The emission of electromagnetic waves in the case of a smooth variation of parameters of a radiating system

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Characteristic properties of electromagnetic radiation are investigated for radiating systems in which variation of parameters is described by model smooth functions permitting derivation of exact solutions. Picked as examples of radiating systems are a moving charge with continuously variable velocity (with respect to magnitude and direction), a dipole with continuously variable dipole moment, and a charge uniformly moving in a medium with permittivity continuously varying in space or time. Exact expressions are derived for the electromagnetic field and spectral energy density of radiation in all problems under consideration. It is shown that the common property of electromagnetic radiation in such systems is exponential decline of the spectrum in the high frequency region.

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CONTENTS

INTRODUCTION

We shall consider a system of charges and currents which produce an electromagnetic field. The parameters of this system will be assumed invariant up to a certain instant. Therefore, a field produced by such a system will be fixed. We shall assume that at a certain instant the system parameters become variable and the variation of parameters occurs in a finite time interval T. During this period the system parameters vary from certain initial to final values and remain unchanged thereafter.

The variation of parameters of a system is accompanied by emission of electromagnetic waves. The emitted wave spectrum is defined by the law of variation of parameters and, consequently, each transition from an initial to a final value of parameters corresponds to a specific emission spectrum. However, despite overall differences the emission spectra have certain properties in common. For example, emission spectra in the low-frequency region are independent of the law of variation of parameters and depend only on the initial and final values of these parameters.[1]

The duration of transition time T does not appear in the expression for the emitted fields and, therefore, the low-frequency spectral region may be obtained under the assumption that the parameters are altered instantaneously.

In this methodological note we wish to comment on another common feature of the emission spectra, this time in the high-frequency region. Specifically, if the dependence of parameters on time is expressed by smooth functions, the emission spectra in the high-frequency region decrease exponentially. In this context we understand smooth functions to be continuous functions as are also all their derivatives.

Inasmuch as it is conceivable that all real changes can be described by smooth functions, it may be assumed that the emission spectra in real physical instances fall off exponentially at high frequencies.

Below we shall present a number of examples which illustrate the common property of radiation in the case of smooth variation of parameters—an exponential drop off in the high-frequency region.

1. CRITERION FOR "INSTANTANEOUS" AND "CONTINUOUS" TRANSITIONS

Bolotovskiĭ and coworkers have examined radiation of electromagnetic waves in the case of an instantaneous change of parameters of a radiating system.[1] Radiation produced as the result of instantaneous variation of the system's dipole moment was considered as an example, with both the magnitude and direction of the dipole moment vector experiencing a sudden discontinuous change. Also considered was radiation occurring as the result

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of instantaneous variation of charged particle velocity. It was assumed in all problems under consideration that the behavior of a parameter causing radiation as the result of variation is as follows: The magnitude of the parameter remains unchanged until a certain time, say $t=0$; at $t=0$, a sudden change occurs in the parameter and the resultant magnitude after a jump remains subsequently unchanged. For example, at $t<0$ a dipole moment $\mathbf{p}$ is assumed to be $\mathbf{p}=\mathbf{p}_1$ and, at $t>0$, $\mathbf{p}=\mathbf{p}_2$, where $\mathbf{p}_1$ and $\mathbf{p}_2$ are time-independent vectors. In the case that the particle velocity $\mathbf{V}$ is assumed to be altered, then at $t<0$, $\mathbf{V}=\mathbf{V}_1$ and at $t>0$, $\mathbf{V}=\mathbf{V}_2$, where $\mathbf{V}_1$ and $\mathbf{V}_2$ are two time-independent vectors.

The concept of an instantaneous change in the parameters of a physical system is an idealization. In real terms, any transition between two states in which the system parameters have steady-state values occurs during a certain finite time interval T. Nevertheless, in many cases of practical interest the transition time T may be considered negligibly short and, therefore, the transition itself instantaneous. Let, for example, a radiating system be traveling as a whole with a velocity $\mathbf{V}$ and the time of variation of a parameter, responsible for radiation, be T. Let a plane wave with a frequency ω be emitted at an angle θ with respect to the direction of the velocity. Then, if the following inequality

$$\omega T \ll \frac{1}{1-(V/c)\cos\theta}, \tag{1.1}$$

is satisfied, the transition may be taken to be instantaneous. Calculations show that in the case of Eq. (1.1) the transition time T does not appear in the formulas which define the radiative field and its spectrum and intensity.

Bolotovskiĭ and coworkers have examined several practical cases in which Eq. (1.1) may be assumed as being satisfied.[1] Equation (1.1) has a simple physical meaning. If a radiating system is at rest (such as is the case, for example, for a dipole whose moment is varying with time), Eq. (1.1) yields the following simple condition

$$\omega T \ll 1, \tag{1.2}$$

which, when satisfied, enables the transition to be regarded as instantaneous. In the physical sense this means that if the period of a radiated wave $2\pi/\omega$ is much greater than the time T, in the course of which the dipole moment undergoes change, details of the transition are unimportant and the transition may be considered instantaneous. The only important quantities are initial and final values of the dipole moment.

If a source is moving, the condition for an instantaneous transition also includes, as can be seen from Eq. (1.1), source velocity and direction of emission. Let a point source traveling with a velocity V generate a plane wave with frequency ω. Let the wave and source propagate in the same direction ($\theta=0$). During the transition time T the source travels a distance VT and the emitted wave, cT. Thus, during the transition time T the phase of the wave at the point where the source is situated changes by an amount

$$\Delta\varphi = k(c-V)T = \omega\left(1-\frac{V}{c}\right)T \quad \left(k=\frac{\omega}{c}\right); \tag{1.3}$$

where we have taken into account the relationship $k=\omega/c$ between the wave factor $\mathbf{k}$ and frequency of the electromagnetic wave ω.

If the emitted wave propagates at an angle θ with the source velocity, the wave phase change $\Delta\varphi$ during the transition time T is

$$\Delta\varphi = \omega\left(1-\frac{V}{c}\cos\theta\right)T = (\omega - (\mathbf{kV}))\,T. \tag{1.4}$$

The condition under which the transition may be considered instantaneous is expressed now as a simple inequality

$$\Delta\varphi = \omega\left(1-\frac{V}{c}\cos\theta\right)T \ll 1. \tag{1.5}$$

The above expression agrees with Eq. (1.1).

Thus far we were considering conditions under which the transition in a radiating system may be taken to be instantaneous. These conditions are satisfied if either Eq. (1.1) or an equivalent Eq. (1.5) holds. It may be concluded from the very form of these inequalities that they cannot be satisfied for sufficiently high frequencies or in a certain range of variations in θ. For example, if radiation in the forward direction ($\theta=0$) is under consideration, the condition cannot be satisfied for the following frequencies

$$\omega \gtrsim \frac{1}{T[1-(V/c)]}, \tag{1.6}$$

and, therefore, the emitted wave amplitude at these frequencies is a function of the system's behavior during the transition period T. In this case the transition clearly can no longer be considered instantaneous. If the radiating system is traveling with a speed close to the speed of light, Eq. (1.6) defines very high frequencies. The radiative field for all the lower frequencies may be determined proceeding from the concept of instantaneous transition.

We shall now consider radiation in the backward direction ($\theta=\pi$). In this case, the factor $1-(V/c)\cos\theta$ is of the order of 1 at all velocities of the radiating system. The condition under which the transition time T must be taken into consideration is

$$\omega \gtrsim \frac{1}{T} \tag{1.7}$$

for all velocities of motion.

The inequalities (1.6) and (1.7) define conditions for which the transition time T must be taken into consideration. Moreover, Eq. (1.6) and Eq. (1.7) refer to forward and backward radiation respectively. It should be noted that both these inequalities may be expressed as follows

$$\Delta\varphi \gtrsim 1, \tag{1.8}$$

where $\Delta\varphi$ is the phase change of the wave emitted during the transition time T at the point where the source is situated.

Evidently, in the case of a relativistic system ($V\approx c$) Eqs. (1.6) and (1.7) differ considerably. There exists a broad frequency range in which the forward radiation may be calculated proceeding from the concept of instantaneous transition, and the backward radiation nearly always depends on the details of transition. This range is defined as follows

$$\frac{1}{T} \leqslant \omega \leqslant \frac{1}{T\,[1-(V/c)]}. \qquad (1.9)$$

In the ultrarelativistic case Eq. (1.6) holds for radiation emitted in the range of angles

$$\Delta\theta \approx \sqrt{1-\frac{V^2}{c^2}}, \qquad (1.10)$$

close to the direction of motion of the radiating system.

When Eqs. (1.6)–(1.8) are satisfied radiation is determined by the behavior of the radiating system during transition. Therefore, to determine the radiative field the nature of the transition must be specified. Below we shall examine several examples selected using the following criteria:

(1) Variation with time of a parameter determining the radiation occurs from an initial steady-state value to a final steady-state value.

(2) The transition occurs during an interval of time which is of the order of T.

(3) The dependence of the parameter on time is expressed by means of a smooth function having continuous derivatives of all orders.

(4) The transition law is selected such that the fields in the case under consideration could be determined exactly.

Requirement (3) is natural if one agrees with the fact that the parameters, themselves, their rates of change, variation in the rate of 'acceleration', etc., are all quantities which should not undergo discontinuous changes.

Requirement (4) is convenient considering the fact that an exact solution written in an analytical form permits by means of asymptotic expansions the evaluation of likely regions in which various approximations ("instantaneous" or "smooth" transition) are valid.

2. RADIATION OF A POINT CHARGED PARTICLE IN THE CASE OF CONTINUOUS VARIATION OF VELOCITY FROM INITIAL $\vec{V}_1$ TO TERMINAL $\vec{V}_2$ VALUES

Let a point charged particle be moving according to the following law. At first (at $t\to-\infty$), its velocity is $\mathbf{V}_1$. Subsequently, the particle velocity changes smoothly from $\mathbf{V}_1$ to $\mathbf{V}_2$, in such manner that the transition between the two values occurs during a certain finite time which is of the order of T.

Proceeding from the considerations made above, we shall specify the law of variation as follows

$$\mathbf{V}(t)=\frac{\mathbf{V}_1+\mathbf{V}_2}{2}+\frac{\mathbf{V}_2-\mathbf{V}_1}{2}\,\mathrm{th}\,\frac{t}{T}. \qquad (2.1)$$

The law of motion [Eq. (2.1)] is characterized by the property that the particle velocity changes from $\mathbf{V}_1$ to $\mathbf{V}_2$ during a time interval of the order of T with the change in velocity being described by a smooth function. These two facts, regardless of the very special form of Eq. (2.1), can be used to make certain general conclusions concerning the emission spectrum of a charged particles subject to the chosen law of motion.

If the velocity is given by Eq. (2.1), dependence of the charge coordinate on time is

$$\mathbf{r}(t)=\frac{\mathbf{V}_1+\mathbf{V}_2}{2}\,t+\frac{\mathbf{V}_2-\mathbf{V}_1}{2}\,T\ln\mathrm{ch}\,\frac{t}{T}. \qquad (2.2)$$

Equation (2.2) was obtained by integrating Eq. (2.1) with respect to time, the integration constant being such that at $t=0$ the moving charge is at the coordinate origin. Moreover, it can be seen from Eq. (2.1) that the charge velocity is equal to the arithmetic mean of the initial $\mathbf{V}_1$ and terminal $\mathbf{V}_2$ velocities.

We shall calculate radiation resulting from the motion of a charged particle according to Eqs. (2.1) and (2.2). This can be done using the known expression for the radiation field of a charged particle at large distances from the region where the radiation is generated (this region lies near the coordinate origin since the particle is accelerated near the coordinate origin).

We shall consider a field with a frequency ω. Then the vector potential of this field at sufficiently large distances from the region, where the emission occurs, has the form of a spherical wave:

$$\mathbf{A}_\omega(\mathbf{r})=\frac{q}{c}\,\frac{e^{ikr}}{r}\,\mathbf{I}; \qquad (2.3)$$

where $\mathbf{A}_\omega(\mathbf{r})$ is the Fourier component of the vector potential $\mathbf{A}(\mathbf{r},t')$ which corresponds to frequency ω.

$$\mathbf{A}_\omega(\mathbf{r})=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{A}(\mathbf{r},t)\,e^{i\omega t}\,dt. \qquad (2.4)$$

The above expression may be inverted as follows

$$\mathbf{A}(\mathbf{r},t)=\int\mathbf{A}_\omega(\mathbf{r})\,e^{-i\omega t}\,d\omega. \qquad (2.5)$$

In Eq. (2.3) q is the charge of a moving particle, c is the speed of light and $k=\omega/c$. The quantity $\mathbf{I}$ is the vector amplitude of the spherical wave of the radiative field. The vector $\mathbf{I}$ is determined by the law of motion of the charged particle. If this law follows the relation $\mathbf{r}=\mathbf{r}(t)$ and the ensuing relation $\mathbf{V}=\mathbf{V}(t)$, the amplitude of the spherical wave $\mathbf{I}$ is expressed as follows:

$$\mathbf{I}=\int_{-\infty}^{\infty}\mathbf{V}(t)\,e^{i(\omega t-\mathbf{k}\mathbf{r}(t))}\,dt; \qquad (2.6)$$

where $\mathbf{k}$ is a vector with the amplitude ω/c and direction from the coordinate origin to the observation point. The direction of the vector $\mathbf{k}$ represents the direction of radiation.

We shall evaluate Eq. (2.6) for $\mathbf{I}$ in the case where the law of motion is expressed by Eqs. (2.1) and (2.2). To accomplish this, we shall substitute in Eq. (2.6) the corresponding expressions for $\mathbf{V}(t)$ and $\mathbf{r}(t)$, and carry out integration with respect to t. Thus, we obtain

$$\mathbf{I}=2^{i[\mathbf{k}(\mathbf{V}_2-\mathbf{V}_1)/2]T-1}\left(\frac{A-B}{A+B}\,\frac{\mathbf{V}_2-\mathbf{V}_1}{2}+\frac{\mathbf{V}_2+\mathbf{V}_1}{2}\right)\frac{\Gamma(iA)\,\Gamma(iB)}{\Gamma(iA+iB)}, \qquad (2.7)$$

where A and B are expressed as follows

$$A=\frac{1}{2}\,T\,(\omega-\mathbf{k}\mathbf{V}_1),\quad B=-\frac{1}{2}\,T\,(\omega-\mathbf{k}\mathbf{V}_2), \qquad (2.8)$$

and Γ is Euler's gamma function.

The quantities A and B, which determine the amplitude of the resultant spherical wave, have a simple physical meaning. Let a point charged particle be moving with velocity $\mathbf{V}$ in the field of a plane electromagnetic wave $e^{i}(\mathbf{kr}-\omega t)$. We shall examine how the wave phase changes at the point where the particle is situated at a given time. If the particle trajectory is described by the equation $\mathbf{r}=\mathbf{V}t$, $\mathbf{r}$ in the index of the exponential function, which defines the wave phase, must be replaced by $\mathbf{V}t$. We then find that the wave phase at the point containing the moving particle increases linearly with time and at a time t it equals $(\mathbf{kV}-\omega)t$. Thus, A and B given by Eq. (2.8) provide the phase advance during the time interval T; moreover, A represents a phase advance (with an accuracy to within the sign and multiplier $\frac{1}{2}$) for a particle moving with velocity $\mathbf{V}_1$, and B a phase advance (with the same provisos with respect to multiplier $\frac{1}{2}$) for a particle moving with velocity $\mathbf{V}_2$.

The meaning of A and B is the same as that of $\Delta\varphi$, a quantity given in Eq. (1.4). Thus, A and B given in Eq. (2.8) are values of phase advance $\Delta\varphi$ for the velocities $\mathbf{V}_1$ and $\mathbf{V}_2$.

The quantity $\mathbf{I}$ [Eq. (2.7)] defines the vector amplitude of spherical radiation wave [Eq. (2.3)] for the case where the velocity of a charged particle changes smoothly from $\mathbf{V}_1$ to $\mathbf{V}_2$ according to the laws in Eq. (2.1) and (2.2). It can be seen from Eq. (2.7) that the amplitude $\mathbf{I}$ depends on the initial and terminal velocities $\mathbf{V}_1$ and $\mathbf{V}_2$; moreover, the latter are a part of the A and B combinations in Eqs. (2.7) and (2.8).

We shall introduce two quantities having the dimension of time

$$t_1=\frac{1}{\omega-\mathbf{kV}_1},\quad t_2=\frac{1}{\omega-\mathbf{kV}_2}. \tag{2.9}$$

The t is in order of magnitude the time during which the wave $\exp[i(\mathbf{kr}-\omega t)]$ gets ahead in phase with respect to the charged particle moving with the velocity $\mathbf{V}_1$, provided the phase advance is of the order of unity. In other words, t_1 is the period of time during which the particle moves in phase with the wave. If the particle velocity is $\mathbf{V}_2$, in-phase motion of the particle and wave occurs during the time t_2. Ter-Mikaelyan calls t_1 and t_2 "effective times."[3] We shall call the same quantities "forming times," bearing in mind the fact that during time, say, t_1 a particle moves in phase with the wave, and the period during which the particle and wave exert effect on each other $t_{1,2}$ has the same sign and, after this period, the particle-wave coherence is distributed (more precisely, the field of the particle moving at a constant velocity $\mathbf{V}_{1,2}$ and the radiative field cease to interfere with each other).

Using the notation in Eq. (2.9) let us rewrite A and B as follows:

$$A=\frac{1}{2}\frac{T}{t_1},\quad B=-\frac{1}{2}\frac{T}{t_2}; \tag{2.10}$$

where T is time of transition from velocity $\mathbf{V}_1$ to $\mathbf{V}_2$, t_1 is the forming time for the initial state, and t_2, the forming time for the final state.

Since A and B determine the amplitude (2.7) of the emitted wave it can be seen that the emission depends on the values of the ratios of the time of transition to the initial and final forming times.

The angular and spectral distribution of radiation is expressed in terms of $\mathbf{I}$ as follows:

$$d\mathscr{E}_\omega=\frac{q^2}{4\pi^2c}\,|\,[\mathbf{k},\mathbf{I}]\,|^2\,d\Omega=W_\omega(\theta,\varphi)\,d\Omega, \tag{2.11}$$

where $d\Omega$ is the element of solid angle and θ and φ are, respectively, the polar and azimuthal angles of a spherical coordinate system which determine the direction of the vector $\mathbf{k}$.

Substitution of Eq. (2.7) into Eq. (2.11) permits us to determine $W_\omega(\theta,\varphi)$ explicitly. In this note, for the sake of simplicity, we shall consider the case $\mathbf{V}_1\|\mathbf{V}_2$. Then the expression for $W_\omega(\theta,\varphi)$ is simplified and has the following form:[4]

$$W_\omega(\theta)=\frac{q^2\omega T\sin^2\theta}{8\pi c^2\cos\theta\,[1-(V_1/c)\cos\theta]\,[1-(V_2/c)\cos\theta]}\times\frac{(V_2-V_1)\,\mathrm{sh}\,\{[(V_2-V_1)/2c]\,\pi\omega T\cos\theta\}}{\mathrm{sh}\,\{(\pi\omega T/2)\,[1-(V_1/c)\cos\theta]\}\,\mathrm{sh}\,\{(\pi\omega T/2)\,[1-(V_2/c)\cos\theta]\}}, \tag{2.12}$$

where θ is the angle between the wave vector $\mathbf{k}$ of emitted wave and the z-axis of a spherical coordinate system which is directed along the charge trajectory. In this case, dependence of W_ω on the azimuthal angle φ vanishes by virtue of the axial symmetry of the problem.

The derivation of Eq. (2.12) involved the use of the following relation:[5]

$$|\,\Gamma(iy)\,|^2=\frac{\pi}{y\,\mathrm{sh}\,\pi y}\qquad(\mathrm{Im}\,y=0). \tag{2.13}$$

We shall now review the asymptotic behavior of the resultant spectrum for the case of large and small values of A and B determined by Eq. (2.8). Let at first $|A|\ll 1$ and $|B|\ll 1$. This situation occurs if the transition time T is considerably smaller than the forming times t_1 and t_2. Moreover, we obtain

$$W_\omega(\theta)=\frac{q^2(V_2-V_1)^2\sin^2\theta}{4\pi^2c^3\,[1-(V_1/c)\cos\theta]^2\,[1-(V_2/c)\cos\theta]^2}\left(1-\frac{\pi^2}{12}\frac{T^2}{t_1t_2}+\ldots\right). \tag{2.14}$$

The quantity in front of the large parentheses in Eq. (2.14) is the radiation intensity in the case of instantaneous variation of the particle velocity from V_1 to V_2.

In the case where the transition time T is much greater than the forming times t_1 and t_2 ($|A|\gg 1$, $|B|\gg 1$), we obtain

$$W_\omega(\theta)=\frac{q^2\omega T\,|\,V_2-V_1\,|\,\sin^2\theta}{4\pi c^2\cos\theta\,[1-(V_1/c)\cos\theta]\,[1-(V_2/c)\cos\theta]}\,e^{-\pi T/\max(t_1,\,t_2)}; \tag{2.15}$$

where $\max(t_1,t_2)$ designates the greater of the two values of t_1 and t_2. The spectrum [Eq. (2.15)] falls off exponentially in the high-frequency region; moreover, the exponential index contains a relation between the transition time and the greater of the two forming times. This is characteristic for relatively smooth particle trajectories (for example, synchrotron radiation spectrum,[2] emission spectrum of a harmonic oscillator with a finite amplitude,[6] etc.). The reason for the exponential decay of the spectrum in the high frequency region is the result of the law of motion in all these cases

$\mathbf{r}=\mathbf{r}(t)$ being a smooth function with continuous derivatives of all orders.

The quantity I [Eq. (2.6)], which defines the radiative field amplitude, is proportional to the Fourier component of current density associated with a moving charge. Moreover, as is known from properties of the Fourier transform, if $\mathbf{r}(t)$ is a function that is continuous, as are all of its derivatives, I falls off at large values of ω faster than any finite power of ω. Thus, the rapid spectral decay with increasing frequency is a common property of radiation for the case of sufficiently smooth trajectories.

In the case where the charge velocity varies not only with respect to magnitude but also direction, the angular and spectral distributions of radiation are also functions of the azimuthal angle φ, since in this case the vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ define a particular plane in space.

An expression for $W_\omega(\theta,\varphi)$ may be obtained from the general Eq. (2.11). The corresponding formula tends to be relatively cumbersome and we do not reproduce it here. We shall simply note that the basic characteristics of the emission spectrum are the same as those for the aforementioned case where the charge velocity was preserved with respect to direction and varied only in magnitude. Namely, the first correction to the spectral and angular energy distributions of radiation in the case of instantaneous change in the charge velocity is proportional to T^2, and the spectrum falls off exponentially at high frequencies as was the situation in the special case considered above.

3. RADIATION IN THE CASE OF CONTINUOUS VARIATION OF THE DIPOLE MOMENT

The variation of the dipole moment of a physical system results in the emission of electromagnetic waves whose characteristics depend on the law which defines the process of variation of the dipole moment in time.

We shall consider a point dipole with the moment

$$\mathbf{p}(\mathbf{r},t)=\mathbf{p}(t)\,\delta(\mathbf{r}). \tag{3.1}$$

For the model function $\mathbf{p}(t)$ we shall assume

$$\mathbf{p}(t)=\frac{\mathbf{p}_1+\mathbf{p}_2}{2}+\frac{\mathbf{p}_2-\mathbf{p}_1}{2}\,\mathrm{th}\,\frac{t}{T} \tag{3.2}$$

(the coordinate origin is coincident with the point in space containing the dipole). According to Eq. (3.2) the dipole moment of a system changes continuously from $\mathbf{p}_1$ (at $t\to-\infty$) to $\mathbf{p}_2$ (at $t\to+\infty$). The characteristic time of a substantial variation in the dipole moment is of the order of T. It can be readily seen that as $T\to 0$, $\mathbf{p}(t)$ changes to

$$\mathbf{p}=\begin{cases}\mathbf{p}_1, & t<0,\\ \mathbf{p}_2, & t>0.\end{cases}$$

The latter case was considered earlier[1] and may be used for a comparison with the results of this work.

The rearrangement of the dipole moment according to Eq. (3.2) is associated with the current density $\mathbf{j}(\mathbf{r},t)$:

$$\mathbf{j}(\mathbf{r},t)=\frac{\partial\mathbf{p}}{\partial t}=\frac{\Delta\mathbf{p}\,\delta(\mathbf{r})}{2T\,\mathrm{ch}^2(t/T)}. \tag{3.3}$$

The Fourier component of $\mathbf{j}_\omega(\mathbf{r})$ is determined from the following formula

$$\mathbf{j}_\omega(\mathbf{r})=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathbf{j}(\mathbf{r},t)\,e^{i\omega t}\,dt=\frac{\Delta\mathbf{p}\,\omega T\delta(\mathbf{r})}{4\,\mathrm{sh}(\pi\omega T/2)} \tag{3.4}$$

[at $T\to 0, \mathbf{j}_\omega(\mathbf{r})\to(\Delta\mathbf{p}/2\Pi)\delta(\mathbf{r})$ which corresponds to Eq. (2.7) in Ref. 1], where $\Delta\mathbf{p}=\mathbf{p}_2-\mathbf{p}_1$ is the total variation of the dipole moment.

As is known, the Fourier component $\mathbf{A}_\omega$ of the vector potential A is related to the Fourier component of the current density as follows:

$$\mathbf{A}_\omega=\frac{1}{c}\int\mathbf{j}_\omega(\mathbf{r}')\frac{\exp(i(\omega/c)\,|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}\,d\mathbf{r}'. \tag{3.5}$$

If $\mathbf{j}_\omega$ from Eq. (3.4) is substituted into Eq. (3.5), we find

$$\mathbf{A}_\omega=\frac{\Delta\mathbf{p}\,\omega T\exp(ikr)}{4cr\,\mathrm{sh}(\pi\omega T/2)},\qquad k=\frac{\omega}{c}. \tag{3.6}$$

If we apply the inverse Fourier transform to Eq. (3.6) we shall obtain an expression for the vector potential:

$$\mathbf{A}(\mathbf{r},t)=\frac{\Delta\mathbf{p}}{2crT\,\mathrm{ch}^2\{[t-r/c)]/T\}}. \tag{3.7}$$

It can be seen from Eq. (3.7) that the field A is basically concentrated in a layer $|r-ct|<cT$ which is $2cT$ thick. When $T\to 0$, the layer's thickness tends to zero and the layer degenerates into a sphere with radius $r=ct$.

In order to study the behavior of the function in Eq. (3.7) at $T\to 0$, we shall consider the course of the function $f_T(x)=\frac{1}{2}T\,\mathrm{ch}^2(x/T)$ as $T\to 0$.

Evidently, $f_T(t)\to 0$ when $x/T\to\pm\infty$ (at $T\to 0$, this requires that $x\neq 0$), while

$$\frac{1}{2T}\int_{-\infty}^{\infty}\frac{dx}{\mathrm{ch}^2(x/T)}=\frac{1}{2}\,\mathrm{th}\,\frac{x}{T}\Big|_{-\infty}^{+\infty}=1,$$

is independent of T.

Thus, it may be assumed that the sequence $f_T(x)$ has the generalized function $\delta(x)$ for its limit. In this case, for a field $\mathbf{A}(\mathbf{r},t)$ we get

$$\mathbf{A}(\mathbf{r},t)=\frac{\Delta\mathbf{p}}{cr}\,\delta\left(t-\frac{r}{c}\right), \tag{3.8}$$

which coincides with the expression (2.10) in Ref. 1 which was calculated for the case of instantaneous change of the dipole moment.

The scalar potential φ of the field under consideration is defined by the charge density

$$\rho=-\mathrm{div}\,\mathbf{p}(\mathbf{r},t) \tag{3.9}$$

in accordance with the wave equation for φ which is obtained in the Lorentz gauge ($\mathrm{div}\,\mathbf{A}=-(1/c)\,\partial\varphi/\partial t$) and has the form

$$\Delta\varphi-\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}=-4\pi(\mathbf{p}(t),\nabla)\,\delta(\mathbf{r}). \tag{3.10}$$

The solution of Eq. (3.10) is

$$\varphi=-\int\frac{(\mathbf{p}[t-(|\mathbf{r}-\mathbf{r}'|/c)],\nabla')\,\delta(\mathbf{r})'}{|\mathbf{r}-\mathbf{r}'|}. \tag{3.11}$$

For $\mathbf{p}(t)$ from Eq. (3.2) we obtain

$$\varphi=\frac{(\Delta p,\ r)}{2r^2cT\ \mathrm{ch}^2\{[t-(r/c)]/T\}}-\left(\mathbf{p}\left(t-\frac{r}{c}\right),\ \nabla\right)\frac{1}{r}. \tag{3.12}$$

It follows from Eq. (3.12) that in the limit when $T\to 0$

$$\varphi\to\frac{(\Delta\mathbf{p},\ \mathbf{r})}{cr^2}\delta\left(1-\frac{r}{c}\right)-\frac{1}{2}\left[1-\mathrm{sgn}\left(t-\frac{r}{c}\right)\right](\mathbf{p}_1,\ \nabla)\frac{1}{r}-\frac{1}{2}\left[1+\mathrm{sgn}\left(t-\frac{r}{c}\right)\right](\mathbf{p}_2,\ \nabla)\frac{1}{r}, \tag{3.13}$$

where $\mathrm{sgn}\,x=x/|x|$.

Equation (3.13) coincides with Eq. (2.13) in Ref. 1.

Expressions for the electric **E** and magnetic **H** fields may be obtained from expressions for the potentials **A** and φ by means of known relations

$$\mathbf{E}=-\mathrm{grad}\,\varphi-\frac{1}{c}\frac{\partial A}{\partial t},\quad \mathbf{H}=\mathrm{rot}\,\mathbf{A}. \tag{3.14}$$

Substitution of Eqs. (3.7) and (3.12) into Eq. (3.14) yields the following

$$\mathbf{E}=-\frac{1}{cr^3}[\mathbf{r},\ [\Delta\mathbf{p},\ \mathbf{r}]]\left[\frac{1}{2rT\ \mathrm{ch}^2\{[t-(r/c)]/T\}}-\frac{\mathrm{sh}\{[t-(r/c)]/T\}}{cT^2\ \mathrm{ch}^3\{[t-(r/c)]/T\}}\right]+\frac{3\{\mathbf{p}[t-(r/c)],\ \mathbf{r}\}\mathbf{r}-\mathbf{p}[t-(r/c)]r^2}{r^5}, \tag{3.15}$$

$$\mathbf{H}=\frac{[\Delta\mathbf{p},\ \mathbf{r}]}{cr^2}\left[\frac{1}{2rT\ \mathrm{ch}^2\{[t-(r/c)]/T\}}-\frac{\mathrm{sh}\{[t-(r/c)]/T\}}{cT^2\ \mathrm{ch}^3\{[t-(r/c)]/T\}}\right]. \tag{3.16}$$

As can be seen from Eqs. (3.15) and (3.16) the energy flow through a sphere with a sufficiently large radius which surrounds a dipole is associated with the spontaneous emission field whose electrical component is represented by the first term in Eq. (3.15) (the second term decreases with increasing value of r as r^{-3} and, therefore, its integral over the spherical surface of a sufficiently large radius tends to zero as r^{-1}).

The spectral density of radiation in the frequency range $(\omega,\omega+d\omega)$ in the direction of unit vector n is

$$dW_{\mathbf{n},\omega}=\frac{|\Delta\mathbf{p}|^2\omega^4T^2}{16c^3\,\mathrm{sh}^2(\pi\omega T/2)}\sin^3\theta\,d\theta\,d\varphi\,d\omega, \tag{3.17}$$

where θ is the angle between the vectors $\Delta\mathbf{p}$ and $\mathbf{n}$, φ is the azimuthal angle in the spherical coordinate system with the origin at the point where the dipole is situated and with the z-axis directed along $\Delta\mathbf{p}$; $d\Omega=\sin\theta d\theta d\varphi$ is the solid angle in the direction of **n**.

It can be seen from Eq. (3.17) that the spectral density of radiation is concentrated primarily in the frequency region $\omega\leqslant 1/T$ and it decreases exponentially beyond the range limits.

In order to determine the total energy emitted in the direction n, we shall integrate Eq. (3.17) over the entire frequency range. As a result of this we obtain

$$dW_{\mathbf{n}}=\frac{12|\Delta\mathbf{p}|^2}{\pi^5c^3T^3}\zeta(4)\sin^3\theta\,d\theta\,d\varphi. \tag{3.18}$$[1)]

When $T\to 0, dW_{\mathbf{n}}$ diverges as T^{-3}. This is natural since in considering the instantaneous jump in **p** Eq. (2.21) was obtained in Ref. 1 for the spectral density:

$$dW_{\mathbf{n},\omega}=\frac{|\Delta\mathbf{p}|^2\omega^2}{4\pi^2c^3}d\omega\,d\Omega.$$

The integral of the above expression diverges as ω_0^3, where ω_0 is the frequency characterizing the upper boundary of the radiation spectrum which in the case of a smooth modification of p(t) is of order ~$1/T$.

4. TRANSITION RADIATION OF A UNIFORMLY MOVING CHARGE AT A DIFFUSE BOUNDARY BETWEEN TWO MEDIA OR IN A CONTINUOUSLY VARIABLE MEDIUM

We shall now consider the radiation which occurs when a moving charge crosses the spatially or temporally diffuse boundary between two media. The properties of radiation of a uniformly moving charge in an inhomogeneous or a variable medium are in many respect similar to the radiation properties of a charge moving irregularly in a homogeneous medium. Actually, when a charge is moving in a medium with the dielectric permittivity ε, radiation results if the parameter $\eta=(V/c)\sqrt{\varepsilon}$ varies along the charge trajectory. Thus, an identical change in the parameter η may be obtained both in a homogeneous steady-state medium ($\varepsilon=$const) by changing the velocity V and in the case of uniform motion of the charge by varying the dielectric permittivity ε. We shall first obtain equations which determine the radiation in the case of a spatial boundary between two media. In this case we shall follow earlier work in which emission at the diffuse boundary was investigated.[7]

Let the permittivity of a medium ε vary in the direction of the z axis. We shall assume that the medium is nonmagnetic ($\mu=1$). We shall consider a charge q moving at a constant velocity **V** along the z axis. In this case the Maxwell equations are as follows:

$$\mathrm{rot}\,\mathbf{H}=\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}+\frac{4\pi}{c}q\mathbf{V}\delta(x)\delta(y)\delta(z-Vt),\quad \mathrm{div}\,\mathbf{H}=0, \tag{4.1}$$
$$\mathrm{rot}\,\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t},\quad \mathrm{div}\,\mathbf{D}=4\pi q\delta(x)\delta(y)\delta(z-Vt).$$

We expand the electric field intensity **E** into the Fourier integral of the following form:

$$\mathbf{E}(\mathbf{r},\ t)=\int\mathbf{E}(\varkappa,\ \omega,\ z)\,e^{i(\varkappa\rho-\omega t)}\,d\varkappa\,d\omega, \tag{4.2}$$

where $\boldsymbol{\rho}$ is a vector perpendicular to the z axis with components x and y. We also carry out a transformation similar to Eq. (4.2) with respect to other field vectors (**D**,**H**), as well as charge and current densities. As a result of this, we shall obtain from Eq. (4.1) an equation for the Fourier component of the magnetic field **H**:

$$\frac{d}{dz}\left(\frac{1}{\varepsilon}\frac{d}{dz}\mathbf{H}\right)+\left(\frac{\omega^2}{c^2}-\frac{\varkappa^2}{\varepsilon}\right)\mathbf{H}=\frac{iq[\mathbf{n},\ \varkappa]}{2\pi^2c\varepsilon}e^{i\omega z/V}, \tag{4.3}$$

where $\mathbf{n}=\mathbf{V}/c$.

If a new function $u(\varkappa,\omega,z)$ is introduced to replace **H** defined as

$$\mathbf{H}=[\mathbf{n},\ \varkappa]\sqrt{\varepsilon}u \tag{4.4}$$

we obtain the following equation for u

[1)]Using the formula

$$\int_0^\infty\frac{x^p\,dx}{\mathrm{sh}^2x}=\frac{\Gamma(p+1)}{2^{p-1}}\zeta(p),$$

where $\zeta(x)$ is the Riemann zeta function.

$$u''+\left[-\sqrt{\varepsilon}\left(\frac{1}{\sqrt{\varepsilon}}\right)''+\frac{\omega^2}{c^2}\varepsilon-\varkappa^2\right]u=\frac{iq}{2\pi^2c\sqrt{\varepsilon}}e^{i\omega z/V}. \qquad (4.5)$$

Let the permittivity vary according to the law of the Epstein transition layer

$$\varepsilon(\omega, z)=1+\frac{\alpha(\omega)e^{az}}{1+e^{az}}, \quad a>0. \qquad (4.6)$$

When $z\to+\infty, \varepsilon=\varepsilon_0=1+\alpha$; when $z\to-\infty, \varepsilon=1$. Thus, Eq. (4.6) describes a diffuse boundary between vacuum and a medium with dielectric permittivity ε_0; moreover, the characteristic width of the diffusion zone is of the order of $1/\alpha$. To simplify Eq. (4.5) we shall assume that the particle is moving at a relativistic velocity. In this radiation is concentrated in a narrow cone around the direction of motion (come aperture angle is $\theta\sim mc^2/E$, where E is the particle energy); moreover, primarily frequencies greater than optical are generated. We may therefore assume that

$$\alpha=-\frac{\omega_p^2}{\omega^2}, \quad |\alpha|\ll 1, \quad \frac{\omega^2\varepsilon}{c^2}-\varkappa^2\approx\frac{\omega^2}{c^2}, \qquad (4.7)$$

where $\omega_p=\sqrt{4\pi Ne^2/m}$ is the plasma frequency.

When the conditions in Eq. (4.7) are satisfied we can neglect the term $\sqrt{\varepsilon}\ (1/\sqrt{\varepsilon})^{11}$ in Eq. (4.5) and, also, set $\sqrt{\varepsilon}$ in the right-hand side of the same equation equal to unity. Finally, Eq. (4.5) becomes in the approximation under consideration

$$u''+\left(\frac{\omega^2}{c^2}\varepsilon-\varkappa^2\right)u=\frac{iq}{2\pi^2c}e^{i\omega z/V}, \qquad (4.8)$$

where ε is defined by Eq. (4.6).

We now introduce a new variable $x=-e^{-\alpha z}$ and a new function $u=-x^{\nu}w(x)$. Thus, Eq. (4.8), without the right-hand side for the function w is as follows:

$$\left.\begin{aligned}&x(1-x)w''+[2\nu+1-(2\nu+1)x]w'-(\nu^2-\mu^2)w=0,\\&\nu=\frac{i\lambda_2}{a}, \quad \mu=\frac{i\lambda_1}{a}, \quad \lambda_1=\sqrt{\frac{\omega^2}{c^2}-\varkappa^2},\\&\lambda_2=\sqrt{\frac{\omega^2\varepsilon_0}{c^2}-\varkappa^2}.\end{aligned}\right\} \qquad (4.9)$$

We now proceed to derive the equation which defines radiation of a charge in a continuously-variable medium.[8] We write down the equation for the electric induction **D**:

$$\Delta\mathbf{D}-\frac{\varepsilon(t)}{c^2}\frac{\partial^2\mathbf{D}}{\partial t^2}=4\pi\left(\operatorname{grad}\rho+\frac{\varepsilon(t)}{c^2}\frac{\partial\mathbf{j}}{\partial t}\right). \qquad (4.10)$$

We expand **D** in a Fourier series

$$\mathbf{D}(\mathbf{r}, t)=\int\mathbf{D}_{\mathbf{k}}(t)e^{i\mathbf{kr}}\,d\mathbf{k}. \qquad (4.11)$$

Then, if we apply to the charge and current densities a transformation which is similar to Eq. (4.11), we obtain from Eq. (4.10) the following equation for $\mathbf{D}_{\mathbf{k}}$

$$\mathbf{D}_{\mathbf{k}}''+\tilde{\varepsilon}(t)\omega_0^2\mathbf{D}_{\mathbf{k}}=i\frac{q}{2\pi^2}[\mathbf{V}(\mathbf{kV})-\tilde{\varepsilon}(t)c^2\mathbf{k}]e^{-i(\mathbf{kV})t}, \qquad (4.12)$$

where $\omega_0=kc$, $\tilde{\varepsilon}(t)=[\varepsilon(t)]^{-1}$.

Since all radiation in a variable medium is determined by the transverse component of the displacement

$$\mathbf{D}_{\mathbf{k}}^{tr}=\mathbf{D}_{\mathbf{k}}-\frac{\mathbf{k}(\mathbf{kD}_{\mathbf{k}})}{k^2},$$

we shall solve the equation for $\mathbf{D}_{\mathbf{k}}$ only. It is as follows:

$$(\mathbf{D}_{\mathbf{k}}^{tr})''+\tilde{\varepsilon}(t)\omega_0^2\mathbf{D}_{\mathbf{k}}^{tr}=\mathbf{B}e^{-i(\mathbf{kV})t}, \qquad (4.13)$$

where

$$\mathbf{B}=\frac{iq}{2\pi^2}(\mathbf{kV})\left(\mathbf{V}-\frac{\mathbf{k}(\mathbf{kV})}{k^2}\right).$$

Let the permittivity $\varepsilon(t)$ vary according to the law governing Epstein's variable layer:

$$\tilde{\varepsilon}(t)=\tilde{\varepsilon}_1+(\tilde{\varepsilon}_2-\tilde{\varepsilon}_1)\frac{e^{t/T}}{1+e^{t/T}}. \qquad (4.14)$$

The permittivity of a medium described by Eq. (4.14) varies smoothly from a value ε_1 as $t\to-\infty$ to ε_2 as $t\to+\infty$; the parameter T characterizes the time during which the permittivity varies from ε_1 to ε_2. We note that the transformation of a plane electromagnetic wave by the Epstein variable layer was considered by Stolyarov.[9]

If we introduce the variable $\xi=-e^t/T$ and make the replacement $\mathbf{D}_{\mathbf{k}}^{tr}=(-\xi)^{\alpha}\mathbf{f}(\xi)$, we obtain for $\mathbf{f}(\xi)$ from Eq. (4.13),

$$(\xi-1)\xi\mathbf{f}''+[-\gamma+(\alpha+\beta+1)\xi]\mathbf{f}'+\alpha\beta\mathbf{f}=T^2\mathbf{B}(-\xi)^{-[i(\mathbf{kV})T+\alpha+1]}(1-\xi). \qquad (4.15)$$

where $a=is\sqrt{\tilde{\varepsilon}_1}$, $b=is\sqrt{\tilde{\varepsilon}_2}$, $\alpha=a+b$, $\beta=a-b$, $\gamma=2a+1$ and $s=\omega_0T$.

Thus, both the transition radiation at the diffuse boundary and the radiation in a continuously-variable medium are described by the hypergeometric Eqs. (4.9) and (4.15). The methods for solving these are generally similar. Therefore, we shall consider in detail the solution of one of these, Eq. (4.15), noting, as we proceed, differences in the solution of Eq. (4.9). Two linearly independent solutions of Eq. (4.15) without the right-hand side, regular in the neighborhood of the singular point $\xi=0$ (note that as $\xi\to0, t\to-\infty$) are described by the following expression:[10]

$$\begin{aligned}&f_1=F(\alpha, \beta, \gamma, \xi),\\&f_2=\xi^{1-\gamma}F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, \xi),\end{aligned} \qquad (4.16)$$

where $F(\alpha,\beta,\gamma,\xi)$ is the hypergeometric function of the argument ξ. It is now convenient to return to Eq. (4.13), since the Wronskian of this equation is a constant quantity that is independent of the time t. Actually, the Wronskian of a linear differential equation is[10]

$$W=\text{const}\cdot e^{a_1(t)t}, \qquad (4.17)$$

where $a_1(t)$ is the coefficient of the first derivative in the equation. Inasmuch as $a_1=0$ in Eq. (4.13), it follows from Eq. (4.17) that its Wronskian is constant. Equation (4.16) yields that Eq. (4.13) without the right-hand side has the following two solutions which are linearly independent:

$$\begin{aligned}&d_1=e^{i\omega_0\sqrt{\tilde{\varepsilon}_1}t}F(\alpha, \beta, \gamma, -e^{t/T}),\\&d_2=e^{-i\omega_0\sqrt{\tilde{\varepsilon}_1}t}F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, -e^{t/T}).\end{aligned} \qquad (4.18)$$

Since $F(\alpha,\beta,\gamma,0)=1$, the solutions d_1 and d_2 in the limit $t\to-\infty$ represent two plane waves propagating toward each other.

Similarly, two linearly independent solutions of Eq. (4.8) without the right-hand side in the limit $z\to-\infty$

represent two plane waves propagating toward each other parallel and antiparallel to the z axis.

We shall solve the inhomogeneous Eq. (4.13) [and Eq. (4.18)] by the method of variation of parameters. This requires calculating the Wronskian of the equation

$$W = d_1 d_2' - d_2 d_1'. \tag{4.19}$$

Since, as was shown above, the Wronskian W_1 is a constant quantity, in order to calculate it one can utilize the asymptotic behavior of solutions of Eq. (4.18) when $t \to -\infty$ ($\lim_{t\to-\infty} F(\alpha,\beta,\gamma,-e^{t/T}) = 1$). In this case

$$W_1 = i2\omega_0\sqrt{\tilde{\varepsilon}_1}. \tag{4.20}$$

We shall now write a general solution of Eq. (4.13):

$$\mathbf{D}_{\mathbf{k}}^{\mathrm{tr}} = -\frac{\mathbf{B}}{W_1} d_1 \int d_2 e^{-i(\mathbf{kV})t}\,dt + \frac{\mathbf{B}}{W_1} d_2 \int d_1 e^{-i(\mathbf{kV})t}\,dt + \mathbf{C}_1 d_1 + \mathbf{C}_2 d_2. \tag{4.21}$$

The integration in Eq. (4.21) is possible if Barnes' integral representations for the hypergeometric function are used[10]

$$F(\alpha,\beta,\gamma,z) = \frac{\Gamma(\gamma)}{2\pi i\Gamma(\alpha)\Gamma(\beta)} \int_{-i\infty}^{i\infty} \frac{\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(-s)}{\Gamma(\gamma+s)}(-z)^s\,ds, \tag{4.22}$$

where the integration path is distorted (if necessary) to place the poles of the functions $\Gamma(\alpha+s)$, $\Gamma(\beta+s)$ and $\Gamma(-s)$ to the right of the integration path. In Eq. (4.22), $\Gamma(x)$ is Euler's gamma function. Since $t < 0$, in order that Jordan's lemma may be used the integration contour must be closed by a circle with an infinitely large radius to the right of the imaginary s axis. Substituting Eq. (4.22) under the integral signs in Eq. (4.21) we shall first carry out elementary integration with respect to t and, subsequently, with respect to s, obtaining

$$\mathbf{D}_{\mathbf{k}}^{\mathrm{tr}} = \frac{\mathbf{B}}{W_1} A e^{-i(\mathbf{kV})t} + \frac{\mathbf{B}}{W_1} M_1 d_1 - \frac{\mathbf{B}}{W_1} M_2 d_2 + \mathbf{C}_1 d_1 + \mathbf{C}_2 d_2, \tag{4.23}$$

where the first term containing the factor $e^{-(\mathbf{k}\cdot\mathbf{V})t}$ is the field of a moving charge which is unrelated to radiation

$$M_1 = \frac{T\Gamma(1-2a)\Gamma(iT(\mathbf{kV}+\omega_0\sqrt{\tilde{\varepsilon}_2}))\Gamma(iT(\mathbf{kV}-\omega_0\sqrt{\tilde{\varepsilon}_2}))\Gamma(-iT(\mathbf{kV}+\omega_0\sqrt{\tilde{\varepsilon}_1}))}{2\Gamma(-\beta)\Gamma(-\alpha)\Gamma(1-2a+i\mathbf{kV}T+i\omega_0\sqrt{\tilde{\varepsilon}_1}T)},$$
$$M_2 = \frac{T\Gamma(1+2a)\Gamma(iT(\mathbf{kV}+\omega_0\sqrt{\tilde{\varepsilon}_2}))\Gamma(iT(\mathbf{kV}-\omega_0\sqrt{\tilde{\varepsilon}_2}))\Gamma(iT(\omega_0\sqrt{\tilde{\varepsilon}_1}-\mathbf{kV}))}{2\Gamma(\alpha)\Gamma(\beta)\Gamma(1+2a+i\mathbf{kV}T-i\omega_0\sqrt{\tilde{\varepsilon}_1}T)}. \tag{4.24}$$

The constants $\mathbf{C}_1$ and $\mathbf{C}_2$ in Eq. (4.23) can be found from the condition of absence of radiation when $t \to -\infty$; we have

$$\mathbf{C}_1 = -\frac{\mathbf{B}}{W_1} M_1, \quad \mathbf{C}_2 = \frac{\mathbf{B}}{W_1} M_2. \tag{4.25}$$

Above, the first difference appears between the solutions of Eq. (4.8) for a diffuse boundary and Eq. (4.13). Specifically, the condition for determining the arbitrary constants in the general solution of Eq. (4.8) is the absence of waves propagating in the direction of the z axis when $z \to -\infty$, and opposite to the direction of the z axis when $z \to +\infty$.

We shall now consider the region $t > 0$. This region corresponds to the neighborhood of the singular point $\xi = \infty$ in Eq. (4.15). In the region $t > 0$ the regular linearly independent solutions of Eq. (4.15) without the right-hand side are the following functions:

$$\delta_1 = e^{i\omega_0\sqrt{\tilde{\varepsilon}_2}t} F(\beta, -\alpha, 1-2b, -e^{-t/T}),$$
$$\delta_2 = e^{-i\omega_0\sqrt{\tilde{\varepsilon}_2}t} F(\alpha, -\beta, 1+2b, -e^{-t/T}). \tag{4.26}$$

The two sets of solutions, d_1 and d_2 and δ_1 and δ_2, are connected by linear relations, since Eq. (4.13), being of the second order, cannot have more than two linearly independent solutions. These relations are as follows:[10]

$$d_1 = a_{11}\delta_1 + a_{12}\delta_2,$$
$$d_2 = a_{21}\delta_1 + a_{22}\delta_2, \tag{4.27}$$

where

$$a_{11} = \frac{\Gamma(1+2a)\Gamma(2b)}{\Gamma(\alpha)\Gamma(1+a+b)}, \quad a_{12} = \frac{\Gamma(1+2a)\Gamma(-2b)}{\Gamma(\beta)\Gamma(1+a-b)},$$
$$a_{21} = \frac{\Gamma(1-2a)\Gamma(2b)}{\Gamma(-\beta)\Gamma(1-a+b)}, \quad a_{22} = \frac{\Gamma(1-2a)\Gamma(-2b)}{\Gamma(-\alpha)\Gamma(1+a-b)}. \tag{4.28}$$

We shall now write the solution of Eq. (4.15) for $t > 0$:

$$\mathbf{D}_{\mathbf{k}}^{\mathrm{tr}} = -\frac{\mathbf{B}}{W_2}\delta_1 \int \delta_2 e^{-i(\mathbf{kV})t}\,dt + \frac{\mathbf{B}}{W_2}\delta_2 \int \delta_1 e^{-i(\mathbf{kV})t}\,dt + \mathbf{A}_1\delta_1 + \mathbf{A}_2\delta_2; \tag{4.29}$$

where the Wronskian $W_2 = \delta_1\delta_2' - \delta_2\delta_1'$ is found in the same way as W_1 above by calculating its asymptotic behavior as $t \to +\infty$. We obtain

$$W_2 = i2\omega_0\sqrt{\tilde{\varepsilon}_2}. \tag{4.30}$$

In Eq. (4.29) $\mathbf{A}_1$ and $\mathbf{A}_2$ are constants, however no longer arbitrary, but related to the constants $\mathbf{C}_1$ and $\mathbf{C}_2$ [see Eq. (4.25)] by definite linear relations. Specifically, from Eq. (4.27), it follows that

$$\mathbf{A}_1 = a_{11}\mathbf{C}_1 + a_{21}\mathbf{C}_2,$$
$$\mathbf{A}_2 = a_{12}\mathbf{C}_1 + a_{22}\mathbf{C}_2. \tag{4.31}$$

Having performed the integrations in Eq. (4.29) in the same way as above, we shall calculate the radiative field $\mathbf{D}_{\mathbf{k}}^{\mathrm{rad}}$ as $t \to \infty$, i.e., approaching the time when the charge will have passed "through" the Epstein layer. Using Eqs. (4.28)–(4.31) we obtain

$$\mathbf{D}_{\mathbf{k}}^{\mathrm{rad}} = \mathbf{a}_+(\mathbf{k})e^{-i\omega_0\sqrt{\tilde{\varepsilon}_2}t} + \mathbf{a}_-(\mathbf{k})e^{+i\omega_0\sqrt{\tilde{\varepsilon}_2}t}, \tag{4.32}$$

where

$$\mathbf{a}_+(\mathbf{k}) = \mathbf{B}T^2\frac{\Gamma(-2b)\Gamma(-iT(\mathbf{kV}+\omega_0\sqrt{\tilde{\varepsilon}_1}))\Gamma(iT(\omega_0\sqrt{\tilde{\varepsilon}_1}-\mathbf{kV}))\Gamma(iT(\mathbf{kV}-\omega_0\sqrt{\tilde{\varepsilon}_2}))}{\Gamma(\beta)\Gamma(-\alpha)\Gamma(1-i(\mathbf{kV})T-i\tilde{\varepsilon}_2^{1/2}\omega_0 T)},$$
$$\mathbf{a}_-(\mathbf{k}) = (\mathbf{a}_+(-\mathbf{k}))^*. \tag{4.33}$$

Similar expressions are obtained for radiation at the diffuse boundary:

$$u_1 = \frac{b}{Wa}\Delta(-\sigma, \nu, -\mu),$$
$$u_2 = \frac{b}{Wa}\Delta(\sigma, \mu, -\nu),$$

where u_1 is the amplitude of the radiation field "forward" as $z \to +\infty$, u_2 is the amplitude of the radiation field "backward" as $z \to -\infty$, $b = iq/2\pi^2 c$, W is the Wronskian of Eq. (4.8), and $\sigma = i\omega/aV$,

$$\Delta(\sigma,\mu,\nu) = \frac{\Gamma(1+2\nu)\Gamma(\sigma-\mu)\Gamma(\mu+\sigma)\Gamma(-\sigma+\nu)}{\Gamma(\nu-\mu)\Gamma(\mu+\nu)\Gamma(1+\nu+\sigma)}. \tag{4.35}$$

We shall now calculate the intensity of the radiation emitted at a frequency $\omega = kc/\sqrt{\varepsilon^2}$ into an element of a solid angle $d\Omega = 2\pi\sin\theta\,d\theta$, where $\cos\theta = (\mathbf{k}\cdot\mathbf{V})kV$ by a charge moving in a continuously variable medium. To obtain the correct expression for the radiation intensity it must be taken into account, as was done in Ref. 11, that in the direction **k** (at angle θ with the z axis) there propagates not only a wave with amplitude $\mathbf{a}_+(\mathbf{k})$, but also one with amplitude $\mathbf{a}_-(-\mathbf{k})$. Using Eq. (4.33) and the known properties of the gamma functions

$$|\Gamma(iy)|^2 = \frac{\pi}{y \,\mathrm{sh}\, \pi y}, \quad |\Gamma(1+iy)\Gamma(1-iy)| = \frac{\pi y}{\mathrm{sh}\, \pi y}, \qquad (4.36)$$

we obtain

$$W_{\theta,\omega}\, d\Omega = \frac{q^2 V^4 \sqrt{\varepsilon_2}\, \omega T(\varepsilon_2 - \varepsilon_1)\cos^2\theta \sin^2\theta}{2\pi c^5 [1-(V^2/c^2)\varepsilon_1 \cos^2\theta][1-(V^2/c^2)\varepsilon_2\cos^2\theta]}$$
$$\times \frac{\mathrm{sh}(\pi\omega_0 T(\sqrt{\tilde{\varepsilon}_1}+\sqrt{\tilde{\varepsilon}_2}))\,\mathrm{sh}(\pi\omega_0 T(\sqrt{\tilde{\varepsilon}_1}-\sqrt{\tilde{\varepsilon}_2}))\,\mathrm{sh}(\pi T(\mathbf{kV}+\omega_0\sqrt{\tilde{\varepsilon}_2}))\, d\Omega}{\mathrm{sh}(2\pi\omega_0 T\sqrt{\tilde{\varepsilon}_2})\,\mathrm{sh}(\pi T(\omega_0\sqrt{\tilde{\varepsilon}_1}+\mathbf{kV}))\,\mathrm{sh}(\pi T(\omega_0\sqrt{\tilde{\varepsilon}_1}-\mathbf{kV}))\,\mathrm{sh}(\pi T(\omega_0\sqrt{\tilde{\varepsilon}_2}-\mathbf{kV}))}. \qquad (4.37)$$

A similar expression, which we shall not derive because of its cumbersomeness is obtained for the radiation intensity at a diffuse boundary.[7]

We shall now consider some specific cases of Eq. (4.37). Let the time T of permittivity variation be sufficiently small, so that the arguments of all hyperbolic sines in Eq. (4.37) are much less than unity. We shall find under these conditions the first two terms of expansion of radiation intensity. The calculations yield

$$W_{\theta,\omega} = W^0_{\theta,\omega}\left[1 - \frac{\pi^2}{3}\omega^2\left(1 - \frac{V}{c}\sqrt{\varepsilon_2}\cos\theta\right)^2 T^2\right], \qquad (4.38)$$

where $W^0_{\theta,\omega}$ is the intensity of radiation in the case of an instantaneous change in the permittivity of the medium:[11]

$$W^0_{\theta,\omega} = \frac{q^2V^4(\varepsilon_1-\varepsilon_2)^2\sin^2\theta\cos^2\theta}{4\pi^2c^5\sqrt{\varepsilon_2}[1-(V^2/c^2)\varepsilon_1\cos^2\theta]^2[1-(V/c)\sqrt{\varepsilon_2}\cos\theta]^2}. \qquad (4.39)$$

As can be seen from Eq. (4.38), the radiation intensity is expanded in terms of the small parameter which corresponds to the ratio of times required respectively for permittivity variation and radiation formation in a medium with permittivity ε_2. The following is the expression for transition radiation forward at a diffuse boundary in the case that the width of the diffuse region is small $z_0 = 1/a$:

$$W_{\theta,\omega} = W^0_{\theta,\omega}\left[1 - \frac{\pi^2}{3}\frac{z_0^2\omega^2}{V^2}\left(1 - \frac{V}{c}\sqrt{1-\varepsilon_0\sin^2\theta}\right)\left(1-\frac{V}{c}\sqrt{\varepsilon_0}\cos\theta\right)\right], \qquad (4.40)$$

where $W^0_{\theta,\omega}$, is transition radiation intensity at a sharp boundary between vacuum and a medium.[12] Upon satisfying the conditions Eq. (4.7) $1-(V/c)\sqrt{1-\varepsilon_0\sin^2\theta} \approx 1-(V/c)\cos\theta$, such that the fundamental correction of radiation intensity at the sharp boundary is proportional to the ratio of z_0^2 to the product of lengths of radiation formation in vacuum and in a medium with permittivity ε_0. We shall now examine high frequency radiation. In this case, the radiation intensities of a charge at a diffuse boundary and in a variable Epstein layer are exponentially small; moreover, the exponential index in the case of a diffuse boundary is

$$-\frac{2\pi z_0\omega}{V}\max\left[\left(1-\frac{V}{c}\cos\theta\right),\ \left(1-\frac{V}{c}\sqrt{\varepsilon_0}\cos\theta\right)\right], \qquad (4.41)$$

i.e., it is proportional to the ratio of the characteristic width of the diffuse zone z_0 to the greater of the forming lengths (in vacuum or a medium with permittivity ε_0). In the case of radiation in the variable layer, the index is

$$-2\pi T\omega\max\left[\left(1-\frac{V}{c}\sqrt{\varepsilon_1}\cos\theta\right),\ \left(1-\frac{V}{c}\sqrt{\varepsilon_2}\cos\theta\right)\right]. \qquad (4.42)$$

Thus, the exponential index [Eq. (4.42)] is proportional to the ratio of the characteristic time of variation of permittivity to the greater of the radiation forming times in media with ε_1 and ε_2.

Equations (4.38) and (4.40) can be used to formulate validity criteria for the approximation of a sharp boundary or a sudden jump. The approximation of a sudden jump in permittivity is admissible for emitted waves whose forming times in a medium with ε_2 are much greater than the characteristic time of variation of permittivity from ε_1 to ε_2. The approximation of a sharp boundary is admissible for emitted waves whose forming lengths in vacuum and in a medium with ε_0 are much greater than the characteristic width of the diffuse zone z_0. Thus, in the case of a relativistic charged particle the spectrum of high frequencies, emitted at small angles for the calculation of which the boundary may be considered sharp (or the change regarded as a sudden jump), broadens substantially in the shortwave direction.

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