Superflows and superfluidity

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The basic concepts of the theory of mass superflows in a superfluid are presented. The stability of such flows with respect to the creation and growth of linear defects (superfluid vortices), the relationship between superflows and the existence of long-range order, and the possible occurrence of persistent flows in one- and two-dimensional systems are discussed. Some analogs of the mass superflows in a superfluid are also examined: spin superflows in magnetically ordered systems having an easy-plane anisotropy and the current states of a Bose condensate of electron-hole pairs. The physical meaning of such "flows" is discussed, and a theory for their stability is derived from the calculated probability for the creation of the linear defects which are analogs of superfluid vortices. There is a discussion of the applicability of the theory of spin superflows to several experiments on the magnetic properties of the *A* phase of superfluid helium-3 and to a possibility which follows from this theory: that domain walls might be generated in the interior of a sample of an easy-plane magnetically ordered material and that the motion of these walls might be controlled by fields applied to the surface of the sample.

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CONTENTS

1. INTRODUCTION

The term "superfluidity" is now used to cover a broad range of phenomena which have been observed in He II (Refs. 1–4), in the superfluid phases of He^3 (Refs. 5–8), and, in the broader sense of the term, in superconductors.[9,10] One of the most remarkable of these phenomena is the nondecaying mass transport in persistent currents. It was this phenomenon, discovered by Kamerlingh Onnes in 1911 for electrons in a metal and by Kapitsa in 1938 for He II, which gave rise to the terms "superconductivity" and "superfluidity." One purpose of the present review is to set forth the present understanding of nondecaying currents ("superflows"), primarily for the two superfluids He^4 and He^3.

A circumstance underlying all superfluid phenomena is that the complex order parameter $\Psi = \eta e^{i\varphi}$, which is the wave function of the bosons or Cooper pairs of fermions in a single quantum state (a Bose condensate), emerges as an additional macroscopic variable of the liquid. The existence of persistent superfluid currents is a consequence of the special topological properties of the order parameter's space of definition, which cause "deformed" states, i.e., states with a spatially inhomogeneous parameter Ψ, to be metastable. The superfluid mass flow $\mathbf{j}_s = \rho_s \mathbf{v}_s$ which exists in these states is proportional to the superfluid velocity $\mathbf{v}_s = (\hbar/m)\nabla\varphi$. The superfluid density ρ_s is determined by the modulus η of the order parameter Ψ, and it is always possible to choose a normalization of Ψ such that $\rho_s = m\eta^2$. In the microscopic theory, however, Ψ is usually normalized to the density of bosons in a single-particle state with a definite momentum: $n_0 = |\langle\Psi\rangle|^2$. In this case, the relationship between ρ_s and Ψ becomes more complicated, especially since long-wave fluctuations of the phase φ may cause the average value of the order parameter over the volume of the liquid, $\langle\Psi\rangle$, to vanish. This occurs in one- and two-dimensional systems, and at one time this circumstance raised doubt that such systems could exhibit superfluidity. However, as we hope to show in this review, on the basis of the results of many studies, a nonzero superfluid den-

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sity ρ_s may exist even if $\langle\Psi\rangle = 0$, since ρ_s must be related to the modulus of the local value of Ψ over the volume. This local value can be introduced if, despite phase fluctuations, correlations in the order parameter are preserved over long if not infinite temporal and spatial scales. It follows that we are completely justified in discussing the possibility of a nondecaying mass or charge transport for systems of any dimensionality. It is important to define at the outset just what we mean by "nondecaying" currents, since in reality these currents do decay but have an anomalously long lifetime. How are we to distinguish between "anomalously long" lifetimes and simply "long" lifetimes? In a ring of a very pure metal in its normal state, for example, an electric current also decays very slowly; does this mean that the current does not decay? We believe it is more reasonable (although not the only possibility) to adopt the following definition of a nondecaying current: In the course of the current relaxation, activation barriers which are large in comparison with the thermal energy kT must be surmounted. These large barriers, which are proportional to the superfluid density ρ_s, determine the large exponential factor in the expression for the lifetime of the superfluid current. The long lifetime of such a current thus results not from the weakness of the momentum-nonconserving interaction, as in a very pure metal, but from an unusual suppression of the effect of this interaction. A goal of the theory and of the present review is to show how and where such large activation barriers appear.

A substantial part of this review will be a discussion of the possibility of nondecaying transport (i.e., of superflows) of physical quantities other than mass and charge, e.g., angular momentum or spin and the number of electron-hole pairs or excitons. Another important question here is a purely semantic one: Is it appropriate to use the term "nondecaying flow" ("superflow") or even simply "flow" in these cases? Keeping this question in mind, we will strive primarily to show just what each term stands for, without imposing a definite terminology. Analogs of superfluid states in this case are metastable states with a deformed field of some order parameter. Transitions to the ground state are hindered by activation barriers, as in a superfluid liquid.

The review is organized as follows: Section 2 deals with mass superflows in He II and the conditions which determine the anomalously slow relaxation of these flows at velocities below the critical velocities. The discussion in Section 2 is not to be regarded as a review of the theory of critial velocities, since it sets forth only the basis ideas of this theory; for a more detailed discussion and for a bibliography on the topic the reader is referred to Chapter 6 in Putterman's book.[3] Section 3 deals with spin superflows and the accompanying metastable helicoidal structures in magnetically ordered systems (ferromagnets and antiferromagnets). Section 4 deals with some extremely unusual superfluid properties of the A phase of He^3, in which a nondecaying transport of both mass and spin may occur. Section 5 deals with electron-hole systems in which a transition to a coherent state is accompanied by the formation of charge or spin density waves. Finally, Section 6 is a discussion of the interrelationship between superflows and the existence of long-range order.

2. SUPERFLUIDITY OF HELIUM II

a) The Landau criterion and Feynman's theory of critical

If there is to be no dissipation as a superfluid flows through capillaries, the current state must correspond to a local minimum of the energy. This statement means that transitions to nearby states which would be produced from the original state through the creation of one or several quasiparticles are unfavorable from the energy standpoint. This condition reduces to the well-known Landau criterion,[11] according to which the energy of Bose quasiparticles, $\varepsilon(p)+pv_s$, in the coordinate system moving at the superfluid velocity $\mathbf{v}_s$ must be positive [$\varepsilon(p)$ and p are the energy and momentum of the quasiparticle in the fluid at rest]. This condition is met if

$$v_s < v_L = \min \frac{\varepsilon(p)}{p}. \tag{2.1}$$

An expression is thus found for the Landau critical velocity v_L.

In addition to the *microscopic* excitations, however, there are *macroscopic* excitations in a moving liquid. These macroscopic excitations alter the state of the liquid in a rather large volume and cannot be described in terms of quasiparticles. Such excitations always allow a reduction of the momentum of the fluid without an increase in its energy, but Landau regarded the creation of these excitations through a direct transition as an improbable event. Experimentally, the critical velocities have turned out to be substantially lower than the Landau values. In this connection Feyman suggested that the relaxation of a superflow occurs through the creation of vortex rings.[12] The spectrum of these rings is of such a nature that $\varepsilon(p)/p$ falls off monotonically with increasing radius (R) of the vortex ring [$\varepsilon(p)/p \sim R^{-1/2}$]. As the critical velocity, Feynman suggested the value of $\varepsilon(p)/p$ at $R \sim d$, where d is the transverse dimension of the capillary. This approach leads to critical velocities in better agreement with those observed experimentally, but the most important point is that the dependence of the critical velocity on the capillary size becomes the same as that observed experimentally.[3] However, it appears that this satisfactory agreement with experiment is misleading. In the first place, Feynman's theory uses the spectrum of vortex rings in an unbounded fluid, while the effect of the capillary wall would have to be considered for vortex rings of large radius. A systematic analysis of this effect shows that the energy of the vortex ring and thus the critical velocity fall off to zero toward the capillary wall (see Refs. 13 and 14; why a nonzero critical velocity was derived in several papers is explained in Ref. 14). The second objection is that the creation of a large-radius vortex ring is an extremely unlikely event (as was pointed out by Vinen[15]), since this process is actually a macroscopic excitation of the sort discarded in Landau's analysis.

Despite these difficulties (to which we will return in Subsection 2c), there is no longer any doubt that the dissipation at velocities above the critical velocity is in fact due to vortices. Quantized vortices thus play a key role in an analysis of the stability of superflows. The Landau criterion, on the other hand, which is a condition for the microscopic stability of a superflow, retains its role as a *necessary* condition which is imposed on the Bose branches of the spectrum.[1)]

In deriving his criterion Landau used Galilean invariance, and for systems which are not invariant in the Galilean sense the condition for the microscopic stability of a superflow may take a form different from the Landau criterion. We will see such a case below when we discuss a spin flow (Subsection 3d). The electrons in a crystal lattice are also not a Galilean-invariant system, generally speaking, and the condition for a positive quasiparticle spectrum in the current state of a superconductor with an inhomogeneous order parameter $\Delta = |\Delta| \exp(2i\mathbf{p}_s\mathbf{r}/\hbar)$ is

$$\varepsilon(\mathbf{k}) = \varepsilon_0(\mathbf{k}) + \frac{\partial\xi(\mathbf{k})}{\partial\mathbf{k}}\mathbf{p}_s > 0, \qquad \mathbf{p}_s = m\mathbf{v}_s,$$

where $\varepsilon_0(\mathbf{k}) = \sqrt{\xi^2 + |\Delta|^2}$ is the quasiparticle spectrum in the ground state, $\mathbf{p}_s = 0$. Here we have generalized to the case of an arbitrary spectrum $[\xi(\mathbf{k})]$ of the electrons in a normal metal the derivation of the relationship between ε and ε_0 which was given at the end of §1 of Chapter 5 of de Gennes's book[9] for the particular spectrum $\xi(\mathbf{k}) = (k^2/2m) - \varepsilon_F$. This condition yields the usual Landau criterion in the approximation of nearly free electrons after the substitution $\partial\xi(\mathbf{k})/\partial\mathbf{k} = \mathbf{k}/m$, but this substitution implies that the quasimomentum $\mathbf{k}$ must lie within the Brillouin zone; we thus cannot add an arbitrary reciprocal-lattice vector to it, despite the fact that spin flip is quite possible during the creation of quasiparticles. In the opposite case, the Landau criterion would always be violated upon the choice of a large reciprocal-lattice vector.

A violation of the Landau criterion for the spectrum of Bose excitations leads to an instability of the gas of excitations; in He II this instability is seen as a Bose condensation of rotons.[17] For the Fermi spectrum, however, a violation of the Landau criterion for a bounded part of $\mathbf{k}$ is not as catastrophic, because the occupation number of each state is finite. In a superconductor the critical velocity v_L for Fermi quasiparticles is called the "depairing velocity." Even if the superfluid velocity slightly exceeds $\mathbf{v}_L$, however, the number of quasiparticles rapidly increases, ultimately leading to an instability (see Sec. 4.4 of Ref. 10). In the *A* phase of He^3 the Landau criterion is violated at zero superfluid velocity, because of the zero gap in the spectrum of quasiparticles with momentum parallel to the orbital angular momentum $\mathbf{l}$. Nevertheless, a mass superflow in the *A* phase may be stable at low velocities (see Subsection 4a of the present review).

b) Quantized vortices and flow quantization

In a superfluid the phase (φ) of the complex order parameter Ψ is a cyclic variable, defined modulo 2π. Consequently, the circulation of the superfluid velocity $\mathbf{v}_s$ taken around any closed contour can assume only the quantized values Nh/m, where N is an integer. This circumstance gives rise to quantized vortices and to flow quantization in a closed annular channel. If there are no vortices, the total superfluid momentum along the axis of such a channel is

$$\mathbf{P}_s = \rho_s \int \mathbf{v}_s\, d\mathbf{r} = \rho_s \frac{\hbar}{m} \int \nabla\varphi\, d\mathbf{r} = \frac{N\rho_s S h \mathbf{t}}{m}, \tag{2.2}$$

where $\mathbf{t}$ is a unit vector along the axis of the channel, ρ_s is the superfluid mass density, and S is the cross-sectional area of the channel. The integration is carried out over the entire volume of the channel. We are assuming that the cross-sectional dimensions are much smaller than the length of the channel and that the curvature of the channel can be ignore. Strictly speaking, in the annular geometry we are dealing with a quantization of angular momentum rather than of linear momentum.

The momentum of a fluid can take on nonquantized values only if there are quantized vortices in the fluid. Let us assume that a vortex with a single quantum of circulation, h/m, arises in some cross section of an annular channel; the vortex is assumed to begin and end at the channel wall (Fig. 1). In order to calculate the momentum it is necessary to introduce cuts with discontinuities of the phase φ so that the phase will be single-valued throughout the volume. One such cut, with a

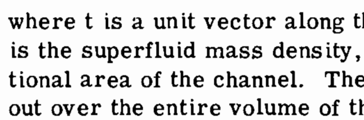

FIG. 1. Phase slippage: the motion of a vortex line across a cylindrical channel. S_v is the area of the section with a phase shift of 2π.

[1)] It should be noted that the *exact* energies of the many-particle system in the excited and ground states of the moving fluid should be compared in the Landau criterion. In the determination of the difference between these energies, however, the theory of a Bose liquid is used, and this theory gives an approximate description of the transition to the excited state as the creation of a single quasiparticle. The energy of a quasiparticle is not determined exactly and is instead distributed with a certain probability density over a certain energy interval. One may sometimes encounter the assertion that this interval goes to zero, so that there always exists some probability for a transition with a change in momentum without an increase in energy, i.e., that the Landau criterion always holds. Actually, the interval of possible quasiparticle energies *cannot* include low energies—lower than $v_L p$—if the Landau criterion holds. If the difference between the *exact* energy of the excited state with momentum p and that of the ground state is greater than $v_L p$, then the Lehmann expansion for the Green's function $G(\omega, \mathbf{p})$ begins with an energy greater than $v_L p$, and the imaginary part of $G(\omega, \mathbf{p})$ vanishes at $\omega < v_L p$. This assertion is confirmed in a paper by Iordanskiĭ and Pitaevskiĭ,[16] who considered phonons, for which the velocity v_L is the sound velocity, as the elementary excitations. Everything stated above is based on the assumption that the average energy of the quasiparticle is a good approximation of the exact energy of the many-particle system. If this assumption is wrong, the entire quasiparticle model of a Bose liquid is also wrong.

phase discontinuity of $2\pi N$, should be made over the entire cross section of the channel. Another cut, with a phase discontinuity of 2π, should be bounded by the vortex line and by the channel wall (as shown in Fig. 1; S_v is the area of this cut). After integration by parts, we are left with only the contributions of the two cuts in the expression for the total momentum:

$$\mathbf{P}_s = \rho_s \frac{\hbar}{m} \int \nabla\varphi \, d\mathbf{r} = \rho_s \frac{h}{m} (NS \pm S_v)\, \mathbf{t}, \tag{2.3}$$

where the sign $\pm$ is determined by the sign of the circulation of the vortex. The quantity $\pm\rho_s S_v h/m$ in (2.3) should be regarded as the momentum of the vortex; it is the same as the Kelvin momentum of a vortex in classical hydrodynamics[2)] (Ref. 18).

It can be seen from (2.3) that as S_v varies from 0 to S there is a continuous change in the momentum by one quantum, $\rho_s Sh/m$. This change corresponds to motion of the vortex line across the flow; the line "cuts" the channel. This motion is none other than the phase slippage introduced by Anderson.[19]

The appearance of a vortex line in a fluid, however, increases the energy of the fluid. Near the vortex the velocity of the fluid is $v_s = \hbar/mr$, so that the energy of the vortex in the still fluid is determined by a logarithmically divergent integral:

$$\varepsilon_v = \rho_s \int \frac{v_s^2}{2}\, d\mathbf{r} = \pi\rho_s H \left(\frac{\hbar}{m}\right)^2 \int_{r_c}^{R} \frac{dr}{r} = \pi\rho_s H \left(\frac{\hbar}{m}\right)^2 \ln\frac{R}{r_c}, \tag{2.4}$$

where H is the length of the vortex line. The upper cutoff parameter R is determined by either the size of the system or the size of the region outside which the velocity v_s begins to fall off more rapidly than $1/r$. For a ring vortex, for example, this would be the radius of the ring. The lower cutoff parameter, r_c, is either the coherence length or the radius of the vortex core. Inside the core, the increase in the kinetic energy due to the velocity increase begins to be offset by a decrease in the superfluid density (the square modulus of the order parameter), which drops to zero directly at the vortex line. The cutoff parameter r_c is determined by equating the kinetic energy density to the condensation energy density (the condensation energy is the difference between the energies of the states having a zero order parameter and the equilibrium value of the order parameter).

In a moving fluid the energy of a vortex should be determined by the following expression, by virtue of Galilean invariance:

$$\tilde{\varepsilon}_v = \varepsilon_v + \mathbf{p}_v \mathbf{v}_s, \tag{2.5}$$

where $p_v = \pm\rho_s S_v h/m$ is the momentum of the vortex [see (2.3)].

Equations (2.3)–(2.5) can be used to derive the dependence of the free energy on the superfluid momentum in an annular channel[14] [Fig. 2(a)]. The quantized values of the momentum correspond to minima in this dependence. To go from one minimum to a neighboring one with a lower momentum required surmounting an activation energy barrier. This barrier will be lowest when the vortex line cutting the channel cross section upon the transition is a circular arc which runs normal to the channel wall where it intersects this wall (Fig. 1). This arc turns out to be a semicircle of radius R if R is much smaller than the radius of curvature of the wall; in this case, $S_v = \pi R^2/2$. The top of the barrier corresponds to the radius $R = (h/4\pi m v_s)\ln(h/m v_s r_c)$, and the height of the barrier is

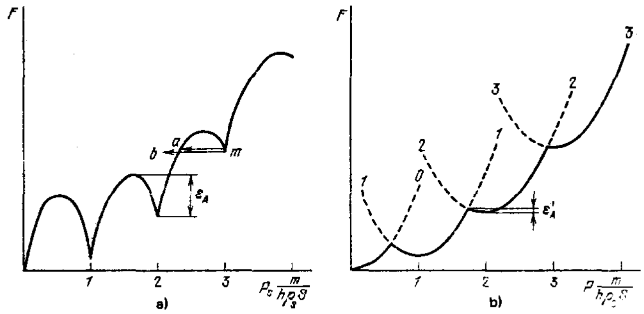

FIG. 2. Momentum dependence of the free energy in an annular channel with cross-sectional area S and length L. a—Dependence on the superfluid momentum (ma is a tunneling to the vortex state corresponding to an extremum of the free energy, while mb is a tunneling to a vortex state corresponding to a maximum of the transition probability); b—dependence on the total momentum [parabolas 0, 1, 2, and 3 (dashed curves) are the free energy $F = [(P - P_s)^2/2\rho_n SL] + (P_s^2/2\rho_s SL)$ of the subensembles with quantized values of the superfluid momentum $P_s m/h\rho_s$ ($s = 0, 1, 2, 3$); the envelope of the parabolas (the solid curve) is the total free energy, determined within quantities of order kT].

$$\varepsilon_A = \frac{\rho_s}{32\pi v_s}\left(\frac{h}{m}\right)^3 \left(\ln\frac{h}{m v_s r_c}\right)^2, \tag{2.6}$$

where the average superfluid velocity $v_s = P_s/\rho_s SL$ is quantized, as is the total momentum P_s, but with a very small quantum, h/mL, which is inversely proportional to the length of the channel, L.

According to (2.6) the activation barrier disappears at a velocity $v_s \sim h/mr_c$ comparable in magnitude to the Landau critical velocity for the roton-phonon spectrum.

Up to this point we have been discussing the dependence of the free energy on the superfluid momentum. It is also possible to derive the dependence of the free energy on the total momentum.[20,21] The derivation method is clear from Fig. 2(b) (vortex states are ignored). There are minima in this dependence only at vanishingly low velocities $v_s < (\rho/2\rho_n) \times (h/mL)$. The activation barriers between these minima are

$$\varepsilon'_A = \frac{h^2 S}{2m^2 L}\frac{\rho^2}{\rho_n}\left(\frac{1}{2} - \frac{\rho_n}{\rho}\frac{v_s L m}{h}\right)^2. \tag{2.6'}$$

The disappearance of the activation barriers from the dependence of the free energy on the total momentum which occurs at higher velocities, however, has no direct bearing on the problem of the stability of superflows, since the unusually long lifetime of superflows is linked to a relaxation of the superfluid momentum, *rather than of the total momentum.*

[2)]In the literature there are differences in the definition of the momentum of a vortex. See Ref. 14 for a discussion of the problem and a corresponding bibliography.

c) The Iordanskiĭ-Langer-Fisher theory and the quantum creation of vortices. Ideal and real critical velocities

The probability for a vortex to cross the channel, overcoming an activation barrier, must determine the intensity of the phase slippage and, correspondingly, the relaxation rate of the superfluid mass flow. Although this conclusion is derived most simply from an analysis of the annular geometry considered in the preceding subsection, it obviously must remain in force when the boundary conditions at the end of the channel are different. Phase slippage is actually a three-dimensional process, and no source—whatever its nature—of a change in phase at the ends of the channel could make a contribution proportional to the length of the channel.

In order to determine the probability for the transfer of a momentum quantum of superfluid motion, Iordanskiĭ[22] examined the appearance of vortex rings as the result of fluctuations and the subsequent growth of these rings due to their interaction with elementary excitations in the superfluid. This theory was pursued by Langer and Fisher[23] and is now known as the "Iordanskiĭ-Langer-Fisher theory." A natural result of this theory is an exponential dependence of the relaxation rate on the activation barrier:

$$\frac{dv_s}{dt} = -\frac{v_s}{\tau_s}, \quad \frac{1}{\tau_s} \sim e^{-\varepsilon_A/kT}. \tag{2.7}$$

In the Iordanskiĭ-Langer-Fisher theory the concept of a critical velocity becomes quite arbitrary, since now it is not a well-defined threshold but some velocity at which the vortex creation rate becomes significant in comparison with some specified scale. In any case, if reasonable values are chosen for this scale and for the coefficient of the exponential function in (2.7) it is possible to describe successfully the experimental values of the critical velocities at temperatures near the λ point, where the activation barrier is low. As the temperature is reduced, the Iordanskiĭ-Langer-Fisher critical velocity increases, reaching values comparable to the Landau critical velocity. At low temperatures, however, the process which primarily determines the probability for the creation of vortex rings should be a direct quantum-mechanical tunneling to a state with a vortex, rather than a fluctuational growth. The probability for such a transition increases exponentially rapidly with decreasing velocity, as follows from Refs. 14 and 24, where the argument of the exponential function in the expression for the tunneling probability was calculated. For this calculation, Volovik[24] used a semiclassical theory, adopting as the Hamiltonian the expression for the energy of a vortex half-ring in an incompressible fluid near a protuberance on the wall. The transition probability was calculated in a different way in Ref. 14. The approach of that paper was based on the model of a slightly nonideal Bose gas in which the many-particle wave function of the N bosons of the condensate can be written out explicitly:

$$\Psi^N = \prod_{i=1}^{N} \Psi(\mathbf{r}_i), \quad \int |\Psi(\mathbf{r})|^2 \, d\mathbf{r} = N, \tag{2.8}$$

where $\mathbf{r}_i$ is the radius vector of the i-th boson. A direct quantum-mechanical transition from the ground state of the moving fluid to a state with a large vortex could occur only as the result of an interaction which disrupts the translational invariance. This interaction might be written, for example, as the sum of random potentials $V(\mathbf{r}_i)$ acting on each of the bosons. In this case the transition probability is found to be

$$\begin{aligned} W &\sim |\langle \Psi_1^N| \sum_{i=1}^{N} V(\mathbf{r}_i) |\Psi_0^N\rangle|^2 \sim \Big| \Big(\sum_{i=1}^{N} \int \Psi_1(\mathbf{r}_i)^* V(\mathbf{r}_i) \, \Psi_0(\mathbf{r}_i) \, d\mathbf{r}_i \Big) \\ &\times \Big(\frac{1}{N} \int \Psi_1(\mathbf{r})^* \Psi_0(\mathbf{r}) \, d\mathbf{r} \Big)^{N-1} \Big|^2 \sim \Big| \frac{1}{N} \int \Psi_1(\mathbf{r})^* \Psi_0(\mathbf{r}) \, d\mathbf{r} \Big|^{2N} \\ &\sim \Big(1 - \frac{\Gamma}{2N} \Big)^{2N} \sim e^{-\Gamma}, \quad \Gamma = 2N - 2 \Big| \int \Psi_1(\mathbf{r})^* \Psi_0(\mathbf{r}) \, d\mathbf{r} \Big|. \end{aligned} \tag{2.9}$$

It was assumed for this derivation that the condensate wave functions $\Psi_0(\mathbf{r})$ and $\Psi_1(\mathbf{r})$ of the initial and final states differ only in a finite volume, so that the scalar product

$$\frac{1}{N} \int \Psi_1(\mathbf{r})^* \Psi_0(\mathbf{r}) \, d\mathbf{r}$$

is approximately unity. The actual shape and height of the potential $V(\mathbf{r})$ affect only the coefficient of the exponential function, which we are not calculating. The condensate wave function in the final state, $\Psi_1(\mathbf{r})$, must be determined from the condition for a maximum of the transition probability W (for a minimum of Γ) under the supplemental condition that the energy of the final state must not exceed that of the initial state.[14] It turns out that the probability reaches an extremum for a transition to a state with a vortex half-ring of radius R which is resting on the wall; in this Ψ_1 differs in modulus from Ψ_0 only at the core of the vortex filament, and the phase difference $\tilde{\varphi} = i \ln(\Psi_1^* \Psi_0)$ between Ψ_1 and Ψ_0 satisfies the equation

$$\Delta\tilde{\varphi} = \frac{\sin\tilde{\varphi}}{l^2}. \tag{2.10}$$

Solving (2.10) (the sine-Gordon equation) we see that during the transition the change in the phase $\tilde{\varphi}(\mathbf{r})$ occurs in a plane region of thickness l which is reminiscent of a domain wall bounded by the vortex half-ring and the channel wall. In the limit of a low velocity v, calculations of logarithmic accuracy yield

$$W = \exp\Big[-\frac{27\hbar^3\rho}{m^4 v^3} \Big(\ln \frac{\hbar}{mvr_c} \Big)^2 \Big], \quad l = \frac{6\hbar}{\pi m v}, \quad R = \frac{3\hbar}{2mv} \ln \frac{\hbar}{mvr_c}. \tag{2.11}$$

These calculations were carried out for $T = 0$, so that we have $\rho = \rho_s$ and $v = v_s$ in (2.11). Since $\Psi_1(\mathbf{r})$ was not determined from the extremum of the energy, the transition does not proceed along the minimum-energy curve but slightly above it [Fig. 2(a)]. Consequently, the incompressibility condition $\operatorname{div}\mathbf{v} = 0$ does not hold in the fluid after the transition, and the fluid begins to contract; the acoustic oscillations which result, however, undergo relaxation without an activation barrier, and they do not affect the argument of the exponential function in the momentum relaxation rate. The argument of the exponential function in (2.11) is equal in order of magnitude to the number of atoms or to the number of degrees of freedom in a region of volume $\sim R^2 l$, in which there is a significant change in the motion of the fluid upon the transition.[3)] This natural result is consistent with the ideas of Landau[11] and Vinen[15]: It is diffucult to alter the state of a fluid in a large volume by a quantum transition alone.

The argument of the exponential function in (2.11) remains very large up to velocities near the Landau critical velocity, at which a transition to a state with a small vortex half-ring becomes allowed from the energy standpoint. It thus follows from the Iordanskiĭ-Langer-Fisher theory and from the quantum theory for vortex creation that the critical velocities should increase with decreasing temperature and should reach values of the order of the Landau critical velocity. Experimentally, however, as the temperature is lowered the critical velocity stops increasing after it reaches values significantly lower than v_L. These values depend on the transverse dimension d of the capillary, satisfying the following empirical law (see §42 in Ref. 3; d is in centimeters and v_{cr} in centimeters per second):

$$v_{cr} d^{1/4} \sim 1.$$

These experimental values of the critical velocity at low temperatures have yet to be fully explained.

Many attempts have been made to resolve this "activation barrier problem," but they have succeeded in reducing the discrepancy by only a small factor, not by the order of magnitude which would be required to explain the experimental data at low temperatures. The low critical velocities observed experimentally are evidence that the vortices are present in a far larger number than would follow from the Boltzmann distribution on which the Iordanskiĭ-Langer-Fisher theory is based. The result is to increase dissipation. It may be that the vortices arise from vibrations, as suggested by Putterman,[3] but no quantitative theory has been worked out for this possibility.

If the unknown mechanism for the creation of vortices is related in some way to the normal component of the fluid, then this mechanism should vanish upon the *complete* disappearance of the normal component, when the momentum of this component with respect to the superfluid part of the fluid, $\rho_n S L |\mathbf{v}_n - \mathbf{v}_s|$ (S and L are the cross-sectional area and the length of the channel), even at the largest relative velocities, of the order of the Landau critical velocities, $|\mathbf{v}_n - \mathbf{v}_s| \sim h/mr_c$, does not exceed the quantum of superfluid momentum, $\rho_s Sh/m$:

$$\frac{\rho_n}{\rho_s} < \frac{r_c}{L}. \qquad (2.12)$$

[3)] If the condensate wave function $\Psi_1(\mathbf{r})$ of the final state were determined from the condition for an energy extremum, then the change in the phase $\tilde{\varphi}$ would satisfy the Laplace equation $\Delta\tilde{\varphi} = 0$; i.e., the incompressibility condition would not be violated. In this case, however, a quantum transition would lead to a change in the state of the liquid in a larger volume, $R^3 \gg R^2 l$, and in the argument of the exponential function for the probability W in (2.11) the larger logarithm would be raised to the third, rather than the second, power. This is the result found by Volovik,[24] who studied an incompressible liquid.

Under this condition we may expect a growth of the observed critical velocities.[14] The critical velocities for the flow of He II through apertures in thin diaphragms[25] have been measured at parameter values near the region defined by inequality (2.12). The critical velocities in these experiments were higher than usual and may be taken as support of the suggestion in Ref. 14; a serious test of this suggestion, however, will require a study of the dependence of the critical velocity on the length or the temperature in this region.

Because of these discrepancies between theory and experiment in terms of the critical velocities, two terms have been adopted for the critical velocities in the literature: the "intrinsic" ("ideal") critical velocity, which follows from the theory, and the "extrinsic" ("real") critical velocity, which is observed experimentally. At present, the only way to interpret the extrinsic critical velocities is to work from the theory of superfluid turbulence derived by Vinen,[26] who postulated the existence of a source which creates vortices (a source of turbulence) and who formulated equations describing the evolution of this turbulence. For a bibliography on this theory and its applications we direct the reader to one of the recent papers on the subject.[27]

d) Collective mode and superfluidity

The transition to the superfluid state involves the appearance of additional hydrodynamic degrees of freedom and the appearance of a new collective-vibration mode. Because of the phase degeneracy of the complex order parameter (gauge invariance), the new mode is a gap-free mode (the Goldstone theorem). This Goldstone mode is second sound for helium II in a large volume and fourth sound in narrow capillaries or in volumes filled with a porous material. Second sound is a vibration of the normal and superfluid parts of the fluid with respect to each other, while the total momentum of the fluid is conserved. In fourth sound, the momentum of the fluid is not conserved; the normal part of the fluid is at rest because of the interaction with the channel wall; and only the superfluid part is vibrating. We will focus on fourth sound here, since we believe that superfluidity is manifested more clearly in this case (there is no relaxation of the superfluid motion *despite* the existence of momentum-nonconserving processes); furthermore, in our discussion of analogs of superfluid flows in the following sections of this paper we will continually run into analogs of fourth sound, but not of second sound.

Fourth sound, with a velocity $v_4 = \sqrt{\rho_s d\mu/d\rho}$, is found from the equations of motion for a pair of canonically conjugate variables, the density ρ and the phase φ. These equations of motion follow from the Hamiltonian

$$\mathscr{H} = \int d\mathbf{r} \left[\rho_s \left(\frac{\hbar}{m}\right)^2 \frac{(\nabla\varphi)^2}{2} + \frac{\rho'^2}{2\alpha} \right], \quad \alpha^{-1} = \frac{d\mu}{d\rho}. \qquad (2.13)$$

Hamilton's equations are

$$\frac{\partial\rho'}{\partial t} = \frac{m}{\hbar} \frac{\delta\mathscr{H}}{\delta\varphi} = -\operatorname{div} \mathbf{j}_s = -\rho_s \frac{\hbar}{m} \Delta\varphi, \qquad (2.14)$$

$$\frac{\hbar}{m} \frac{\partial\varphi}{\partial t} = -\frac{\delta\mathscr{H}}{\delta\rho} = -\mu = -\frac{\rho'}{\alpha}, \qquad (2.15)$$

where ρ' is the deviation of the total density from equilibrium, μ is the chemical potential, and $\mathbf{j}_s = \rho_s \mathbf{v}_s = \rho_s (\hbar/m)\nabla\varphi$ is the superfluid mass flow.

As long as there are no vortices, all possible dissipation mechanisms must make an imaginary contribution to the frequency which falls off no more slowly than k^2 in the limit $k \to 0$ (**k** is the wave vector). To show this, we add a dissipation to the equations of motion in (2.14) and (2.15):

$$\frac{\partial \rho}{\partial t} = \frac{m}{\hbar} \frac{\delta \mathcal{H}}{\delta \varphi} - \frac{\delta R}{\delta (\delta \mathcal{H}/\delta \rho)}, \qquad (2.16)$$

$$\frac{\hbar}{m} \frac{\partial \varphi}{\partial t} = -\frac{\delta \mathcal{H}}{\delta \rho} - \frac{\hbar}{m} \frac{\delta R}{\delta (\delta \mathcal{H}/\delta \varphi)}, \qquad (2.17)$$

where the dissipation function R is a homogeneous function of second degree of the derivatives $\delta\mathcal{H}/\delta\rho$ and $\delta\mathcal{H}/\delta\varphi$ and their gradients. In the limit $k \to 0$, we need retain only the lowest-order gradients. The cross terms $(\delta\mathcal{H}/\delta\rho)\delta\mathcal{H}/\delta\varphi$ actually lead to a renormalization of the dissipationless terms in the equations of motion and may be omitted. The contribution of $(\delta\mathcal{H}/\delta\rho)^2$ to R gives rise to terms $\sim\rho'$ on the right side of (2.16) (since $\delta\mathcal{H}/\delta\rho \sim \rho'$) and violates the conservation of the total number of particles. Therefore, this term must also be omitted. As a result, we are left with the following leading terms in the dissipation function:

$$2R = a_\rho \left(\nabla \frac{\delta \mathcal{H}}{\delta \rho}\right)^2 + a_\varphi \left(\frac{\delta \mathcal{H}}{\delta \varphi}\right)^2. \qquad (2.18)$$

Because of the phase degeneracy, $\mathcal{H}$ does not depend on φ, but only on its gradients, so that we have $\delta\mathcal{H}/\delta\varphi \sim \Delta\varphi$. At this point it is not difficult to see from Eqs. (2.16) and (2.17) that the dissipative terms provide an imaginary increment in the frequency $\sim k^2$. The vanishing of the dissipation in the limit $k \to 0$ is a direct consequence of gauge invariance and of the existence of the phase as a well-defined hydrodynamic variable; in other words, the vanishing of dissipation is a consequence of the very properties which give rise to the steady-state mass superflows.

If, on the other hand, quantized vortices arise in a fluid, then the phase becomes a multivalued function. The motion of the vortices across streamlines causes a "superfluid friction force" in the Euler equation for the superfluid component, which is found by taking the gradient of both sides of Eq. (2.15):

$$\frac{\partial v_s}{\partial t} = -\nabla\mu - \frac{v_s}{\tau_s}. \qquad (2.19)$$

The friction force transforms the fourth sound into a damped mode at a frequency $\omega < 1/\tau_s$. According to the Iordanskiĭ-Langer-Fisher theory, however, the superfluid friction force is exceedingly weak at low velocities v_s and in the limit $\tau_s \to \infty$, in which[4)] $v_s \to 0$. At any rate, we can expect the damping of the fourth sound to be slight until the velocity v_s in the sound wave reaches the critical velocities or perhaps higher values, since there may not be time for vortex formation over an oscillation period at velocities above the critical values. The existence of superflows thus does in fact mandate the existence of an undamped long-wave collective mode.

[4)] The same is true of second sound in He II. In a solid, on the other hand, a lower limit is placed on the frequency interval in which second sound exists by the finite inverse time for flip processes.[28]

e) One-dimensional and two-dimensional superfluidity

Superfluid systems of fewer than three dimensions became the subject of an active discussion after Little[29] suggested using one-dimensional organic chains to search for a high-temperature superconductivity. The possibility of superfluidity in such systems subsequently came under doubt, because of the vanishing of long-range order. These systems will be discussed at the end of the present subsection and also in Section 6. What we are interested in at this point is the stability of superfluid flows upon the transition to one- and two-dimensional systems, under the assumption that such systems can be described by two-velocity hydrodynamics with some finite superfluid density $\rho_s(d)$, where d is the dimensionality of the space.

The stability of a two-dimensional superflow, like that of a three-dimensional flow, is determined by an activation barrier to motion of the vortices across the flow, but now the vortex lines degenerate to a point. Determining the energy of a vortex, and minimizing it in terms of the distance from the vortex to the wall, we find

$$\varepsilon_A = \pi\rho_s(2)\left(\frac{\hbar}{m}\right)^2 \ln\frac{\hbar}{m v_s r_c}, \quad R = \frac{\hbar}{2 m v_s}, \qquad (2.20)$$

where R is that distance from the vortex to the wall at which the vortex energy is at a maximum and equal to the height of the activation barrier, ε_A.

In a one-dimensional system with periodic boundary conditions, i.e., in a one-dimensional ring, the change in the phase shift along the length of the system (a jump in the phase or phase slippage) can occur only upon the appearance of points at which the order parameter vanishes, as was first pointed out by Little.[30] As in three and two dimensions, therefore, the dependence of the free energy on the superfluid momentum has minima at quantized values of the momentum, and these minima are separated by activation barriers (Fig. 2). The height of a barrier is determined by the energy required to reduce the modulus of the order parameter to zero at some point, and in the limit of low velocities v_s the height is given in order of magnitude by

$$\varepsilon_A \cong \frac{\hbar^2 \rho_s(1)}{m^2 r_c}. \qquad (2.21)$$

In contrast with two- and three-dimensional systems, in which the barrier height is determined by the velocity fields v_s in the London region (i.e., where the modulus of the order parameter is constant; this region contributes a large logarithm to the barrier height), an accurate calculation of the barrier height in the one-dimensional case requires knowing the change in the modulus of the order parameter over distances of the order of the coherence length r_c from the phase slippage center (the slip center). Accordingly, these calculations were carried out by Langer and Ambegaokar[31] only in the region where the Ginzburg-Landau theory is applicable (see also Sec. 7.1 in Tinkham's book[10]). The same expression for the barrier height can be derived in analogs of the Ginzburg-Landau theory for superfluid systems: the Ginzburg-Pitaevskiĭ and Gross-Pitaevskiĭ theories.

Comparing (2.6), (2.20), and (2.21), we see that the activation barriers and, correspondingly, the relaxation time τ_s [see (2.7) and (2.19)] become infinite in the limit $v_s \to 0$ only for three- and two-dimensional systems, not for one-dimensional systems. This result is undoubtedly a consequence of the absence of a phase transition in the one-dimensional system: If the time τ_s, which determines the friction force which is linear in the velocity, $\sim v_s/\tau_s$, is finite at high temperatures, it cannot become infinite at a nonzero temperature. At low temperatures, however, it may become exponentially long, and we may speak of "nondecaying" currents in a restricted sense at least in the sense which we defined for this term in the Introduction.

Equations (2.20) and (2.21) for the activation barriers are equally valid for genuinely two-dimensional and one-dimensional systems, which completely lack certain degrees of freedom in the motion of the particles. The same is true of three-dimensional systems of the thin-film or capillary (thin-wire) type, with a thickness b and a diameter D shorter than the coherence length r_c. In the latter case, we must replace $\rho_s(d)$ by $\rho_s b$ and $\rho_s \pi D^2/4$ in Eqs. (2.20) and (2.21), respectively.[5)] What values should we expect for the activation barriers for genuinely one-dimensional systems? In the theory of a Bose liquid, the difference between the total and superfluid densities at low temperatures is the normal phonon density

$$\rho_n = \frac{\pi (kT)^2}{3c^3\hbar}, \tag{2.22}$$

where c is the sound velocity. In a slightly nonideal Bose gas with an excitation spectrum $\varepsilon(p) = \sqrt{c^2p^2 + (p^2/2m)^2}$, expression (2.22) holds at $kT \ll mc^2$; at $kT \gg mc^2$, the normal mass is $\rho_n = kTm/\hbar c$. It follows that since $\rho_s(1) \sim \rho(1)$, i.e., $\rho_n \ll \rho(1)$, or $kT \ll \rho(1)\hbar c/m \sim \rho(1)\hbar^2/m^2 r_c$, the argument (ε_A/kT) of the exponential function in the expression for the time τ_s [see (2.7)] is very large. If we substitute into (2.21) the expression for the coherence length from the Bardeen-Cooper-Schrieffer (BCS) theory, $r_c \sim \hbar v_F/\Delta$ (Δ is the superconducting gap), we find the argument to be $\varepsilon_A/kT \approx \Delta/kT$.

The use of the phonon spectrum to determine the normal density for a one-dimensional Bose gas, or the use of the BCS theory for a one-dimensional Fermi gas, requires justification, however, since systems of limited dimensionality do not have an average order parameter. In Ref. 20 a justification of this type was constructed from the exactly solvable model of Lieb and Liniger for a one-dimensional Bose gas with a δ-function repulsive potential between particles.[33,34] Lieb also showed[34] that for short excitation pulses the spectrum in this model is an acoustic spectrum; with a weak repulsion, the sound velocity for the spectrum agrees in magnitude with the sound velocity determined from the Bogolyubov theory, i.e., from the theory of a self-consistent field. It is also confirmed that there are energy minima in the limit $T = 0$ at the quantized values of the momentum, as in Fig. 2(a). It was shown in Ref. 20 that an ensemble which includes microstates of the Lieb model with a given momentum and which lie in the vicinity of one minimum, the main minimum, say, is an ensemble with a given superfluid velocity $\mathbf{v}_s = 0$, and its free energy density $\rho_n v_n^2/2$ determines the normal density ρ_n. The latter, as it turns out, is precisely equal to the normal phonon density, (2.22), in the limit[6)] $T \to 0$.

A justification for the use of the phonon spectrum (a hydrodynamic Hamiltonian) to determine the normal density at low temperatures was also offered by Berezinskiĭ,[35] in a two-dimensional model of plane rotors, and by Popov,[36] for one- and two-dimensional Bose gases with a weak interaction, through the use of a path-integral method.

A complete solution of the question of whether nondecaying one- and two-dimensional currents exist will require evaluating the tunneling probability in addition to the barrier height. In the model of a slightly nonideal Bose gas, these calculations can be carried out by the same method as is used in three dimensions (Subsection 2c). The change in phase upon the transition, $\tilde{\varphi}$, which satisfies Eq. (2.10), occurs in a region of thickness l which is bounded in the two-dimensional case by the wall and by the vortex, at a distance R from the wall. Following the same calculation procedure as in the three-dimensional case, we find, instead of (2.11),

$$d=2:\ W = \exp\left(-\frac{16\hbar^2\rho(2)}{\pi m^3 v^2}\ln\frac{\hbar}{mvr_c}\right), \quad l = \frac{4\hbar}{\pi m v},$$
$$R = \frac{\hbar}{mv}\ln\frac{\hbar}{mvr_c}; \tag{2.23}$$

$$d=1:\ W = \exp\left(-\frac{8\hbar\rho(1)}{\pi m^2 v}\right), \quad l = \frac{2\hbar}{\pi m v}. \tag{2.24}$$

Corresponding to tunneling in the model of Lieb and Liniger is the creation of a type II excitation (in the terminology of Ref. 34) with a low energy but a large momentum. Although this probability cannot be calculated directly, it is argued in Ref. 20 that in the thermodynamic limit (in which the number of particles becomes infinite, $N \to \infty$) this probability falls off in the limit $v \to 0$ as $[a/(a + \rho(1)/m)]^N$, where a is the reciprocal of the length determining the repulsive boson potential, $V(r) = (\hbar^2/m)a\delta(r)$. For a gas of impenetrable bo-

5) Many papers have recently appeared on the theory of the resistive state of thin superconducting wires, in which the currents (superfluid velocities) are so high that the activation barrier for their relaxation is either low or completely absent.[32] We believe that there are many questions in this theory which have not yet been completely clarified, but a discussion of these questions would go beyond the scope of the present review. The theory of the resistive state also links a finite resistance with the formation of slip centers.

6) This agreement was established in Ref. 20 only in the limit of strong repulsion. However, it has been found possible to prove (2.22) in the Lieb-Liniger model for an interaction of arbitrary strength. We hope to publish this proof in the near future.

sons, the probability for a transition accompanied by the loss of a quantum of momentum is thus not small, but this is a special case in the thermodynamic limit. From the thermodynamic standpoint, a gas of impenetrable bosons is indistinguishable from a one-dimensional Fermi gas.[33,34] In particular, in the latter case there are again minima in the momentum dependence of the energy, and barriers separate these minima.[20] These barriers occur because, in the transition of a fermion from one half of the Fermi boundary to the other, the momentum can change only by an amount $2p_F$ without increasing the energy. The smaller momentum transfers in a one-dimensional Fermi gas result from the appearance of a hole inside the one-dimensional Fermi sphere, with a corresponding increase in the energy. Any factors which make large momentum transfers $\sim 2p_F$ improbable can suppress the decay of the mass flow in a one-dimensional Fermi gas. This circumstance is related in a sense to the Frohlich superconductivity in a Peierls dielectric, where transitions between the two halves of the Fermi boundary are suppressed even further by the appearance of a gap in the spectrum.

f) Mechanical analog of a superfluid mass flow

In Sec. 3 of this review we will take up the possible existence of nondecaying flows of spins and electron-hole pairs, which are analogous to the superfluid mass flow. Before we go onto that topic we wish to point out an extremely simple and obvious mechanical analog of a mass superflow: the flow of momentum or angular momentum in a deformed solid.

Let us twist a long elastic rod. The twisting angle φ will increase linearly along the rod, reaching values many times 2π. If we bend the rod into a ring and connect the ends rigidly, we find a mechanical model for superflow in a ring channel. The analog of the velocity here is the strain $\nabla\varphi$, which determines the elastic energy, whose density is $\sim(\nabla\varphi)^2$. The only way to transfer this energy, i.e., to relieve the strain, is through plastic displacements, which in turn can occur only when the strain reaches some threshold, which is the analog of the critical velocity. Plastic displacements occur through the appearance and motion of dislocations, which are reminiscent in many ways of vortices in a superfluid. Instead of a mass flow we have a flow of angular momentum in our twisted rod. If we do not bend the rod into a ring, but if we fix its ends rigidly, then the angular momentum flow is manifested in the effect of the torsional moment on whatever we are using to fix the ends of the rod. An ideally elastic rod can transmit a torsional moment over an arbitrary distance; if the torsional moment performs work, then energy can also be transported over any distance. In summary, two concepts which might appear to be at opposite poles (the "stiffness" of a rod and the "fluidity" of angular momentum) fit together quite nicely in the same phenomenon. It is pertinent to recall here that London[1] used the concept of the "stiffness" of boson wave function to explain superfluidity.

3. SPIN SUPERFLOWS AND HELICOIDAL STRUCTURES IN MAGNETICALLY ORDERED SYSTEMS

a) Helicoidal structure in easy-plane ferromagnets and antiferromagnets. Stability and magnetic vortices

If the strong anisotropy energy confines the magnetization of a ferromagnet or the magnetization of the sublattices of an antiferromagnet in a definite crystallographic plane (an easy plane), and if the directional anisotropy within this plane can be ignored, then the angle through which the magnetization rotates is the same sort of cyclic, continuous degeneracy parameter as the phase of the complex order parameter in helium II. There are accordingly many analogies between these systems, as has been pointed out frequently elsewhere. Halperin and Hohenberg[37] studied spin hydrodynamics by making use of its similarity to the hydrodynamics of a superfluid. Berezinskiĭ[35] worked from the analogy between a superfluid and the model of plane rotors (i.e., the planar magnetically ordered systems mentioned above) in a study of phase transitions. In this subsection of the review we will take this analogy even further and examine the possible existence of metastable helicoidal structures in which there is a nondecaying spin transport[7)] analogous to superfluid mass transport in He II.

Our analysis can be carried out for both a planar antiferromagnet and a ferromagnet, but in the latter case we will ignore the long-range dipole-dipole interaction and the demagnetization fields which this interaction causes. The state of our planar system is determined by specifying two canonically conjugate variables: the angle φ through which the magnetization has rotated in the easy plane and the projection m_z of the magnetization onto the "hard" z axis. The phenomenological Hamiltonian for the pair of variables φ, m_z differs only in notation from the Hamiltonian in (2.13), for the density and phase in a superfluid:

$$\mathscr{H} = \int d\mathbf{r}\left(A\frac{(\nabla\varphi)^2}{2} + \frac{m_z^2}{2\chi}\right), \tag{3.1}$$

where A is the inhomogeneous-exchange constant, and χ is the magnetic susceptibility along the z axis. Hamilton's equations again yield a gap-free Goldstone mode: magnons, which constitute an analog of fourth sound in He II.

We now assume that we have deformed the spin system in such a manner that as we move along the sample the magnetization rotates in the easy plane through some (generally large) number of rotations. The magnetization at the ends of the sample is held in a certain direction, and there is no precession in the plane ($m_z = 0$). This structure will be metastable, for the same reason as for the state with a mass superflow in He II or for the defomed state in a twisted rod.[39,40] How can we remove one turn of the spiral traced out by the magnetization? As in He II, the cross section of the sample must be "cut" by the motion of a singular line,

[7)] If the helicoidal structure corresponds to a ground state,[38] there is no spin flux.

along which the angle φ is not determined, but around which the angle φ changes by 2π. This line may be called a "magnetic vortex." The entire procedure of Subsection 2c for calculating the activation energy barrier opposing the motion of vortices across a sample can be repeated here. The final result for the barrier height is

$$\varepsilon_A = \frac{\pi^2}{4}\frac{A}{|\nabla\varphi|}\left(\ln\frac{1}{|\nabla\varphi|\, r_c}\right)^2. \tag{3.2}$$

The only differences from the He II case are in the definition of the radius r_c of the core of the magnetic vortex. In He II the increase in the energy due to the increase in $\nabla\varphi$ at the core of a vortex is offset by a decrease in the modulus of the order parameter, to zero. In a magnetic vortex this increase can be offset by an excursion of the magnetization out of the easy plane, since the constant A in (3.1) is proportional to the square of the projection of the magnetization onto the easy plane. Correspondingly, r_c is determined by equating the inhomogeneous-exchange energy $A(\nabla\varphi)^2 \sim A/r_c^2$ to the energy E_A of the anisotropy which confines the magnetization to the easy plane; we find[39,40]

$$r_c = \sqrt{\frac{A}{E_A}}. \tag{3.3}$$

It can be seen from (3.2) that the activation barrier can be quite high, with correspondingly long times for transitions to the ground state.

If the ground state also corresponds to some helicoidal structure,[38] then our "deformed" state differs from the ground state in that the spiral has a different pitch; the strain $\nabla\varphi$ determines the difference between the wave vectors of the two spirals.

b) Spin nonconservation and lifting of the angular degeneracy: fixing of the angle. Stability of helicoidal structures

The existence both of helicoidal structures, and of undamped superfluid mass flows, is intimately related to the degeneracy of the ground state in terms of angle (or phase), which is in turn a consequence of the conservation law for the quantity which is the canonical conjugate of the angle (or phase). In He II, this would be the number of atoms, while in a planar ferromagnet or antiferromagnet this would be the spin projection onto the hard axis. However, while this conservation law is exact in the former case, in the latter the spin is not an exact integral of motion, and there always exists an energy anisotropy (although it is usually slight) in the easy plane, which lifts the degeneracy in terms of the angle φ. A similar phenomenon in an excitonic dielectric (more on this in Sec. 5) has been labeled "phase fixing" by Guseĭnov and Keldysh.[41] Because of this phase (or angle) fixing, a gap appears in the collective Goldstone mode, and the mode may cease to be a Goldstone mode. For magnons this gap corresponds to the frequency of the homogeneous magnetic resonance. The Hamiltonian of the system for the pair of canonical variables φ, m_z can be written as follows for the case in which there is an anistropy in the easy plane with an energy density E_Φ:

$$\mathcal{H} = \int d\mathbf{r}\left(A\frac{(\nabla\varphi)^2}{2} + \frac{m_z^2}{2\chi} - \frac{E_\Phi}{n}\cos n\varphi\right), \tag{3.4}$$

where n is the order of the symmetry axis, which is the z (hard) axis. The equations of motion (Hamilton's equations) are the Landau-Lifshitz equations[42] in cylindrical coordinates. After we add dissipative terms we find [cf. Eqs. (2.14)–(2.17)]

$$\frac{1}{\gamma}\frac{\partial m_z}{\partial t} = -A\Delta\varphi + \underline{E_\Phi \sin n\varphi} + \underline{\frac{m_z}{\gamma T_1}}, \tag{3.5}$$

$$\frac{1}{\gamma}\frac{\partial\varphi}{\partial t} = \frac{m}{\chi} + a_\varphi(+A\Delta\varphi - \underline{E_\Phi \sin n\varphi}), \tag{3.6}$$

where γ is the gyromagnetic ratio. The wavy underscoring shows terms which give rise to processes which violate spin conservation. The dissipation function R may now include a term $\sim(\delta H/\delta m_z)^2$ [cf. (2.18) for He II]. As a result, a term corresponding to longitudinal Bloch relaxation, m_z/T_1, appears in (3.5). A dissipative term appears in Eq. (3.6) because of the contribution $(a_\varphi/2)(\delta H/\delta\varphi)^2$ to the dissipation function R, but now, because of the lifting of the degeneracy with respect to the angle φ, this dissipative term includes, in addition to $\Delta\varphi$, a term which depends explicitly on φ (with the wavy underscoring). This term is an analog of the relaxation mechanism proposed by Leggett and Takagi[43] for the longitudinal magnetization in the superfluid phases of He^3 (Section 4).

A steady-state helicoidal structure ($m_z = 0$) corresponds to the solutions of the sine-Gordon equation,

$$\Delta\varphi = \frac{\sin n\varphi}{l^2}, \tag{3.7}$$

where

$$\frac{1}{l^2} = \frac{E_\Phi}{A}. \tag{3.8}$$

For one-dimensional problems, in which the angle φ varies along only one coordinate, say the x axis, Eq. (3.7) becomes the equation of a physical pendulum if we replace the distance x by t and the angle φ by φ/n. This nonlinear equation has two types of solutions. The solutions of the first type are oscillations of the pendulum around its equilibrium position and are of no interest here. The solutions of the second type correspond to a rotation of the pendulum, with the angle φ increasing monotonically over time. The solutions of this type are parameterized by the average "rotation velocity" $\langle\nabla\varphi\rangle$, which determines the period of the motion—in our case, the spatial period of the helicoidal structure, $\bar{x}$:

$$\bar{x} = \frac{2\pi}{n\,|\langle\nabla\varphi\rangle|}. \tag{3.9}$$

At low average gradients, the structure is a periodic chain of domain walls of width $\sim l$, which separate domains with different easy directions within the easy plane. With increasing average gradient $\langle\nabla\varphi\rangle$ the density of these domain walls increases, and they begin to overlap; at $|\langle\nabla\varphi\rangle| \gg 1/l$ ($\bar{x} \ll l$), the anisotropy which fixes the angle in the plane contributes only small increments, which oscillate over space, to the constant component of the gradient $\nabla\varphi$. In this region we may speak in terms of a "depinning" of the phase or the angle φ by the large inhomogeneous-exchange energy, and the stability of the structure is determined by the activation barrier ε_A, found without consideration of the anisotropy in the easy plane [Eq. (3.2)].

At small values of $|\langle\nabla\varphi\rangle| \gg 1/l$ the relaxation of the helicoidal structure to the ground state reduces to the elimination of the domain walls from the volume of the sample. This may occur through a continuous contraction of the area of a domain wall; for this to happen, the walls must acquire a boundary which does not coincide with a boundary of the sample. A magnetic vortex is such a boundary. Since the angle changes by $2\pi/n$ in each domain wall, and since the angle φ must change by 2π as we go around a vortex, then n domain walls should terminate at a magnetic vortex. The areas of these walls should simultaneously contract. This process is illustrated in Fig. 3 for the case $n=4$. The magnetic vortices are a generalization of Bloch lines, which separate subdomains in domain walls.[44]

In a region in which a helicoidal structure breaks up into isolated domain walls, the activation barrier for the motion of vortices across the sample becomes independent of the average gradient $\langle\nabla\varphi\rangle$. This barrier can be calculated by analogy with the calculation of the energy of the vortex state to which the tunneling from a homogeneous state with a mass superflow occurs (see Subsection 2c and Refs. 14, 39, and 40). The result is

$$\varepsilon_A = \frac{\pi^3\sqrt{n}}{16} Al\left(\ln\frac{l}{r_c}\right)^2. \tag{3.10}$$

Using (3.3) and (3.8), which relate the lengths l and r_c to the anisotropy energy densities E_Φ and E_A, and using a qualitative estimate of the inhomogeneous-exchange constant, $A\sim J/a$ (a is the lattice constant, and J is the exchange energy per cell of the crystal lattice), we find

$$\varepsilon_A \sim J\sqrt{\frac{J}{E_\Phi a^3}}\left(\ln\frac{E_A}{E_\Phi}\right)^2. \tag{3.11}$$

Accordingly, although the anisotropy E_Φ which fixes the rotation angle in the plane does in fact reduce the stability of the helicoidal structure, the activation barrier ε_A may be very large, because of the condition $E_A \gg E_\Phi$, which holds for many easy-plane ferromagnets and antiferromagnets.

c) Unattenuated spin flows and ways to produce them

Up to this point we have described metastable helicoidal structures, representing them as certain elastic deformations of the original spin systems, but avoiding

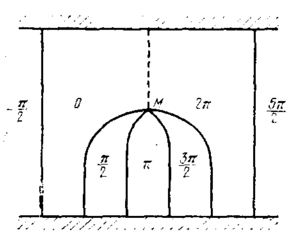

FIG. 3. Annihilation of domain walls in the case $n=4$. Solid lines—90° domain walls; the point M—magnetic vortex (the numbers are the angles through which the magnetization is rotated); dashed lines—cut with a jump of 2π in the angle.

saying anything about a spin flow. Evidence that the "elastic deformation" $\nabla\varphi$ is proportional to a spin flow is the continuity equation for the spin projection m_z in (3.5). According to this equation the spin flow is

$$\mathbf{j}_s = \gamma\frac{\delta H}{\delta\nabla\varphi} = \gamma A\nabla\varphi. \tag{3.12}$$

Such a spin flow could exist only in an ordered spin system below the Curie or Néel point.

In addition to the unattenuated spin flow there is the possibility of a spin transport through an attenuated spin-diffusion flow $D\nabla m_z$. To separate a spin flow into unattenuated and attenuated components is analogous to separating a mass flow in a superfluid into superfluid and normal components or a momentum flow in a solid elastic and viscous components.

In a purely steady-state helicoidal structure, there is no precession of the magnetization, we have $m_z=0$, and there is no dissipation of any sort. Such a structure can exist without an external agent only in an annular geometry (a long sample is bent into a ring, and the wave vector of the helicoidal structure is directed along the resulting circle). At first glance it is difficult to see that the flow $\mathbf{j}_s$ does in fact transport spin under these conditions. If spin transport is to be detectable, something must vary over time. If, for example, we rapidly cut through a ring in which a spin flux $\mathbf{j}_s$ is circulating then we should note an increase and a decrease in the magnetization m_z on each side of the cut.

In an open geometry, in a sample of finite length, it will be necessary to continuously pump spin across a boundary of the sample in order to maintain a helicoidal structure and an unattenuated spin flow. One possibility for injecting spin is to pass a current of spin-polarized electrons.[40] The presence of an angle-focusing anisotropy in a plane imposes the requirement that the spin pump be strong enough to achieve phase (or angle) depinning in the volume. If, on the other hand, the spin flux $\mathbf{j}_s=\gamma A\nabla\varphi$ created by the pump is smaller than the spin flux at the center of an isolated domain wall ($\sqrt{2/n}\gamma A/l$), then the effect of the pump may be only a partial penetration of a domain wall into the sample near the boundary at which the pumping is carried out, while the entire volume remains unperturbed. (See ref. 40 for details). The method of using a spin pump to produce unattenuated flows may be called *incoherent*, since in this case the effect on the medium is achieved by changing the density of the spin, rather than the phase (or angle).

We might note that in the case of spin pumping in a sample of finite length a strictly steady-state helicoidal structure is impossible in principle; there is always some sort of precession in spin space, and we have $m_z\neq 0$. Indeed, a solution of Eqs. (3.5), (3.6) in the volume for the steady state (Subsection 3b), in which there is no Bloch relaxation $m_z/T_1=0$), shows that if there is a spin flux at one end of the sample because of the pump there must necessarily be a spin flux at the other end also. At that other end, however, the flux must vanish if that end borders on a vacuum, or it must transform into a diffusive flow if it borders on a paramagnetic medium. If we are to maintain a diffusive flow we must have a nonequilibrium magnetization: $m_z \neq 0$. Then we must have $m_z \neq 0$, and the time-dependent equations in (3.5) and (3.6) must be solved with the boundary conditions. Let us assume that at one end of the sample ($x = 0$) there is spin pumping, and the spin flow is $j_s(0) = j_H$. Domain walls must accordingly form here, and they must move toward the other end of the sample, $x = L$. If the pump is intense enoudh to satisfy $j_H/\gamma A \sim |\nabla\varphi| \gg 1/l$, then the density of domain walls will be high, the spin flow will be essentially uniform, and the anisotropy in the plane can be ignored. A solution for this case[8)] was derived in Ref. 40; the result for a boundary with a vacuum is

$$j_s(x) = j_H\left(1 - \frac{x}{L}\right), \quad m_z = \frac{j_H T_1}{L}. \tag{3.13}$$

The magnetization $m_z \sim \delta\mathcal{H}/\delta m_z$ must also remain constant in the volume of the sample, just as the chemical potential $\mu = \delta\mathcal{H}/\delta\rho$ is constant in the steady state of a superfluid. If this condition is not met, the magnetization will precess at different rates at different points, and there will be an unbounded increase in the deformation in the spin system ("acceleration" of the spin flow, $\gamma A\nabla\varphi$).

The appearance of a nonequilibrium spin density far from the site of the pumping is evidence of a real spin transport by an unattenuated flow. A linear law for the decay of the spin flow, on the one hand, and a constant nonequilibrium magnetization throughout the sample volume, on the other, radically distinguish an experiment with spin pumping in a planar ferromagnet or antiferromagnet from a corresponding experiment with a paramagnet, in which the only spin-transport mechanism is diffusion, and in which the spin flow and the magnetization decay exponentially with distance from the site of the spin pumping.

It follows from Eqs. (3.5) and (3.6) that the energy flux associated with the flow $\mathbf{j}_s$,

$$\mathbf{Q} = \frac{\delta\mathcal{H}}{\delta m_z}\,\mathbf{j}_s = \frac{m_z}{\chi}\,\mathbf{j}_s, \tag{3.14}$$

is also different from zero only in the case $m_z \neq 0$. Bloch relaxation occurs in this case, and energy must be supplied in order to compensate for the loss due to this relaxation.

There is also the possibility of a *coherent* method for producing a helicoidal structure (this was proposed by Aronov in 1978): One end of the sample is placed in a strong magnetic field, and rotation of this field can rotate the magnetization in a plane. Because of the stiffness of the spin system, this rotation is transmitted to the spins throughout the volume, which is not subjected to the direct effect of the magnetic field. This method for producing an unattenuated spin flow has no analog in the superfluidity of a liquid, since in the latter case there is no field linked to an order parameter. On the other hand, we can clearly see an analogy with the twisting of an elastic rod, especially since there is a flow of angular momentum in both cases.

We thus see that these metastable helicoidal structures are, on the one hand, deformed states of a system of ordered spins, while on the other they are states with a spin flow (or an angular momentum flow). This flow might be called a spin "superflow" by analogy with mass superflow, but there is also another possible name, which highlights the analogy with the angular momentum flux in a solid: "elastic" or "stiff" spin flow. Regardless of the words we use, and regardless of whether we refer to the anomalously low attenuation of the spin flow over space and time as a "superproperty," the effect in question is a *real, observable* spin transport.

The absence of a rigorous spin-conservation law and the consequent fixing of the phase (or the rotation angle in the easy plane) are important circumstances, which distinguish the unattenuated transport of spin from the superfluid transport of mass. This distinction, however, is not as radical as it might seem at first glance. It is possible to devise an experimental arrangement with a superfluid liquid in which there is something similar to phase fixing.[9)] Let us assume, for example, that a thin film of He II is flowing over a plate which separates the film from some helium which is at rest. If we fabricate some narrow apertures in this plate and arrange conditions such that the helium in the resulting channels always has the lowest kinetic energy compatible with the phases at the two ends of the channels, then the expression for the energy of the film will acquire a term which depends explicitly on the phase, with a period of 2π. The periodic change in the velocity of the film along the flow direction which results from this term is a consequence of the flow of helium into and out of the channels.

d) Superflows and topology of the order-parameter degeneracy space

We have seen that the cyclic nature of the phase (or the rotation angle) plays a decisive role for the existence of unattenuated transport of both mass and spin. This property is a consequence of the *topology of the degeneracy space of the order parameter*, which is usually called "R space." Research on the topology of this space has expanded considerably in recent years in connection with research on liquid crystals and superfluid He^3 (see Refs. 45–48). This R space is the set of all values of the order parameter which correspond to the ground state. To go from the R space to a more general space of possible values of the order parameter involves an increase in the energy, and under ordinary conditions this occurs only in small regions of the real space (at the core of a vortex, for example). For He II and for an easy-plane ferromagnet of antiferromagnet, R space is a circle on a complex plane and on the easy plane, respectively (the customary designation is S_1). The most convenient way to illustrate the relationship

8) The more general problem of the motion of domain walls of arbitrary density has been solved for the A phase of He^3, but without boundary conditions (see the end of Subsection 4b). A detailed study has also been made of the dynamics of isolated domain walls.[44]

9) This possibility was pointed out to the author by Iordanskiĭ (1976).

between the topology of R space and the existence of metastable structures is to consider a closed geometry: the medium filling a ring. We consider all possible mappings of the contour around the entire ring in the real space onto R space. These mappings, which are contours in R space, may be assigned to various *homotopic classes*. All the contours which belong to a given class are *homotopically equivalent*; i.e., they can be transformed into each other through a continuous deformation. The set of these classes forms the fundamental group π_1. For He II and for easy-plane ferromagnets and antiferromagnets, the group π_1 is isomorphic to the group of integers (the group Z). These integers show the number of times the contour runs around the circle S_1, which is the R space in this case. If this number is not zero, the contour can be transformed into a point corresponding to a uniform order parameter in the ring only if, during the transformation, the order parameter in some part of the real space leaves R space (leaves the circle S_1), and this would unavoidably mean a large energy of the intermediate states, i.e., an activation barrier for the transition to the ground state.

These arguments show how topological analysis can be used to generate predictions regarding the stability of inhomogeneous structures. The role played by this analysis here is reminiscent of the role played by group-theory analysis in optics; the latter cannot tell us the actual transition probabilities but it can indicate which probabilities should be small. It should also be noted that topological analysis generates quite reliable predictions of the stability of structures, but the absence of stability in the topological sense does not necessarily mean that the given structure is in fact unstable. There may be some energy barriers of nontopological origin[47] which prevent deformation of the contour *within* R space. In particular, such nontopological barriers could exist in the superfluid A phase of He^3 (see Subsection 4a).

For an isotropic ferromagnet or antiferromagnet the degeneracy space R is a surface: a three-dimensional sphere (usually designated S_2). Any contour on such a sphere can be shrunk to a point; i.e., the fundamental group π_1 for the space S_2 is trivial, including only a single element. The topology thus indicates that a helicoidal structure in an isotropic ferromagnet or antiferromagnet may be unstable. If, for example, the magnetization is gradually brought out of the plane in which it is rotating in the original helicoidal structure, the energy will decrease monotonically.

At the time at which the magnetization everywhere becomes directed perpendicular to the original plane the state becomes uniform. This process corresponds to the contraction to a point of a contour which runs many times around the spere S_2 in the equatorial plane.

To conclude this section we will examine the condition for the microscopic stability of a helicoidal structure—an analog of the Landau criterion. For this purpose we must consider the spectrum of excitations (magnons) in the helicoidal structure. While the superflow makes a correction to the spectrum which is linear in the velocity $v_s=(\hbar/m)\nabla\varphi$ $(\varepsilon\to\varepsilon+\mathbf{p}\mathbf{v}_s)$, the unattenuated spin flow makes a correction to the magnon spectrum which is quadratic in $\nabla\varphi$. This can be seen most simply in the case of a ferromagnet. Here the inhomogeneous-exchange constant A in Hamiltonian (3.4) is porportional to the square of the magnetization projection onto the easy plane: $A\sim m_\perp^2=m^2-m_z$. A nonlinear increment $-(\nabla\varphi)^2$ thus appears in the inverse susceptibility $1/\chi$; this increment may make the square of the magnon frequency negative, which would mean an instability. For an isotropic ferromagnet the inverse susceptibility $1/\chi$ is zero, and the helicoidal structure is unstable at any $\nabla\varphi$.

When there is an anisotropy which singles out an easy plane, the condition for microscopic stability is violated at those values of $\nabla\varphi$ at which the activation barrier to the motion of the magnetic vortices vanishes.

4. SUPERFLUIDITY OF THE A PHASE OF He^3

a) Mass superflows

Important progress toward an understanding of superfludity followed the experimental discovery of the superfluid phases of He^3, which stimulated further development of the theory. This topic is now the subject of a fair number of comprehensive reviews,[5-8] so here we will simply review certain properties of the superfluid A phase which are required for understanding the discussion below. The reason we have selected the A phase for discussion instead of the other well-known superfluid phase, the B phase, is that the superfluid properties of the B phase differ only slightly from those of He II. In contrast, the properties of the A phase are genuinely unique, and by studying these properties we can get a view of superfluidity in a new light.

The Cooper pairs of He^3 atoms which form below T_c are in a triplet state (the resultant spin is 1). The orbital angular momentum of the pair must therefore be odd, and it is assumed to be 1. The wave functions for a unit angular momentum, either a spin or an orbital angular momentum, have three components, so that the order parameter has $3\times3=9$ complex components $A_{\alpha i}$ ($\alpha, i=1,2,3$; α and i are the spin and orbital indices). The free energy in the superfluid state depends on all possible invariants which can be constructed from these components and their gradients. For the discussion below we will need only that part of the free energy which depends on the gradients. It includes three invariants (spin-orbit coupling is being ignored):

$$\mathscr{H}_{\text{grad}}=\frac{1}{2}\left[K_1(\nabla_i A^*_{\mu j})(\nabla_i A_{\mu j})+K_2(\nabla_i A^*_{\mu j})(\nabla_j A_{\mu i})+K_3(\nabla_i A^*_{\mu i})(\nabla_j A_{\mu j})\right]\left(\frac{\hbar}{2m}\right)^2. \tag{4.1}$$

In the A phase the order matrix is the direct product of two vectors:

$$A_{\alpha i}=(\Delta_{1i}+i\Delta_{2i})\,d_\alpha. \tag{4.2}$$

The pair of orthogonal vectors Δ_1,Δ_2 determines the orbital part of the wave function. The orbital vector $\mathbf{l}=\Delta_1\times\Delta_2$ gives the direction of the orbital angular momentum of the Cooper pair. The real vector $\mathbf{d}$ in spin space determines the direction along which the projection of the spin of the pair is precisely zero. For any other direction, the spin projection is zero on the average.[10)]

As is easily shown, the phase φ of the gauge transformation in the A phase is equal to one of the parameters of the rotation group for a reference system of three vectors $(\Delta_1, \Delta_2, \mathbf{l})$, specifically, the angle of rotation around $\mathbf{l}$. Since the three-dimensional rotation group is not commutative, this rotation angle cannot be determined unambiguously throughout the space. Only infinitesimally small changes of the phase φ are well-determined, so that, as in the He II case, we may introduce a superfluid velocity $\mathbf{v}_s = \frac{\hbar}{2m} \cdot \frac{\delta\varphi}{\delta\mathbf{r}}$. In the present case, however, the quantity

$$\frac{\delta\varphi}{\delta r} = \frac{\varphi(\mathbf{r}+\delta\mathbf{r}) - \varphi(\mathbf{r})}{\delta r}\Big|_{\delta r \to 0}$$

is not the gradient of a single-valued scalar function, so that in general we have $\operatorname{rot}\mathbf{v}_s \neq 0$ (rot = curl). Here m is the mass of the He^3 atom.

The density of the mass superflow in the A phase is determined in the usual way. If $\mathbf{d}$ = const, then

$$j_{si} = \frac{\delta \mathcal{H}_{\text{grad}}}{\delta (v_{si})} = \frac{\hbar}{2m} [K_1 (\Delta_{1j} \nabla_i \Delta_{2j} - \Delta_{2j} \nabla_i \Delta_{1j}) + K_2 (\Delta_{1j} (\nabla_j \Delta_{2i}) - \Delta_{2j} (\nabla_j \Delta_{1i})) + K_3 (\Delta_{2i} (\nabla_j \Delta_{1j}) - \Delta_{1i} (\nabla_j \Delta_{2j}));$$ (4.3)

or

$$\mathbf{j}_s = (2K_1 + K_2 + K_3)\mathbf{v}_s - (K_2 + K_3)(\mathbf{v}_s \mathbf{l})\mathbf{l} + (K_2 + K_3)\frac{\hbar}{2m}(\mathbf{l} \operatorname{rot} \mathbf{l})\mathbf{l} - K_3 \frac{\hbar}{2m} \operatorname{rot} \mathbf{l}. \quad (4.4)$$

We see that in the A phase the superfluid density (the coefficient of the proportionality between $\mathbf{j}_s$ and $\mathbf{v}_s$) is not a scalar but a tensor, and it depends on the orientation of $\mathbf{l}$ with respect to $\mathbf{v}_s$. In the Ginzburg-Landau theory, in the weak-coupling approximation, we have $K_1 = K_2 = K_3 > 0$, and the energy is at a minimum when $\mathbf{l}$ is parallel to $\mathbf{v}_s$. The existence of a dependence of the total superfluid mass $M_s = \int \rho_s d\mathbf{r}$ on the orientation of $\mathbf{l}$ requires to be approached cautiously in the case of a Galilean transformation.[11)] If, for example, we produce a texture which is inhomogeneous with respect to $\mathbf{l}$ and which has an energy ε and a momentum $\mathbf{p}$ in a still liquid in the ground state, then upon a transformation to another coordinate system which is moving with a velocity $\mathbf{v}_0$ the energy of the texture will transform by a law which differs from that for the transformation of the energy of excitations in He II [see (2.5)]:

$$\tilde{\varepsilon} = \varepsilon + \mathbf{p}\mathbf{v}_0 + \Delta M_s \frac{v_0^2}{2}, \quad (4.5)$$

where ΔM_s is the difference between the total superfluid mass for the texture and that for the original ground state of the liquid.

The absence of a well-defined phase of the order parameter in the A phase leads to a remarkable result: States with mass superflows are not topologically stable in large volumes.[52] Let us see how this conclusion follows from the topology of the degeneracy space (R space) of the order parameter. We will ignore the spin degrees of freedom (the vector $\mathbf{d}$), as we are completely justified in doing in two limiting cases: those in which the gradient energy is much larger than or much smaller than the dipole-dipole energy, which depends on the angle between $\mathbf{d}$ and $\mathbf{l}$. In the former case we may ignore the spin-orbit coupling and treat the orbital part of the wave function independently of its spin part. In the latter case, the "dipole-locked" case, the dipole-dipole interaction causes the vector $\mathbf{d}$ to follow the vector $\mathbf{l}$ everywhere, remaining parallel to it, so that the order parameter is determined completely by a specification of the position of the reference system $(\Delta_1, \Delta_2, \mathbf{l})$. In both cases the mass superflow is determined from (4.3) or (4.4), but the free-energy constants are different, since in the dipole-locked case the gradients of the vector $\mathbf{d}$ contribute to the energy, while in the first case the vector $\mathbf{d}$ may be assumed constant.

The R space here is the same as the projection space, P_3, which is the space of the parameters of the three-dimensional rotation group.[8] This space is a sphere of radius π. A vector connecting any point on the sphere with the center of the sphere specifies the direction of the rotation axis, while the length of the vector specifies the angle of the rotation around this axis. Obviously, rotations through π and $-\pi$ around oppositely directed axes are identical, so that diametrically opposed points on the surface of the sphere are equivalent. Let us again consider the current states in a ring which correspond to a rotation of the reference system $(\Delta_1, \Delta_2, \mathbf{l})$ around $\mathbf{l}$ in going around the ring through an angle $2\pi N$ (N is any integer). The mapping of one rotation of the reference system onto R space is the diameter of a sphere, which coincides with the axis parallel to the vector $\mathbf{l}$. This is a closed contour, since the ends of the diameter are equivalent. Furthermore, N rotations around $\mathbf{l}$ in real space correspond to an N-fold traversal of the same diameter. It turns out that all contours with even N are homotopically equivalent to a contour with $N = 0$, i.e., to a homogeneous state without a current, while all contours with odd N are equivalent to the contour with $N = 1$. The fundamental group π_1 thus consists of only two elements.[46] Figure 4 shows how the $N = 2$ contour is contracted by a continuous deformation to the point corresponding to the homogeneous state. The only topologically stable current state with one quantum, $N = 1$, corresponds to a vanishingly low superfluid velocity in a large channel. In contrast, states with a macroscopic number of quanta become topologically stable only as a consequence of a magnetic field and the effect of boundaries which alter the topology of the R space.[8,48]

[10)]A more accurate analysis shows that the orbital angular momentum of the rotation of a Cooper pair induces a slight electron orbital magnetic momentum[49] and also a slight nuclear spin momentum[50] along the direction of the orbital angular moment $\mathbf{l}$. In the A phase there is accordingly a very weak spontaneous magnetization, which has been observed experimentally.[51] In most cases, however, this magnetization may be ignored.

[11)]Here we mean a Galilean transformation of not the entire liquid but only of its superfluid part, i.e., the transformation $\mathbf{v}_s \to \mathbf{v}_s + \mathbf{v}_0$, $\mathbf{v}_n \to \mathbf{v}_n$. The relative velocity $\mathbf{v}_s - \mathbf{v}_n$ is assumed to be small before and after the transformation, and the effects which are nonlinear in $\mathbf{v}_s - \mathbf{v}_n$ are assumed to be unimportant. The assumption $\mathbf{v}_n = 0$ is adopted in the equations of this section of the review.

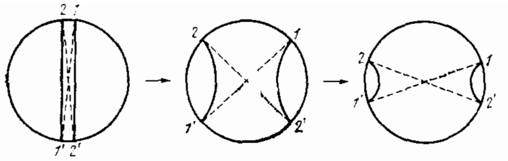

FIG. 4. A continuous deformation in R space for the A phase which transforms the contour for the current state in a ring with $N=2$ (two quanta of the circulation) into a point which is the mapping of the ground state.[6]

However, before we conclude that mass superflows are unstable in large volumes of the A phase without a magnetic field, we must check to see whether activation barriers of *nontopological* origin arise, as a result of an increase in the energy during a continuous deformation of the contour *inside* the R space. By analyzing the stability of current states, Bhattacharyya, Ho, and Mermin[53] showed that nontopological barriers do arise in the dipole-locked case.[12)] If barriers are present, however, their order of magnitude is determined from dimensionality considerations. Barriers result from an increase in the gradient energy in (4.1), which does not contain any scale lengths, and the sole scale length of this problem is h/mv_s. Accordingly, approximate heights of the nontopological barriers are found from expressions (2.6), (2.20), and (2.21) for topological barriers in He II by replacing the coherence length r_c (the size of the vortex core) by the length h/mv_s:

$$\begin{aligned} &\text{3 dimensions:} \quad \varepsilon_A \sim \left(\frac{h}{m}\right)^3 \frac{\rho_s}{v_s}, \\ &\text{2 dimensions:} \quad \varepsilon_A \sim \left(\frac{h}{m}\right)^2 \rho_s, \qquad (4.6) \\ &\text{1 dimension:} \quad \varepsilon_A \sim \frac{h}{m} \rho_s v_s, \end{aligned}$$

where the superfluid densities ρ_s for the three-, two-, and one-dimensional cases are of the order of the largest of the constants K_1, K_2, K_3 in (4.1).

The reason for the increase in the gradient energy upon the loss of one quantum of the flow is a deviation from a parallel arrangement of $\mathbf{l}$ and $\mathbf{v}_s$. The barrier ε_A thus becomes higher with increasing v_s in the one-dimensional case. In two and three dimensions, in contrast, the growth of the gradient energy can be slowed by restricting to a length h/mv_s the size of the region which is inhomogeneous in terms of $\mathbf{l}$ and in which $\operatorname{rot} \mathbf{v}_s \neq 0$. This length might be called the core of a "nonsingular" vortex. In the intermediate state corresponding to the largest energy (the top of the barrier), however, the size of this core is of the same order of magnitude as the size of the "London region" in the field of the vortex, in which $\operatorname{rot} \mathbf{v}_s = 0$. Consequently, a large logarithm does not appear in (4.6), as it does in (2.20) and (2.6).

When the superfluid velocity reaches values greater than $h/m\xi_d$ (ξ_d is the dipole length, $\xi_d \sim 10\,\mu$m), the gradient energy becomes larger than the dipole-dipole energy, and the dipole-locked situation becomes unfavorable from the energy standpoint.[54] A disruption of the coupling of the vectors $\mathbf{d}$ and $\mathbf{l}$ causes the nontopological activation barriers to disappear. At velocities $v_s > h/m\xi_d$, homogeneous mass superflows in the A phase are thus unstable.[13)]

b) Unattenuated spin flows

If there is an energy which keeps the spin vector $\mathbf{d}$ in the easy plane (e.g., the energy of interaction with a magnetic field), then the spatial rotation of $\mathbf{d}$ within this plane implies a transport of the spin magnetization which is longitudinal with respect to the magnetic field. The resulting spin "superflow" may be described at the microscopic level as consisting of two oppositely directed superflows of Cooper pairs with spin projections +1 and −1 onto the direction of the magnetic field. This flow is therefore sometimes called a "magnetic countercurrent." It was first studied theoretically by Vuorio.[55] In interpreting their experiments on the longitudinal magnetic relaxation in the A phase, Corruccini and Osheroff[56] suggested that the extremely important shortening of the relaxation time below the point of the transition to the superfluid state results from a rapid transport of spin by the superflow from the interior to the wall, where the spin undergoes its ultimate relaxation with magnetic impurities. The decrease in the magnetization in the volume occurs linearly, and this result can be explained in the same way as the flow of superfluid helium films at a constant velocity: The gradient of the magnetization "accelerates" the spin superflow until a strong dissipation mechanism comes into play and stops the growth of the spin flow.[55,56]

The dipole-dipole interaction, which puts the vectors $\mathbf{d}$ and $\mathbf{l}$ in a parallel orientation, lifts the energy degeneracy in terms of the angle between $\mathbf{d}$ and $\mathbf{l}$. In certain cases, the changes in the direction of $\mathbf{l}$ can be ignored; one case is that in which $\mathbf{l}$ is fixed by a boundary effect, and another is that in which the vector $\mathbf{d}$ but not $\mathbf{l}$ is twisted during the formation of an inhomogeneous state. The latter case may occur in experiments on longitudinal relaxation. In such a case the dipole-dipole interaction plays the same role as that which is played by the magnetic anisotropy in the easy plane for planar ferromagnets and antiferromagnets. It renders a weak spin superflow very nonuniform: The texture is broken up into isolated domain walls or d-solitons.[57] The analysis of the stability of spin superflows for planar ferromagnets and antiferromagnets (Sec. 3) can also be applied to the A phase; the expressions for the activation barriers will differ from (3.2) and (3.10) only in the notation. The spin superflows turn out to be stable if the energy keeping $\mathbf{d}$ in the plane (the interaction with the magnetic field) exceeds the dipole-dipole energy which is opposing the free rotation of $\mathbf{d}$ in the plane. This condition requires fields $H > 30$ G (Ref. 58). This conclusion is confirmed by the experiments of Sager *et al.*[59] As they reduced the magnetic field, they observed, in the field interval 30–85 G, a transition from

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[12)] They showed that the current state is a minimum of the energy, rather than a saddle point.

[13)] Helicoidal textures with a spatial rotation of the vector $\mathbf{l}$, in which there may be a nondecaying mass transport, may, however, be stable (see Ref. 54 and the references cited there).

Corruccini-Osheroff relaxation (a linear decrease in the magnetization) to a different type of relaxation (a linear decrease in thesquareof the manetization), which follows from the theory of Leggett and Takagi,[43] who studied uniform relaxation in a volume.

Proof that large supercritical spin flows arise in a volume during longitudinal relaxation might be the observation of spin turbulence: a large number of magnetic vortices near which the vector d departs from the plane perpendicular to the field H. A turbulence of this sort has been used in several papers[59,60] to explain a nonmonotonic relaxation, in which the magnetization may increase during short time intervals in the course of relaxation.[14)]

The propagation of solitons (domain walls) over substantial distances has recently been detected experimentally in the *A* phase, and the propagation velocity *has been measured.*[62] *These experiments confirm the* possibility of an unattenuated spin transport, since each moving soliton carriers a spin. In Ref. 63, equations similar to (3.5)–(3.6), which were reduced to a time-dependent sine-Gordon equation with dissipative terms after the elimination of m_z, were used to derive a hydrodynamic theory which described the motion of solitons in terms of their density and velocity. Comparison of this theory with the experiment of Ref. 62 on soliton propagation showed that spin turbulence might play a major role in that experiment.

5. UNATTENUATED FLOWS AND METASTABLE STRUCTURES IN ELECTRON-HOLE SYSTEMS

a) Preliminary comments

The problem of the Bose condensation and superfluidity of electron-hole pairs or excitons has been the subject of discussion for a rather long time now.[64-66] It has been suggested that an exciton concentration sufficient for Bose condensation could be produced by optical pumping. If the exciton lifetime is long enough, the resulting state may be treated approximately as an equilibrium state. At high exciton concentrations, however, what usually happens instead of a Bose condensation is the onset of a condensation into electron-hole drops.[67] Nevertheless, there are indications in the literature that Bose condensation of excitons or biexcitons has been achieved experimentally.[68] The Bose condensation of excitons, however, can also occur at equilibrium, in a semiconductor with a gap width smaller than the exciton binding energy or in a semimetal with overlapping valence and conduction bands.[69-72] The state which arises is called an "exciton dielectric" or "exciton phase."[15)]

[14)] However, another explanation has also been offered for this effect,[61] on the basis of the Leggett-Takagi exchange mechanism for relaxation.

[15)] The model proposed by Lomer[73] to explain spin-density waves in chromium is also essentially an exciton-phase model. Chromium, however, does not become a dielectric, since there are parts of the Fermi surface which do not participate in electron-hole pairing.

The superfluidity of an exciton dielectric was first discussed by Kozlov and Maksimov.[74] They suggested that it would take the form of "thermal superconductivity," but they worked from an incorrect (as was shown later[75]) expression for the energy flux. A nonzero energy flux can exist in an exciton phase only under nonequilibrium conditions (Subsection 5c). In the He II case, thermal superconductivity is usually understood as entropy transport without a temperature gradient.

Like second sound, the latter is possible only if momentum is conserved. Accordingly, thermal superconductivity is impossible not only in an exciton phase but also in a superconductor, because of impurities. The "superproperties" of the exciton phase must therefore be manifested in some other way. To see just what these superproperties are and to see whether it is meaningful to label them as such, we must examine two important questions: 1) the role of interband transitions, which lift the phase degeneracy and rule out states with a uniform flow of electron-hole pairs[41]; 2) just which real physical quantity is carried by the superflow in a Bose-condensed electron-hole system, and whether this superflow can be observed.

The question of the physical meaning and observability of a superflow of electron-hole pairs has a simple answer (theoretically, at least) for two-dimensional systems with spatially separated electrons and holes: two films with free electrons in one and holes in the other, separated by a dielectric layer. Kogan and Tavger[76] pointed out that the transport of bound, neutral electron-hole pairs in such systems might be accompanied by the flow of nondecaying electric current if the two films are connected in a series circuit (see Fig. 5, which is taken from Ref. 76).

The theory for such systems was developed further in Refs. 77–80, but the problem of fixing the phase remains an urgent one for systems with spatially separated electrons and holes.

In the preceding sections we have seen that superproperties are phenomena which occur over long temporal and spatial scales, so we have used phenomenological equations everywhere: the hydrodynamic equations, the Landau-Lifshitz equations, and the Leggett-Takagi equations.

In this sense, the present section is a deviation from this rule, since here we are talking in terms of a *microscopic* model: the model of an exciton dielectric. In making this exception we are, on the one hand, paying tribute to established tradition, thereby facilitating a discussion of the work which has been done on this subject.

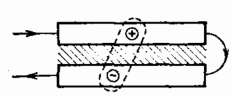

FIG. 5. Flow of an electric current in the case of motion of bound electron-hole pairs.[6] The electrons and holes are in two films, which are separated by an insulating layer.

On the other hand, the conditions under which the model of an exciton dielectric is applicable overlap considerably with the conditions under which superproperties exist. However, this overlap by no means rules out the possibility that these superproperties may also hold for some other model of the dielectric. Furthermore, it is possible to describe the phenomenon in phenomenological terms, without resorting to such concepts of the microscopic theory as an electron-hole pair. We will return to this question in Subsection 5d.

b) Physical meaning of the order parameter of an exciton state. Phase fixing

The order parameter in an exciton state can be written as a 2×2 matrix $\Delta_{\alpha\beta}$, which, on the one hand, is the wave function of the electron-hole pair (α and β are the spin indices of the electron and the hole), while on the other it determines the amplitude of the charge- or spin-density waves which arise in the exciton state:[72]

$$\langle\delta\rho\rangle = \mathrm{Re}\,\{\Delta\Phi(\mathbf{r})\}, \tag{5.1}$$

$$\langle\delta\sigma\rangle = \mathrm{Re}\,\{\sigma_{\alpha\beta}\Delta_{\alpha\beta}\Phi(\mathbf{r})\}, \tag{5.2}$$

where the Fourier expansion for $\Phi(r)$ may include both the wave vectors of the original reciprocal lattice and vectors which are or are not commensurable with them. The components of the vector σ are the Pauli matrices. The singlet pairing $\Delta_{\alpha\beta}=\delta_{\alpha\beta}\Delta\exp(i\varphi)$ leads to charge-density waves ($\langle\delta\rho\rangle\neq 0$, $\langle\delta\sigma\rangle=0$). In the case of triplet pairing, $\Delta_{\alpha\beta}$ is a zero-trace matrix whose components can be expressed in terms of the three-dimensional vector **d**:

$$\Delta_{\alpha\beta}\rightarrow\begin{pmatrix} d_z & d_x-id_y \\ d_x-id_y & -d_z\end{pmatrix}e^{i\varphi}. \tag{5.3}$$

The vector **d** is the polarization vector of the spin-density wave which arises upon triplet pairing. The model of a triplet exciton state is actually the model of band antiferromagnetism, and the vector **d** is equivalent to the antiferromagnetic vector of a Heisenberg antiferromagnet.

The overall phase of the order parameter, φ (the exciton phase in the terminology of Refs. 39 and 40) determines the displacement of the charge-density and spin-density waves with respect to the basic lattice.[16)] The free energy depends on such displacements only if interband transitions are taken into account.

In the exciton state, equations of motion must exist for a pair of canonical variables, the exciton phase φ and the electron-hole pair density n_{eh}; in the triplet state, there must also be equations of motion for the vector **d** and for the magnetization (the Landau-Lifshitz equations). These (phenomenological) equations can be written in a form which incorporates both the theoretical requirements of symmetry and the conservation laws. In order to find the theoretical values of the coefficients in these equations, however, it is necessary to derive them from some specific theoretical model. Such a derivation has been carried out for an exciton dielectric, for a low-density exciton gas, and for systems with spatially separated electrons and holes[17)] (Refs. 39, 40, 79, and 81–83). Because of the interband transitions, a term which depends on the exciton phase φ, i.e., which fixes the phase, appears in the free energy. For interactions which are invariant with respect to spin flip, the phase fixing can be described by

$$E_\Phi(\varphi)=\int d\mathbf{r}\,[a_1\Delta_{\alpha\alpha}(\mathbf{r})+a_2\Delta_{\alpha\beta}(\mathbf{r})\Delta_{\beta\alpha}(\mathbf{r})\ldots+\mathrm{c.c.}]. \tag{5.4}$$

Here we are assuming that the order parameter $\Delta_{\alpha\beta}(r)$ chosen in (5.1)–(5.3) is constant for the ground state; a perturbation of the ground state may cause this parameter to vary over space, over scale distances greater than the wavelength of the structure which appears as a result of the phase transition.

Terms of first order in $\Delta_{\alpha\beta}$ appear in expansion (5.4) only if the electron and hole extrema lie at the same point in k space. A phase transition involving a disruption of the symmetry does not occur, however; the only possibility is a diffuse transition between regions with small and large values of Δ. Accordingly, upon a phase transition accompanied by a violation of translational symmetry (the appearance of new wave vectors of the structure, which do not belong to the original reciprocal lattice) or a violation of the point-group symmetry, expansion (5.4) cannot begin with terms of first order Δ (Volkov and Kopaev, 1978). In antiferromagnetic chromium the extrema are separated in k space by half the reciprocal-lattice vector,[84] and the expansion begins with terms of order Δ^2, regardless of whether a structure commensurable or incommensurable with the basic lattice appears. The coefficient a_2 is determined by flip processes in this case.[40]

c) Unattenuated flows and deformation of charge and spin density waves

The gradients of the exciton phase, $\nabla\varphi$, determine the flows of electron-hole pairs, while gradients of the vector **d** determine the spin flows. If, for example, d rotates in the xy plane, then it can be seen from (5.3) that the changes which occur in the phases of off-diagonal elements of the order-parameter matrix are opposite in sign. These elements may be interpreted as the order parameters for electron-hole pairs whose z spin projections are +1 and −1. A spatial rotation of **d** thus corresponds to a counterflow of pairs with oppositely directed spins, so that there is spin transport.[18)] In

[16)] Each harmonic of the spin-density and charge-density waves is shifted by an amount φ/k. There are also changes in other electron parameters, e.g., the dipole moment.

[17)] We will use this opportunity to point out that Klyuchnik and Lozovik's assertion (see their notes in Refs. 82 and 83) that in Ref. 40 I erroneously compared an equation for the rotation angle of the magnetization (the vector **d**) with an equation derived for the exciton phase (the phase of the condensate in Refs. 82 and 83) is based on a misunderstanding. If these papers are read carefully, expecially (Russian) page 2108 in Ref. 40, the reader will see that the equations found by Klyuchnik and Lozovik for the exciton dielectric are actually the same as those in Refs. 39 and 40. The only discrepancies are in the terminology: Klyuchnik and Lozovik object to the use of the term "superproperty" (see subsection 5d).

[18)] Here there is an analogy with spin transport in the A phase of He^3 (see Refs. 39 and 85 and subsection 4b of the present review).

contrast, the electrons in a Heisenberg antiferromagnet are localized, and spin transport is not accompanied by transitions of electrons from one site to another. These differences in the microscopic nature of magnetism and in the spin-transport mechanism, however, should have absolutely no effect on the form of the phenomenological equations which describe the motion in spin space (the Landau-Lifshitz equations). For this reason, all the conditions for the existence of, and methods for the generation of, spin flows which are attenuated only slightly over space and time ("superflows") which were derived in Sec. 3 from these equations can be extended entirely to the triplet exciton state, i.e., to a band antiferromagnetic.

A study of the stability of structures with spatial gradients of the exciton phase φ is also basically the same as the study of structures with a spatial rotation of the magnetization. The equations of motion for the pair of canonical variables φ, n_{eh} (the exciton phase and the density of electron-hole pairs) differ from those for the pair φ, m_z (the angle through which the spins are rotated around the z axis and the z projection of the spin density) [m_z must be replaced by n_{eh} in Eqs. (3.5) and (3.6)], and they also differ in the physical meaning of the coefficients in the equations. Instead of a magnetic anisotropy [the terms $E\Phi \sin n\varphi$ in Eqs. (3.5) and (3.6)] we would now have the phase-fixing energy (5.4). The number n, which gives the order of the symmetry of the anisotropy in the plane, must now correspond to the power of Δ at which the expansion in Δ in (5.4) begins. In the equation for $\partial n_{eh}/\partial t$, the term corresponding to the Bloch relaxation in (3.5), m_z/T_1, will describe actual interband transitions which do not depend on the phase, in contrast with the virtual transitions, which fix the phase. We will not repeat the entire analysis of Sec. 3; we will simply examine the basic results of this analysis for states which are inhomogeneous in terms of exciton phase.

1) Phase fixing leads to a gap in the collective-oscillation spectrum and rules out the existence of homogeneous stationary states with an electron-hole pair flow[41] $j_{eh} \sim \nabla\varphi$; there may, however, be inhomogeneous states with a periodically varying flow j_{eh} (Refs. 79, 81, and 86). If the "kinetic energy" of the flow (the deformation energy $\sim\nabla\varphi^2$) is much smaller than the phase-fixing energy, then such states are a periodic chain of solitons (domain walls), which separate domains of a fixed constant phase φ.

2) The relaxation of a "deformation" $\nabla\varphi$ can occur only in the case of motion of vortices across the flow, $j_{eh} \sim \nabla\varphi$. This will require surmounting an activation barrier of topological origin, determined by the same expressions as for a spin superflow, (3.2) and (3.10). The size of the vortex core, r_c, which appears in the logarithm in these expressions is equal to the coherence length in this particular case, since the order parameter must vanish on a vortex line if the phase is to change by 2π when this line is circumvented.

3) Sufficiently high activation barriers exist in the case $|\langle\nabla\varphi\rangle| < 1/r_c$ if the condensation energy (or pairing energy) is much larger than the phase-fixing energy; in other words, the distance l over which the phase is fixed [see the sine-Gordon equation in (3.7)] must be far larger than the coherence length r_c.

The stability of those deformed structures in the exciton state which transport the flow of electron-hole pairs should be manifested in experiments on the injection of electron-hole pairs. Experiments of this type have been proposed for observing an exciton Bose condensation (see Refs. 87 and 65). It was suggested in Ref. 39 that an electric current be used to inject electron-hole pairs. Because of the difference between the conductivities of the conduction and valence bands and also the difference between the probabilities for an electron to reach the two bands at the contact, a nonequilibrium concentration of electron-hole pairs arises near the contact. If the injection exceeds a certain threshold determined by the phase-fixing energy, the unattenuated flow of electron-hole pairs rapidly distributes the injected pairs over the entire volume, so that the density of pairs turns out to be constant, while the superflow decays linearly along the length of the sample (see the corresponding problem involving spin injection in Subsection 3c).

As usual, the energy flow associated with the superflow is proportional to the chemical potential $\mu_{eh} = \delta\mathscr{H}/\delta n_{eh}$ of the electron-hole pairs:

$$Q = \mu_{eh}\, \mathbf{j}_{eh}. \tag{5.5}$$

At a total equilibrium, $\mu_{eh} = 0$, no energy is transported. During the injection of pairs into a sample of finite length, however, the condition $\mu_{eh} \neq 0$ always holds, although the values of μ_{eh} in the exciton dielectric in this case are small, of the order of the reciprocal of the sample's length (again we refer the reader to Subsection 3c, where, there is a more detailed explanation of the same phenomenon for the case of spins). On the other hand, when excitons are pumped into a semiconductor with a large energy gap E_h, there is a rapid rotation of the order parameter in the complex plane, at a rate $\partial\varphi/\partial t \sim \mu_{eh} \sim E_h$. The phase-fixing energy thus vanishes on the average; the phase is not fixed; and the superflow transports a large energy.[66,72]

d) Does an exciton dielectric differ from an ordinary dielectric?

The electron-hole pair flow j_{eh}, which is proportional to the "deformation" $\nabla\varphi$, is attenuated rapidly in space and time, and it exists only below the phase-transition point. It may thus be called a "superflow" quite legitimately, and the unattenuated transport itself (i.e., the transport with a very low dissipation) may be called a "superproperty." Other terms could of course be used and in fact might be more appropriate. A more interesting question, however, is this: Are the "superproperties" (or whatever they are called) something which fundamentally distinguish an exciton dielectric from an ordinary dielectric? The impulse is to say no,[41] since superproperties are phenomena which occur at the macroscopic level, where the various particular details of the theoretical models at the microscopic level would not seem to be important. One such detail is the very concept of a "hole" or that of an "electron-hole pair." In contrast with spin magnetization, for which there is a clear operator expression, which applies to any medium, the number of holes or pairs is an approximate concept, meaningful only in the particular basis chosen for the electron wave functions. We would therefore like to describe "superproperties" without resorting to the concept of a "hole," and at this point we wish to show that this can be done.

Actually, a collective mode associated with a change in the density of electron-hole pairs is none other than an optical mode corresponding to the motion of some electron subsystem with respect to the rest of the crystal. As we have already mentioned, the exciton phase determines the displacement of the spin-density and charge-density waves with respect to the crystal. The deformation $\nabla\varphi$ then determines (as in an elastic solid) the stresses (the momentum flux) in the subsystem which we have singled out, and these stresses appear in the model of an exciton dielectric as a flux of electron-hole pairs.[19] Such stresses are attenuated only slightly over space and time if the coupling of the particular subsystem with the rest of the crystal is sufficiently weak. This conclusion means that the optical mode must be very soft, and the part of the energy which depends on the displacement (the phase-fixing energy) must be quite small in comparison with the elastic strain energy near the threshold for the apperance of plastic phenomena (the motion of vortices or dislocations). The condition for the existence of a soft mode in the model of an exciton dielectric reduces to the condition that the energy of the interband transitions must be small. On the other hand, this condition requires that the model of an exciton dielectric be of general applicability, and for this reason the model of an exciton dielectric occupies a special position with respect to other models for the dielectric state. On the other hand, there are no fundamental distinctions between an exciton dielectric and an ordinary dielectric. It may be suggested that there are other physical reasons for the appearance in a dielectric (and not only in a dielectric) of a soft, slowly damped optical mode, which would allow us to introduce a new phenomenological parameter in the hydrodynamic description. If the space in which this new parameter is defined has the appropriate topology, then the metastable states discussed in this review are completely feasible.

6. SUPERFLUIDITY CRITERION AND LONG-RANGE ORDER

Ever since superconductivity and superfluidity were discovered there has been an unceasing effort to identify some key property of a condensed system which determines whether it will exhibit superfluid properties. When London[1,88] revealed the relationship between superfluidity and Bose condensation, he took us a long way toward an understanding of the superfluidity of He II. Bose condensation automatically implies the appearance of an off-diagonal long-range order (ODLRO) in the one-particle density matrix,[89,90]

$$\rho(\mathbf{r}, \mathbf{r}') = \langle \psi(\mathbf{r})^+ \psi(\mathbf{r}') \rangle \to \langle \psi(\mathbf{r})^+ \rangle \langle \psi(\mathbf{r}') \rangle \neq 0, \quad |\mathbf{r} - \mathbf{r}'| \to \infty; \tag{6.1}$$

here $\Psi(\mathbf{r})$ and $\Psi(\mathbf{r})^+$ are the boson annihilation and creation operators. In a superfluid Fermi liquid, an ODLRO arises in the two-particle density matrix.

Yang[91] has suggested that ODLRO might be considered a *necessary* and *sufficient* condition for superfluidity. Yang understood superfluidity (or superconductivity) to be the quantization of flux in a ring. According to Yang, the flux quantization results from a periodic repetition of minima in the dependence of the free energy on the *total* current. Actually, as we saw in Subsection 2b, flux quantization and the associated very slow attenuation of supercurrents result from minima in the dependence of the free energy on the superfluid current, not on the total current. In contrast, the barriers separating the minima in the dependence of the free energy on the total current vanish even at temperatures and superfluid velocities so low that quantization must definitely occur [see Eq. (2.6′) and also the paper by Eggington and Leggett[92]]. It followed immediately from Yang's concept that superfluidity would be impossible in one-dimensional and two-dimensional systems, since in these cases the long-wave phase fluctuations destroy the ODLRO. This was pointed out by Rice,[93] and a rigorous proof of the vanishing of ODLRO was offered by Hohenberg,[94] who worked from Bogolyubov's $1/k^2$ theorem.[95] We saw in Subsections 2c and 2e, however, that superfluidity is disrupted not by the phase fluctuations but by the fluctuations of the modulus of the order parameter, which give rise to regions with a zero modulus: vortex lines or phase-slippage centers. The absence of an ODLRO thus does not necessarily mean that there is no quantization and that there are no nondecaying currents. Another argument in favor of this conclusion is the fact that the ODLRO vanishes not only in systems with fewer than three dimensions but also in three-dimensional systems which have dimensions which are greatly different in different directions,[20,96-98] e.g., systems in which the length of the object, L, exceeds $S\lambda_B^2\rho_s/m$, where S is the cross-sectional area of the object, and λ_B is the thermal de Broglie wavelength. In such elongated objects the energy barriers separating the minima on the total-current dependence of the free energy become vanishingly low [see (2.6′)], while the ε_A barriers in the dependence of the energy on the superfluid current do not depend on the shape of the three-dimensional object, provided that all the dimensions are greater than the coherence length. If the spanning of the entire length of the sample by the ODLRO were a necessary condition for superfluidity, this would mean that there would be no superfluidity in very long capillaries or in films. This is not what we observe experimentally, however, although experiments have been carried out with very long helium films[99,100] and very long thin superconductors,[101] with a length greater than that over which the ODLRO extends. Eggington and

[19] Since a change in phase leads not only to shifts (see footnote[16]), flows of other physical quantities may also arise. Just which physical quantities will be singled out will be determined by a more detailed analysis of the particular type of exciton pairing.

Leggett[92] have also suggested some special experiments which would confirm that superfluidity is not suppressed with increasing length of a channel.

A generalized ODLRO has been proposed,[102] in which the requirement on the long-range order is relaxed: Although distance over which the off-diagonal correlations in the density matrix in (6.1) extend tends toward infinity in the thermodynamic limit, it may do so more slowly than the length of the system. This means that the Bose condensation does not proceed to a single quantum-mechanical level but to a group of levels, in a small region in momentum space with a width inversely proportional to the smallest dimension of the system.[20)] When this definition is adopted, the ODLRO and Bose condensation exist in a three-dimensional system of any shape with macroscopic dimensions in all directions, but again they do not occur in one- and two-dimensional systems. Consequently, a generalized ODLRO would also be an overly stringent necessary condition for the existence of nondecaying currents.

Berezinskiĭ[35] has suggested an even weaker requirement as a necessary condition for superfluidity: Although the correlations do not extend an infinite distance, they must fall off in a power-law manner (in contrast with the exponential decrease in normal systems). This approach allows a two-dimensional superfluidity,[21)] but a one-dimensional superfluidity is still ruled out. If we work from the argument that a phase transition is required for the appearance of superfluidity, then this condition would be completely justified. However, the existence of only slightly attenuated currents (i.e., a superfluidity in the sense in which we are using this term) is not directly related to the occurrence of a phase transition, as we mentioned in Subsection 2e, although the absence of a phase transition in one-dimensional systems does have definite consequences for a one-dimensional superfluidity (or, perhaps, a "quasi-superfluidity" if we wish to speak more cautiously).

In summary, the requirement that the long-range order extend over infinite distances (in the form of correlations which do not decay or which decay in a power-law manner) is overly stringent from the standpoint of the broader interpretation of a "nondecaying current" which we have adopted in this review. Undoubtedly, however, even in this interpretation the pronounced stiffness of the order parameter which gives rise to the stability of the "nondecaying currents" necessarily means that the scale distance over which the correlations propagate is quite large. What should this scale distance be in a one-dimensional system? As the scale dimension for the decrease in the correlations we should choose a distance which determines an exponential decay of the correlations and is $\lambda \sim \rho_s(1)\hbar^2/mkT$ [see, for example, Ref. 93 or Eq. (A.5) in Ref. 20, where $A=\frac{\rho_s\hbar^2}{m^2k}$ and $\rho_s(1)=\rho_s L_y L_z$]. The condition for the stability of a one-dimensional superflow, $\varepsilon_A/kT \gg 1$ [see (2.7) and (2.21)], thus reduces to the condition that the length λ must be greater than the coherence length r_c. The ODLRO is subject to criticism not only as a necessary condition for superfluidity but also as a sufficient condition. In addition to the trivial example of an ideal Bose gas, in which there is an ODLRO but no superfluidity, we might mention systems which have interactions which fix the superfluid phase. Phase fixing, by introducing a gap in the Goldstone mode, eliminates the destructive effect of longwave phase fluctuations on the long-range order. A strong phase fixing, however, makes the superflow unstable. Another example is the A phase of He^3, if its parameters are such that there are no nontopological activation barriers for the relaxation of mass superflows (Subsection 4a). In both these cases, the superflows are unstable with respect to the creation of vortices (of nonsingular vortices in the case of the A phase).

However, the definition of "superfluidity" as the possible existence of nondecaying currents, which we have adopted throughout this review, has its own disadvantages. The superfluid A phase of He^3 may not be a superfluid at all. It is also unclear, for example, how we are to deal with He II in broad channels, where—for reasons which are not at all clear at present—the critical velocities for the nondecaying currents are vanishingly small, although all the other superfluid properties are present. There are accordingly adherents to a different definition of the term "superfluidity." Putterman,[3] for example, suggests identifying superfluidity with Bose condensation. In this approach the problem of the relationship between the ODLRO and superfluidity is resolved at once. ODLRO is superfluidity. Accordingly, in Putterman's interpretation an ideal Bose gas is a superfluid. At present, therefore, we do not have a firmly established agreement on the meaning of the term "superfluidity," and this term is used in two meanings: 1) in a thermodynamic meaning—the system undergoes a phase transition accompanied by a change in the law by which the off-diagonal elements of the density matrix decay, although nondecaying currents may not exist below the transition point—and 2) in a kinetic meaning—nondecaying currents with finite critical velocities exist. Historically, the term "superfluidity" arose in the second of these meanings, and it is the second which is preferred from the semantic standpoint.

While nondecaying flows of charge and mass are associated with the extension of off-diagonal long-range order in the density matrix over large (although not necessarily infinite) scale dimensions, the nondecaying transport of other physical quantities is related to other types of long-range order. Spin superflows, for example, are related to a long-range order in the spin correlations. For nondecaying transport of electron-hole pairs, the long-range order must appear in the two-particle density matrix for the electron-hole pair,[103] not that for a pair of electrons, as in superconductors. For this reason, Kohn and Sherrington's assertion[104] that the Bose condensation of an electron-hole system does not lead to the appearance of a superfluidity, on

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[20)] London[88] understood Bose condensation in this generalized sense.

[21)] A power-law decay of the correlations in a two-dimensional system was established by Rice[93] in a study of fluctuations in the Ginzburg-Landau theory.

the basis that there is no ODLRO in the two-electron density matrix in this case, is correct only if "superfluidity" means a nondecaying transport of mass and charge. As was shown in Sec. 5, however, there may be "superproperties" of a different type in such systems.

7. CONCLUSION

We will briefly point out some conclusions which can be drawn from this analysis and which would be pertinent to experiment. It has been shown that the present theoretical understanding of the nature of nondecaying currents does not rule out the possibility that they will exist, to some extent or other, in systems of fewer than three dimensions. Such systems have been widely discussed and continue to be discussed in connection with attempts to find high-temperature superconductivity.[105] It is, of course, a long way from the conclusion that one- or two-dimensional superconductivity is possible *in principle* to specific indications of where and how to seek it. The role of the analysis above is restricted to the assertion that such searches are completely justified and reasonable.

Also justified and reasonable would be a search for analogs of nondecaying mass and charge flows, i.e., "superflows," of other physical quantities, in particular, spin. There are some weighty pieces of experimental evidence for the existence of spin superflows in superfluid He^3 (Subsection 4b). To detect this phenomenon in an ordinary magnetically ordered system would undoubtedly be a feasible experiment right now, and it would furnish a new possibility for the transmission of a signal over a large distance without damping, e.g., controlling the motion of domain walls by a field applied far from these walls.

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