Nonlinear waves and one-dimensional turbulence in nondispersive media

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The main results of the theory based on the solution of Burgers' equation for large Reynolds numbers are reviewed. The basic properties of the arising stochastic regime, which is an example of strong turbulence, and its relation to hydrodynamic turbulence are discussed. Different stages in the evolution of a nonlinear wave are interpreted from the point of view of a flow of noninteracting particles. The statistical properties of Riemannian waves are analyzed for the stage of single-stream propagation. Methods for describing and the characteristics of the turbulence of sawtooth waves, forming at the many-stream stage, are examined. The self-preserving nature of this regime is demonstrated. The coupling of regular and random waves at different stages of propagation is examined. The possibility of describing the evolution of the average velocity with the help of turbulent viscosity is analyzed. Possible generalizations of the theory to related problems are discussed.

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CONTENTS

1. INTRODUCTION

Nature provides many examples of nonlinear random fields and waves. There is no need to comment on the importance of studying them. This is already clear from a brief listing: turbulence in liquids and gases, chaotic motions of plasma, intense acoustical noise, and random waves on the sea surface. One trend in the development of the theory of nonlinear random waves consists of identifying a small number of concepts and ideas that, on the one hand, permit describing in a unified manner the behavior of nonlinear random waves of different physical nature and, on the other, classifying clearly nonlinear random waves according to the nature of the interactions characteristic of them. This includes primarily the concept of weak and strong turbulence. Weak turbulence is characteristic of weakly linear waves in media with strong dispersion, when the energy of interaction of the spatial harmonics is much smaller than the total energy. The random phase approximation, which assumes that the interaction between the harmonics is noncoherent, permits, in this case, a closed statistical description of the turbulent state.[1,2] If the dispersion of the waves is small or is absent, then the properties of the turbulence are determined by the strong interaction of a large number of coherent harmonic waves. In such cases, it is customary to talk about strong turbulence. The best-known example of strong turbulence is the eddy turbulence of a low-viscosity liquid.[3]

The extreme difficulty of analyzing nonlinear waves, especially strong turbulence, has given rise to another trend in the development of the theory of such waves: the transition from complicated equations of nonlinear random waves to simpler model equations. One of such model equations of strong turbulence is Burgers' equation (BE)

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial x^2}. \tag{1.1}$$

The solution of BE with the random initial condition

$$u(x,\ t=0)=u_0(x) \tag{1.2}$$

takes into account the combined action of two very important mechanisms, which form the properties of real hydrodynamic turbulence: inertial nonlinearity and viscosity. Equation (1.1) is deservedly named after Burgers,[1)] who not only proposed it as the simplest model of hydrodynamic turbulence,[4a] but also clarified many characteristics of the behavior of the model turbulence.[4b]

It is now increasingly recognized that the formation of locally coherent nonharmonic profiles plays an important role in strong turbulence. The nature of these structures, which have been studied primarily for spatially one-dimensional waves, depends primarily on a basic factor that limits nonlinearity in this system. Where dispersion fulfils the role of this factor, strong turbulence can be described as a gas of solitons distributed in space.[5]

A different picture emerges if weak dissipation fulfils the role of the limiter of nonlinearity, as in Burgers' turbulence (BT), which can be viewed as a gas of large, adjoining quasiparticles.[4,6,7] In this case, it is especially interesting that the description of the turbulent regime that appears can be related to the initial conditions and it is thus possible to follow the complete pattern of evolution of random disturbances in the system. A detailed analysis of the stochastic regimes arising here was made by Burgers himself, as well as in work performed at the University of Kyoto, Gor'kiĭ State University, and the Institute of Physics of the Atmosphere of the USSR Academy of Sciences.

The value of BE as a model equation for eddy turbulence is sometimes questioned in view of the enormous physical difference between the two problems. In this respect, a more adequate model, discussed in recent years, could be the relation between hydrodynamic turbulence and stochastic oscillations of systems with a small number of degrees of freedom.[8] It is, however, impossible to deny the similarity between BT and hydrodynamic turbulence. In both cases, the strong nonlinear interaction establishes universal power-law asymptotic behavior of the energy spectrum, self-preserving properties of the turbulence, etc. In view of the undoubted generality of the problems of closing the equations for the statistical characteristics, BT could be useful for preliminary testing of approximate methods of closure and description of hydrodynamic turbulence.[7,9-11]

Often, the simplest model descriptions of complex phenomena occurring in nature in due time find an increasing number of applications and acquire an increasingly deeper meaning. This is precisely what has occurred with BE. It became clear that the description of a wide class of nonlinear acoustical waves reduces to BE.[12-16] It turned out that the solutions of BE adequately describe the processes of nonlinear steepening and subsequent viscous dissipation of waves of different physical nature in nonlinear nondispersive media. It has been found that BE is related to the ray description of wave propagation, flows of noninteracting particles, and a gas of inelastically colliding particles.[7] Finally, it has become clear that BE is a standard equation for a wide class of waves in nonlinear nondispersive media, worthy of occupying a place alongside the classical linear hyperbolic equation.

Under closer scrutiny, the relations between BT and the properties of strong hydrodynamic turbulence turned out to be even closer. Models of large quasiparticles in BT may turn out to be close to models of a type of eddy turbulence, when its structural nature is followed distinctly.[18] The problems of describing turbulence are related to the BE, as Struminskiĭ indicates,[19] in another respect as well. Burgers equation describes in the simplest approximation the potential part of the fluctuation component of eddy turbulence. The next approximation gives the three-dimensional BE, which is likewise an interesting object for research. Thus the study of BT is apparently a necessary step along the path of formulating models of hydrodynamic turbulence and strong turbulence in general.

In this review, we present the basic ideas of the theory of one-dimensional nonlinear waves in nondissipative media from a unified point of view, we discuss the physical applications of BE, and we analyze in detail methods for describing the statistical properties of BT as well.

2. NONLINEAR WAVES IN A NONDISPERSIVE MEDIUM

a) Dispersive and hyperbolic waves

From the large variety of wave motions, it is possible to single out, with a certain degree of arbitrariness, the classes of dispersive and hyperbolic waves.[17] Dispersive waves, existing in a medium with its own temporal and spatial scales, are characterized by the dependence of the velocity of propagation on the frequency or wave number. Hyperbolic waves arise in media without inherent scales or if the magnitude of these scales are incommensurate with the scales of the wave.

Weak, plane hyperbolic waves propagate without distortions with a single velocity c and, outside the region of the source (to the right of it), they satisfy the equation

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=0. \tag{2.1}$$

The velocity c, however, is the same only for weak waves. In nondispersive media, it can depend on the amplitude, as a result of which, nonlinear distortions of the shape of the wave appear.

The difference between dispersive and hyperbolic waves is manifested in the basic model, serving for their interpretation. For dispersive waves, the basic

[1)]Jan Burgers (1895–1981) studied in Leiden with N. Bohr, A. Einstein, H. Lorentz, and P. Ehrenfest. He was a professor at the universities in Delft (Holland, up to 1955) and Maryland (USA). He first worked on N. Bohr's model of the atom, but after meeting T. Karman, he turned his attention to the problems of fluid mechanics. In this field, Burgers obtained fundamental results and wrote a number of books. His other works concern the physics of crystals and suspensions and the philosophy of science (see the obituary in Physics Today, No. 1, 1982).

model is a collection of weakly interacting oscillators, while for hyperbolic waves, it is a flow of weakly interacting particles. The potential energy of the system plays an important role in the first case, while in the second case, the potential energy is much smaller than the kinetic energy.

There is another aspect to the difference under consideration. Dispersive waves propagate in pre-existing structures, while nonlinear hyperbolic waves can often be related with the process of formation of structures.

Of course, the juxtaposition of the two types of wave motion is not absolute. In real problems of hydrodynamics, astrophysics, and the theory of plasma, the variation of not one, but several local parameters of the medium (velocity, density, temperature, etc.) can be examined. In this case, terms describing their mutual influence must be introduced into the corresponding equations, which can lead to the appearance of a mixed type of motion. This is especially true for multidimensional systems. However, in these cases as well, as a rule, characteristic times and scales, on which one or another simplified approach is valid, can be identified.

b) Riemann waves

So-called finite-amplitude waves, in the analysis of which the appearance of a wave moving in the opposite direction or, in other words, self-backscattering of the nonlinear wave, can be neglected, are of considerable interest in the analysis of nonlinear distortions in hyperbolic systems. It is convenient to study waves of this type in a system of coordinates moving together with the wave.

The basic equation of nonlinear finite-amplitude waves in nondispersive media is called the equation of the simple wave or the equation of Riemann, who obtained it from the equations of gas dynamics:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0. \qquad (2.2)$$

The simplest physical example, which gives a clear interpretation of Riemann waves, is the hydrodynamic flow of noninteracting particles, each of which moves along the x axis with a constant velocity. If the velocity profile of the particles $u_0(x)$ is given at time $t=0$, then the velocity field $u(x,t)$ satisfies Riemann's equation with the initial condition (1.2). Without repeating Refs. 1 and 20, wherein this example is examined in detail, we shall present the velocity profiles, illustrating how the smooth profile (Fig. 1a) first becomes steeper due to the fact that some particles overtake others (Fig. 1b)

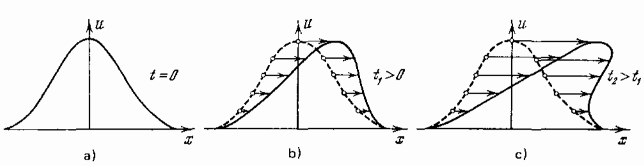

FIG. 1. Evolution of the velocity profile $u(x,t)$ of a Riemann wave. a) initial profile; b) steepening of profile; c) profile in the region of many-stream propagation.

and then, after passing them, the wave topples over and becomes a many-stream wave (Fig. 1c). The toppling is accompanied by a coalescence of nearby particles, which is accompanied by gradient catastrophes: appearance of infinite gradients of the velocity field. The onset of toppling can be found from the condition

$$t_0=-\frac{1}{\min u_0'(x)}. \qquad (2.3)$$

c) Shock waves

Before toppling, Riemann's wave behaves as a flow of noninteracting particles. After toppling, waves of different physical nature separate into two types: waves with resolved many-stream motion (the same flows of noninteracting particles, many-stream motion of cold plasma, multiray propagation of light waves) and waves which are inherently single-stream waves. Pressure waves in a gas are a typical example of the latter. For these waves, at the instant of toppling, one must take into account in the vicinity of the gradient catastrophe the nonlocal interaction of the sharply varying wave with the medium, leading to the formation of steep drops in the wave profile, the so-called discontinuities or shock fronts, as before and after which the smooth profile of the wave satisfies Riemann's equation as before.

There are two approaches to describing waves in a nondispersive medium, taking shock fronts into account (see, for example, Refs. 17, 21, and 22). The first approach, which almost ignores the problem of the mechanism of the formation of shock waves, assumes that the fronts are infinitely thin and singles out from the possible mathematically equivalent solutions of Riemann's equation, the physically true solutions, satisfying the fundamental conservation laws and the thermodynamic inequalities. Here, use is made of the fact that the differential equations describing the waves follow from more general integral laws, which are also valid in the regions of the gradient catastrophes.

When a Riemann wave topples over, weak shock waves arise, whose description is based on the law of conservation of momentum. In analogy to the flow of noninteracting particles, the formation of discontinuities can be viewed as being a result of absolutely inelastic collisions between particles, while the discontinuity itself can be viewed as a heavy particle, formed as a result of attachment of light particles. Weak shock waves do not take into account backscattering by the discontinuities (partial elasticity of collisions).

The second approach explicitly takes into account the nonlocal nature of the interaction of the wave with the medium in the vicinity of shock fronts and leads to more complicated, compared with (2.2), equations. Inclusion of dissipation in the simplest approximation leads to Burgers' equation, which describes the absorption of energy in the region of the discontinuity as a process occurring with a finite rate. As a result of this, the region of the front can be viewed as having extent and structure.

d) Applications of Riemann's and Burgers' equations

In application to waves in nondispersive media, Riemann's and Burgers' equations arise as abridged equations that take into account the slowly accumulating nonlinear and dissipative distortions. The method for deriving such equations was developed by Khokhlov for the example of waves in long nonlinear radio transmission lines.[23] Analogous equations for electromagnetic waves in nonlinear media and long transmission lines were obtained by Ostrovskiĭ.[24,25] The BE and RE approximations to the analysis of nonlinear electromagnetic waves are discussed in Refs. 26–28.

Burgers' equation is encountered even more often in nonlinear acoustics, where it is derived from the equations of hydrodynamics of a viscous heat-conducting medium[14-17] and is generalized to the case of cylindrical[29] and spherical[30] waves and waves in media with relaxation.[31] The analysis of the propagation of intense acoustical beams also reduces to such equations if diffraction and nonlinear distortions are spatially separated.[32,33] Thus, if diffraction of the beam is initially more significant, then it is calculated using the linear theory, taking into account the subsequent nonlinear steepening with the help of the nonlinear acoustics of spherically diverging waves (see, for example, Ref. 34).

The analysis of excitation of an acoustical wave by intense modulated optical radiation reduces to the inhomogeneous Burgers' equation in acoustics.[35,36] Burgers' equation is also used to describe nonlinear wave processes in thermoelastic media.[37]

RE and BE are used to calculate high-frequency acoustical waves in inhomogeneous media by the method of nonlinear geometrical acoustics. In this case, the nonlinear distortion of the wave is described by RE, taking into account the discontinuities, in a system of coordinates fixed to the rays in the linear inhomogeneous medium.[38,39]

We also note that for weakly linear waves in a compressible fluid, when the interaction with countermoving waves is small due to high-frequency averaging, the perturbations of the density and velocity of the fluid can be represented as a superposition of waves each of which in its accompanying system of coordinates is approximately described by BE.[40,41] The approximation of weak interaction turns out to be valid not only for countermoving waves, but also for quasiperiodic waves, traveling at not too small angles relative to one another.[42,43] Thus BE can also be used to describe one-dimensional turbulence of a compressible fluid[40,41] and waves in nonlinear acoustical resonators and waveguides.[42,43]

Burgers' and Riemann's equations are usually associated with waves in nondispersive media. However, very similar equations also arise naturally in the analysis of short-wavelength radiation in dispersive media, when the scales of the inhomogeneities in the medium and of the slowly varying frequency and local wave vector are large compared to the dispersive or diffractive nonlocal behavior of the waves. In this case, the evolution of the wave parameters is described by the equations of geometrical optics (acoustics, etc.), close to and sometimes coinciding with RE (see, for example, Refs. 44, 45). As Riemann's equation in acoustics, the equations of geometrical optics are not valid near caustics: gradient catastrophes, where the nonlocal behavior of the waves must be taken into account.

As examples of other applications of RE and BE, we note also the broad spectrum of kinematic waves: surges, motion of glaciers, waves in traffic flow, etc.[17]

3. DYNAMIC OF NONLINEAR WAVES IN A DISSIPATIVE MEDIUM

a) Basic stages of evolution

The dynamics of a nonlinear wave, including dissipative effects, is determined by the law of conservation of momentum following from BE

$$M=\int_{-\infty}^{+\infty} u(x,\ t)\,\mathrm{d}x=\int_{-\infty}^{+\infty} u_0(x)\,\mathrm{d}x=\mathrm{const},\tag{3.1}$$

and the relation for the rate of dissipation of energy

$$\frac{\partial}{\partial t}\frac{1}{2}\int_{-\infty}^{+\infty} u^2(x)\,\mathrm{d}x=\nu\int_{-\infty}^{+\infty}\left(\frac{\partial u}{\partial x}\right)^2\mathrm{d}x.\tag{3.2}$$

The change in the wave profile is characterized by the toppling and dissipation times: $t_n = l_0/u_0$ and $t_d = l_0^2/\nu$, where u_0 and l_0 are the amplitude and scale of the initial perturbation. The ratio of these times gives the value of the acoustical Reynolds number:

$$R_0=\frac{t_d}{t_n}=\frac{u_0 l_0}{\nu}.$$

For $R_0 \ll 1$, nonlinear effects are not important and the behavior of the wave is determined by linear dissipation.

In the more interesting case $R_0 \gg 1$, the wave passes through three stages:

I. The initial stage before the formation of discontinuities ($t < t_n$), in which the coherent nonlinear interaction of the harmonics in the initial perturbation with conservation of energy appears.

II. Stage of discontinuous waves ($t_n < t < t_1$, where t_1 is determined by the same equation as t_d, but, in this case, generally speaking, it is necessary to take into account the change in scale of the wave during propagation). At the stage of discontinuous waves, shock fronts form in the wave. The position of the fronts can be found from the conservation law (3.1), leading to the rule of equal areas well known in acoustics.

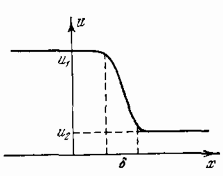

FIG. 2. Profile of a shock front. $\delta = \nu/(u_1 - u_2)$ is the width of the shock front.

The structure of the shock fronts can be judged from the stationary solution of BE:

$$u(x-u_+t)=u_+-u_-\,\mathrm{th}\left[\frac{u_-}{2\nu}(x-u_+t)\right], \tag{3.3}$$

representing a jump in the field u with amplitude u_- and width δ propagating with velocity u_+, where

$$u_+=\frac{u_2+u_1}{2},\quad u_-=\frac{u_2-u_1}{2},\quad \delta=\frac{\nu}{u_-}.$$

Any initial perturbations with fixed $u_0(-\infty)=u_1$ and $u_0(\infty)=u_2$ approach a stationary wave of the form (3.3), but it is especially important that for small ν, the shape of the shock front is established more rapidly than the changes in u_1 and u_2 and, for this reason, describes the local profile of the shock front, where energy dissipation occurs. Between discontinuities the velocity field at this stage varies according to a linear law, corresponding to the self-preserving solution of BE. On the whole, the wave represents a saw whose teeth have the same slopes.

III. Stage of linear damping ($t>t_1$). At this stage, the width of the dissipation zone is of the same order of magnitude as the scale of the wave, the nonlinear interaction of the harmonics disappears, and energy is lost due to linear dissipation.

The behavior of a nonlinear wave for $R_0\gg 1$ is interpreted well by two models that coincide initially: model of noninteracting particles and model of particles undergoing inelastic collisions.

b) Relation to flow of noninteracting particles

A remarkable property of BE is that it admits an interpretation from the point of view of a flow of noninteracting particles, not only in the single-stream region, but also in the many-stream region of the motion. However, in contrast to the problem of the propagation of light, in which the streams interfere, here, we must talk about their competition. At the stage $t_n<t<t_1$, it turns out that of the streams arriving at a point, for $R_0\gg 1$, only the stream with minimum action need be included, since the other streams are absorbed by the discontinuities.

The possibility of such an approach follows from the exact solution of BE, which with the nonlinear substitution[46,47]

$$u(x,\,t)=-2\nu\frac{\partial}{\partial x}\ln\varphi(x,\,t) \tag{3.4}$$

reduces to the linear diffusion equation, and this permits obtaining a solution of the initial problem (1.1) and (1.2) in the form

$$u(x,\,t)=\frac{\int_{-\infty}^{+\infty}(x-y)\exp[-s(x,\,y,\,t)/2\nu]\,dy}{t\int_{-\infty}^{+\infty}\exp[-s(x,\,y,\,t)/2\nu]\,dy}, \tag{3.5}$$
$$s(x,\,y,\,t)=s_0(y)+\frac{(x-y)^2}{2t};\quad s_0(y)=\int^{y}u_0(x)\,dx.$$

The properties of the exact solutions of BE and tables of these solutions are presented in Ref. 48.

For $R_0\gg 1$, the integrals in (3.5) can be calculated sufficiently accurately by the saddle-point method[4b,7]:

$$u(x,\,t)=\frac{\sum_m u_m(x,\,t)\,|j_m(x,\,t)|^{-1/2}\exp[-s_m(x,\,t)/2\nu]}{\sum_m |j_m(x,\,t)|^{-1/2}\exp[-s_m(x,\,t)/2\nu]}. \tag{3.6}$$

The collection of quantities entering into (3.6) can be viewed as multivalued functions: the velocity of the particles in the stream, satisfying RE and (1.2); the action of the stream, satisfying the Hamilton-Jacobi equation

$$\frac{\partial s}{\partial t}+\frac{1}{2}\left(\frac{\partial s}{\partial x}\right)^2=0,\quad s(x,\,0)=s_0(x), \tag{3.7}$$

and, the divergence of the stream, satisfying the equation

$$\frac{\partial j}{\partial t}+u\frac{\partial j}{\partial x}=j\frac{\partial u}{\partial x},\quad j(x,\,0)=1. \tag{3.8}$$

A graphic interpretation relates (3.6) to the hydrodynamic flow of noninteracting particles. Each term in the sum corresponds to a particle, with an initial Lagrangian coordinate y_m, arriving at the point (x,t). The summation extends over all particles arriving at the point for which $j>0$. In the Lagrangian representation, the parameters of the flow in the vicinity of a fixed particle with initial coordinate y are described by a system of characteristic equations, whose solution has the form

$$\left.\begin{aligned} &x(y,\,t)=y+u_0(y)\cdot t,\quad u(y,\,t)=u_0(y),\\ &s(y,\,t)=s_0(y)+\frac{1}{2}u_0^2(y)\,t,\\ &j(y,\,t)=\frac{\partial x}{\partial y}=1+qt,\quad q(y,\,t)=\frac{\partial u}{\partial y}=u_0'(y).\end{aligned}\right\} \tag{3.9}$$

Equation (3.6) can be viewed as an average over trajectories arriving at a point with the help of a sum analogous to a partition function. At the initial stage, the sum contains one term, corresponding to the only solution of RE. At the shock-wave stage, the sum can include several terms, but, at almost every point there exists one dominant term, which has minimum action. For $t>t_1$, the action of many terms (particles) becomes comparable, so that many terms must be included in the sum.

It is of greatest interest to use the model of competing streams to describe the properties of discontinuous waves. In this case,

$$u(x,\,t)=\frac{x-y(x,\,t)}{t}, \tag{3.10}$$

where y is the initial coordinate of the particle whose action is the least of all particles arriving at the point. Equation (3.10) gives a simple graphical rule for finding the values of $u(x,t)$ according to the point of minimum action $y(x,t)$.[4b,7] For this, by varying H, it is necessary to find the first point at which the initial action $s_0(y)$ is tangent to the parabola (Fig. 3)

$$\alpha=H-\frac{(x-y)^2}{2t}. \tag{3.11}$$

At times exceeding the toppling time, the entire stream of particles separates into partial streams, bounded by discontinuities, arising at the point x_k where the actions of the two dominant particles are equal:

$$x_k=\frac{y_{k+1}^-+y_k^+}{2}+V_kt,\qquad V_k=\frac{s_0(y_{k+1}^-)-s_0(y_k^+)}{y_{k+1}^--y_k^+}. \tag{3.12}$$

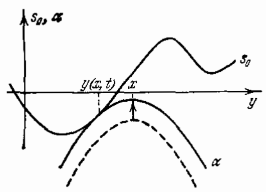

FIG. 3. Graphical determination of the point of the absolute minimum of the action. The initial coordinate, corresponding to the absolute minimum of the action $y(x, t)$ for fixed coordinate of the point of observation x coincides with the point at which the initial distribution function of the action $s_0(y)$ is tangent to the parabola $\alpha = [-(x-y)^2/2t] + H$, where H assumes the minimum of possible values.

Here y_k^+ and y_{k+1}^- are the initial coordinates of particles arriving at the point of discontinuity from the left and from the right (see Fig. 4). Taking into account the competition of the actions of two particles, the profile of the shock front assumes the form (3.3), where

$$u_+ = V_k, \quad u_- = \frac{y_{k+1}^- - y_k^+}{2t}.$$

For $t \gg t_n$, the initial coordinates y_k^- and y_k^+, bounding one partial stream, turn out to be in a small neighborhood of the point y_k, where s_0 has a minimum. In this case, for all points of the given flow, we can set $y = y_k$ in Eq. (3.10), and from this it follows that between discontinuities the velocity field is described by the self-preserving solution of RE and BE: $u = (x - y_k/t)$. The set of linear segments as a whole forms a sawtooth wave with teeth having a slope $1/t$ (Fig. 5).

In the case that the minima of the initial action are of different depth, the discontinuities move toward shallower minima, i.e., partial streams related to increasingly deeper minima predominate, as a result of which, the scale of the teeth in the sawtooth wave increases.

c) Colliding-particle model

If we examine a stream of particles with uniform initial density $\rho = 1$, then the evolution of the wave leads to the formation of regions with high and low density. In low-density regions (between discontinuities), the density varies as $\rho \sim 1/j$, while the partial momentum of the region has the form

$$P_k = s_0(y_k^+) - s_0(y_k^-).$$

Particles with the common mass

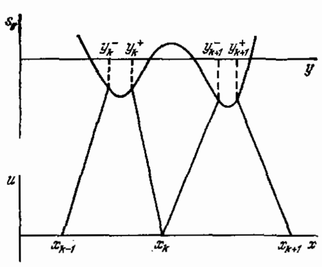

FIG. 4. Formation of discontinuities. Position of the discontinuity x_k corresponds to the point at which the actions of streams emanating from the neighborhoods of two minima s_0 are equal.

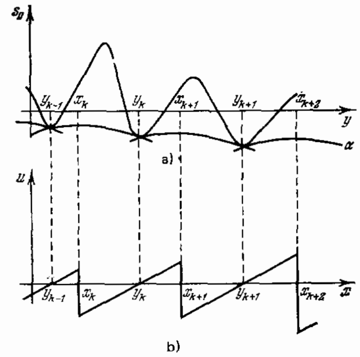

FIG. 5. Formation of the profile of a sawtooth wave. The centers of the teeth y_k correspond to minima of the initial distribution of the action s_0.

$$m_k = y_{k+1}^- - y_k^+ = [u_0(y_{k+1}^-) - u_0(y_k^+)]\, t,$$

accumulate in high-density regions (on the fronts), and from this we have the following expression for the momentum of such a heavy particle:

$$\tilde{P}_k = m_k V_k = s_0(y_{k+1}^-) - s_0(y_k^+).$$

Evaluating the sum, we find

$$\sum_k (P_k + \tilde{P}_k) = M = \text{const}, \qquad (3.13)$$

i.e., in this interpretation, (3.1) represents the law of conservation of momentum for a gas consisting of two types of particles. Momentum is transferred from the light particles to the heavy particles according to the law of absolutely inelastic collisions.

For $t \gg t_n$, the entire mass is practically concentrated in the regions of the discontinuities, which can be viewed as a gas of heavy particles with masses $m_k = y_{k+1} - y_k$ and velocities

$$V_k = \frac{s_0(y_{k+1}) - s_0(y_k)}{y_{k+1} - y_k}.$$

The difference between the velocities of the discontinuities also leads to their inelastic collisions, as a result of which they coalesce. The amplitudes ($u_k = m_k/2t$) and velocities of coalescing discontinuities are determined by the laws of conservation of mass and momentum.[7]

More rigorously, a discontinuity must be viewed as a quasi-particle with internal structure and finite size, whose spreading is limited by pressure from the low-density region. We note that the structure of the discontinuity is quasi-one-dimensional. Using this fact, it is possible to obtain a selection rule for the dominant particle according to the minimum of the action and the velocity field for $t \gg t_n$, avoiding the exact solution of BE.[21,22] This makes it possible to generalize the results obtained to equations with a more general type of nonlinearity of the form $Q(u)\partial u/\partial x$.

We should point out that there is a definite analogy between the formation of discontinuities in a shock wave and caustics in a light wave. The latter likewise represent high-density regions of the field, whose structure is independent of the initial conditions and which arise at a well-defined stage in the evolution of the initial distribution.

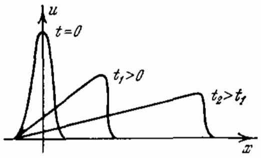

FIG. 6. Evolution of a unipolar initial perturbation.

d) Unipolar and periodic waves

We shall illustrate the evolution of the waves through the three stages indicated in subsection 3a using two examples that are important for understanding the properties of BT.

1. Unipolar initial pulse ($M \neq 0$, Fig. 6). At times $t > t_n$, it approaches the self-preserving solution of BE, following from (3.5) with $u_0(x) = M\delta(x)$,[17] whose main characteristic is that its instantaneous Reynolds number $R = M/\nu \sim R_0$ remains constant in time. For this reason, for $R_0 \gg 1$, the wave does not reach stage III of linear dissipation and at times $t \gg t_n$, it has the form of a triangle:

$$u(x, t) = \begin{cases} \frac{x}{t}, & 0 < x < \sqrt{2Mt}, \\ 0, & x < 0,\ x > \sqrt{2Mt}, \end{cases}$$

whose leading edge is described by Eq. (3.3) with $u_- = u_+ = \sqrt{2Mt}$.

2. Sinusoidal initial field (Fig. 7)

$$u_0(x) = u_0 \sin \frac{2\pi x}{l_0}, \quad R_0 = \frac{u_0 l_0}{\nu} \gg 1.$$

Its evolution is discussed, for example, in Ref. 14. Here, we point out that for $R_0 \gg 1$, the wave will pass through all three stages. When $t < t^* = t_n/2\pi$, the nonlinearity efficiently generates higher-order harmonics, but there are still no shock fronts and dissipation is practically absent. At stage II ($t_n \ll t \ll t_g$), the wave has the form of a sequence of triangular pulses with stationary shock fronts with width $\delta(t) \sim \nu t/l_0$ and amplitude $u_- = l_0/t$. We emphasize that at this stage, the periodic wave and, therefore, its energy $u^2 \sim l_0^2/t^2$ forget the initial amplitude u_0. At stage III ($t \geqslant t_d$), the width of the shock fronts equals the period of the field and the wave again approaches a sinusoidal shape, but it no longer depends on u_0.

For a quasiharmonic initial field, the wave likewise must pass through all three stages. However, due to the possible difference between the velocities of shock fronts, from time to time they will coalesce and the average distance between them $l(t)$ will increase. Ultimately, the instantaneous Reynolds number $R(t) \sim l(t)/\delta(t)$ decreases more slowly than for a periodic wave and attainment of the linear stage is delayed. This result is especially valid for BT with a broad-band initial spectrum. In this sense, the evolution of BT is closer to the evolution of a unipolar pulse. The case of quasiharmonic initial waves, which is important for acoustical applications, is examined in Refs. 14 and 50–52.

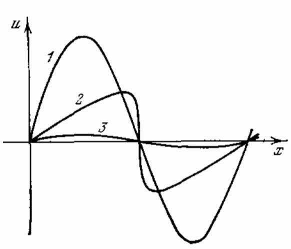

FIG. 7. Evolution of a sinusoidal wave. 1) Initial distribution; 2) sawtooth wave; 3) wave at the damping stage.

4. RANDOM WAVES IN A NONDISPERSIVE MEDIUM

a) Methods of analysis

The statistical description of random nonlinear waves represents a special problem, even if the exact solution of the dynamic problem is known, since it leaves unresolved difficulties related with averaging. One way to overcome these difficulties is to formulate closed equations for the moments, starting from general considerations concerning the nature of the statistical relations in the given problem.[3]

If we examine the evolution of a regular wave packet against the background of a noise wave, then it is natural to attempt to close the equation for the average field by introducing the turbulent viscosity, which describes transfer of energy from the coherent component into the noncoherent component.[53,54] The justification for this approach is discussed in Sec. 6.

An equation, analogous to the Karman-Horvath equation in the theory of hydrodynamic turbulence,[3] for the correlation function of statistically homogeneous Burgers turbulence

$$B(\rho, t) = \langle u(x, t)\, u(x+\rho, t)\rangle \tag{4.1}$$

follows from BE. This equation is not closed, since it contains the third moment function $\langle u^2(x+\rho, t)u(x, t)\rangle$, the equation for which contains a fourth moment, etc. The main techniques for closing the equation involve truncating this infinite chain of equations, setting the higher moments or cumulants equal to zero. These assumptions are valid for small Reynolds numbers or when the probability characteristics of the turbulence are nearly Gaussian (Millionshchikov's hypothesis). However, for $R_0 \gg 1$, such closure leads to physically contradictory results, for example, to the appearance of negative values of the energy spectrum,[57] which is a result of the sign-alternating model distributions with a finite number of cumulants.[55,56]

A detailed review of such methods in application to turbulence in a different number of dimensions is given in Ref. 10 (see, also, Refs. 58 and 59).

In Kraichnan's "direct interaction" approximation[60–63] and in the method of "Markov stochastic models,"[64–65] the infinite chain of equations for the correlation function is replaced by a model equation, which takes into account correctly the trend in the evolution of $B(\rho, t)$, namely, the breakdown in analyticity as $\nu \to 0$ and energy dissipation due to formation of discontinuities, but does not permit following the process over long periods of time.[66,67]

In analyzing BT, use is often made of an expansion in stochastic orthogonal functions (Wiener-Cameron-Martin method), the first term of which describes the Gaussian component and the remaining terms describe the non-Gaussian effects. The nonlinearity of BE leads to an infinite chain of equations for the kernel of the series, truncation of which does not permit studying BT for $R_0 \gg 1$.

We recall also Refs. 76 and 77, where expansion of BT into modes was used to construct a cascade model of turbulence.

A more systematic approach to the problem involves an analysis of the equations for the characteristic Hopf functional.[78,79] The one-dimensionality and the existence of an exact solution of BE permit more progress along this line of thought[9,80-82] than in eddy turbulence.

Summarizing what has been said above, we note that attempts to describe BT by solving approximate equations for the correlation function or by other similar methods are not successful for large Reynolds numbers.

Another approach to the statistical description of nonlinear systems is based on attempts to represent them as an ensemble of stable and weakly interacting elements. The ensemble of particles describing a rarefied gas or the ensemble of mode-oscillators describing weak turbulence is an ensemble of this type. For waves satisfying the RE and BE, such an ensemble can be based on models of noninteracting or weakly interacting particles.[4,7,49,83-86]

b) Lagrangian approach

The relation of the solutions of the RE and BE equations to the flow of noninteracting particles, indicated in Sec. 3, makes the Lagrangian approach a natural approach to the analysis of the statistics of BT. The discussion concerns the investigation of Lagrangian statistics and its use for subsequent reconstruction of the statistics of Eulerian fields. The key relations here are the universal coupling equations, which are of interest in themselves for comparing different types of experiments.

These equations were obtained for a compressible fluid in Refs. 90–92. Analogous equations for an incompressible fluid are presented in Refs. 87–89. If we examine a fluid particle with initial coordinate y, then W_l, the distribution of its parameters [instantaneous coordinate $x(y,t)$, velocity $u(y,t)$, and divergence $j(y,t)$], is related to W_e, the distribution of the fields $u(x,t)$ and $j(x,t)$ at the point x, by the equation

$$W_e(j,\ u;\ x,\ t) = |j| \int_{-\infty}^{+\infty} W_l\,(j,\ u,\ x;\ t|y)\,dy. \tag{4.2}$$

For a statistically homogeneous fluid, this equality goes over into

$$W_e\,(j,\ u;\ t) = |j|\,W_l\,(j,\ u;\ t). \tag{4.3}$$

Thus the Eulerian and Lagrangian distributions of the same parameters of a statistically homogeneous fluid differ by $|j|$, since a point in space has a higher probability of being located in an expanded particle than in a compressed particle (see Fig. 8).

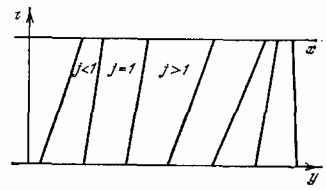

FIG. 8. Difference between the Lagrangian and Eulerian velocity distributions, due to expansion and compression of fluid particles. j is the expansion factor, y is the initial coordinate of the particle, and x is the instantaneous coordinate of the particle.

The above equations take into account the many-stream nature of the motion. For an average number of flows $\langle N\rangle$ arriving at a point, we have

$$\langle |j| \rangle_l = \langle N\rangle. \tag{4.4}$$

Equations analogous to (4.2) and (4.3) establish connections between many-point distributions, as well as spectra.[93]

Applying the coupling equations to the flow of noninteracting particles and using the equations describing the change in the parameters s, u, q, and j along Lagrangian trajectories, we obtain the Eulerian distribution in the form

$$W_e(s,\ u,\ j,\ q;\ x,\ t) = |j|\,w_0\left(s - \frac{1}{2}u^2 t,\ u,\ q;\ x - ut\right)\delta(j - 1 - qt), \tag{4.5}$$

where $w_0(s,u,q;x)$ is the single-point distribution of the initial action and the velocity and its derivative $[s_0(x);\ u_0(x);\ u_0'(x)]$.

c) Initial stage of turbulence

At the initial stage of evolution of the perturbation ($t \ll t_n$), when the velocity field satisfies the RE and the motion may be assumed to be a single-stream motion, the statistical characteristics of the wave can be described using expression (4.5) and the many-point distributions analogous to it. If the initial perturbation is a statistically homogeneous Gaussian velocity field with a correlation function of the form

$$\langle u_0(x)\,u_0(x+\rho)\rangle = B_0(\rho), \quad B_0(0) = \delta^2,$$
$$-B''(0) = k_1^2 = \frac{\sigma^2}{l_0^2}, \quad B^{\mathrm{IV}}(0) = k_2^2 \sim \frac{\sigma^2}{l_0^4},$$

then the dependence of the average number of flows $\langle N\rangle$ on $z = k_1 \cdot t$ is shown in Fig. 9.

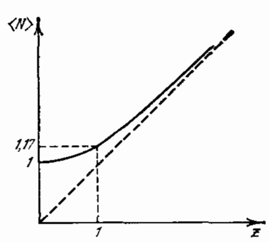

FIG. 9. Dependence on the dimensionless time of the average number of particle streams arriving at a given point.

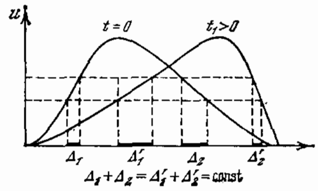

FIG. 10. Conservation of the velocity distribution for BT before the wave topples. Δ_1 and Δ_2 are the spatial intervals, corresponding to the velocity interval Δu in the initial perturbation. Δ_1' and Δ_2' are the same intervals after steepening of the profile.

For $t \lesssim t_n = l_0/\sigma$, it may be assumed that the number of flows at a point does not exceed three and it is possible to obtain an equation for the average number of discontinuities per unit length:

$$n(t) = \frac{k_2}{2\pi k_1} \exp\left(-\frac{1}{2\tau^2}\right). \qquad (4.6)$$

Expression (4.6) gives a criterion for neglecting discontinuities. In the region where such a procedure is relatively valid $t \ll t_n$, the velocity distribution in the wave is obtained from (4.5) by integrating over all remaining parameters. For a statistically homogeneous initial field, the velocity distribution is conserved: $W(u,t) = w_0(u)$.[83,94,95] This is explained by the fact that compression of steepening fronts is compensated by an equal expansion of stretching sections (Fig. 10). With the appearance of discontinuities, compensation breaks down and $W(u,t)$ begins to change.[96]

For the spectrum of the velocity field

$$g(k, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} B(\rho, t) \cos k\rho \, d\rho$$

with a Gaussian initial velocity field, the Lagrangian approach gives

$$g(k, t) = \frac{1}{2\pi k^2 t^2} \exp(-\sigma^2 k^2 t^2) \int_{-\infty}^{+\infty} \{\exp[B_0(\rho) k^2 t^2] - 1\} \exp(ik\rho) \, d\rho. \qquad (4.7)$$

The characteristics of the evolution of the spectra of Riemannian waves, obtained independently in Refs. 97 and 98, were discussed for different physical situations in Refs. 14, 21, and 98–100. We shall indicate some of the characteristic features of the spectrum (4.7).

The nonlinear interaction of harmonics leads to the fact that the spectrum grows most rapidly for large k. If the initial spectrum $g_0(k)$ has a peak at $k_0 \neq 0$, then generation of difference harmonics causes the spectrum to flow over to the side $k < k_0$ (Fig. 11).[97]

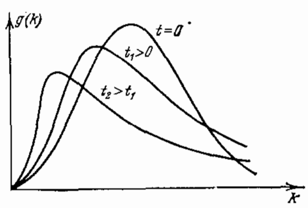

FIG. 11. Evolution of the BT spectrum before toppling of the wave.

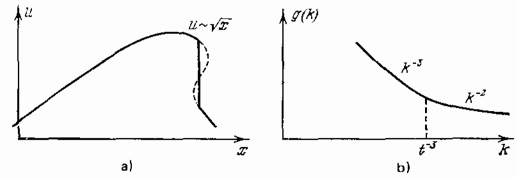

FIG. 12. Characteristics of a random wave after the appearance of a discontinuity. a) Characteristic profile, b) turbulence spectrum. Immediately after toppling, the wave has a singularity $u \sim \sqrt{x}$, which corresponds to the spectrum $g \sim k^{-3}$. The presence of a discontinuity leads to the asymptotic spectrum $g \sim k^{-2}$.

As $k \to \infty$, we obtain from (4.7) the universal asymptotic form $g \sim k^{-3}$;[97,98,101] related to singularities of a field of the type $u \sim \sqrt{x}$ accompanying toppling of the wave (see Fig. 12a).[98] The appearance of discontinuities causes the spectrum to fall off more slowly: $g \sim k^{-2}$.[6] However, the asymptotic form $g \sim k^{-3}$ remains at the initial stage of the formation of discontinuities in the interval

$$\frac{1}{l_0} \ll k \ll \frac{1}{l_0} \left(\frac{t_n}{t}\right)^3$$

and is replaced by the asymptotic form $g \sim k^{-2}$ as the amplitude and the number of discontinuities increase (Fig. 12b).[96]

d) Inclusion of discontinuities

Within the framework of the general analogy to the flow of noninteracting particles, inclusion of discontinuities means taking into account the competition between streams by selecting the dominant stream. The probability distribution of the velocity at a given point in this case can be represented in the form

$$W(u; x, t) = \int_{-\infty}^{+\infty} W_e(s, u; x, t) \tilde{W}(s) \, ds, \qquad (4.8)$$

where W_e is the joint distribution of the action and velocity for a Riemann wave, while the cutoff factor $\tilde{W}$ represents the probability that when a particle with parameters s arrives at the point of observation, all other particles arriving at the point have action $s' > s$. The general expression for $\tilde{W}$ can be found as the solution of the problem of the absence of crossings of a given level by a random process in the form of a continuous integral. In practice, the continuous integral can be reduced to a finite-dimensional integral, which can be calculated numerically. The analytical results taking into account the selection rule for the dominant particle, valid for any t, can be obtained by using simplified expressions for the cutoff factor.[48,49] Rigorous analytic results can be obtained in two limiting cases, when the number of streams included is small or very large.

e) Hypothesis of self-preserving behavior

At the stage of sawtooth waves $t \gg t_n$, when, as noted in Sec. 3, an analogy exists between discontinuities and a gas of particles with masses $m_k = y_{k+1} - y_k$ separated by distances $l_k = x_{k+1} - x_k$, it is attractive to analyze BT by the method of kinetic equations for the probability distribution of the parameters m_k and l_k.[7] Taking into account the conservation of mass and momentum in collisions between particles, a chain of equations for the distribution functions was obtained in Ref. 7. It was possible to close the equations with the help of the hypothesis of self-preserving behavior of probability distributions for BT, which, according to the hypothesis, depend on the time only through the self-preserving scale $l(t)$, which represents the average distance between particles (discontinuities). The hypothesis of self-preserving behavior leads to equations for the average number of discontinuities per unit length $\langle n\rangle$ and for the scale $l(t) = \langle n\rangle^{-1}$:

$$\frac{\mathrm{d}\langle n\rangle}{\mathrm{d}t} = -\frac{\alpha}{t}\langle n\rangle, \qquad \frac{\mathrm{d}l}{\mathrm{d}t} = \frac{\alpha}{t} l, \tag{4.9}$$

where α is a constant, determining the collision frequency.

Additional considerations are required to find α. The statistical invariant, following from the dynamic invariant (3.1) and analogous to Loĭtsyanskiĭ is invariant in hydrodynamic turbulence,

$$D = 2\pi g(0, t) = 2\pi g_0(0) = \int_{-\infty}^{+\infty} B_0(\rho)\,\mathrm{d}\rho \tag{4.10}$$

is especially useful here.

Physically, expression (4.10) describes the conservation of the constant component in the energy spectrum of BT.

According to the hypothesis of self-preserving behavior $D \sim l^3/t^2$, which leads to $\alpha = 2/3$. The case $D = 0$ is more complicated. In this situation, α was determined in Ref. 7 with the help of additional, inadequately justified hypotheses, which lead, however, to the correct result $\alpha = 1/2$.

Somewhat different considerations were used in Ref. 85, where the relation $v \sim l/t$ was obtained from the assumption that $\mathrm{d}n/\mathrm{d}t \sim -n/\tau$, where $\tau \sim l/v$ is the time of free flight of a discontinuity (v is the rms velocity of the discontinuity). On the other hand, Eq. (3.12) permits relating the velocity v with the distribution of the initial action:

$$v^2 \approx \frac{1}{l^2}\int_0^l (l-\rho) B_0(\rho)\,\mathrm{d}\rho. \tag{4.11}$$

For times $t \gg t_n (l \gg l_0)$, v depends strongly on the behavior of the initial energy spectrum as $k \to 0$. Indeed, for $l \gg l_0$, we have

$$\begin{aligned} v^2 &\sim \frac{1}{l}\int_0^\infty B_0(\rho)\,\mathrm{d}\rho = \frac{2D}{l} \qquad (D \neq 0),\\ v^2 &\sim -\frac{1}{l^2}\int_0^\infty \rho B_0(\rho)\,\mathrm{d}\rho \sim \sigma^2 l_0^2 l^{-2} \quad (D = 0). \end{aligned} \tag{4.12}$$

Substituting $v \sim l/t$, we find from (4.12)

$$l(t) \sim \sqrt[3]{Dt^2} \quad (D \neq 0), \qquad l(t) \sim \sqrt{\sigma l_0 t} \quad (D = 0). \tag{4.13}$$

From (4.13) follow the laws governing the drop in the average number of discontinuities, correcting Eq. (4.6) for $t \gg t_n$ (see Fig. 13).

The rate of energy dissipation, as follows from (3.2) and the hypothesis of self-preserving behavior, is related to the scale $l(t)$ by the expression

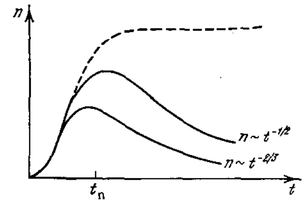

FIG. 13. Average number of discontinuities per unit length as a function of time. After the wave topples, $t > t_n$, the number of discontinuities gradually begins to decrease, due to their coalescence. The asymptotic laws for the decrease are $t^{-1/2}$ ($D = 0$) and $t^{-2/3}$ ($D \neq 0$).

$$\frac{\mathrm{d}E}{\mathrm{d}t} \sim \nu \frac{l^2}{t^2 \cdot l \cdot \delta} \sim \frac{l^2}{t^3} \quad (E(t) = \langle u^2(x, t)\rangle),$$

from which, substituting (4.13), we have

$$\begin{aligned} E(t) &\sim \left(\frac{D}{t}\right)^{2/3} \quad (D \neq 0),\\ E(t) &\sim \frac{\sigma l_0}{t} \quad (D = 0). \end{aligned} \tag{4.14}$$

We note that if the discontinuities were not to coalesce, then the energy would decrease according to the law $E \sim t^{-2}$, which is well known to specialists in nonlinear acoustics.[14] The decrease in the rate of dissipation of energy is related to the motion and coalescence of discontinuities.

The self-preserving scale plays the role of the *external scale* of BT. *The internal scale can be taken as* $\delta \sim \nu t/l$. Their ratio gives the time-dependent instantaneous Reynolds number:

$$R(t) = \frac{l}{\delta} = \frac{l^2}{\nu t}. \tag{4.15}$$

Thus the coalescence of discontinuities in the case of $D \neq 0$ increases the Reynolds number, as a result of which the linear stage is never attained. The same conclusion can be reached from (4.15) for the case $D = 0$, but, as rigorous theory shows, this result turns out to be incorrect. For $D = 0$, BT still reaches the linear stage, but the time at which this happens is large: $t_1 \gg t_d = t_n R_0$.

The hypothesis of self-preserving behavior and its basic consequences are confirmed, as will be demonstrated in Sec. 5, by a more accurate analysis. As far as the other assumptions of the kinetic theory are concerned,[7] they are not justified and lead to partially incorrect results.

5. TURBULENCE OF SAWTOOTH WAVES

a) Scale of correlations and the asymptotic theory

Turbulence (BT) of sawtooth waves is the most interesting stage of BT, combining features of stochastic and ordered behavior. The growth of the correlation scale leads at this stage to the appearance of a large parameter l/l_0, due to which the starting velocity field may be assumed to be delta correlated. The asymptotic theory following from this, however, turns out to depend strongly on the nature of the initial action field, since the dominant particles are associated with the minima of precisely this field. As noted above, the properties of BT are different for $D \neq 0$ and $D = 0$, which is directly related to the differences in the action fields.

For $D \neq 0$, the initial action can be viewed as a Wiener process. This case was analyzed in detail by Burgers.[4b]

For $D = 0$, from the statistical homogeneity of the initial velocity field also follows the statistical homogeneity of the initial action. Since the selection rule for particles with minimum action permits including for $t \gg t_n$ only large negative values of the initial action, the theory of overshoots of stationary random processes can be used to analyze BT.[102,103] An ideologically similar, but technically somewhat different analysis of statistically homogeneous BT was given in Refs. 84–86.

Aside from the two basic types of BT ($D = 0$, $D \neq 0$) examined in the present section, there also can exist intermediate types of BT, which depend on the specific form of the initial velocity spectrum and which are realized at well-defined times.

b) Properties of turbulence in the case of $D = 0$

We shall discuss in greater detail the properties of BT at the stage $t \gg t_n$, assuming that the initial velocity field is statistically homogeneous and Gaussian (although this is not necessary) and that its correlation function has the characteristic scale l_0. We shall also assume that the initial spectrum $g_0(k) \to k^n (n > 2)$ as $k \to 0$, from where it follows that the initial action has the variance

$$\sigma_s^2 = \int_{-\infty}^{+\infty} g_0(k) \frac{dk}{k^2} = \frac{\sigma^2}{\Omega^2}, \quad \Omega \sim \frac{1}{l_0}.$$

The properties of BT at a point are determined by the particle that has the minimum action of all the particles arriving there. For $t \gg t_n$, the competing particles are associated with large negative overshoots s_0, which in the case of a single-scale function $B_0(\rho)$ may be assumed to be uncorrelated, which permits obtaining the velocity distribution of the dominant particles with least action:[49]

$$W(u;\, t) = \int_{-\infty}^{\infty} W_e(s,\, u;\, t) \exp\left[-\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} W_e(s,\, u;\, t)\, ds\right] ds. \qquad (5.1)$$

Here $W_e(s,u;t)$ is the Eulerian joint distribution of action and velocity following from (4.5). For $l \gg l_0$, the dominant particles will be particles that leave small neighborhoods of the overshoots of the initial action beyond a very low level $H_0 = -\sigma_s \xi (\xi \gg 1)$, determined by the condition that the integrand in expression (5.1) has an extremum with respect to s:

$$\sqrt{\frac{\gamma}{\xi}} \exp\left(-\frac{\xi^2}{2}\right) = 1, \quad \xi \approx \sqrt{\ln \gamma}, \quad \gamma = \frac{\sigma \Omega t}{2\pi} \sim \frac{t}{t_n}. \qquad (5.2)$$

Evaluating the integral (5.1) by the saddle-point method, we obtain a Gaussian distribution for the single-point probability density of BT:

$$W(u;\, t) = \frac{1}{\sqrt{2\pi E}} \exp\left(-\frac{u^2}{2E}\right), \qquad (5.3)$$

$$E(t) = \frac{l^2(t)}{t^2} \approx \frac{\sigma}{\Omega t \sqrt{\ln \gamma(t)}}, \qquad (5.4)$$

$$l^2(t) = \frac{\sigma t}{\Omega \xi} \approx \frac{\sigma t}{\Omega \sqrt{\ln \gamma(t)}}. \qquad (5.5)$$

The relative motion of the discontinuities leads to two main differences between the behavior of the energy of BT (5.4) and the energy of a periodic wave $E \approx l_0^2/t^2$ at the same stage $t \gg t_n$ (see subsection 3d). The first difference is the already mentioned slower dissipation of the energy of BT, described by the law $E \sim t^{-1}$ (4.14), which also follows from qualitative estimates, to within a logarithmic correction.[84–86] The second difference is that E depends on the initial amplitude of BT $\sigma \sim u_0$. The memory of the initial amplitude present in the energy of BT, which is absent in the periodic wave, is due to the dependence of the velocity of the discontinuities on σ: the higher σ, the more often discontinuities coalesce, the weaker the dissipation, and the higher the energy of BT.

Analogously, for $l \gg l_0$, asymptotic expressions can be found for the joint probability density of BT at two points separated by a distance 2ρ. Just as the single-point distribution (5.3), it is self-preserving, i.e., it depends on the single spatial scale $l(t)$ (5.5), the external scale of BT. For this reason, it is convenient to switch to dimensionless variables:

$$v_1 = \frac{t}{l(t)} u(-\rho,\, t), \quad v_2 = \frac{t}{l(t)} u(\rho,\, t), \quad s = \frac{\rho}{l(t)}. \qquad (5.6)$$

The joint distribution of v_1 and v_2 has the form[85]:

$$W(v_1,\, v_2;\, 2s) = \frac{\delta(v_2 - v_1 - 2s)}{\Phi(-v_1) \exp[(1/2) v_1^2] + \Phi(v_2) \exp[(1/2) v_2^2]}$$
$$+ \exp\left(-\frac{v_1^2}{2} - \frac{v_2^2}{2}\right) \int_{-s-v_1}^{s-v_2} \frac{2s\, dz}{[\Phi(s+z) \exp(zs) + \Phi(s-z) \exp(-zs)]^2}, \qquad (5.7)$$

where Φ is the probability integral. The first term here describes the properties of BT with the condition that there are no discontinuities in an interval of length 2ρ, while the second term describes the properties of BT with the condition that at least one discontinuity be present in that interval. In contrast to the single-point distribution, the two-point distribution (5.7) is not Gaussian, but, of course, it approaches a Gaussian distribution for separations of the points of observation that are large and small, compared to $l(t)$:

$$W(v_1,\, v_2;\, s) = \begin{cases} \delta(v_1 - v_2) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{v_1^2}{2}\right) & (s \to 0), \\ \frac{1}{2\pi} \exp\left(-\frac{v_1^2}{2} - \frac{v_2^2}{2}\right) & (s \to \infty). \end{cases} \qquad (5.8)$$

From (5.6) and (5.7) it is not difficult to find the correlation function and the spectral energy density of BT (Fig. 14). The discontinuities of sawtooth waves lead to nonanalyticity of the correlation function and a power-law falling off of the spectrum

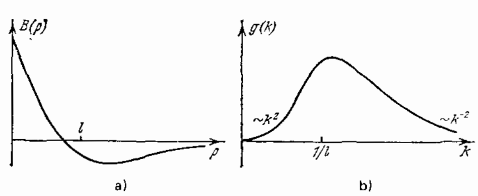

FIG. 14. Statistical characteristics of BT in the sawtooth wave regime. a) Correlation function; b) spectrum, l is the self-preserving scale of the turbulence.

$$g(k,\ t)=\frac{2l(t)}{k^3\pi^{3/2}t^2}\left(k\gg\frac{1}{l(t)}\right). \tag{5.9}$$

For small k, the spectrum also has a power-law asymptotic form

$$g(k,\ t)\sim\frac{k^2l^5}{t^2}\sim k^2\sqrt{t}, \tag{5.10}$$

and the maximum of the spectrum is displaced toward small k in accordance with $1/l$.

We note that the laws $E\sim t^{-1}$ and $g\sim k^{-2}$, following from the asymptotic theory, agree with the actual and numerical experiments (see Refs. 104 and 105).

It should be added that the statistics of the parameters of "heavy particles" (discontinuities) m_k and l_k, which also give complete information about the properties of BT, are also of interest.[84]

c) Dissipation field

The characteristics of BT obtained above essentially do not take into account the finite magnitude of the viscosity. When the thickness of the shock fronts is taken into account, the dissipation field, i.e , the gradient of BT, has the form (Fig. 15)

$$q(x,\ t)=\frac{\partial u}{\partial x}=\frac{1}{t}-\frac{1}{t}\sum_k\frac{m_k}{\delta_k}\mathrm{ch}^{-2}\left(\frac{x-x_k}{\delta_k}\right),$$
$$\delta_k=\frac{4\nu t}{m_k}. \tag{5.11}$$

As long as the instantaneous Reynolds number is large, the dissipation field, just as in hydrodynamic turbulence,[3] has an "alternating character": it is concentrated in randomly distributed clusters, between which there is practically no dissipation. It is easy to conclude from (5.11) that the dissipation field is strongly non-Gaussian.

The finiteness of the thickness of the shock fronts destroys the power-law asymptotic behavior of the spectrum of BT and leads to a more rapid falling off of the spectrum in the dissipative interval $k\cdot\delta\gg1$ (Ref. 85):

$$g(k,\ t)=\frac{16\pi\nu^2}{\sqrt{3}\cdot l}\sqrt[3]{\pi\delta k}\cdot\exp\left[-3(\pi k\delta)^{2/3}\right],\qquad \delta=\frac{\nu t}{l}. \tag{5.12}$$

We note that the spectrum of BT drops off more rapidly than for a periodic wave, which is explained by the fluctuations of the internal scale of BT.

d) Degeneracy of turbulence

The asymptotic expressions presented above are valid for times $t_1\gg t\gg t_n$, where t_1 is the time at which BT reaches the final linear stage of degeneracy. We shall estimate it from the condition that the instantaneous Reynolds number is equal to 1. Using (5.5), we have

$$R(t)=\frac{l^2}{\nu t}\sim R_0\left(\ln\frac{t}{t_n}\right)^{-1/2};$$
$$t_1\sim t_n\exp R_0^2. \tag{5.13}$$

The transition to the linear stage in BT occurs much later than for a periodic wave, for which $t_1\sim t_d\sim t_nR_0$.

At the linear stage, it is more convenient to investigate the properties of BT by using the exact solution of the diffusion equation, to which the BE reduces by a substitution, since at these times, its average value greatly exceeds the fluctuations.[106]

Analysis shows that for $t\gg t_1$, the energy spectrum of BT is likewise self-preserving:

$$g(k,\ t)\approx4\nu^2\sqrt{\frac{2}{\pi}}\frac{\exp(R_0^2/4)}{\Omega R_0}\exp(-2\nu k^2t^2), \tag{5.14}$$

and the energy drops off according to the law

$$E(t)\sim t^{-3/2}.$$

The large slope of the BT spectrum at the linear stage is explained by the long time it takes to reach the linear regime.

e) Stationary turbulence

An important, for the theory of turbulence, example of the appearance of stationary turbulence is BT excited by a random external force and satisfying the inhomogeneous BE:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\nu\frac{\partial^2u}{\partial x^2}+f(x,\ t),$$

where $f(x,t)$ is a Gaussian field with the correlation function

$$\langle f(x,\ t)f(x+\rho,\ t+\tau)\rangle=\theta(\rho)\,\delta(\tau).$$

It is not difficult to show, using, for example, the Furutsu-Novikov equation, that the following equality is satisfied:[98]

$$g(0,\ t)=\frac{1}{2\pi}(D+bt), \tag{5.15}$$

which generalizes the invariant (4.10) to the case of BT excited by an external force. Here

$$b=\int_{-\infty}^{\infty}\theta(\rho)\,d\rho.$$

It is evident from (5.15) that BT can approach a strictly stationary regime only if $b=0$. Physically, this is related to the fact that for $b\neq0$, pumping of energy into the long-wavelength part of the spectrum does not have time to compensate viscous dissipation.

It is demonstrated in Ref. 81 with the methods of the theory of similarity that in the stationary regime the spectral energy distribution of BT has the form

$$g_\infty(k)=\varepsilon^{2/3}k^{-5/3}\psi(kl,\ kL), \tag{5.16}$$

where $l=\nu^{3/4}\varepsilon^{-1/4}$ is the microscale of turbulence, L is the correlation length of the exciting force $f(x,t)$, and $\varepsilon=\theta(0)/2L$ is the rate of dissipation of energy.

In Ref. 48, the properties of stationary BT were investigated with the help of an analysis of the statistics

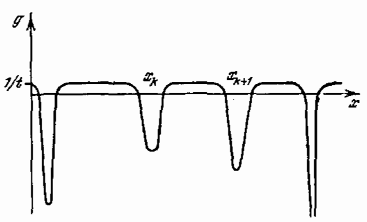

FIG. 15. Dissipation field of BT. $q(x,t)$ is the gradient of the velocity field; x_k are the locations of the shock fronts.

of velocities of dominant particles. Their selection in Ref. 48 depended on the hypothesis that the action and the velocity are statistically independent. In this case, it was possible to show that the stationary BT has a Gaussian distribution and it was possible to calculate the energy and some other characteristics of stationary BT. Thus, for a BT spectrum in the inertial interval, it was found that $g_\infty(k) \sim \varepsilon^{2/3}/k^2L^{1/3}$, which agrees with (5.16), but contradicts the conclusion in Ref. 81 that there exists an inertial interval with a universal spectrum $k^{-5/3}$. It should be noted, however, that the results obtained in Ref. 48 are not entirely rigorous and require justification.

f) Turbulence in the absence of degeneracy ($D\neq 0$)

The fundamental difference between BT with $D\neq 0$ and the case with $D=0$, analyzed above, lies in the growth of nonlinear effects and the unlimited growth of Reynolds number with the passage of time. This is related to the difference in the statistics of the initial action and the more rapid, for $t \gg t_n$, growth of the self-preserving scale according to the law $l\sim (Dt^2)^{1/3}$.

The linear stage of BT with $D\neq 0$, can be realized only for $R_0 \ll 1$ and only at comparatively short times. Indeed, if $R_0 \ll 1$, then nonlinear effects are at first small, but smoothing of the field by viscous dissipation leads at times $t > t_d$ to a growth of its spatial scale $l = 2\sqrt{\nu t}$. Since for $D \neq 0$, the energy of the field dissipates according to the law $E\sim \sigma^2 l_0/l$, the instantaneous Reynolds number increases as

$$R(t) = \frac{\sqrt{E}\,l}{\nu} \sim R_0\left(\frac{t}{t_d}\right)^{4},$$

and for $t > t_d/R_0^4$, it exceeds one, after which nonlinear effects begin to affect the propagation of the wave. Thus, at sufficiently long times, BT with $D\neq 0$ behaves as a sawtooth wave, although with strongly smeared fronts.

Transforming to normalized coordinates $\bar{x} = x/l(t)$ and $\bar{y} = y/l(t)$, we find that the initial velocity field

$$\bar{u}_0(\bar{x}, t) = \frac{t}{l(t)} u_0(\bar{x} l(t))$$

has a correlation function

$$\langle \bar{u}_0(\bar{x}, t)\, \bar{u}_0(\bar{x}+\lambda, t)\rangle = \left(\frac{t}{l}\right)^2 B_0(l\lambda)$$

with correlation length $\lambda_0 = l_0/l$. At times $t \gg t_n$, when $l \gg l_0$, $\bar{u}_0$ approaches a delta-correlated field. As far as the initial action distribution is concerned, giving it in the form

$$s_0(\bar{x}, \bar{y}, t) = \int_{\bar{x}}^{\bar{y}} \bar{u}_0(\bar{x}, t)\, d\bar{x},$$

we find that for $l \gg l_0$, it approaches a Wiener process, all statistical properties of which are determined completely by its first two moments

$$\langle s_0\rangle = 0, \quad \langle s_0^2\rangle = |\bar{x} - \bar{y}| \tag{5.17}$$

and are independent of time.

It follows from here that for $R(t) \gg 1$ and $t \gg t_n$, all statistical characteristics of BT become self-preserving and are determined by the probability properties of the Wiener process. In particular, the correlation function and the probability distribution of BT can be represented in the form

$$W(u;\ t) = \frac{t}{l} w_\infty\left(\frac{ut}{l}\right),$$
$$B(\rho,\ t) = \left(\frac{l}{t}\right)^2 \langle y^2\rangle R\left(\frac{\rho}{l}\right) \quad (R(0) = 1), \tag{5.18}$$

where

$$\langle y^2\rangle = \int_{-\infty}^{+\infty} y^2 w_\infty(y)\, dy.$$

The specific form of the functions entering into (5.18) can be determined numerically by the method of statistical sampling or by using the theory of Markov processes. Burgers[4b] obtained a relation between the function ω_∞ and the probability that the Wiener process not attain a parabolic boundary, which in its turn is expressed in terms of the solution of the diffusion equation. Using the representation obtained, Burgers calculated the probability distributions of the amplitudes of the discontinuities and distances between them.

g) Self-preserving behavior and dissipative structure

BT, just as eddy turbulence, is an example of a system that reaches the self-preserving regime, arising as an intermediate asymptotic state for $t_n \ll t \ll t_1$.

The sawtooth waves arising at this stage exhibit the property of local self-preserving behavior. In addition, the statistical characteristics of the random sawtooth waves, i.e., developed BT, are self-preserving. In this case, the self-preserving behavior of BT can be complete or incomplete,[107] i.e., determined only by the dimensionality or dependent on the initial conditions, depending on the form of the initial spectrum (see Ref. 4b, 7, 49, 84–86). The self-preserving scale of BT plays the role of an external scale of turbulence and indicates the average dimensions of the locally ordered regions (domains).

The domain structure of BT can be viewed as an example of a dissipative structure.[108-110] The narrow zones of dissipation, into which energy enters, are situated on the boundaries between the domains. Each domain is related to some partial wave, which can be identified in the initial perturbation, characterized by the magnitude of the action, playing in this problem the role of an order parameter. As a result of the competition between domains, their scale, i.e., the correlation radius of BT, increases. As the viscosity of BT increases, something like a phase transition can occur: the regime of domains is replaced by a regime with weakly coupled harmonics; however, this transition has a continuous character.

The regions of dissipation, which serve as domain walls, can be interpreted as regions of high density, forming as a result of inelastic collisions of dispersing noninteracting particles. We note that analogous phenomena can also arise with other types of interactions between particles, if the interaction energy is much less than their kinetic energy. The problem of dispersion of particles, taking into account their gravitational interaction, leads to a cellular or domain structure,[111] which is very similar to the structure that may be expected in a system described by three-dimensional generalization of the Burgers equation. It should be kept in mind here that if the gravitational interaction is assumed to be significant only in regions where particles cluster, then its role at a certain stage is analogous to the role of dissipation. It transforms the energy of translational motion of particles into the energy of their vibrational and chaotical eddy motion in the zone of the front. For this reason, the initial stage of formation of the cellular gravitational structure (while the evolution of domain walls is not yet manifested) is apparently in many ways similar to phenomena described by the model of inelastic collisions.

A somewhat different point of view of BT is also possible. For finite ν, quasi-one-dimensional structures, which have the nature of traveling waves with slowly varying parameters, arise in the regions of the fronts. These structures are joined with one another with the aid of separate saw teeth, which can be viewed as a Riemannian pedestal related to the initial state. The energy of the pedestal is gradually transferred to the region of the fronts, where it is dissipated. This transfer of energy can be viewed as resulting from collisions of some quasiparticles with the background. A similar approach was developed in Ref. 112 for systems that permit the formation of solitons.

In connection with what was said above, the analogy between strong turbulence in continuous systems and the stochastic motion of systems with a small number of degrees of freedom becomes clearer.[8] For the latter, together with structures such as strange attractors, etc., trajectories combining sections of stochastic and ordered motion are also characteristic.[8b] Burgers turbulence is also an example of this type of motion. Examining the behavior of the solution of BE with t = const, we can say that "switching type" chaotic free oscillations occur.

6. WAVE COUPLING IN NONDISPERSIVE MEDIA

a) Qualitative classification

So far, we have mainly assumed that the initial perturbation is single-scale noise. However, for example, in acoustical applications multiscale nonlinear waves are often realized.[14] Typical examples are a quasi-monochromatic disturbance or a signal interacting with noise with a nonoverlapping spectrum. The phenomena arising in this case are determined by the set of nonlinear times of separate components and their interaction times. The basic physical laws governing the behavior of multiscale waves can be understood by investigating the propagation of the sum of two disturbances with different scales. Such an analysis is performed below for the example of a periodic signal with a zero constant component, interacting with stationary noise.[113]

In contrast to the linear problem of scattering of a wave by fixed inhomogeneities, here the self-action of the components plays an important role. Due to the nonlinear distortions and the formation and coalescence of discontinuities, the properties of interacting waves vary as they propagate, which leads to fundamentally different mechanisms of interaction at different stages. In addition, in contrast to dispersive media, where satisfaction of the conditions of synchronization is more likely to be the exception than the rule and, with interaction of waves, coherence sooner or later breaks down,[114] in a nondispersive medium, breakdown of the coherence of interaction can be caused only by the self-action of sufficiently intense noise.

We shall single out the main problems that arise in the analysis of the interaction of the signal and noise: 1) effect of noise on the damping of the average field of a regular wave and the closely related problem of turbulent viscosity; 2) redistribution of the energy of the signal and of the noise over the spectrum due to the interaction; 3) effect of an intense signal on the transformation of the statistical properties of noise.

To characterize the interaction of two components of the initial perturbation, it is convenient to introduce the concept of the turbulent diffusion length l_d (Ref. 113), a scale describing the smearing of the first component of the initial field $u_1(x)$ due to the interaction with the second component $u_2(x)$ $[u_0(x) = u_1(x) + u_2(x)]$. We shall denote the characteristic amplitudes, spatial scales, and nonlinear times of the components of the initial perturbation as a_1, l_1, t_1 and a_2, l_2, t_2, respectively. As previously, it is appropriate to characterize the stages of evolution of the waves here in the language of hydrodynamic streams of noninteracting particles.

In single-stream propagation, the diffusion length is determined by the displacement of the initial coordinate of the particle arriving at a given point due to the influence of the component u_2. In many-stream propagation, l_α depends on the displacement of the dominant particle, which is limited by the condition for compensation of the regular increase in the action by its possible decrease due to s_2:

$$\frac{l_d^2}{2t} = \Delta s_2(l_d).$$

The equations obtained from these considerations for the turbulent diffusion length of the signal u_1 due to the noise u_2 have the form

$$l_d = a_2 t \quad (t < t_2), \tag{6.1}$$

$$l_d = \begin{cases} \sqrt{a_2 l_2 t} & (D = 0), \\ \sqrt[3]{Dt^2} & (D \neq 0) \end{cases} \quad (t > t_2). \tag{6.2}$$

The interaction time of u_1 with u_2 is determined from the condition $l_1(t) = l_d(t)$, where $l_1(t)$ takes into account the possible change in the scale of u_1 due to self-action.

We shall describe the basic types of interactions of the components of the initial perturbation, assuming u_1 is a large-scale component $(l_1 > l_2)$.[113]

1. Breakdown of the coherent structure of the small-scale component due to phase modulation by the large-scale component. If u_2 is the signal, then the interaction time is $\tau = l_2/a_1$; if u_2 is the noise, then the index of the modulation attains the value one only if $a_1 l_1 > a_2 l_2$ and, in this case, $\tau = l_2(\tau)/a_1$.

2. Breakdown of the spectral structure of the small-

scale component and transfer of its energy into the low-frequency range. This process can be roughly characterized by the time for toppling of the large-scale component $\tau \sim t_1$. If u_2 is the noise, then, as in the preceding case, the breakdown occurs efficiently only if $a_1 l_1 > a_2 l_2$.

3. Breakdown of the spectral structure of the large-scale component and transfer of its energy into the small k range due to residual modulation by the small-scale component. This type of interaction appears if u_1 is the signal and is characterized by the time

$$\tau = \frac{l_1^2}{l_2 a_2} \quad (D=0), \qquad \tau = \sqrt{l_1^3 D} \quad (D \neq 0).$$

As a comparison of the toppling and interaction times shows, the following basic situations can arise in the combined propagation of a periodic signal and stationary noise:

1. Interaction of a large-scale signal with weak small-scale noise ($a_1 l_1 > a_2 l_2$). In this case, the noise is modulated by the signal, leading to a breakdown of the structure of the noise and establishment of sawtooth waves of the signal, weakly modulated by the noise. At $\tau = t_1$, the modulation smears the structure of the signal and the transition to the BT regime with a scale increasing with time begins.

2. Interaction of large-scale signal with strong small-scale noise ($a_1 l_1 < a_2 l_2$). At the initial stage, the noise is weakly modulated by the signal. After the noise topples, its scale begins to grow, which leads to breakdown of the large-scale component before it topples at $t = l_1^2 / l_2 a_2$ and transfer to a purely noisy turbulent regime.

3. Interaction of large-scale noise with a small-scale signal. The breakdown of the coherent structure of the signal occurs at $t = l_2 / a_1$; its final smearing and transition into the noise regime occur at $t = l_1 / a_1$.

As we can see, the physical picture of the interaction of the signal and noise is different for different ratios of the spatial scales. The answer to the question, often arising in the theory of nonlinear waves and the theory of scattering, concerning the physical meaning of the separation of the field into regular and fluctuating components

$$u(x,\ t) = \langle u(x,\ t)\rangle + \delta u(x,\ t) \qquad (\langle \delta u\rangle = 0)$$

is also different. The physical interpretation of the fluctuations δu and, what is more, the possibility of measuring them experimentally depend on the ratio of l_s and l_n. In the presence of large-scale noise, the damping of the mean field, a purely ensemble effect, is due to the random drift, differing in different realizations, of the regular wave $u_s(x,t)$ due to the slow flow u_n. The change in the variance $\langle(\delta u)^2\rangle$ here is also determined by the random drift of the regular wave and $\langle(\delta u)^2\rangle \neq \langle u_n^2\rangle$. In the opposite case, $l_s \gg l_n$, the change in $\langle(\delta u)^2\rangle$ is related to the true distortion of the statistical properties of the noise due to the interaction with the signal and can be found by averaging one of the realizations over an interval $l_n \ll \Delta x \ll l_s$. For $l_s \ll l_n$, the spectral properties of the noise itself do not change and it is natural to talk about the phase modulation of the signal by the noise. In the case $l_s \gg l_n$, however, the interaction indeed leads to redistribution of the noise energy over the spatial spectrum.

b) Interaction of Riemann waves

We shall discuss the possibility of describing quantitatively the interaction of waves at the initial stage. Before the wave topples, the interaction of its components can, as indicated above, be viewed as a modulation of the small-scale component by the large-scale component. For $l_1 \gg l_2$, the total wave at this stage can be represented in the form[117,113]

$$u(x,\ t) = u_1(x,\ t) + u_2(\tilde{x},\ \tilde{t})\left(1 - \frac{\partial u_1}{\partial x}\tilde{t}\right), \tag{6.3}$$

where u_1 and u_2 are the solutions of BE for the separate components,

$$\tilde{x} = x - u_1(x,\ t)\,t, \quad \tilde{t} = t\left(1 - \frac{\partial u_1}{\partial x}\,t\right).$$

Assume now that the initial perturbation is given in the form $u_0 = u_s(x) + u_n(x)$, where u_n is statistically homogeneous noise. As follows from (4.5) or (6.3), the single-point probability density $W(u;x,t)$ has the form

$$W(u;\ x,\ t) = \int_{-\infty}^{\infty} w_0(v)\,\delta\,[u - v - u_s(x - vt,\ t)]\,dv, \tag{6.4}$$

where u_s is the field of the signal in the absence of noise and w_0 is the initial probability density of the noise.

One property of (6.4) is that $W(u;x,t)$ does not depend on the nature of the variation of the noise as a function of x.

For the mean field, we have from (6.4)

$$\langle u(x,\ t)\rangle = \int_{-\infty}^{+\infty} w_0(v)\,u_s(x - vt,\ t)\,dv, \tag{6.5}$$

from where follows, for the case when $u_s = a_s \sin k_0 x$ and u_n is Gaussian noise with variance σ^2, an expression indicating the nature of the damping of the harmonics of the mean field[83]:

$$A_n \sim \exp\left[-\frac{1}{2}(\sigma n k_0 t)^2\right].$$

With combined propagation of signal and noise, their interaction dissipates the energy in the signal into new spectral regions and changes the noise spectrum at the same time. The influence of the long-wavelength component on the short-wavelength component is most noticeable, as follows from (6.3). In the limiting case of spectrally strongly separated signal and noise, their interaction can be examined in the approximation of a fixed (but varying due to self-action) large-scale field. The general equation for the energy spectrum of the total wave was obtained in Ref. 115.

Let us consider in greater detail the scattering of a small-scale signal by large-scale noise ($l_s \ll l_n$). In this case, the spectra of the scattered component near the harmonics of the signal have a similar structure to the spectrum of a harmonic signal with a random phase modulation,[116] which is explained by expression (6.3). The important parameter here is the index of phase modulation of the harmonic $\gamma_n = \sigma n k_0 t$, characterizing the ratio of the shift of the wave by the noise $\sim\sigma t$ to the period of the n-th harmonic of the signal. For $\gamma_n \ll 1$, when the displacement is small, the scattered components repeat the form of the long-wavelength noise. We note, however, the fundamental asymmetry of the form of the scattered signal relative to nk_0, which follows from a more accurate analysis.[115] Its short-wavelength wing increases somewhat more rapidly, which also follows from the Manley-Rowe relations.[114] For $\gamma_n \gg 1$, the long-wavelength modulation leads to a universal spectrum of the scattered component, independent of the fine structure of the noise spectrum[117]:

$$g_n(k,\ t) \approx \frac{J_n^2(a_c k t)}{k^3 t^3 \sigma k_1} \sqrt{\frac{2}{\pi}} \times \exp\left(-\frac{2(k-nk_0)^2}{\sigma^2 k^2 t^2 k_1^2}\right), \tag{6.6}$$

where k_1 is the width of the noise spectrum. As is evident from (6.6), the width of the spectrum of the scattered harmonic increases both with the number n and with time, due to which, for sufficiently short wavelengths, the harmonics overlap and form a continuous power-law tail, whose boundary shifts with increasing t toward increasingly smaller k. Thus, due to the interaction with noise, the energy of the initially harmonic signal has a tendency to spread out over all k, and, in addition, the nonlinear distortion of the signal itself increases the efficiency of this process (Fig. 16). The interaction of the signal and noise, distorting the signal, also changes the statistical properties of the noise, as is evident from (6.4). However, if $\sigma \ll a_s$, then the form of the distribution remains the same and only its variance changes. It increases on the steep sections of the signal $[u_s'(x) > 0]$ and decreases on the extended sections $[u_s'(x) < 0]$. Thus, when $k_0 x = 0,\ \pi$, for the harmonic signal we have

$$\sigma_1^2\left(kx = \begin{Bmatrix} 0 \\ \pi \end{Bmatrix},\ t\right) = \sigma^2 \times \begin{cases} (1+\sigma k_0 t)^{-2}, \\ (1-\sigma k_0 t)^{-2}. \end{cases}$$

The increase in dispersion for $kx = \pi$ is limited at times $t \sim t_s$. In this case, the form of the noise distribution also changes. We note, however, that the overall picture of the interaction of the short-wavelength signal with the noise will not change after the signal topples,

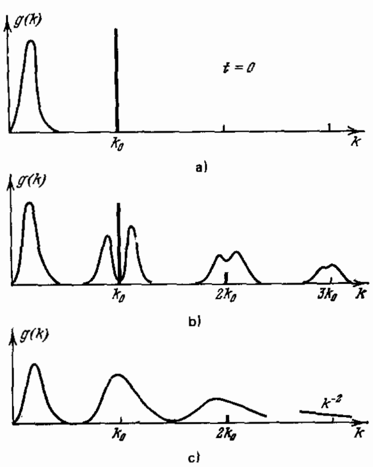

FIG. 16. Spectrum of the signal interacting with large-scale noise. a) Starting spectrum; b) deformation of spectrum at short times; c) spectrum in the region of overlap of the harmonics.

but rather when $t < t_n$. Here, as before, the concept of modulation of the signal by a long-wavelength flow is valid and Eqs. (6.3)–(6.5) are satisfied, where now $u_s(x,t)$ is the field of the signal including discontinuities.[117,113] Equation (6.6) also remains valid if we set $J_n^2(a_s k_0 t) = 1$.[117]

c) Noise and flow

The mechanism of the distortion of the spectral properties of the noise due to interaction with a large-scale signal is graphically illustrated by the interaction of noise with a regular flow $u_s(x) = \beta(x - x_*)$. The field here equals[117] (see 6.3)

$$u(x,\ t) = \beta\frac{x-x_*}{1+\beta t} + u_n\left(\frac{x+\beta t x_*}{1+\beta t},\ \frac{t}{1+\beta t}\right)\frac{1}{1+\beta t}, \tag{6.7}$$

where the first term is the flow in the absence of noise and $u_n(x,t)$ is the noise in the absence of flow. For the energy spectrum of the noise distorted by the flow, we have from (6.11)

$$\bar{g}(k,\ t) = \frac{1}{1+\beta t}\, g\left(k(1+\beta t),\ \frac{t}{1+\beta t}\right), \tag{6.8}$$

where $g(k,t)$ is the spectrum of the undistorted noise. It is evident from (6.11) and (6.12) that the interaction of noise with the flow leads to three effects: a change in the variance of the noise $\bar{\sigma}^2(t) = E[t/(1+\beta t)]/(1+\beta t)^2$, its spatial scale $\bar{l}(t) = l[t/(1+\beta t)](1+\beta t)$, and the nonlinear self-action time $\bar{t}_n = t_n/(1-\beta t_n)$. Here, $t_n = l_0/\sigma$ and $E(t)$ and $l(t)$ are the previously studied variance and spatial scale of noise in the absence of flow (energy and outer scale of BT).

In the presence of an interaction with a compression wave ($\beta < 0$), whose gradient becomes infinite at time $t_T = |\beta|^{-1}$, the variance of the noise increases and t_n decreases, as a result of which, independent of $R_0 = \sigma l_0/\nu$, the noise attains the regime of sawtooth waves over a time less than t_T.

In the presence of an interaction with a rarefaction wave ($\beta > 0$), due to the decrease in the variance and the increase in the spatial scale of the noise, its nonlinear self-action is stabilized and for $t > \beta^{-1} = t_T$, the form of the noise spectrum remains unchanged, coinciding with the form of the BT spectrum at $t = t_T$. Only nonlinear stretching of the scale and decrease of the noise variance due to the interaction occur.

We emphasize once again that the analysis of the interaction of noise with a rarefaction wave, which is of interest in itself, is also useful in studying the interaction of noise with a harmonic signal, since the zeros of its extended sections determine the statistics of the dominant particles and their order parameters.

d) Problem of turbulent viscosity

For large-scale noise and $t \gg t_n$, the wave practically forgets the signal and goes over into the turbulent regime. If the noise is small-scale noise, then the turbulent regime, forming at times $t > t_n$ if $t_n < t_s$ and $t > l_s^2/l_n\sigma$ if $t_s < t_n$, retains a memory of the signal to a higher degree. In the case of noise with $D = 0$, a method analogous to the one presented in Sec. 5b can be used to describe the turbulent state.

Using the selection rule for particles with minimum action, in the region of developed turbulence ($t \gg t_n$), we obtain the following probability distribution of the field $u(x,t)$[49]:

$$W(u;\ x,\ t)=\frac{\Phi(u;\ x,\ t)}{\int_{-\infty}^{\infty}\Phi(u;\ x,\ t)\,du},$$
$$\Phi(u;\ x,\ t)=\exp\left\{-\frac{\xi}{\sigma l_n}\left[s_s(x-ut)+\frac{u^2t}{2}-s_s(0)\right]-\frac{u_s^2(x-ut)}{\sigma^2}\right\}, \qquad (6.9)$$

where $s_s(x)$ is the initial action of the signal, and ξ is determined by the solution of a transcendental equation analogous to (5.2), from which it follows that $\xi \approx \sqrt{\ln A}$, while A can depend on the time t and the amplitude and scale of the noise and signal, depending on the ratios of the parameters.[49]

Using (6.9), it is possible to investigate the behavior of the mean and fluctuation components of the field $u(x,t)$ at different stages of propagation. In the interval $t_n < t < t_s$, the mean field coincides with the Riemann wave of the signal, while the noise transforms into a sawtooth wave modulated by the signal. For $t_s < t < l_c^2/\sigma l_n$, the mean field has the form of a sawtooth wave with smeared fronts and scale l_s. Here pulses in the neighborhoods of the fronts of the mean field make the main contribution to the noise. For $t > l_s^2/\sigma l_n$, the mean field decays according to the law

$$\langle u\rangle \sim \exp-\frac{k_0^2 t\sigma l_n}{\xi}.$$

For the characteristics of the noise at this stage, we have

$$\sigma^2(t)=\frac{\sigma l_n}{2t\xi},\quad l_n(t)\sim\sqrt{\frac{\sigma l_n t}{\xi}}.$$

As is evident, the mean field passes through the same stages in the course of its evolution as the signal in the absence of noise. This parallelism leads to the possibility of introducing a turbulent viscosity, which permits closing the equation for the mean field:

$$\frac{\partial\langle u\rangle}{\partial t}+\frac{1}{2}\frac{\partial\langle u\rangle^2}{\partial x}=\nu\frac{\partial^2\langle u\rangle}{\partial x^2}.$$

It is possible to close this equation by introducing a turbulent viscosity if the following relation is satisfied:

$$\frac{1}{2}\frac{\partial\langle u\rangle^2}{\partial x}=\langle u\rangle\frac{\partial\langle u\rangle}{\partial x}-\nu_T\frac{\partial^2\langle u\rangle}{\partial x^2}. \qquad (6.10)$$

For $t \gg t_n$, for a signal interacting with small-scale noise, from (6.9) follows (6.10) with

$$\nu_T=\frac{\sigma l_n}{2\xi}. \qquad (6.11)$$

In this case, the turbulent viscosity also correctly reflects the decay of the mean field and the noise-induced of fronts in the state with sawtooth waves. We note that in contrast to molecular viscosity, the turbulent viscosity introduced in this manner depends weakly on the time via the ξ and, in general, on the properties of the signal as well.[49]

In the presence of an interaction between the signal and large-scale noise, relation (6.10) is valid for $t \ll t_s$ and $t \gg t_{sn} = l_s/\sigma$.[111] The turbulent viscosity in this case equals

$$\nu_T=\sigma^2 t. \qquad (6.12)$$

This result, as also Eq. (6.5), was obtained using the functional method in Ref. 82, but without any indication of the limits of applicability.

The turbulent viscosity (6.12) correctly describes the damping of the signal, interacting with large-scale noise $\sim\exp(-k_0^2\int_0^t \nu_T dt/2)$. At the stage $t_s < t < l_s/\sigma$, it does not give a completely correct structure of the mean field near the shock front, but permits estimating its width $\sigma_T \sim \nu_T t/l_s \sim \sigma^2 t^2/l_s$.

On the whole, by introducing the turbulent viscosity (6.12) for $t < t_n$ and (6.11) for $t > t_n$ (for $t \sim t_n$ both equations give a quantity with the same order of magnitude), it is possible to describe qualitatively correctly the evolution of the mean field. Here, it turns out to be convenient to use as well the turbulent Reynolds number $R_T = a_s l_s/\sigma l_n$. When $R_T \gg 1$, the average velocity successively passes through the stages of the Riemann wave, sawtooth wave, and linear damping. When $R_T < 1$, only the linear stage remains. In all stages, the turbulent viscosity is related in a natural manner to the previously introduced turbulent diffusion length l_d.

The limits of applicability of the mean-field method,[53,54,118-122] based on expanding the field with respect to the fluctuation component and taking into account the effect of fluctuations on the mean field in the first approximation only, are narrower than the limits of applicability of expression (6.11) and the concept of turbulent viscosity. As is evident from (6.4) and (6.9), this method is valid for $t < t_{sn} = l_s/\sigma$, t_s, t_n. Physically, this means that the change in the average and fluctuation component over the diffusion length must be small.

As far as the turbulent viscosity is concerned, it can be introduced not only in the presence of a weak perturbation of the signal by the noise, but also in the opposite case of strong noise weakly modulated by the signal. The spatial inhomogeneity of the noise, created in this case by the signal, is compensated by the viscosity. The introduction of turbulent viscosity in this case is in many ways analogous to the introduction of molecular viscosity in nonequilibrium thermodynamics. It is interesting to note that in this case both (6.11) and (6.12) lead to the same relation $\nu_T = E(t)t$, where $E(t)$ is the noise energy in the absence of the signal.

We note also that any regular periodic regime turns out to be unstable relative to perturbations which have large-scale spectral components into which energy is gradually transferred. As far as the turbulent state is concerned, it may be assumed to be stable, i.e., its characteristics change little under the influence of the additional noise perturbation if the turbulent Reynolds number of the initial state is large compared to the Reynolds number of the perturbed state.

7. CONCLUSIONS

It is evident from this review that, in recent years, a great deal of progress has been achieved in studying Burgers turbulence, one of the important physical examples of strong turbulence, which, in this case, assumes the form of an ensemble of locally coherent dissipative structures.

The analysis of BT by the method of singling out the dominant particle can apparently be extended successfully to a wide class of similar problems. These include the problem, touched upon in this review, of BT in the presence of an external force and the analysis of potential turbulence, satisfying the three-dimensional Burgers equation, as well as the investigation of random waves satisfying the modified Burgers equation[123]

$$\frac{\partial u}{\partial t}+Q(u)\frac{\partial u}{\partial x}+a(u,\ t)=\nu\frac{\partial^2 u}{\partial x^2}. \qquad (7.1)$$

Equation (7.1) describes nonlinear acoustical waves, including the cylindrical or spherical divergence of the front, inhomogeneity of the medium, and low-frequency dissipation, as well as waves in active nondispersive media, etc.

We shall indicate further paths for research directly related to the problems of nonlinear acoustics, which were omitted in this review. First of all, there is the analysis of the interaction of waves moving in different directions, which can be conducted using methods similar to those examined in this review, taking into account at least weak coupling of the countermoving waves. Second, there is the transition from the state of Burgers turbulence to the weakly turbulent state of an ensemble of waves propagating at small angles.[124] The basic ideas of this approach are indicated in Refs. 125 and 126, but it deserves additional study.

Another path for research involves the development of a unified statistical theory of one-dimensional strong turbulence (and some of its multidimensional generalizations) for waves in media with nonlinearity limited by different types of interactions, arising in regions with strong gradients. As a result of these interactions, which can be represented as a form of collision, after the wave topples or is focused, its energy is transferred into the energy of the other type of motion. A complete solution can be constructed here by joining the quasi-one-dimensional structures that arise with the residual background.

The most important problem remains the problem of the relations between BT and eddy turbulence. There exists a number of results that have been rigorously justified for BT, such as self-preserving behavior of statistical characteristics, the presence of an inertial interval in the spectrum, dependence of degeneracy on the invariant D, possibility of closing the equations for the average velocity with the help of turbulent viscosity, etc., which is of undoubted heuristic value for solving problems in statistical hydrodynamics. It should also be mentioned that apparently there is no basis for assuming that there is only one mechanism for the onset and development of turbulence. In recent years, there is increasing talk about turbulence having the structure of locally coherent eddies, close to the structure of BT.[18] Such is the nature of the turbulence at the stage when "zones of alternation" exist. Models of separate interactions of eddies are successfully used in the theory of two-dimensional turbulence. From this point of view, attempts to develop further the Lagrangian description of eddy turbulence could be of interest. Viewing turbulence as a structure combining stochasticity with local coherence presumes further in-depth study of different types of random motion of systems with a small number of degrees of freedom. It is also useful to proceed further with the investigation of the analytic relations between eddy and potential turbulence, begun in Ref. 19.

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