medium in the absence of magnetic activity[16)] is of fundamental importance. In fact, the ordinary and extraordinary waves in nongyrotropic anisotropic media are linearly polarized ($q^2 = \infty$) and, consequently, are not coupled when the orientation of the optical axes of the medium is fixed in space [coupling coefficient $\Psi = 0$ in (2.17), since $q =$ const]. Wave coupling becomes possible only in the case of rotation of the anisotropy axes which occurs, for example, in liquid crystals, ferromagnetic structures (Sec. 6), in mechanically stressed (twisted) bodies,[42,112,113] and under other conditions.

A. Helical waves in a cholesteric crystal

Liquid crystals retain orientational order despite the partial or complete absence of spatial order. This means that the long axes of the molecules are preferentially aligned in the direction of the unit vector **L**—the director. In cholesteric liquid crystals (including smectic crystals) containing chiral molecules without mirror symmetry, the director $\mathbf{L}(z)$ describes a helix of pitch $g = 2\pi/|\psi'|k_0$ along the z axis (this is the "cholesteric helix"). The pitch and the nonuniformity of this helix can be controlled by placing the cholesteric crystal in external electric, magnetic, or ultrasonic fields, or by varying the temperature, pressure, or chemical composition of the crystal. The rotation of the optical axes that is connected with the rotation of the director has a radical effect on the optical properties of the crystal. An example of this is the Bragg reflection of light of wavelength $\lambda \approx g$.[55,114,115]

16)The natural (molecular) optical activity is, in most cases, weak[41,109] and can change the structure of the normal waves,[110] including helical waves,[111] only in the course of propagation at a small angle to the anisotropy axis.

We shall now consider the coupling of helical waves propagating in the same direction,[88] which occurs during propagation along the z axis in a smoothly inhomogeneous cholesteric helix (inhomogeneity scale $\Lambda \gg \lambda$, pitch $g \gg \lambda$, ratio of Λ to g can be arbitrary).[17)] Helical waves, i.e., normal waves in a medium with a uniform helix ($\psi' =$ const), have very close dispersion branches by virtue of the weak anisotropy of the liquid crystal ($\Delta\varepsilon = \varepsilon_{\parallel} - \varepsilon_{\perp} \ll \varepsilon$, usually $\Delta\varepsilon/\varepsilon \sim 10^{-1}$) and slow rotation of the director plane ($|\psi'|/\sqrt{\varepsilon} \ll 1$). For such waves,

$$\tilde{n}_1 - \tilde{n}_2 = 2\psi' \sqrt{\tilde{q}^2 + 1}, \quad \tilde{q} = \frac{n_0 - n_e}{2\psi'} \approx -\frac{\Delta\varepsilon \sin^2\alpha}{4\sqrt{\varepsilon}\,\psi'}, \tag{5.1}$$

where α is the angle of inclination of the director **L** to the z axis (see Figs. 1 and 9). The quantity $\tilde{q}$, determined by the local properties of the cholesteric crystal, defines the refractive indices $\tilde{n}_{1,2}$, given by (2.12), and the polarization coefficient $\tilde{K}_{1,2}$ of the helical waves, given by (2.13). When $\tilde{q}^2 \gg 1$, so that the helix is not highly twisted, the helical waves are polarized linearly and are actually indistinguishable from the ordinary and extraordinary waves ($\tilde{n}_{1,2} \approx n_{e,0}$). When $\tilde{q}^2 \ll 1$, i.e., when the spiral is highly twisted, the polarization of the helical waves is circular. The director then executes a large number of revolutions in one period of the beats between the ordinary and extraordinary waves. As a result, the optical anisotropy of the liquid crystal is lost, and the light propagates just as in an isotropic medium with refractive index $\sqrt{\varepsilon} = (\tilde{n}_1 + \tilde{n}_2)/2 = (n_e + n_0)/2$.

The coupling conditions for an inhomogeneous helix are

$$|\Delta\eta| \sim 1, \quad G = \left|\frac{4\sqrt{2}\,\psi'}{\tilde{q}'}\right|_{\tilde{q}^2 \sim 1} \lesssim 1 \tag{5.2}$$

[see Sec. 3C and the definitions of η and $\Delta\eta$ given by (3.10)–(3.12)], so that it is clear that effective helical-wave conversion in inhomogeneous liquid crystals can occur only in transition from a highly twisted to a slightly twisted helix, or vice versa (i.e., when $|\Delta\tilde{q}| \sim 1$ in the region $\tilde{q}^2 \sim 1$). According to (5.2), this transition must be sufficiently rapid. In particular, within the pitch of the helix, there must be an appreciable change in $\tilde{q}$, i.e., the degree of anisotropy $\Delta\varepsilon$, in the inclination of the vector to the axis of the helix, or in the pitch of the helix itself. The propagation of light along a uniform helix with $\tilde{q} \approx$ const is not accompanied by wave coupling and is fully described by the geometrical optics of helical waves.[45,61,86,114-116]

B. Polarization properties of an inhomogeneous cholesteric helix

We shall now consider specific examples of light-wave coupling in cholesteric cells in which a thin layer of the liquid crystal (thickness $d \sim 10^{-3} - 10^{-2}$ cm) lies between two transparent plates (Fig. 9). The inhomo-

17)Numerical calculations on the propagation of polarized light in a number of specific models of a highly twisted inhomogeneous helix with pitch $g \sim \lambda$ are reported in Ref. 80. Even in a smoothly inhomogeneous helix with $\Lambda \gg \lambda$, it turns out that Bragg reflection gives rise to coupling of waves propagating in opposite directions, whereas the conversion of helical waves propagating in the same direction is weak.

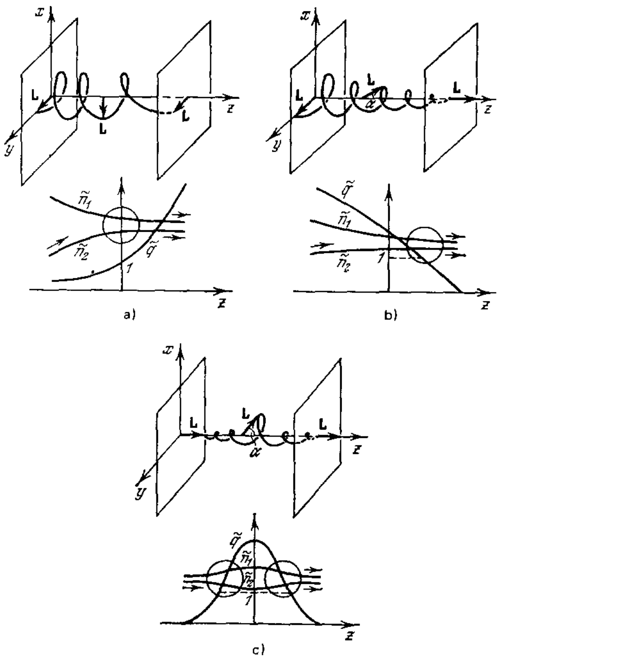

FIG. 9. Inhomogeneous liquid-crystal cells of the cholesteric type (schematic): a) homogeneous (with variable pitch); b) hybrid; c) homeotropic. Circles show regions of helical-wave coupling.

geneity can be associated with either boundary conditions or with external fields.[114,115,117,118]

Suppose that the pitch of the helix varies linearly, i.e., $g(\zeta) = g' \cdot \zeta$, $g' =$ const, and the director is $\mathbf{L} \perp \mathbf{z}$ ($\alpha \equiv \pi/2$, Fig. 9a). For this transition from a highly twisted to a slightly twisted helix, exact solution yields (4.12) with coupling parameter $G = 8\sqrt{2}\pi/|q'|k_0$, from which it follows that, for a highly inhomogeneous helix for which the derivative g' is large, the coupling is strong, whereas, in a weakly inhomogeneous helix, the conversion is exponentially small. Qualitatively similar results are obtained by solving other standard problems,[88,119] for example, the problem of a layer in which $\psi'(\zeta) = \psi''\zeta$, i.e., $g \propto \zeta^{-1}$, although the coupling parameter is now different:

$$G = \sqrt{2}\,(n_e - n_0)^2\,|\psi''|^{-1}. \tag{5.3}$$

Let us now suppose that the angle between the director and the axis of the helix is not constant, but is represented by the function $\alpha(\zeta)$. Figure 9b shows a hybrid cell with $\mathbf{L} \perp \mathbf{z}(\alpha = \pi/2)$ on the entrance wall and $\mathbf{L} \parallel \mathbf{z}(\alpha = 0)$ on the exit wall. The variation in $\tilde{q}$ can now be qualitatively described by[119]

$$\tilde{q}^2 = -1 + \left[(\tilde{q}_{\rm in}^2 + 1)^{-1/2} + G^{-1}\int\limits_{\zeta_{\rm in}}^{\zeta}(n_e - n_0)\,d\zeta\right]^{-2}, \tag{5.4}$$

where $\tilde{q}_{\rm in} = \tilde{q}(\zeta_{\rm in}) > \tilde{q}_{\rm out} = \tilde{q}(\zeta_{\rm out}) \geqslant 0$. In this case, the function (3.14) is $G(\eta) = G =$ const within the interval $\Delta\eta$ of (3.11), and exact solution is elementary:

$$Q = (G^2 + 1)^{-1}\sin^2\left(\Delta\eta\sqrt{G^2+1}\right), \tag{5.5}$$

$$\operatorname{tg}\gamma = -\operatorname{tg}\varphi = \frac{G}{\sqrt{G^2+1}}\operatorname{tg}\left(\Delta\eta\sqrt{G^2+1}\right). \tag{5.6}$$

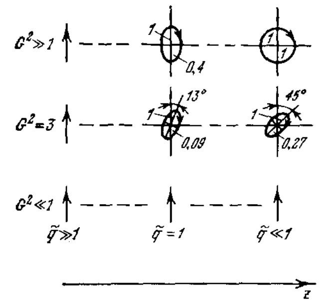

FIG. 10. Transformation of the polarization ellipse in a hybrid cell in the approximation in which $G(\eta) =$ const for weak ($G^2 \gg 1$), effective ($G^2 = 3$), and strong ($G^2 \ll 1$) coupling. Type 2 wave is incident from the left.

These two formulas describe the transition of waves from one dispersion branch ($\tilde{n}_2$) to another ($\tilde{n}_1$), and the corresponding conversion of polarization (Fig. 10).

In the homeotropic cell shown in Fig. 9c, the molecular axes are parallel to the axis of the helix on both walls ($\alpha = 0$). This cell can be modeled by a layer in which

$$\tilde{q}(\zeta) = \frac{\tilde{q}_0}{\operatorname{ch}\left[4\sqrt{2}\,\psi(\zeta)/p\right]}. \tag{5.7}$$

We have given the explicit dependence of $\tilde{q}$ on the angle ψ of rotation of the plane of the director because, according to (5.2), it is precisely the nature of this dependence that determines the effectiveness of the coupling. The quantity $\tilde{q}$ is a maximum at the center of the cell, where $\psi = 0$. Along the edges of the cell, $\tilde{q}_{\rm in} \approx \tilde{q}_{\rm out} \approx 0$. In this case, the conversion coefficient is given by[119]

$$Q = \sin^2\left(\frac{\pi\tilde{q}_0 p}{4\sqrt{2}}\right)\operatorname{ch}^{-2}\left(\frac{\pi p}{4\sqrt{2}}\right). \tag{5.8}$$

It depends not only on the coupling parameter $p > 0$, which characterizes the degree of inhomogeneity of the layer, but also on the interference factor $\sin^2(\pi\tilde{q}_0 p/4\sqrt{2})$, which involves the geometric path difference between the two coupling regions with $\tilde{q}^2 \sim 1$ [see (3.28a) and Fig. 9c]: $\pi\tilde{q}_0 p/4\sqrt{2} \approx (\frac{1}{2})\int(\tilde{n}_2 - \tilde{n}_1)d\zeta$. It follows that, in a highly inhomogeneous homeotropic cell ($p \ll 1$, $\tilde{q}_0 \gg 1$), the conversion process is not necessarily complete ($Q \neq 1$), since coupling in these two regions may cancel out as a result of interference.

Coupling between helical waves leads to a redistribution of the luminous flux among the waves, and thus gives rise to a "nonadiabatic" change in the polarization of light in inhomogeneous liquid crystals. The polarization of light can be varied by varying the conversion efficiency through a variation in the external conditions and fields that determine the inhomogeneous structure of the cholesteric helix. Special polarizers that transmit a particular helical wave can then be used for helical-wave selection at entry to and exit from the liquid-crystal cell.

We now draw attention to the fact that the coupling parameter G and, consequently, the degree of conversion Q are functions of frequency [cf. (4.7)]. For example, when (5.3) applies, we have $G \propto \omega^4$ and, for approximate estimates, we can put $|n_e(\omega) - n_0(\omega)| \propto \omega$. The result of this is that the inhomogeneous liquid-crystal helix between the two polarizers transmitting different helical waves can be used as an optical frequency filter.[88,119] This filter will not transmit light in the frequency region for which $G(\omega) \gg 1$; the filter will be partially transparent outside this region and will transmit light with polarization corresponding to the polarization of the helical wave produced as a result of conversion for $G(\omega) \lesssim 1$.

C. Orientational phase transition and the Mauguin cutoff

The theory of wave coupling is of particular interest in connection with the problem of the relationship between the polarization cutoff threshold and the phase transition point in liquid-crystal structures.[117,118] These transitions, usually referred to as Frederiks transitions, occur when there is a change in the magnitude of the applied electric or magnetic field, and are connected with the reorientation of the long axes of the liquid-crystal molecules under the influence of the fields. Figure 11a shows the so-called nematic twist structure. The Frederiks transition occurs in this system in a longitudinal electric field E: when a certain critical field E_{crit} is exceeded, the molecules find it energetically more convenient to align themselves gradually along the field. Depending on the field strength, the angle α between the director and the helix axis, for example, at the center of the cell, will vary as shown in Fig. 11b. The result of this is that the twist structure becomes inhomogeneous, and helical-wave coupling becomes possible. When polarizers are placed on either side of the system and transmit only one of the helical waves, an increase in the electrical field eventually results in the extinguishing of the transmitted light, and this is sometimes referred to as the adiabatic Maugin cutoff,[81,117] i.e., the cutoff of the geometrical optics propagation of light. It occurs at a certain threshold field E_0 (optical threshold, see Fig. 11b) and is due to the transition of the light wave from one dispersion branch to another. When $E > E_0$, this transition is complete, and the system consisting of the twist structure and the two polarizers becomes opaque.

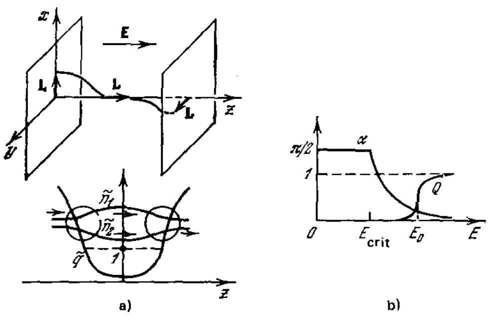

FIG. 11. Nematic twist structure with twist angle $\psi_0 = \pi/2$ in a longitudinal field **E**: a) characteristic form of the distribution along the layer of the quantity $\tilde{q} \propto \sin^4\alpha$ and of the $\tilde{n}_{1,2}$ dispersion curves given by (5.1) (circles show helical wave conversion region); b) threshold characteristics.

Since the Frederiks transitions are phase transitions of the second order,[114,118] the attendant changes in the optical properties of liquid-crystal structures are very small when the field exceeds the critical value by a small amount. On the other hand, a substantial deformation of the structure is necessary to extinguish it altogether. This means that the optical threshold must substantially exceed the critical field for the transition. A similar situation occurs not only for the twist structure, but also in a number of other cases (see Refs. 88, 114, 117–120). The connection between the optical threshold E_0 and the critical field for the phase transition E_{crit} remained unclear for a long time. However, the relationship was elucidated as a result of a qualitative analysis of the coupling phenomenon. The observed cutoff field E_0 corresponds to the appearance in the structure of a coupling region $\tilde{q}^2 \sim 1$ in which $|\Delta\tilde{q}| \sim 1$ and the coupling parameter is $G(E) \sim 1$. Hence, for the Maugin cutoff in the twist structure with $\psi_0 = \pi/2$ (Fig. 11), we have[18]

$$E_0 = E_{crit}\left(\frac{2}{\pi\sqrt{3}}\right) \ln\left[\frac{8\pi\tilde{q}_0}{\ln(8\pi\tilde{q}_0)}\right], \tag{5.9}$$

where $\tilde{q}_0 = \omega|n_e - n_0|d/\pi c$. For example, for typical values[81] $d = 13\ \mu m$, the critical field is $E_{in} = 0.6E_0$. Equation (5.9) has been used to explain the measured[81,117] relationship between the polarization cutoff threshold and the thickness d of the twist structure and light frequency ω. In particular, (5.9) shows that E_0/E_{crit} is a logarithmic function of the thickness of the nematic twist structure. The dependence on frequency and twist angle ψ_0 is more complicated, and is discussed in Ref. 88. Accurate formulas such as (5.9) for optical thresholds can be used to investigate the structural properties of liquid crystals in terms of their optical properties, and to measure the liquid-crystal parameters.[88,119,120] The first step is, of course, to find the phase transition point E_{crit} and the optical threshold E_0 from independent measurements. Thus, the wave coupling phenomenon in liquid-crystal structures of the cholesteric type can also be exploited for polarization-based light conversion and to investigate the properties of liquid crystals.

D. Polarization diagnostics of dynamic processes

Light-wave conversion in liquid crystals can be used to investigate various dynamic processes that modify the inhomogeneity of the helix, and to measure the corresponding conversion coefficients. These processes include diffusion, heat flow, hydrodynamic flows, orientational waves, chemical reactions, and so on.[114,117,118,121] The point is that, in contrast to geometrical optics, light transmitted by an inhomogeneous liquid crystal structure after linear coupling carries information on the inhomogeneity of the structure.[82] This opens up the possibility of polarization diagnostics of dynamic processes in liquid crystals.

Let us illustrate this by a specific example. Figure 12 illustrates the diffusion of chiral molecules in a nematic liquid crystal. The presence of the chiral molecules results in the appearance of the cholesteric helix, whose pitch is inversely proportional to the concentra-

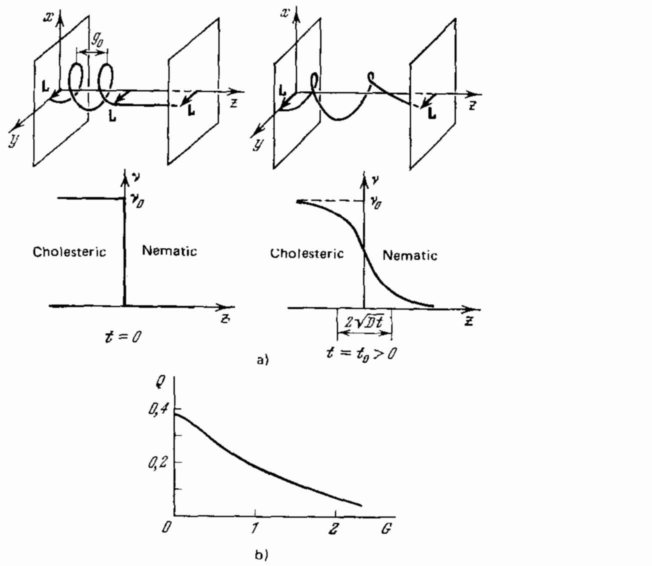

FIG. 12. Transitional diffusion layer: a—concentration of chiral molecules $\nu(z)$ and cholesteric structure of the director $\mathbf{L}(z)$ at times $t=0$ and $t_0>0$; b—conversion coefficient Q as a function of the coupling parameter $G\propto\sqrt{Dt}$ for a high initial concentration of chiral molecules (the graph was constructed numerically for $\tilde{q}_0^2=[g_0k_0(n_0-n_e)/4\pi]^2=1/16$).

tion of these molecules.[114] Suppose that, at $t=0$, there is a sharp boundary between the cholesteric and the nematic crystals. In the course of diffusion of the chiral molecules, the scale of the inhomogeneity of the helix will vary in accordance with the diffusion law in proportion to $\sqrt{Dt}$, and there will be an accompanying variation in the coupling parameter $G=2\sqrt{Dt}k_0|n_e-n_0|$ and conversion efficiency $Q(G)$ (Fig. 12). Hence, by following the polarization of light and recording the function $Q(t)$, it is possible to observe the diffusion process and then determine the diffusion coefficient D. All that needs to be done is to record the time t_0 at which the conversion efficiency falls by a factor of two since, as is shown in Ref. 82, this should occur for $G=1$. We then have

$$D=c^2[4t_0\omega^2(n_e-n_0)^2]^{-1}. \tag{5.10}$$

The time t_0 at which the diffusion coefficient is measured amounts to only a few seconds rather than the hours or days that are necessary in other currently used methods.

6. LINEAR CONVERSION IN MAGNETIC MEDIA

All that we have said about the coupling of helical electromagnetic waves in magnetoactive plasmas and liquid crystals will also apply to magnetic media. In magnetically ordered crystals,[124,125] the relative anisotropy and gyrotropy in the optical and infrared ranges are up to 10^{-3}–10^{-2} so that, in accordance with the foregoing, we have the possibility of wave coupling even in the case of a continuous inhomogeneity of the medium[18)] $\Lambda\gg\lambda$. The necessary condition is that the inhomogeneity along the beam must take the form of either a nonuniform helical rotation of magnetization $\mathbf{M}$, or of a transition from quasilongitudinal to quasitransverse (relative to $\mathbf{M}$) propagation, or vice versa. Either inhomogeneity is inherent in the internal structure of magnetic media (domain walls, helicoidal and sinusoidal structures, spin waves), but can also be produced by an external magnetic field. Of known examples, we note magnetic semiconductors[125-128] (for example, those with wurtzite or spinel structure), ferromagnetic materials, helicoidally ordered ^{3}He (see Ref. 129), and ferromagnetic liquids (suspensions of ferromagnetic particles) in a magnetic field.[130] A nonuniform distribution of a magnetic impurity or mechanical stress introduced while the crystal is being grown will also introduce an inhomogeneity into the nematic structure.

[18)]Apart from optics, coupling effects can be observed in the microwave, millimeter, and submillimeter ranges of inhomogeneously magnetized ferrites and, apparently, in the ultraviolet range in the case of metallic ferromagnetic materials when they become transparent.

A. Optical properties of Bloch-type domain walls

The optical polarization method is very convenient and sometimes the only way of studying inhomogeneous domain structures,[123-125,131-135] especially in the investigation of the dynamics of domain walls in pulsed processes.[136-138] The domain structures used in magnetooptics are usually in the form of thin ferrodielectric films (10^{-3}–10^{-2} cm) that are transparent to optical or infrared radiation.[19)] We shall now consider the propagation of light in this type of film with a Bloch-type domain wall.[88,131,135] We shall suppose that the film is a plane-layered structure in the direction of the z axis. We shall also suppose that the magnetization vector $\mathbf{M}(z)$ in the film rotates on a helix through a finite angle ψ_0 in a region with characteristic scale 2ξ (Fig. 13) in accordance with the following law:

$$\psi=\frac{2\psi_0}{\pi}\,\mathrm{arctg}\exp\left(\frac{z}{\xi}\right)-\frac{\psi_0}{2} \qquad (-\infty<z<+\infty). \tag{6.1}$$

The domain wall inhomogeneity gives rise to coupling between the helical light waves with conversion coefficient[47,88]

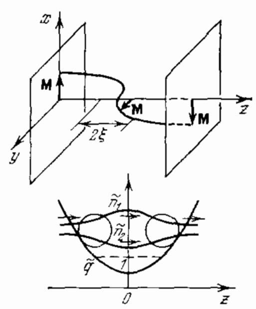

FIG. 13. Bloch-type domain wall and characteristic behavior of dispersion branches of $\tilde{n}_{1,2}$ and quantity $\tilde{q}$. Circles show helical-wave conversion regions.

[19)]In practice, absorption is considerably even in such thin films. Because of dichroism, it can then influence the nature of the conversion of geometrical optics waves, for example, when the difference $\int \mathrm{Im}\,\tilde{n}_1 d\xi-\int \mathrm{Im}\,\tilde{n}_2 d\xi$ is of the order of unity in the coupling region (see Refs. 39, 75, 94, 116, and 139). This effect will be neglected below.

$$Q = \sin^2\psi_0 \,\mathrm{ch}^{-2}\left[\frac{\pi}{2} k_0 (n_e - n_0)\,\xi\right] \tag{6.2}$$

(when $\varepsilon_{xy}^{-1} = 1$). In fact, this expression gives the conversion coefficient for ordinary and extraordinary waves because the latter are identical with the helical waves outside the domain wall, where $\psi' = 0$. It is clear from (6.2) that there will be no conversion in a domain wall with $\psi_0 = \pi$ and, consequently, in a sequence of such domain walls, even when there is rapid rotation over the small scale ξ. This result is related to the symmetry of the layer (6.1): interference between the waves ensures that coupling effects in the first half of the layer are compensated by coupling effects in the second half [see Fig. 13 and Eq. (3.28a)[20]].

This cancellation will not occur in the case of rotation through, for example, the angle $\psi_0 = \pi/2$ (see Fig. 11). This type of domain wall can evidently be produced in an external magnetic field under the appropriate boundary conditions. According to (6.2), effective conversion will occur when the domain-wall thickness is $\xi \lesssim \xi_{in} = |\pi k_0(n_e - n_0)/2|^{-1}$, and light propagation will then be subject to geometrical optics $\xi \gg \xi_{in}$. When crossed polarizers are placed on either side of the nematic film and transmit light that is linearly polarized in the plane of the magnetization vector **M**, the domain structure will be transparent for $\xi \gg \xi_{in}$. When, on the other hand, $\xi \ll \xi_{in}$, the structure will be opaque and, under the conditions of strong coupling, the linearly polarized light leaving the first polarizer will maintain constant plane of polarization as it propagates through the magnetic crystal. It will therefore be extinguished by the second polarizer, which is crossed with the first. Clearly, this system is capable of operating as a shutter controlled by the external magnetic field.

B. Highly twisted helix

A common case encountered in magnetooptics[46,125,131] and liquid-crystal optics[55,114-117] is that where the pitch of the helix produced by the rotating magnetization **M** or director **L** is small in comparison with the period of spatial beats between the ordinary and extraordinary waves: $|\psi'| \gg |n_e - n_0|$. If the medium is weakly anisotropic, so that $|n_e - n_0| \ll n_e + n_0$, the pitch of the helix $g = 2\pi/|\psi'|k_0$ can be either less than or greater than the wavelength $\lambda = 2\pi/\sqrt{\varepsilon}\,k_0$. We now turn to the properties of helical waves in this kind of system.[21] For such systems,

$$\tilde{n}^2_{1,2} = n^2 + \psi'^2 \pm 2\psi' n \sqrt{\left(\frac{n_e^2 - n_0^2}{4\psi' n}\right)^2 + 1}, \quad n^2 = \frac{1}{2}(n_e^2 + n_0^2), \tag{6.3}$$

$$\tilde{K}_{1,2} = \frac{n_0^2 - \psi'^2 - \tilde{n}^2_{1,2}}{2\psi' \tilde{n}_{1,2}}, \quad (\tilde{K}_1\tilde{K}_2)^2 = \frac{n_0^2 - \psi'^2}{n_e^2 - \psi'^2}. \tag{6.4}$$

It is clear from (6.4) that, when $|\psi'| \gg |n_e - n_0|$, the polarization will be nearly circular ($\tilde{K}^2_{1,2} \approx 1$) and, consequently, will not vary even on an inhomogeneous helix. Effective coupling between waves propagating in the same direction is therefore impossible. It follows from (2.15) that the coupling of waves propagating in opposite directions will then occur just as in the isotropic medium with refractive index $n = \sqrt{(n_e^2 + n_0^2)/2}$, and can be described by (2.18). According to Sec. 3D and Ref. 76, reflection will become effective only in a highly inhomogeneous medium when $|n'|/(2n^2) \gtrsim 1$ and $|\Delta n| \sim 1$.

C. Above-barrier reflection near Bragg resonance

The above conclusion does not apply to the case of Bragg resonance,[46,56,131,140] in the neighborhood of which there is effective coupling of waves propagating in opposite directions, even in a smoothly inhomogeneous periodic structure (here, we are concerned with above-barrier reflection). According to (6.3) and (6.4), Bragg resonance will occur in a helix when $n_0 \leqslant \psi' \leqslant n_e$, and one of the helical waves will not propagate: $\tilde{n}_2^2 \leqslant 0$, $\tilde{K}_2^2 \leqslant 0$ (to be specific, we assume that $\psi' > 0$, $n_e > n_0$). We are dealing with a wave that is polarized near the Bragg resonance, so that the end point of the vector **E** describes a helix that is identical in shape with the helix described by the magnetization **M** or the director **L**. (At the same time, throughout the Bragg resonance region $\psi' \sim n$, the other wave has a helical structure with opposite rotation, and propagates freely; for this wave, $\tilde{n}_1^2 \approx (2n)^2$, $\tilde{K}_1 \approx -1$.) In the neighborhood of the Bragg resonance, when $n_0 - \psi' \lesssim n_e - n_0$, the refractive index for the wave of type 2 is nearly zero: $\tilde{n}_2^2 \approx (n_0 - \psi')(n_e - \psi')$. This wave will therefore be effectively coupled to its reflected wave. According to (2.15), the coupling equations are given by (2.18) with coupling coefficient $\Omega = i\tilde{K}_2'/(2\tilde{K}_2) \approx i\tilde{n}_2'/(2\tilde{n}_2)$ [in the last equation, we have taken into account the fact that $\tilde{K}_2 \approx (n_e - n_0)^{-1}\tilde{n}_2$ (see (6.4)]. They are analyzed qualitatively in Sec. 3D where the conditions for effective reflection are also elucidated.

As an example, let us consider a hyperbolic transition layer of finite thickness ζ_{in}, ζ_{out}, in which, in the neighborhood of the Bragg resonance ($n_0 - \psi' \ll n_e - n_0$), the refractive index falls smoothly from the low value $\tilde{n}_2(\zeta_{in})$ to a still lower value $\tilde{n}_2(\zeta_{out})$: $\tilde{n}_2(\zeta) = \tilde{n}_2(\zeta_{in})/[1 + (\zeta - \zeta_{in})/k_0\Lambda]$ (this can be assured by a smooth reduction in the pitch of the spiral as the Bragg resonance boundary is approached $\psi' \to n_0 - 0$). The increment in the variable $\theta = -\ln\sqrt{\tilde{n}_2}$ is the Riccati equation (3.29) is then given by

$$\Delta\theta = \theta_{out} - \theta_{in} = \ln\sqrt{\frac{\tilde{n}_2(\zeta_{in})}{\tilde{n}_2(\zeta_{out})}} \tag{6.5}$$

and the coupling parameter is

$$\mathcal{G} = 2\tilde{n}_2(\zeta_{in})\,k_0\Lambda, \tag{6.6}$$

since $\mathcal{G}(\theta) \equiv 2\tilde{n}_2^2/\tilde{n}_2' = \mathrm{const}$. This enables us to obtain an exact solution of (3.29) and to determine the reflection coefficient:

---

[20]This is obvious in the case of narrow domain walls, where $k_0|n_e - n_0|\xi \ll 1$. According to Sec. 3, light propagation occurs in this case just as in the isotropic medium, independently of the nature of the magnetization. The result is that, if the angle ψ_0 is arbitrary, direct composition of the linearly polarized wave incident on the domain wall into two linearly polarized waves at exit from the domain wall yields $Q = \sin^2\psi_0$—in complete agreement with (6.2).

[21]For simplicity, we are neglecting the natural gyrotropy and assume that $n_e^{-2} = \varepsilon_{yy}^{-1}$, $n_0^{-2} = \varepsilon_{xx}^{-1}$. The restrictions on ε_{xy}^{-1}, ε_{xx}^{-1}, ε_{yy}^{-1} and ψ' that this imposes can be derived from (2.9) and (2.10).

$$R = \left|\frac{f_4}{f_2}\right|^2_{\zeta=\zeta_{in}}$$

$$= \begin{cases} \sin^2(\Delta\theta\sqrt{\mathcal{G}^2-1})\,[\mathcal{G}^2-1+\sin^2(\Delta\theta\sqrt{\mathcal{G}^2-1})]^{-1} & \text{for} \quad \mathcal{G}\geqslant 1, \\ \mathrm{sh}^2(\Delta\theta\sqrt{1-\mathcal{G}^2})\,[1-\mathcal{G}^2+\mathrm{sh}^2(\Delta\theta\sqrt{1-\mathcal{G}^2})]^{-1} & \text{for} \quad \mathcal{G}\leqslant 1. \end{cases} \tag{6.7}$$

Hence, it follows that there is weak conversion of waves propagating in opposite directions when the increment $\Delta\theta$ is small: $R \approx (\Delta\theta)^2 \ll 1$ for $\Delta\theta \ll 1$. When $\Delta\theta \gtrsim 1$, reflection is still weak for $\mathcal{G} \gg 1$ ($R \lesssim \mathcal{G}^{-2} \ll 1$). Reflection becomes effective when the coupling parameter is $\mathcal{G} \sim 1$. When $\mathcal{G} \to 1$, the reflection coefficient R tends to $(\Delta\theta)^2/[1+(\Delta\theta)^2]$, so that $R \sim 1/2$ when $\Delta\theta \gtrsim 1$. Finally, strong reflection occurs for $\mathcal{G} \ll 1$, and there is a rapid variation in the refractive index and $\Delta\theta \gtrsim 1$. The reflection coefficient is then $R \approx \mathrm{th}^2\Delta\theta \sim 1$ [cf. (3.30)]. We also note that, in the case of effective coupling with $\mathcal{G} > 1$, the reflection coefficient exhibits oscillations as a function of the increment $\Delta\theta$ and of the coupling parameter $\mathcal{G}$. These oscillations vanish for $\mathcal{G} < 1$.

7. COUPLED MODES IN WAVEGUIDES AND LIGHTGUIDES

So far, we have been concerned with the propagation of waves in an unbounded medium. On the other hand, there is often considerable interest in waveguide propagation in situations where the structure of normal waves (modes) and the nature of their propagation are determined by the properties of the guiding (lateral) surfaces.

A. Mode coupling in waveguide systems

Wave (mode) coupling will again arise when the boundary conditions on the above surfaces vary along the propagation path. Such problems are widely known in electrical engineering (inhomogeneous transmission lines),[32,170] electrodynamics (irregular waveguides),[4-6,22] acoustics (stratified ocean),[7,8,22] and integrated optics (planar and fiber lightguides).[23,57,141,142] The coupling phenomenon is also of considerable interest for the physics of surface waves in which considerable advances have been made in recent times.[96,143-146]

Despite the fact that mode conversion in the above cases occurs as a result of the presence of inhomogeneous boundaries, the coupling equations and the corresponding qualitative analysis are similar to those developed for the unbounded medium in Secs. 2 and 3. The only difference is that the mode-coupling coefficients a_{ij} in the coupling equations are determined by the local properties of the boundaries. Thus, a Riccati equation of the form given by (3.29) can be obtained[89] for the reflection coefficient for the principal mode and the corresponding resonantly coupled to it reflected mode in a plane corrugated dielectric waveguide of variable thickness (inhomogeneous Bragg mirror, Fig. 14). In this case, the variable θ and the function $\mathcal{G}(\theta)$ are determined by the parameters of the corrugated waveguide and not by the refractive index. The effect of the waveguide thickness gradient on the reflecting properties of the Bragg mirror is investigated in Ref. 89 by considering the standard problem of a corrugated waveguide with a linear thickness profile, using the

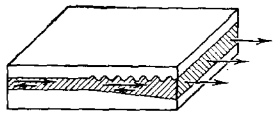

FIG. 14. Planar dielectric waveguide with inhomogeneous Bragg mirror.

perturbation theory method. The conclusions are in agreement with the qualitative analysis of the coupling of two modes of opposite direction of propagation that can be made on the basis of the Riccati equation, as described in Sec. 3D.

Similar problems on mode conversion in waveguides and systems of coupled waveguides have been investigated in the literature.[4,24,36,87,147-152,165] The corresponding problems relating to lasers with distributed feedback have also been examined.[90,153,154] However, all this lies outside the framework of the present review. We therefore refer the reader to the literature, and consider in detail only the case of the single-mode twisted birefringent waveguide. We shall use this example to show that the analysis of the coupling phenomenon described above can be used to investigate mode conversion in waveguide systems.[22)]

B. Helical modes in a twisted lightguide

There has been increased interest in the course of the last five years in the polarization properties of single-mode fiber lightguides. This is connected with the introduction of fiber technology that ensures that the polarization of radiation remains unaffected over fiber lengths of hundreds of meters or more, and with the likely applications of such fibers in optical communication systems, optical devices, gyroscopes, hydrophones, various transducers, and so on.[142,155-161]

The regular birefringent single-mode lightguide is analogous to an anisotropic medium: it supports two principal modes (with amplitudes E_x and E_y) with different phase velocities (c/n_x and c/n_y) and practically linear and orthogonal polarizations (these are the so-called LP-modes).[155,158] Mode degeneracy is removed in the manufacture of fibers of circular cross section by the unavoidable ellipticity of the core cross section, bending, optical anisotropy of glass due to internal or external mechanical stresses, and so on.[156-160] The period of the spatial-mode beats is then $2\pi/k_0|n_x - n_y|$, and is usually of the order of 0.1–10^4 cm. The propagation equations for the coupled LP modes in a weakly directional and weakly anisotropic lightguide can be written in the form[157,158]

[22)]Situations in which the mode coupling coefficients and the corresponding propagation constants do not vary along the waveguide have been examined in the literature (see, for example, Refs. 4, 32, 141, and 155). The coupling phenomenon becomes trivial under these conditions because it is then always possible to transform to new normal modes that are not coupled. The interaction phenomenon discussed below is due not simply to coupling, but to irregular coupling between modes along the waveguide.

$$
\begin{aligned}
E_x' + in_x E_x &= \alpha E_y,\\
E_y' + in_y E_y &= -\alpha E_x,
\end{aligned}
\tag{7.1}
$$

where x, y are the local optical axes of the fiber (in the transverse cross section) along which the principal axes of the polarization ellipses of two normal modes are directed (see below).

The coupling coefficient α is real and is determined by the deformation of the fiber core, the effect of external fields, and the anisotropy and gyrotropy of the glass. In a twisted lightguide, it is also determined by the azimuthal rotation of the optical axes. The last factor ensures that $\alpha \approx \psi'$ and is the most important in real situations because it ensures that the coupling coefficient α can reach values of the order of $|n_x - n_y|$ (see Refs. 157–160). In this case, the properties of the normal modes in the lightguide are highly variable and (this is particularly important) their polarization becomes elliptic. This can be seen from the expressions for the refractive indices $\tilde{n}_{1,2}$ and polarization coefficients $\tilde{K}_{1,2}$ of the normal modes of a uniformly twisted fiber, which are readily found from (7.1) for $n_{x,y}$ = const:

$$
\begin{aligned}
&\tilde{n}_{1,2} = \frac{1}{2}(n_x + n_y) \pm \psi' \sqrt{\tilde{q}^2+1},\\
&\tilde{K}_{1,2} = \tilde{q} \mp \sqrt{\tilde{q}^2+1}, \quad \tilde{q} = \frac{n_x - n_y}{2\psi'}.
\end{aligned}
\tag{7.2}
$$

The existence of these normal modes has long been established experimentally.[159] As in the unbounded medium [see (2.12) and (2.13)], we shall refer to them as helical modes.

The helical modes will become coupled in an irregular and, in particular, inhomogeneously twisted lightguide in which n_x, n_y, $\psi' \neq$ const. This can be described in precisely the same way as in Secs. 3, 5, and 6 because the geometrical optics substitution (1.2) [see also (2.14)], i.e., the transformation to coupled helical modes, ensures that (7.1) again leads to (2.19), (3.13), and (3.23) for the complex amplitudes $f_{1,2}$ of the helical waves, their ratio $P = -if_1/f_2$, and the polarization coefficient $K = -iE_y/E_x$, respectively. All the above conclusions about the nature of the various interactions will therefore remain in force. As in the case of liquid crystals (Sec. 5A), effective conversion of helical modes will occur only in those segments of the waveguide in which there is a transition from a highly twisted (on the scale of the mode beat period) to weakly twisted fiber, or vice versa. In this type of transition, there must be an appreciable change in $\tilde{q}$ in one pitch, i.e., there must be a change either in the birefringence (see Sec. 7C) or in the pitch itself (see Sec. 7D).

Mode conversion can be undesirable under certain conditions, for example, in optical communication cables, because it produces dispersive spreading and other changes in pulsed optical signals. It can be suppressed by removing irregular coupling regions in which $\tilde{q}^2 \sim 1$. This is done either by enhancing the birefringence of the fiber by increasing $\tilde{q}$ throughout, or by highly twisting the fiber and thus reducing $\tilde{q}$ along its entire length (see Refs. 158, 160).

There are also conditions where effective conversion is of considerable interest, for example, as a way of measuring the local optical characteristics of a fiber,[161,162] or as a way of controlling the polarization of radiation transmitted by a fiber without breaking the lightguiding circuit.[163] Thus, it has been shown[163] that the variation in the polarization of a light signal depends both on its input polarization, i.e., on the relative intensity of the modes, and on the position of the perturbing source in the lightguide, i.e., on the phase difference with which the mode arrives in the irregular coupling region. All this requires detailed analysis of the amplitude and phase characteristics of the conversion effect. Whereas, in the case of weak interaction, this analysis was performed relatively simply in a general form (see Sec. 3 and, for example, Ref. 164), in the case of effective coupling, one must turn to the various standard problems (see Sec. 3E).

C. Lightguide with irregular birefringence

Let us begin by considering a uniformly twisted (ψ' = const) lightguide in which there is a smooth transition from a birefringent nondegenerate fiber to a degenerate isotropic fiber. Suppose that the birefringence falls to zero in accordance with the linear relationship $n_x - n_y = \Delta n \cdot \zeta$. This means that there is also a linear relationship of the form $\tilde{q} = (\Delta n / 2\psi')\zeta$, and this corresponds to a smooth transition from linear polarization of the helical modes at one edge of the coupling region ($\tilde{q}^2 \to \infty$ for $\zeta \to -\infty$) to circular polarization at the other ($\tilde{q}^2 = 0$ at the point $\zeta = 0$). This type of irregularity profile occurs, for example, in the case of a linear variation in the mechanical stress or ellipticity of the lightguide core. It follows from the exact solution[88] (Fig. 15) that the conversion matrix (3.2) and, in the final analysis, all the characteristics of the conversion effect in this linear model, are determined by the following coupling parameter:

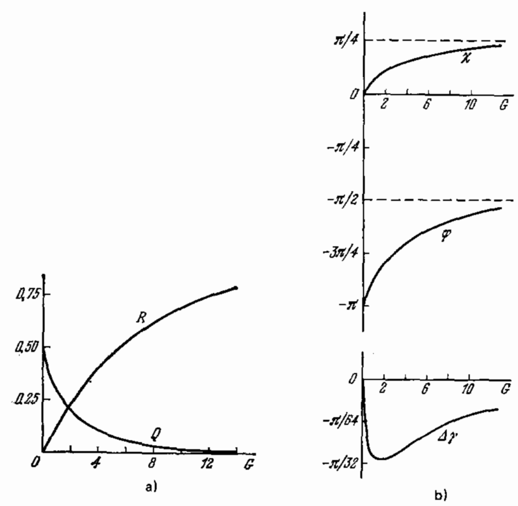

FIG. 15. Linear model of transition from birefringent to degenerate lightguide. A single linearly polarized mode is present at entry to the transition region: a—conversion coefficient Q and axial ratio R of the polarization ellipse; b—phases φ, $\Delta\nu = \nu - \nu_0$ and inclination $\chi = (\pi + \varphi)/2$ of the polarization ellipse to the y axis at the degeneracy point.

$$G = 8\sqrt{2}\,\frac{(\psi')^2}{\Delta n}. \tag{7.3}$$

Figure 15a shows the relative intensity Q of the created mode and the ratio of the axes of the polarization ellipse of the resultant radiation $R = 1 - 2\sqrt{Q(1-Q)}$ at $\zeta = 0$ on the assumption of single-mode degeneracy well away from this point. When $G \ll 1$, i.e., in the case of a sharp irregularity and weak twisting, the conversion coefficient is a maximum and is given by $Q \approx 1/2 - (\pi G/32\sqrt{2})^{1/2}$. In this case, linearly polarized radiation corresponding to one mode at entry traverses the coupling region up to $\zeta = 0$ practically without change in polarization, and $R \approx \pi G/16\sqrt{2} \ll 1$. Since the helical modes are circularly polarized at $\zeta = 0$, the linear polarization at this point corresponds to the superposition of modes of equal amplitude. Consequently, $Q_{max} = 1/2$. As the coupling parameter G increases, the degree of conversion is reduced, and the degree of circular polarization increases. Finally, when $G \gg 1$, i.e., in the case of weak irregularity, a helical mode of a given type will reach the point $\zeta = 0$ in one pitch practically without conversion ($Q \approx 2/G^2 \ll 1$). The polarization of the resultant radiation will therefore be nearly circular for this mode: $R \approx 1 - 2\sqrt{2}/G$.

Figure 15b shows the phase characteristics of the conversion process. The phase ϕ of the created mode varies monotonically from $-\pi$ to $-\pi/2$, i.e., for strong coupling, the new mode appears in antiphase, whereas for weak coupling it appears in quadrature with the incident wave (this property is typical of the coupling phenomenon; see Sec. 3b). The difference $\Delta\gamma = \gamma - \gamma_0$ between the mode phases γ as compared with the geometrical optics situation is small and is of interest only in the case of effective coupling for which $G \sim 1$.

In the case of a symmetric irregular layer, for which the degeneracy segment in the fiber is bounded by birefringent regions on both sides, and the quantities $n_x - n_y$ and $\tilde{q}$ vary linearly between infinite limits [see (3.18)], the conversion coefficient is $Q = \exp(-\pi G/4\sqrt{2})$ and can reach unity (complete conversion). A similar symmetric linear model is also encountered in the problem of two coupled lightguides.[165] However, it was investigated in the greatest detail in connection with nonadiabatic transitions in quantum systems (see Refs. 50 and 93 and the literature cited therein).

D. Nonuniformly twisted lightguide

We now turn to the nonuniformly twisted single-mode lightguide in which the irregularity scale is $\Lambda = |\psi/(d\psi/dz)| = \text{const}$ and the birefringence is $n_x - n_y = \text{const}$. It is clear that we then have

$$\psi(\zeta) = \psi(0)\exp\left(-\frac{\zeta}{k_0\Lambda}\right), \tag{7.4}$$

i.e., the twist period is $g(\zeta) = 2\pi/k_0\psi'$ and the quantity $\tilde{q}(\zeta) = \frac{1}{2\pi}g(\zeta)k_0(n_x - n_y)/2\pi$ that determines the local characteristics of the helical mode (7.2) varies exponentially. The solution of (3.13) can then be shown to reduce to Bessel functions.[119] Analysis of this solution shows that the helical-mode coupling region is localized near $\tilde{q}^2 = 1$, and its characteristic size is equal to the irregularity scale Λ. Hence, it follows that the coupling process is unaffected by regions of highly twisted lightguide, where $g/2 \ll 2\pi/k_0|n_x - n_y|$, or regions of lightly twisted lightguide, where $g/2 \gg 2\pi/k_0|n_x - n_y|$. By taking the asymptotic representations of the Bessel functions for $\tilde{q}^2(\zeta_{in}) \to 0$ and $\tilde{q}^2(\zeta_{out}) \to \infty$, we obtain the elements of the conversion matrix (3.2):

$$Q = \left[1 + \exp\left(\frac{\pi G}{2\sqrt{2}}\right)\right]^{-1}, \tag{7.5}$$

$$\sqrt{1-Q}\,e^{i\gamma} = \sqrt{\frac{\pi}{2}}\left\{\operatorname{ch}\left(\frac{\pi G}{4\sqrt{2}}\right)\Gamma\left(\frac{1}{2} - \frac{iG}{4\sqrt{2}}\right)\left[-\frac{\psi(\zeta_{out})}{2}\right]^{iG/4\sqrt{2}} e^{i\psi(\zeta_{in}) - \pi G/8\sqrt{2}}\right\}^{-1}, \tag{7.6}$$

$$\sqrt{\frac{Q}{1-Q}}\,e^{i\varphi} = -\Gamma\left(\frac{1}{2} - \frac{iG}{4\sqrt{2}}\right)\left[\Gamma\left(\frac{1}{2} + \frac{iG}{4\sqrt{2}}\right)\right]^{-1} \times \left[-\frac{\psi(\zeta_{out})}{2}\right]^{iG/2\sqrt{2}} e^{-\pi G/4\sqrt{2}}, \tag{7.7}$$

where Γ is the gamma function. As can be seen, the character of the optical polarization phenomena determines the coupling parameter given by $G = 2\sqrt{2}|n - n_y|k_0\Lambda$. When $G \gg 1$, mode conversion is exponentially small, and it follows that a simple transition from a highly twisted to a lightly twisted segment of the fiber lightguide (or vice versa) is insufficient for effective coupling. The transition must, in fact, be rapid, on the scale of the mode beats: we then have $G \lesssim 1$ and $Q \sim 1/2$ [see (7.5)].

According to (7.6) and (7.7), coupling will also determine the phase relationship between the modes.[119] Comparison of the phase shift γ with its geometrical optics value γ_0 given by (3.7) shows, as expected (see Sec. 3B), that the two are appreciably different only for $G \sim 1$ when the size of the coupling region Λ is of the order of the spatial mode beat period (see Fig. 16a). Figure 16b shows the other phase relationship $\Delta\varphi = \varphi - \int_{\zeta_{min}}^{\zeta_{out}}(\tilde{n}_2 - \tilde{n}_1)d\zeta$ that characterizes the difference between the phase ϕ of the mode created as a result of coupling from the geometrical optics phase. The latter was calculated on the interval between the section ζ_{min}, in which the departure from geometrical optics of the helical modes is particularly strong $G(\zeta_{min}) = \min G(\zeta)$, up to the exit point ζ_{out}. In other words, the phase ϕ does not contain the geometrical optics phase difference produced after the coupling region. It is clear from $\Delta\varphi(G)$ that, when the single mode $\tilde{n}_2$ arrives at the irregular segment of the lightguide, then under conditions of effective conversion ($G \lesssim 1$), the second mode

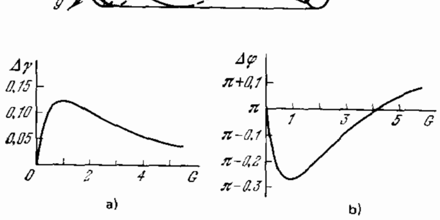

FIG. 16. Exponentially twisted fiber lightguide. The phases $\Delta\gamma = \gamma - \gamma_0$ and $\Delta\varphi$ are plotted as functions of the coupling parameter G.

$\tilde{n}_I$ will appear approximately in antiphase with the first: $\Delta\varphi \approx \pi$.

By determining the conversion coefficient Q and the phases φ and γ from polarization measurements, it is possible to deduce information on the coupling parameter G and on mode interference and, consequently, to determine the irregularity scale Λ, the birefringence, and the type of twist of the fiber. This possibility of nondestructive diagnostics of local lightguide parameters and of the radiation emitted by external local sources (probes) is available for all irregular lightguides in which mode coupling takes place (including time-dependent cases; cf. Sec. 5D).

Problems relating to constant polarization, regular conversion of polarization, and the diagnostics of optical properties are also encountered in the case of multimode lightguides when depolarization produced in them is small.[166,167] Multimode lightguides have large core diameters as compared with single-mode lightguides, and their properties are therefore closer to those of an unbounded anisotropic medium. It may be convenient to use the quasiisotropic approximation of geometrical optics (see Sec. 2D) without introducing the waveguide mode in the analysis of the propagation of light in such systems. One way or another, there is no doubt that the coupling phenomenon connected with polarization degeneracy has the same character in the case of the multimode lightguide as in the examples of unbounded medium and single-mode lightguide, considered above. In particular, it may be expected that conversion effects will be found as a result of recently initiated experiments[168] on the polarization properties of twisted multimode lightguides excited by a parallel beam of light propagating at an angle to their axis.

CONCLUSIONS

In the last few years, studies of the propagation of polarized electromagnetic waves and their mutual linear conversion have ranged well beyond the framework of classical microwave physics, or plasma physics, and have attracted considerable interest in connection with the discovery and application of unusual polarization properties in various inhomogeneous anisotropic media and waveguide structures. All this refers to the wide interval between the radio- and x-ray frequencies and to most types of medium, including magnetoactive plasma (laboratory and space), the "plasma + magnetized vacuum" system, liquid crystals, ferroelectrics, magnetic semiconductors, dielectric waveguides, and so on. Although there are specific reasons for the increased interest in each special case, there is a general tendency to use polarization effects associated with linear wave conversion (and the corresponding polarization techniques) to investigate the inhomogeneous anisotropic structures themselves, and hence develop new polarization devices and instruments. In particular, the phenomenon of linear wave conversion is beginning to play an appreciable role in optics—above all, in the optics of liquid crystals and lightguides—where the polarization properties of inhomogeneous helical structures have been intensively investigated in the last few years.

In these and many other cases, qualitative analysis of linear wave coupling is necessary (and occasionally sufficient) as a basis for the investigation and application of this phenomenon. An important point that has frequently been noted above, but which we must emphasize once again, is that the linear coupling phenomenon is determined not only by the type of behavior of wave dispersion branches but, to a comparable extent, by the type of behavior of their polarization. Qualitative analysis and, above all, determination of the coupling parameter that takes this circumstance into account can be used to elucidate the possible occurrence and the degree of effectiveness of wave coupling in any particular situation. Moreover, it will be clear from this review that qualitative analysis will also point to the characteristic features of the conversion effect as a function of the properties of the inhomogeneous medium and of waves propagating within it. (Examples were examined in Secs. 4–7; they do not, of course, exhaust the entire range of phenomena connected with linear wave conversion.) All this can be done without obtaining a direct solution of the coupling equations (this is particularly valuable because direct solution is, in most cases, rather difficult). Our review suggests that a sufficiently fundamental understanding of linear wave conversion effects can be achieved by relatively simple means. This is particularly important at present in view of the expanding range of application of linear wave conversion effects.

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