A. I. Larkin and D. E. Khmel'nitskii. *Anderson localization and anomalous magnetoresistance at low temperatures.* Consider a conductor in which the path length l is much greater than the wavelength. Let us calculate the probability of diffusion in a time t from a point A to a point B (see Fig. 1). We can imagine that at the point A there is a source which coherently emits wave packets that propagate along rays of thickness λ (they begin at A and end at B). According to the general rules of quantum mechanics, to find the probability w we must add the probability amplitudes of diffusion along each trajectory and calculate the square of the modulus of this sum: $w \sim |\sum_i A_i|^2 = \sum_i |A_i|^2 + \sum_{i \neq j} A_i A_j^*$.

The first term on the right-hand side describes the sum of the probabilities referring to each individual ray, and the second term is the interference term. The interference of most of the amplitudes is not important, since the lengths of the trajectories and hence their phases are very different, and the mean value of the interference term is equal to zero. The exceptions are the trajectories which intersect themselves. Each such trajectory can be associated with two amplitudes A_1 and A_2, corresponding to passage around the closed loop in opposite directions. These two amplitudes are mutually coherent, so that their interference cannot be neglected: $A_1A_2^* + A_2A_1^* = 2|A_1|^2$. Neglect of the interference corresponds to the classical description (the Boltzmann equation), and inclusion of the interference corresponds to the quantum corrections to the classical kinetics.

Let us estimate the relative value of the quantum correction $\delta\sigma/\sigma_0$. This quantity (it is negative) is proportional to the probability of self-intersection of a ray with cross section λ^{d-1} for classical diffusion. Therefore

$$\frac{\delta\sigma}{\sigma_0} \sim -\int_{\tau}^{\tau_\varphi} \frac{v\,dt\,\lambda^{d-1}}{(Dt)^{d/2}}. \qquad (1)$$

The integration in Eq. (1) is taken over the range $\tau < t < \tau_\varphi$, where τ_φ is the time of destruction of the phase owing to inelastic scattering or scattering with spin flip. As a result, we have[1,2]

$$\delta\sigma \sim -\frac{e^2}{\hbar}\begin{cases} L_\varphi & d=1, \\ \ln\dfrac{L_\varphi}{l} & d=2,\ L_\varphi=\sqrt{D\tau_\varphi}, \\ \text{const}-\dfrac{1}{L_\varphi} & d=3 \end{cases} \qquad (2)$$

It can be seen from (2) that the resulting corrections, albeit small in the parameter λ/l, determine singular dependences on the temperature ($\tau_\varphi \sim T^{-p}$) or frequency ω [for $\omega\tau_\varphi \gg 1$ in Eq. (2) we must replace τ_φ by $1/\omega$, and for $d=3$, for example, we obtain $\delta\sigma \sim \sqrt{\omega}$]. If a film or a wire has transverse dimension a and $a \ll L_\varphi$, then the diffusion has a two-dimensional (one-dimensional) character, and the corrections to the resistivity of a wire of unit length ($d=1$) or of a film ($d=2$) can be estimated according to the formula[1-3]

$$\left(\Delta\frac{1}{R}\right)\left(\frac{1}{R}\right)^{-1} \sim -\int_{a^2/D}^{\tau_\varphi} \frac{v\,dt\,\lambda^{d-1}}{(D_d^* t)^{d/2}}, \qquad (1')$$

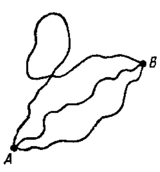
where

$$D_d^* = \begin{cases} D_s, & d=1, \\ D_a, & d=2. \end{cases}$$

If an external magnetic field is switched on, the amplitudes A_1 and A_2 acquire additional factors:

$$A_1 \to A_1 \exp\left(i\frac{e}{\hbar c}\oint \mathbf{A}\,d\mathbf{r}\right) = A_1 \exp\left(\frac{2\pi i HS}{\Phi_0}\right),$$
$$A_2 \to A_2 \exp\left(-\frac{2\pi i HS}{\Phi_0}\right),$$

where S is the projection of the area of the loop onto a plane perpendicular to the direction of the magnetic field. As a result, Eq. (1) can be rewritten in the form[4]



$$\frac{\delta\sigma}{\sigma_0} \sim -\int_{\tau}^{\tau_\varphi} \frac{\lambda^{d-1} v\, dt}{(Dt)^{d/2}} \int w(S, t) \cos\left(\frac{4\pi HS}{\Phi_0}\right) dS, \qquad (3)$$

where $w(S, t)$ is the probability that a loop with length vt has area S. In the two-dimensional case, the whole of the last integral can be replaced by $\cos(HDt/\Phi_0)$. This gives

$$\Delta\sigma(H) = \sigma(H) - \sigma(0) \sim \frac{e^2}{\hbar} \begin{cases} \left(\frac{eHD\tau_\varphi}{\hbar c}\right)^2, & \frac{eHD\tau_\varphi}{\hbar c} \ll 1, \\ \ln \frac{eHD\tau_\varphi}{\hbar c}, & \frac{eHD\tau_\varphi}{\hbar c} \gg 1. \end{cases} \qquad (4)$$

In the three-dimensional case, we can (for an estimate) assume that all the trajectories lie in a plane. The angle θ between this plane and the direction of the magnetic field is determined by the fact that $\cos\theta = S/Dt$; therefore

$$\int w(s, t) \cos \frac{4\pi SH}{\Phi_0} dS \sim \int_{-Dt}^{Dt} \frac{dS}{Dt} \cos \frac{4\pi HS}{\Phi_0} \sim \frac{\Phi_0}{DHt} \sin \frac{DHt}{\Phi_0}.$$

As a result, we obtain

$$\delta\sigma \sim -\frac{\Phi_0}{D^{3/2} H} \int_{\tau}^{\tau_\varphi} \frac{dt}{t^{5/2}} \sin \frac{DHt}{\Phi_0}, \qquad (3')$$

$$\Delta\sigma(H) \sim \frac{e^2}{\hbar} \sqrt{\frac{eH}{\hbar c}} \begin{cases} \left(\frac{eHD\tau_\varphi}{\hbar c}\right)^{3/2}, & \frac{eHD\tau_\varphi}{\hbar c} \ll 1, \\ \text{const}, & \frac{eHD\tau_\varphi}{\hbar c} \gg 1. \end{cases} \qquad (4')$$

The basic properties of the phenomenon are as follows:

1. For weak fields, $\Delta\sigma \sim H^2 \tau_\varphi^2 (H^2 \tau_\varphi^{3/2})$ (a large coefficient).

2. There is "saturation" for $\Omega_H \tau \sim (\lambda/l)(\tau/\tau_\varphi) \ll 1$, i.e., in the region of classically weak fields.

3. There is no dependence on the angle between the field and the current ($d = 3$).

4. For films, the effect exists only for a field normal to the plane of the film.

5. The sign of the correction is positive (the field "favors" the conductivity).

If the resistance of a hollow body (cylinder or ring) is measured, this resistance oscillates as a function of the magnetic flux passing through the cavity, with period $\Phi_0/2 = \pi\hbar/ec$.[5] This Aharonov-Bohm effect has been observed in experiments by Sharvin and Sharvin.[6]

Another oscillation effect can be observed in a normal metal in contact with two superconductors S_1 and S_2.[7] It turns out that, owing to Andreev reflection of the electrons at a boundary with a superconductor, the resistance of a normal metal is sensitive to the phase difference $\varphi = \chi_1 - \chi_2$ of the order parameters in the superconductors and oscillates with period π.

The additional terms $A_1 A_2^*$ which occur when allowance is made for the electron spin carry information about not only the phase of the electron, but also its spin polarization. If the intial and final states have wave functions φ_α and φ_β, respectively, the additional term can be written in the form $C = (\varphi_\alpha^1 \varphi_\beta^2 \varphi_\beta^{1*} \varphi_\alpha^{2*})/2$. Going over to the representation involving the total angular momentum of the two particles, $\Psi_{1,\pm1} = \varphi_\pm^1 \varphi_\pm^2$, $\Psi_{1,0} = (1/\sqrt{2})(\varphi_+^1 \varphi_-^2 + \varphi_-^1 \varphi_+^2)$, $\Psi_0 = (1/\sqrt{2})(\varphi_+^1 \varphi_-^2 - \varphi_-^1 \varphi_+^2)$, we have $C = 1/2 \sum_{m=-1}^{+1} |\Psi_{1m}|^2 - 1/2 |\Psi_0|^2$.

The wave function Ψ_{1m} carries information about the spin and disappears in the presence of a spin-orbit interaction after the relaxation time of the spin, τ_{s0}. The wave function Ψ_0 is responsible for only the phase and persists for a time $t \sim \tau_\varphi$. Therefore[8,9]

$$\frac{\delta\sigma}{\sigma_0} \sim -\int_{\tau}^{\tau_\varphi} \frac{\lambda^{d-1} v\, dt}{(Dt)^{d/2}} \left(\frac{3}{2} e^{-t/\tau_{s0}} - \frac{1}{2}\right). \qquad (5)$$

As a result, for $d = 2$, for example, we have

$$\delta\sigma \sim \frac{e^2}{\hbar} \begin{cases} -\ln \frac{\tau_\varphi}{\tau}, & \tau_{s0} \gg \tau_\varphi, \\ -\frac{3}{2} \ln \frac{\tau_{s0}}{\tau} + \frac{1}{2} \ln \frac{\tau_\varphi}{\tau}, & \tau_{s0} \ll \tau_\varphi, \end{cases} \qquad (6)$$

from which we see that a strong spin–orbit interaction and a fast relaxation of the spin lead to a change of sign in the quantum correction and therefore to a change of sign in the magnetoresistance. A dependence of the sign of the anomalous magnetoresistance on the magnitude of the field has been observed in films of copper.[10] A direct qualitative experiment can be performed in cubic semiconductors of p type (p-Ge, p-Si, etc.).[11] The complicated structure of the valence band leads to fast relaxation of the hole angular momentum in elastic scattering. Therefore a positive magnetoresistance should be observed in these substances. In deformed crystals, the degeneracy is removed for $\mathbf{k} = 0$, and fast relaxation of the spin can be eliminated. Therefore the theory predicts a negative magnetoresistance in sufficiently strongly deformed crystals. Experiments carried out before the advent of the theory confirm this qualitative prediction.[12]

The theory also predicts a suppression of the quantum corrections in a high-frequency external field, which leads to an additional destruction of the phase.[13]

The calculation of the quantum corrections and their dependences on the frequency, temperature, magnetic field, and spin–orbit interaction plays an important role in the construction of a theory of localization of electrons in disordered substances.[9,14]

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