Laboratory observation of caustics, optical simulation of the motion of particles, and cosmology

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Laboratory experiments on the observation of caustics when light rays are refracted by a "random" smooth surface are described. The analogy between caustics in geometrical optics and the singularities that arise in irrotational flows in a cold collisionless medium is discussed. The analogy is then extended to a very important cosmological process—the formation of the large-scale structure of the Universe. It is shown that in an expanding Universe the evolution of smooth irrotational perturbations, which grow under the influence of the gravitational force, leads to the formation of structures geometrically similar to generic optical caustics.

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CONTENTS

1. INTRODUCTION

A familiar phenomenon in optics is that of caustics, when converging light rays produce a sharp increase in the light flux at certain points. In modern language, one can speak of catastrophes in geometrical optics. This phenomenon can be reproduced in the laboratory. The characteristic singularities of various types predicted by the theory are clearly shown in the photographs.

Geometrical optics can be regarded as a special case of mechanical motion of noninteracting particles. Therefore, our photographs are a good aid for illustrating some results obtained in mechanics.

Moreover, these results still hold good in the theory of the collective motion of particles in their own gravitational field. Such a theory is an inseparable part of modern cosmology—it describes the occurrence of structure in the Universe. That is why the word cosmology appears in the title.

The deep reason for the similarity of the optical and mechanical phenomena considered here is to be sought in the well-known opticomechancial analogy (see, for example, Ref. 1).

The simple laboratory experiments are not intended to replace exact theory or laborious numerical calculations with high-speed computers. However, these experiments can help the intuitive understanding of the essence of the considered phenomena. A simulating experiment may also be helpful in pedagogical and popularizing lectures.

The material is presented below in an order that differs from the Introduction. First, we consider the mechanics of noninteracting particles (Sec. 2); we then describe the optical models and the results obtained with them (Sec. 3) in the one-dimensional (a) and two-dimensional (b) cases. Finally, in Sec. 4 we make a giant extrapolation to consider the question of the large-scale structure of the Universe.

2. NONINTERACTING PARTICLES

Modern mechanics makes wide use of the concepts of a vector field and transformations of it. A systematic exposition of such an approach is given in Arnol'd's book.[1]

The basic idea is due to Lagrange; the motion of matter can be described by specifying the instantaneous position of every particle $\mathbf{r}$ (its "Eulerian" coordinates x, y, z) as a function of its initial position ψ (as a function of its "Lagrangian" coordinates ξ, η, ζ) and the time t:

$$\mathbf{r} = \mathbf{r}(\psi, t).$$

One can say that the motion consists of a continuous transformation of the vector field $\mathbf{r}$ as a function of the continuously varying scalar parameter t. The law of this transformation is determined by specifying the velocity field $\mathbf{u}$:

$$\frac{d\mathbf{r}}{dt} = \mathbf{u}.$$

In its turn, the velocity field can be specified by some law depending on the acting forces; however, here we shall not be concerned with this aspect of the problem.

We take the simplest case when no external forces act on the matter. Moreover, we omit the internal forces, i.e., all interactions of the particles of matter with each other. In other words, in the equations we ignore the stresses or, in the case of a liquid or gas, the pressure and viscosity. Then every particle of the matter moves with constant velocity.

It would appear that the problem has been simplified so much that nothing interesting can be expected from

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its solution. We are now dealing with simple uniform and rectilinear motion of individual particles of matter,

$$\mathbf{r} = \boldsymbol{\psi} + (t - t_0)\,\mathbf{u},$$

$\mathbf{u}$ being constant for each particle. This means that $\mathbf{u} = \mathbf{u}(\boldsymbol{\psi})$ but there is no time dependence.

Nevertheless, it can be shown that the study of a continuum, i.e., not one but an entire ensemble of particles, leads to interesting and nontrivial results even in this simple case.

To be specific, we assume that the initial density of the medium at the time $t = t_0 = 0$ is everywhere constant, $\rho = \rho_0$. The motion is given in the Lagrangian system by the simple formula

$$\mathbf{r} = \boldsymbol{\psi} + t\mathbf{u}(\boldsymbol{\psi}).$$

We pose the problem of how the density distribution changes with the passage of time.

A specific feature of the problem is that the trajectories of noninteracting particles can intersect.

We take the simplest case of one-dimensional motion:

$$x = \xi + tv(\xi).$$

This case is considered in detail in the book of Zel'dovich and Myshkis.[2]

Two particles whose initial positions are ξ_1 and ξ_2 are at the same point x' if the equation

$$x' = x_1 = x_2 = \xi_1 + t'_{1,2}v(\xi_1) = \xi_2 + t'_{1,2}v(\xi_2)$$

holds at the time

$$t'_{1,2} = \frac{\xi_2 - \xi_1}{v(\xi_1) - v(\xi_2)}.$$

Such a time exists in the future if for $\xi_2 > \xi_1$

$$v(\xi_2) < v(\xi_1).$$

For two neighboring particles (we omit the index of t')

$$\xi_2 = \xi_1 + d\xi, \quad t' = -\left(\frac{dv}{d\xi}\right)^{-1}.$$

The amount of matter in the small integral $d\xi$ is

$$dm = \rho_0 d\xi.$$

The quantity dm is conserved during the motion. We can write

$$dm = \rho\, dx,$$

where dx must be chosen in accordance with the choice of $d\xi$.

Thus, we obtain

$$\rho = \frac{dm}{dx} = \frac{dm}{d\xi}\cdot\frac{d\xi}{dx} = \rho_0\left(\frac{dx}{d\xi}\right)^{-1},$$
$$\rho = \rho_0\left(1 + t\,\frac{dv}{d\xi}\right)^{-1}.$$

It follows that the time of intersection t' of the neighboring trajectories is a time at which the matter density becomes infinite. The curves of the dependence of the density on the coordinate x with infinite peaks are considered in Ref. 2.

As was shown in Ref. 3, allowance for pressure eliminates the caustic singularity in a mechanical system. Similarly, the finite wavelength of light eliminates the infinite light flux in an optical caustic. Strictly speaking, caustics exist only in the limit of infinitesimally small wavelength or infinitely low pressure. Therefore, the caustic representation is helpful when the wavelength of the light or, respectively, the pressure can be ignored.

A theoretician may say that the above expressions are rather simple and transparent. In principle, the matter could be clearly demonstrated by putting the corresponding distributions on a computer display and taking a corresponding film. However, at the present time it is impossible to append such a film or a video tape to each copy of Uspekhi and invite the reader to examine it.

However, there is a way—which we shall describe in the following section—of achieving visualization in the literal sense of this word, i.e., seeing with the eyes of a distribution of brightness of light corresponding to the density distribution in the above expressions.

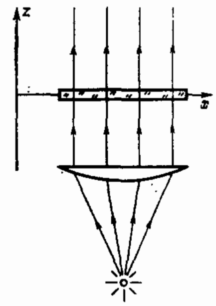

FIG. 1.

3. OPTICAL MODEL

a) One-dimensional case

We replace the particle trajectories by light rays. All the following treatment is in the framework of geometrical optics.

We imagine a horizontal plate illuminated from below by a parallel pencil of rays, for example, from a point source after passage through a converging lens (Fig. 1).

If the plate is plane-parallel, the rays remain vertical. We now take a plate with a smoothly varying thickness (Fig. 2). When the rays pass through such a plate, they are deflected somewhat differently at different points.

We denote the deflection angle by α; regarding it as a small quantity, we replace $\tan\alpha$ by α. For each ray, α is given: α is a function of the point $\alpha = \alpha(\xi)$.

Thus, we now have $x = \xi + z\alpha(\xi)$.

We have obtained complete analogy with the one-dimensional inertial motion of particles! The vertical coordinate z plays the role of the time.

We place a screen along the path of the rays. Depending on the value of z the illumination of the screen shows an entirely different picture.

Close to the plate, at small z, the illumination is almost constant. The differences in the illumination at small z increase in proportion to z. We shall denote the illumination by the same letter ρ as the density in

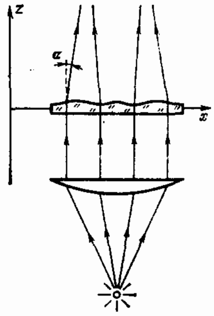

FIG. 2.

a mechanical problem:

$$\rho(x,\ z)=\rho_0\left(1-z\frac{d\alpha}{d\xi}+\ldots\right),$$

The first term retained in this expression corresponds to linear perturbation theory, when $\delta\rho=\rho-\rho_0\ll\rho_0$.

However, with a further increase in z, i.e., the distance between the screen and the plate, a bright point—a caustic—suddenly appears.[1)] This point then divides into two, other caustics appear, and so forth.

It is assumed that the surface of the refracting plate is given by a fairly smooth random function.

Figure 3 shows a series of photographs illustrating what we have said. Note further that in the absence of sharp breaks on the surface of the refracting plate, i.e., for bounded $d\alpha/d\xi$, there is nowhere complete shadow on the screen.

At some parts of the screen we have $\rho>\rho_0$ and at individual points $\rho=\infty$, but this infinity is integrable, and $\int\rho dx$ always converges. However, we nowhere have $\rho=0$ and, *a fortiori*, nowhere the absurd $\rho<0$.

The abrupt appearance of points with $\rho=\infty$, caustics, is an example of the phenomena treated by catastrophe theory. In this connection, we recommend to the reader the elementary but very informative booklet by Arnol'd[4] published by Zannie and his popular paper in the journal Priroda.[5] To those who wish to make a thorough study of catastrophe theory we can recommend the books of Refs. 6 and 29.

As an exercise, we recommend the reader to construct the curves $\rho(x,\ z)$ for $z<z'$ and $z>z'$, before and after the appearance of the first caustic. We also suggest the reader would consider what happens when the converging lens in Fig. 2 is removed. Then a diverging beam of rays is incident on the plate from below.

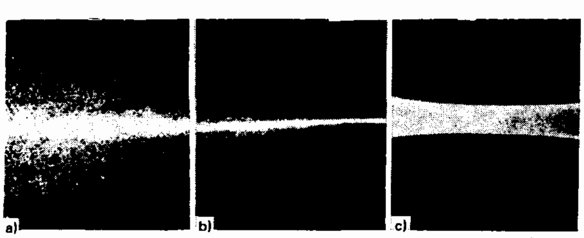

FIG. 3.

[1)]If the homogeneous problem is reproduced by means of a plate in which one of the surfaces is bent only in one direction, $z=z(x)$, and the width of the plate is equal to Δy, the point is transformed into a strip of length Δy.

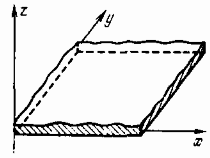

FIG. 4.

Finally, as a last problem we consider the case when the surface has a break of a small angle. Then there is also a discontinuity of the ray angle, namely, two neighboring rays with infinitesimal separation have slopes differing by a finite amount $\alpha(x-0)\neq\alpha(x+0)$. The reader is invited to consider the two cases corresponding to different signs of the difference between the inclinations of the rays.

b) Two-dimensional case

We now turn to the two-dimensional case (Fig. 4). We consider a plate lying in the x, y plane for $z=0$ (assuming approximately a thin plate). It is illuminated from below by a pencil of rays parallel to the z axis. The corresponding arrangement—the light source and lens (cf. here Figs. 1 and 2)—is not shown in Fig. 4. The screen is arranged parallel to the plate, i.e., at different (but in each individual experiment constant) values of z.

Each individual ray passes through the plate at a point $\psi_2=(x=\xi,\ y=\eta,\ z=0)$; here, ψ_2 is a two-dimensional vector $(\xi,\ \eta)$. In the general case, a ray is slightly deflected from the direction parallel to the z axis, so that subsequently both x and y change. We have for the ray

$$x=\xi+z\alpha_x(\xi,\ \eta),$$
$$y=\eta+z\alpha_y(\xi,\ \eta),$$

or, in vector form,

$$\mathbf{r}_2=\boldsymbol{\psi}_2+z\boldsymbol{\alpha}_2(\boldsymbol{\psi}_2),$$

where $\mathbf{r}_2$, ψ_2, α_2 are two-dimensional vectors lying in planes perpendicular to z, and the z coordinate again plays a distinguished role. The vector $\boldsymbol{\alpha}_2=(\alpha_x,\ \alpha_y)$ is the two-dimensional velocity, and z plays the part of the time. Once again, with the passage of time, i.e., with growth of z, the initially small contrasts in the illumination increase, and then bright sections—caustics—appear. This time, however, the observed picture has several singularities, which can be clearly seen in photographs, but they are not so trivial (Fig. 5).

The caustics arise in the form of short sections. When they have only just appeared, it can be readily

FIG. 5.

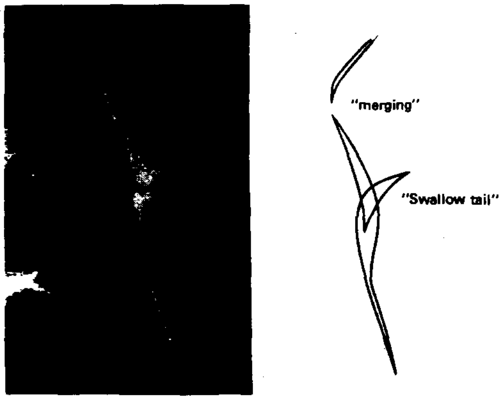

FIG. 6.

FIG. 8.

seen that the bright boundary is not at all similar to a circle or an ellipse, even an elongated one; rather it is similar to the contour of what is called a Cupid's bow.

Analysis shows that with the passage of time, i.e., with increasing distance of the screen in the z direction from the critical z' at which the caustic is generated, the thickness of the illuminated region increases as $d \propto (z-z)^{3/2}$, and the length as $l \propto (z-z')^{1/2}$. Therefore, the ratio l/d, which characterizes the degree of elongation, increases with decreasing $(z-z')$: $l/d \propto (z-z')^{-1}$. This means that the caustic is generated as a line and not as a spot.

Subsequently there is intersection of the caustics and the occurrence of the more complicated structures shown in Figs. 6–9.

Figure 6 shows two typical events—the merging of two "bows" and a rearrangement of swallow tail type. The names "swallow tail," "purse," and "pyramid" (see below) for the singularities arise from the characteristic shape in three-dimensional space. The photograph in Fig. 6 has a sketch that "deciphers" the image. It is not easy to photograph some of the singularities. For example, at the extreme points of "bows," at which two bright boundaries converge and touch, the brightness is particularly great (in the approximation of geometrical optics it is infinite), but it is also large in a certain neighborhood of this point (but finite), where there is as yet no caustic but there soon will be. The image of such points in the photograph is smeared and amorphous.

The rigorous classification of such structures is not a simple problem; it requires professional mathematical knowledge. We refer the reader to the papers of Arnol'd—alone[7–9] and together with members of our group.[10]

The structure in Fig. 7 is called a "pyramid"; Fig. 8 demonstrates a "purse."

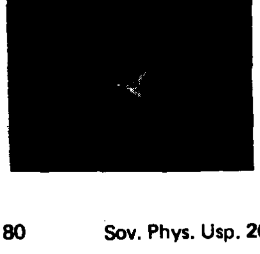

FIG. 7.

We note a general tendency to the formation of dark regions separated by a mesh of comparatively bright bands (see Fig. 9). This can be seen particularly well in photographs of the screen deliberately taken with the camera out of focus. Such a photograph smooths the details but makes it easier to see the general structure.

The general and strongest impression is that we take a smooth plate with smoothly varying thickness. The absence of breaks, notches, etc., is proved by the fact that the screen near the plate is uniformly illuminated.

Bright regions and the sharp boundaries of these regions arise only at a sufficient distance from the plate. The abruptness of their appearance and the sharp boundaries are common to the one- and two-dimensional problems. A new feature in the two-dimensional problem is the definite structure—pattern—produced by the rays (see Figs. 5–9). There are no such patterns in the orginal plate, in which the lines of equal thickness have smooth shapes!

We note one further fundamental aspect of the formulation of the experiment. We describe in more detail the mechanism by which the rays are curved, for which we consider the shape of the plate.

Suppose its lower surface is strictly flat, $z=0$, while the upper is defined by the equation

$$z = z_0 + \zeta(x, y),$$

where z_0 is the small mean thickness and ζ is a random but smooth function of x and y. Ideally, we would wish to specify ζ by means of the Fourier integral

$$\zeta = \iint f(|\mathbf{k}|)\, \zeta_{\mathbf{k}} \cos(k_x x + k_y y + \varphi_{\mathbf{k}})\, d^2k;$$

here $\varphi_{\mathbf{k}}$ is the random phase, for which all values from 0 to 2π are equally probable; $\zeta_{\mathbf{k}}$ is a random variable distributed in accordance with the normal law $\langle \zeta_{\mathbf{k}} \rangle = 0$, $\langle \zeta_{\mathbf{k}}^2 \rangle = 1$; and $f(|\mathbf{k}|)$ is a regular function that defines

FIG. 9.

the spectrum, for example, $k^m e^{-k^2}$, which decreases sufficiently rapidly both as $k \to 0$ and as $k \to \infty$.

In practice, we used cells, in which a layer of water was poured between a flat plastic plate and an upper plastic plate slightly bent by skilled hands. It is very difficult to avoid the formation of axisymmetric convexities and concavities, which are naturally obtained when a finger is pressed onto the heated and softened plate. Axisymmetric deformations of the plate produce nontypical axisymmetric structures. However, after many trials a certain skill was acquired, and we believe the structures of the illuminated regions shown in the photographs are fairly typical.

However, irrespective of the skill in bending the plates and their actual shape, the arrangement of the experiment is such that one always simulates a definite—irrotational—type of motion of particles.

Indeed, the deflection of a ray in accordance with the laws of geometrical optics lies in the plane passing through the normal to the surface and the undisturbed ray. For rays originally propagating along the z axis to the surface given by the expression $z = z_0 + \zeta(x,y)$, we obtain

$$\alpha_x = -(n-1)\frac{\partial\zeta}{\partial x}, \quad \alpha_y = -(n-1)\frac{\partial\zeta}{\partial y}$$

(n is the refractive index), or, in vector form,

$$\boldsymbol{\alpha}_2 = -(n-1)\,\mathrm{grad}_2\,\zeta.$$

We recall that $\boldsymbol{\alpha}_2$ is the analog of the initial velocity. If a velocity can be expressed as the gradient of a function ζ, then the curl of the velocity vanishes identically[2)] :

$$\frac{\partial\alpha_x}{\partial y} - \frac{\partial\alpha_y}{\partial x} = 0.$$

Unfortunately, it is impossible to simulate by a sufficiently simple and convenient method a three-dimensional structure or caustics in three-dimensional space.

To conclude this section, we mention some publications in which the reader can learn about the latest results of the use of catastrophe theory in geometrical optics: Refs. 28 and 30.

4. LARGE-SCALE STRUCTURE OF THE UNIVERSE

It remains to explain how the word "cosmology" came to appear in the title of the paper. The scale of the experiments are different—a few centimeters and billions of light years in the Universe—and the materials are different—light rays in the laboratory and atoms or heavy neutrinos in the Universe. In fact, we have become used to such differences; the aim of modeling is to have different scales!

For small deflections of the rays, the velocity of their motion in the x, y plane is small, and the z coordinate does indeed play the role of time measured along the trajectory in the x, y plane for a Newtonian particle.

We aim to interpret experiments with light rays and caustics as a model of the formation of the large-scale structure of the universe.

The initial deflection of the light rays after passage through the plate is the analog of the initial perturbations, and the formation of the brightly illuminated regions simulates, in our view, the formation of dense clouds of gas. We assume that the occurrence of an atom-plasma or heavy-neutrino giant cloud with a strongly flattened shape is accompanied by or precedes the formation of individual galaxies. Galaxies do not occupy the whole of space. To a large extent, galaxies are combined in clusters, and the clusters of galaxies and isolated galaxies, taken together, retain for a long time—to the present epoch—a "recollection" of the cloud in which they were born, and they are distributed principally near the surfaces of superclusters and around the lines of their intersections. More and more frequently observers speak of a cellular structure of the matter distribution in the universe.[11–14] What is the similarity and what is the difference between the Universe and our modest models? The similarity is that at the stage of formation of the structure in the Universe one can ignore the plasma pressure and one can ignore the random thermal velocities of the heavy neutrinos.

The neglect of the pressure means that the motion of neighboring "particles" in the Universe is independent, like the motion of neighboring light rays. Here it is necessary, perhaps, to be more precise: The motion of neighboring rays is independent, and each ray is a straight line with given point of intersection $\psi_2(\xi = x_0, \eta = y_0)$ of the plane $z = 0$ and given slope $\alpha_2(x_0, y_0)$. However, the motion of neighboring rays is correlated in the sense that if they are neighbors their slopes, i.e., the vectors $\boldsymbol{\alpha}_2$, are necessarily nearly the same. This is not a general property of rays but a property of the plate, the condition of smoothness of the plate. We encounter the same smoothness of the initial conditions in cosmology for very deep reasons for which we do not have space to discuss here.

However, there is a deep and, at the first glance, insuperable obstacle—the rays in the model do not interact with one another at all. The particles in the Universe do not interact through pressure, but for them the gravitational interaction is decisive.

Taken together, they produce a definite coordinate- and time-dependent gravitational field, and they move in this field. Since the time of Newton (see his famous letter to Bentley) the instability of uniformly distributed matter has been known at the qualitative level. At the beginning of this century, Jeans[15] developed a quantitative Newtonian theory of the instability. In the middle of our centry, Lifshitz[16] considered small perturbations of a homogeneous expanding Universe. The decisive role of gravitational instability in the formation of structure of the Universe is beyond doubt.

2) Strictly speaking, we prove this for the initial velocity distribution on $z = 0$ as a function of $x = \xi$ and $y = \eta$. It can be shown that later for $z > 0$, when $x = \xi$, $y = \eta$, the relation $\partial\alpha_x/\partial y - \partial\alpha_y/\partial x = 0$ holds as before. However, this assertion becomes meaningless within regions bounded by caustics, where three or more different sets ξ, η correspond to one x and y.

How can this be reconciled with our proposed simulation?!

In 1970, one of the authors (Ya. Z.) proposed an approximate nonlinear theory of the gravitational instability of cold matter in an expanding Universe[17] (see also Ref. 14).

The solution can be written in Lagrangian form:

$$\mathbf{r} = a(t)[\boldsymbol{\psi} + b(t)\,\mathbf{u}(\boldsymbol{\psi})]$$

here, $a(t)$ is an increasing function of the time which describes the general expansion of the Universe, and $b(t)$ is an increasing function of the time that describes the growth of the perturbations. The spatial distribution of the perturbations is given by a function of the Lagrangian coordinates $\mathbf{u}(\psi)$. All details relating to the conditions of applicability of this solution, its accuracy, etc., can be found in our publications in Refs. 18–25. Here, we restrict ourselves to demonstrating the similarity of the approximate solution and the optical model.

We write the solution in the form

$$\mathbf{R} = \boldsymbol{\psi} + \tau\,\mathbf{u}(\boldsymbol{\psi}),$$

where we have introduced the new variables

$$\tau = b(t), \quad \mathbf{R} = \frac{\mathbf{r}}{a(t)} = \frac{\mathbf{r}}{a(t(\tau))}.$$

For a definite choice of the scale of the spatial picture (transition from $\mathbf{r}$ to $\mathbf{R}$) and the effective time τ the solution in the coordinates $\mathbf{R}$, ψ, τ is exactly equivalent to the problem of the motion of free entirely noninteracting particles or rays.

The gravitational interaction has been hidden in the definition of the function $\tau = b(t)$.

The equivalence of the two problems is approximate. However, the approximation is sufficiently good for the structure that arises in the model to describe the structure of the Universe.

We note further the fact that the gravitational interaction is characterized by a *potential* and therefore the effective velocity is *irrotational*, i.e., curl $\mathbf{u} = 0$. Irrotationality in the sense of the derivatives of the components of the vector $\mathbf{u}$ with respect to the components of the Lagrangian coordinate ψ ensures irrotationality in the Eulerian space as well. This property of $\mathbf{u}(\psi)$ is common to our optical model and the Universe.

As we have already noted, a restriction of the optical model is that it is two-dimensional (plus the coordinate z, which plays the role of time), whereas the Universe is three dimensional (plus the time).

The approximate formula describes a period that goes far beyond the limit of the theory of small perturbations and includes the formation of caustics.[3)] However, its region of applicability is also restricted. In the regions of maximal density phenomena rapidly arise that the approximate theory does not describe: shock waves, many-stream configurations, break up into galaxies, etc.

But nevertheless the structure of the Universe today still carries the imprint of the heroic period of caustic formation![14]

We recognize that our optical model belongs methodologically to the time when physics was done with sealing wax and string. This is still true if the light source is a laser.

Today, the three-dimensional picture corresponding to the approximate solution can be obtained on a computer display. Moreover, the corresponding pictures can, using density coding by means of color, be expressed as a film.

But recognition of the far reaching possibilities of modern technology does not diminish the joy of the direct observation of cuastics appearing abruptly as the screen is displaced in a perturbed beam in a darkened laboratory.

[3)]The formation of caustics in problems of the general theory of relativity for dust particles ($p = 0$) moving in a synchronous frame of reference was noted earlier in Refs. 26 and 27.

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