The energy problem in Einstein's theory of gravitation (Dedicated to the memory of V. A. Fock)

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The review is devoted to a discussion of the definition and properties of energy in Einstein's theory of gravitation. Asymptotically flat space-time is defined in terms of admissible asymptotically Cartesian coordinates and a corresponding group of coordinate transformations. A Lagrange function is introduced on such a space-time, and a generalized Hamiltonian formulation of the theory of gravitation is constructed in accordance with Dirac's method. The energy is defined as the generator of displacement with respect to the asymptotic time. It is shown that the total energy of the gravitational field and the matter fields with normal energy-momentum tensor is positive and vanishes only in the absence of matter fields and gravitational waves. The proof follows Witten's proof but contains a number of corrections and improvements. Various standard criticisms of the energy concept in general relativity are discussed and shown to be without substance.

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CONTENTS

INTRODUCTION

The energy concept plays a central role in modern theoretical physics. The law of conservation of energy (and also momentum and angular momentum) is a consequence of the homogeneity of time (respectively, the homogeneity and isotropy of space). In this sense, the energy concept is associated with the fundamental structure of space-time. A characteristic property of energy is its positivity, reflecting stability of a physical system.

The traditional method for determining the energy and momentum in relativistic field theory is based on the introduction of the energy–momentum tensor. This tensor is defined as the variation of the action with respect to an external gravitational field. Such a method is not valid in the case when the gravitational field itself is regarded as a dynamical variable, since the resulting tensor vanishes identically by virtue of the equations of motion. As a result, the energy concept in the theory of gravitation requires further discussion.

The problem of determining the fundamental integrals of the motion—the energy, momentum, and angular momentum—arose immediately after the final formulation of the theory of gravitation by Einstein and Hilbert at the end of 1915, and it was essentially solved by Einstein by 1918 (see Ref. 1). His proposal originally evoked many questions and objections from his contemporaries, who included Lorentz, Levi-Civita, Schrödinger, and others. This discussion is well reflected in Pauli's review article of Ref. 34. However, the situation was gradually clarified and a conception formulated that has found its way into the textbooks and monographs (see, for example, Refs. 2–5); this can be formulated as follows.

1. The energy (and also the other fundamental integrals of the motion) of the gravitational field interacting with a system of masses and matter fields can be introduced if space-time is asymptotically flat, i.e., becomes identical with Minkowski space asymptotically at spatial infinity.

2. The energy of the gravitational field is not localized, i.e., a uniquely defined energy density does not exist.

The asymptotic condition 1) replaces the homogeneity of time in ordinary relativistic field theory. It makes it possible to define a dynamic displacement in time as a displacement with respect to an observer far from the gravitating matter, and to associate energy with the displacement. In contrast, in cosmological models there is no natural time displacement and accordingly no energy concept.

The nonlocalizability of the energy of the gravitational field is due to a specific property of the relativity principle in the theory of gravitation. It is not the metric of space-time that is a physical quantity but the class of equivalent metrics differing by an arbitrary coordinate transformation consistent with the asymptotic conditions. The value of the metric at a given point of space-time does not have absolute significance, and the

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theory of gravitation itself is in this sense fundamentally nonlocal.

These matters are discussed in detail and critically in the quoted monographs.

The above two propositions must be augmented by a third:

3. The total energy of the gravitational field and gravitating matter is positive and vanishes only in the absence of matter and gravitational waves, when the metric becomes identical with the flat Minkowski metric.

However, this result has not yet found its place in the monographs. The proof is a difficult problem of mathematical physics, and it has been solved only very recently. A particularly elegant solution had just been found by Witten,[6] and it is given in the main text.

During the 60 years that Einstein's theory of gravitation has existed, Einstein's solution to the energy problem has continued to be doubted. The criticism has crystallized in a number of fixed ideas, which have appeared periodically in the publications of various authors. In a recent series of papers by Logunov *et al.* (see Refs. 7 and 8), this criticism was the stimulus for the construction of a new theory of gravitation. In the main text, we mention some of the main arguments of this criticism, and we show where they are defective.

In the present paper, we review the energy question in Einstein's theory from the point of view of Hamiltonian dynamics. In such an approach, the energy plays the part of a dynamical observable—it is the generator of displacement in time. Together with the energy P_0, the momentum P_k, the angular momentum M_{ik}, and the Lorentz moments M_{0k} form the 10 generators of the Poincaré group, which act on the phase space of the system consisting of the matter fields and the gravitational field. From this point of view, the dynamical group of the theory of gravitation in the case of asymptotically flat space-time does not differ from the dynamical group of any other relativistic dynamical system. Fock[2] particularly insistently emphasized the distinguished role of the Poincaré group in the theory of gravitation.

The history of our approach began with Dirac's studies[9] in 1958–1959, in which he applied to the gravitational field the general theory of dynamical systems described by singular Lagrangians that he had created earlier in Ref. 10. In the sixties, this approach was adopted by many theoreticians, among whom we mention the Arnowitt–Deser–Misner team,[11] Schwinger,[12] DeWitt,[13] and Regge and Teitelboim.[14] Although this method is fundamental, it has not yet found its place in the textbooks and is regarded by many as something exotic rather than a basic method of exposition of gravitational theory. Its importance has been widely recognized only in the field of the quantum theory of gravitation (see, for example, the review of Ref. 15).

I hope that this review will serve to popularize the Hamiltonian approach to the theory of gravitation. The satisfactory solution to the energy problem by means of this approach convincingly illustrates the power of the method and demonstrates once more that the theory of gravitation itself is in need of neither revision nor modification.

We give a brief summary of the review. In Sec. 1, we give the fundamentals of the generalized Hamiltonian dynamics of Dirac for systems defined by means of a singular Lagrangian. In Sec. 2, we introduce the concept of asymptotically flat space-time and discuss its group of transformations. In Sec. 3, the generalized Hamiltonian formulation will be given for the theory of gravitation, and the generators of the Poincaré group, with the energy among them, arise naturally. Finally, in Sec. 4 we discuss briefly the history of the problem of the positivity of the energy and give Witten's proof. Some lengthy calculations are given in Appendices I and II.

We use the usual relativistic notation: $\mu = (0, i)$ is a coordinate index, $a = (0, \alpha)$ a local Lorentz index, and $\eta^{\mu\nu}$ is the metric tensor of Minkowski space with signature $(-+++)$.

1. GENERALIZED HAMILTONIAN FORMULATION

The concept of a generalized Hamiltonian formulation of the dynamics of a mechanical system appeared in Dirac's paper Ref. 10 and lectures Ref. 16 devoted to singular Lagrangians. Dirac himself extended this formulation to field theory and applied it to the theory of gravitation in Refs. 9 and 17.

We clarify the idea first using the example of a mechanical system with a finite number of degrees of freedom. Suppose the system is described by n pairs of variables p_i, q^i, $i = 1, \ldots, n$, and by a further m variables λ^a, $a = 1, \ldots, m$, the Lagrangian having the form

$$l = \sum_i p_i \dot{q}^i - \sum_a \lambda^a \varphi_a(p, q) - h(p, q), \tag{1.1}$$

where φ_a, $a = 1, \ldots, m$, and h are certain functions of p and q. We use the so-called "first-order formalism," in which the derivatives of the independent dynamical variables appear linearly in the Lagrangian. The Lagrangian (1.1) is singular, since the equations of motion do not contain the derivatives of the variables λ^a.

It is natural to call p_i and q^i canonical variables, λ^a Lagrangian multipliers, φ_a constraints, and h the Hamiltonian. By $\{f, g\}$ we denote the ordinary Poisson brackets:

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} \right). \tag{1.2}$$

The equations of motion that follow from the variational principle are

$$\dot{p}_i = -\frac{\partial h}{\partial q^i} - \sum_a \lambda^a \frac{\partial \varphi_a}{\partial q^i}, \quad \dot{q}^i = \frac{\partial h}{\partial p_i} + \sum_a \lambda^a \frac{\partial \varphi_a}{\partial p_i}, \quad \varphi_a = 0. \tag{1.3}$$

We shall say that the Lagrangian (1.1) defines a generalized Hamiltonian formulation if the constraints and the Hamiltonian satisfy the conditions

$$\{\varphi_a, \varphi_b\} = \sum_d c_{ab}^d \varphi_d, \ \{\varphi_a, h\} = \sum_b c_a^b \varphi_b, \tag{1.4}$$

where c_{ab}^d and c_a^b are arbitrary functions of p and q. The condition (1.4) means that the Poisson brackets of the constraints with one another and with the Hamiltonian vanish on the constraint surface $\varphi_a = 0$. It guarantees that this surface remains invariant during the motion for any choice of the time dependence of the Lagrangian multipliers $\lambda^a(t)$. The condition $m < n$ is necessary for (1.4) to hold.

The generalized Hamiltonian formulation reduces to the ordinary one if the constraint equations are solved and the solution substituted in the Lagrangian (1.1). We then obtain the new Lagrangian

$$l^* = \sum_k p_k^* \dot{q}^{*k} - h^*(p^*, q^*), \tag{1.5}$$

where $k = 1, \ldots, n - m$ and the Hamiltonian h^* is equal to h restricted to the constraint surface:

$$h^* = h\,|_{\varphi=0}. \tag{1.6}$$

Indeed, on the constraint surface, the Hamiltonian h does not, by virtue of (1.4), depend on the m variables canonically conjugate to the constraints φ_a; for $\varphi_a = 0$, the same variables disappear from the kinetic form $p_i\dot{q}^i$. In other words, constraints φ_a satisfying the conditions (1.4) decrease the number of variables in the Lagrangian (1.1) by $3m$ and lead to an ordinary Hamiltonian system with $n - m$ degrees of freedom. The function $h^*(p^*, q^*)$ plays the part of the Hamiltonian—the energy—of this system.

In practice, it is difficult to solve a constraint equation, and one must learn to work with the generalized Hamiltonian formulation. A certain experience has by now been gained in this field. For example, in Ref. 18 the present author showed how a mechanical system defined in terms of a generalized Hamiltonian formulation can be quantized. For the energy problem discussed in the present paper, it is important that the numerical values of the physical Hamiltonian $h^*(p^*, q^*)$ are equal to the values of the generalized Hamiltonian $h(p, q)$ on the constraint surface $\varphi_a = 0$. Therefore, to discuss general properties of the energy such as positivity, it is sufficient to know $h(p, q)$ and the constraints and not necessarily to calculate $h^*(p^*, q^*)$.

The transition to field theory is made in the standard manner. The spatial variables x^k are the "numbers" or "labels" of the degrees of freedom, and Σ_i is replaced by $\int d^3x$. We illustrate the generalized Hamiltonian formulation of a field-theory system for the example of electrodynamics, i.e., for a system consisting of the electromagnetic field interacting with a charged field, which we shall not particularize.

To describe the electromagnetic field, we use the potentials $A_\mu(x)$ and the field intensities $F_{\mu\nu}(x)$. The part of the variables q and p is played by $A_k(x)$ and $E_k(x) = F_{0k}(x)$ and the canonical variables φ_α and π^α of the charged field; A_0 is a Lagrangian multiplier, and F_{ik} can be regarded as explicit functions of A_k, $F_{ik} = \partial_i A_k - \partial_k A_i$. The Lagrangian has the form

$$L = \int \Big[E_k \frac{d}{dt} A_k + A_0 (\partial_k E_k + \rho(\varphi, \pi)) - \frac{1}{2}(E_k^2 + G_k^2) + \pi^\alpha \frac{d}{dt} \varphi_\alpha - h_c(\varphi, \pi, A_k) \Big\} d^3x, \tag{1.7}$$

where h_c is the energy density of the charged field in the external electromagnetic field, which enters h through the spatial covariant derivatives $\nabla_k = \partial_k + iA_k$; $\rho(\varphi, \pi)$ is the charge density and $G_i = \varepsilon_{ijk} F_{jk}/2$ is the magnetic field. We have set the coupling constant e equal to 1. The function $h_c(\varphi, \pi, A_k)$ is positive.

It is obvious that (1.7) has the form (1.1) and we must make the identifications

$$h \equiv \int \Big\{ \frac{1}{2}(E_k^2 + G_k^2) + h_c(\varphi, \pi, A_k) \Big\} d^3x, \tag{1.8}$$

$$\varphi(x) \equiv \partial_k E_k + \rho = 0, \tag{1.9}$$

so that x also plays the role of a constraint label. The Lagrangian (1.7) is associated with the Poisson brackets

$$\{E_i(x), A_k(y)\} = \delta_{ik}\delta^{(3)}(x-y), \ \{\pi^\alpha(x), \varphi_\beta(y)\} = \delta_\beta^\alpha \delta^{(3)}(x-y). \tag{1.10}$$

The constraint $\varphi(x)$ has a perspicuous geometrical meaning. For arbitrary function $\Lambda(x)$, we introduce the functional

$$Q(\Lambda) = \int \varphi(x) \Lambda(x)\, d^3x \tag{1.11}$$

and consider the canonical transformation which it generates. We have

$$\delta_\Lambda A_i = \{Q, A_i\} = -\partial_i \Lambda, \quad \delta_\Lambda E_i = 0, \tag{1.12}$$

and $\delta_\Lambda \varphi = \{\varphi, Q\}$ is the infinitesimal phase transformation of the charged field generated by the charge density. Thus, $Q(\Lambda)$ is the generator of a gauge transformation for the electromagnetic field and the charged field. In electrodynamics, the group of such transformations is commutative, and the Hamiltonian is invariant with respect to it. This is expressed in the relations

$$\{\varphi(x), \varphi(y)\} = 0, \quad \{h, \varphi(x)\} = 0, \tag{1.13}$$

which can be readily verified directly. These relations realize the determining property (1.4) of the generalized Hamiltonian formulation.

The equations of motion

$$\dot{A}_k = \{H, A_k\} - \int A_0(y) \{\varphi(y), A_k\}\, d^3y = E_k + \partial_k A_0, \qquad \dot{E}_k = \Delta A_k - \partial_k \partial_l A_l - \frac{\partial h_c}{\partial A_k} \tag{1.14}$$

together with (1.9) give all the nontrivial Maxwell equations, so that the Lagrangian (1.7) does indeed correspond to electrodynamics. In fact, it is simply identical to the manifestly relativistically invariant Lagrangian in the first-order formalism

$$L = (\partial_\mu A_\nu - \partial_\nu A_\mu) F_{\mu\nu} - \frac{1}{2}(F_{\mu\nu})^2 + L_c \tag{1.15}$$

after the magnetic field F_{ik} has been expressed in terms of the vector potential A_k.

In the considered simple example, the constraints can be solved explicitly. The role of the variables q^* and p^* is played by φ_α, π^α and the three-dimensionally transverse components A_k^T and E_k^T of the fields A_k and E_k. The longitudinal component A_k^L of the field A_k is canonically conjugate to the constraint and does not occur in the Hamiltonian h. The longitudinal component E_k^L of the field E_k can be expressed in terms of p^* and q^* by means of the constraint equation. If we set

$$E_k = \partial_k \chi + E_k^{\mathrm{T}}, \quad \partial_k E_k^{\mathrm{T}} = 0, \tag{1.16}$$

then Eq. (1.9) takes the form of the Poisson equation

$$\Delta \chi = -\rho, \tag{1.17}$$

which can be solved explicitly. The contribution $\frac{1}{2}\int (E_k^L)^2 d^3x$ of the longitudinal field to the Hamiltonian gives the instantaneous Coulomb interaction of the charges:

$$h_{0c} = \frac{1}{8\pi}\int \rho(x)\frac{1}{|x-y|}\rho(y)\, d^3x\, d^3y. \tag{1.18}$$

However, it is not necessary to solve the constraint. For example, the positivity of the total energy, which is made up of the energy of the electromagnetic waves, the energy of the waves of the charged field, and the Coulomb energy,

$$h^* = \int \left\{\frac{1}{2}[(E_k^{\mathrm{T}})^2 + G_{kl}^2] + h_c\right\} d^3x + h_{0c}, \tag{1.19}$$

follows from the explicit positivity of the generalized energy (1.8).

We have intentionally gone into such detail for the standard and noncontentious example of electrodynamics to have the possibility, when analyzing the theory of gravitation, to draw a parallel with this more simple case. In the following sections, we shall see that the Hamiltonian formulation of the theory of gravitation differs little from the formulation of electrodynamics that we have just given. The only important difference will be the circumstance that in the case of gravitation the energy in which we are interested is itself the source of the graviational field, whereas in electrodynamics the charge is the source.

2. ASYMPTOTICALLY FLAT SPACE-TIME

There exist several definitions of asymptotically flat space-time, differing in the level of covariance and mathematical rigor. We shall use here the most naive but at the same time perspicuous definition based on the introduction of admissible coordinates. As is customary in geometry, the invariance of this definition is ensured by the introduction of an admissible group of transformations.

Physically, asymptotically flat space-time corresponds to situations when the gravitating masses and matter fields at finite times are effectively concentrated in a finite region of space. Then far from such a region, in the spatial directions, there exists only the Newtonian tail of the graviational field due to all the masses and the energy of all the wave fields, including the energy of gravitational waves. Clearly, to describe such a situation it is sensible to use coordinates that match this picture.

We shall consider the case of a topologically simple space-time whose points can be uniquely parametrized by four coordinates x^μ, $-\infty < x^\mu < \infty$. One sometimes considers the more general case when such coordinates can be introduced only in the asymptotic region. However, as follows from the results of Ref. 32, our restriction is not fundamental.

Let $x^\mu = (x^0, x^i)$ be one such system, $g_{\mu\nu}$ be the pseudo-Euclidean metric with signature $(-+++)$, and $\Gamma^\sigma_{\mu\nu}$ be the components of the connection. These quantities define an asymptotically flat space-time if in the limit $r \to \infty$ and for finite t

$$g_{\mu\nu} = \eta_{\mu\nu} + O\left(\frac{1}{r}\right), \quad \partial_\sigma g_{\mu\nu} = O\left(\frac{1}{r^2}\right), \quad \Gamma^\sigma_{\mu\nu} = O\left(\frac{1}{r^2}\right), \tag{2.1}$$

where

$$r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2, \quad t = x^0, \tag{2.2}$$

and $\eta_{\mu\nu}$ is the metric tensor of flat Minkowski space:

$$\eta_{00} = -1, \quad \eta_{0k} = \eta_{i0} = 0, \quad \eta_{ii} = 1. \tag{2.3}$$

The conditions (2.1) indicate, in particular, that in the limit $r \to \infty$ the coordinates x^i are space-like and Cartesian, and the coordinate x^0 is timelike. The condition on the masses and matter fields which ensures their effective localization in a compact region of space can be formulated as

$$T_{\mu\nu} = O\left(\frac{1}{r^4}\right), \tag{2.4}$$

where $T_{\mu\nu}$ is the corresponding energy–momentum tensor.

The condition (2.1) does not restrict the coordinate transformations

$$x'^\mu = \eta^\mu(x) \tag{2.5}$$

in a finite region; however, at large r the functions $\eta^\mu(x)$ must have the asymptotic behavior

$$\begin{gathered}\eta^\mu(x) = \Lambda^\mu_\nu x^\nu + a^\mu + O\left(\frac{1}{r}\right),\\ \partial_\nu \eta^\mu = \Lambda^\mu_\nu + O\left(\frac{1}{r^2}\right), \quad \partial_\nu\partial_\sigma\eta^\mu = O\left(\frac{1}{r^{2+\alpha}}\right), \quad \alpha > 0,\end{gathered} \tag{2.6}$$

where Λ^μ_ν is the matrix of Lorentz transformations, and a^μ is an arbitrary constant vector. We shall assume that these transformations act on the set of metrics and connections referred to a fixed coordinate system. The corresponding infinitesimally small transformations are given by

$$\begin{gathered}\delta g_{\mu\nu} = -\partial_\mu \varepsilon^\sigma g_{\sigma\nu} - \partial_\nu \varepsilon^\sigma g_{\mu\sigma} - \varepsilon^\sigma \partial_\sigma g_{\mu\nu},\\ \delta\Gamma^\lambda_{\mu\nu} = -\partial_\mu\varepsilon^\sigma\Gamma^\lambda_{\sigma\nu} - \partial_\nu\varepsilon^\sigma\Gamma^\lambda_{\mu\sigma} + \partial_\sigma\varepsilon^\lambda\Gamma^\sigma_{\mu\nu} - \varepsilon^\sigma\partial_\sigma\Gamma^\lambda_{\mu\nu} - \partial_\mu\partial_\nu\varepsilon^\sigma,\end{gathered} \tag{2.7}$$

where ε^μ is a vector field having in the limit $r \to \infty$ the asymptotic behavior (2.6) with infinitesimal $\omega^\mu_\nu = \Lambda^\mu_\nu - \delta^\mu_\nu$ and a^μ.

We denote by G the infinite-dimensional group generated by these transformations. The group G has a normal subgroup G_0 generated by the transformations (2.7) for which $\Lambda^\mu_\nu = \delta^\mu_\nu$ and $a^\mu = 0$, i.e., they are identify transformations as $r \to \infty$. The factor group

$$P = G/G_0 \tag{2.8}$$

is identical with the Poincaré group—the 10-parameter group of displacements and rotations in Minkowski space.

The group G_0 is the gauge group of the theory of gravitation in asymptotically flat space-time. Two metrics that differ by transformations of this group describe the same physical situation, provided, of course, a corresponding transformation of the matter fields is made as well. At the same time, the symmetry group of the Lagrangian, the equations of motion, and the boundary conditions is the group G. This means that the Poincaré group acts nontrivially on the space of interacting matter and gravitational fields. In particular, the time displacement defined by

$$x^i \to x^i, \quad x^0 \to x^0 + a^0 \tag{2.9}$$

makes it possible to define the energy up to transformations in G_0.

Thus, from the point of view of dynamics the theory of gravitation in asymptotically flat space-time does not differ from other relativistic field theories, since the Poincaré group plays the part of the dynamical group in it. In this sense, the expression "general relativity" does not apply to the dynamics but to the definition of the gauge group. This point of view was formulated in Ref. 19 in the terms just introduced. It is close to many parallel formulations of other authors, in particular Fock's.[2]

The two last paragraphs might appear rather peremptory. However, they summarize in a few words the results that will be given in the following two sections.

We now discuss the Lagrange function of the gravitational field. As is already clear from Sec. 1, we shall find it convenient to use the first-order formalism, in which $g_{\mu\nu}$ and $\Gamma^\sigma_{\mu\nu}$ are regarded as independent dynamical variables. The density of the Lagrange function is usually taken to be the scalar density

$$\sqrt{-g}\,R = \sqrt{-g}\,g^{\mu\nu}R_{\mu\nu}(\Gamma), \tag{2.10}$$
$$R_{\mu\nu} = \partial_\sigma\Gamma^\sigma_{\mu\nu} - \partial_\mu\Gamma^\sigma_{\nu\sigma} + \Gamma^\lambda_{\mu\nu}\Gamma^\sigma_{\lambda\sigma} - \Gamma^\sigma_{\mu\lambda}\Gamma^\lambda_{\nu\sigma},$$

or the function

$$L = \sqrt{-g}\,R + \partial_\mu(\sqrt{-g}\,g^{\mu\nu}\Gamma^\sigma_{\nu\sigma} - \sqrt{-g}\,g^{\nu\sigma}\Gamma^\mu_{\nu\sigma}), \tag{2.11}$$

which differs from $\sqrt{-g}R$ by a total divergence and therefore gives the same equations of motion. Adherents of the second variant usually adopt a defensive position, recognizing the fundamental role of $\sqrt{-g}R$ and invoking only the formal convenience of L (for example, the absence in L of the second derivatives of $g_{\mu\nu}$ which occur in the second-order formalism). However, in asymptotically flat space-time L is the only admissible density in the definition of the action

$$S = \int L \, d^3x \, dt, \tag{2.12}$$

where the integration is over the whole of space and a finite time interval. Because of the conditions (2.1), we have in the limit $r \to \infty$

$$\sqrt{-g}\,R = O\left(\frac{1}{r^3}\right), \quad L = O\left(\frac{1}{r^4}\right), \tag{2.13}$$

where the first estimate is correct only if it is assumed in addition to (2.1) that $\partial_\lambda\Gamma^\sigma_{\mu\nu} = O(1/r^3)$. Therefore, it is S and not

$$\tilde{S} = \int \sqrt{-g}\,R \, d^3x \, dt \tag{2.14}$$

which is invariant with respect to the group G. Indeed, it follows from (2.7) that

$$\delta(\sqrt{-g}\,R) = -\partial_\sigma(\varepsilon^\sigma\sqrt{-g}\,R), \tag{2.15}$$
$$\delta L = -\partial_\sigma(\varepsilon^\sigma L + \sqrt{-g}\,g^{\mu\nu}\,\partial_\mu\partial_\nu\varepsilon^\sigma - \sqrt{-g}\,g^{\mu\sigma}\partial_\mu\partial_\nu\varepsilon^\nu).$$

On integration by parts, the integrated terms for δS disappear at spatial infinity on account of (2.13) and (2.6). But in the analogous integral for $\delta\tilde{S}$ they do not in general vanish. All this shows that the correct variational principle for the gravitational field in asymptotically flat space-time must be based on the action S. The use of the Lagrange function density L is sharply criticized in Ref. 20 on account of its noncovariance. However, as we have just shown, the Lagrange function $\int L d^3x$ is invariant with respect to the allowed transformations of the group G. Thus, the objections are without substance, since they do not take into account boundary effects in the noncompact asymptotically flat space.

In the definition of asymptotically flat space-time, one cannot essentially relax the condition of decrease of the remainder terms in (2.1) and (2.6). For example, if instead of (2.1) we use

$$g_{\mu\nu} = \eta_{\mu\nu} + O\left(\frac{1}{r^\alpha}\right), \quad \partial_\sigma g_{\mu\nu} = O\left(\frac{1}{r^{1+\alpha}}\right), \tag{2.16}$$

then for $\alpha \leq \frac{1}{2}$ the action (2.12) becomes meaningless, since the spatial integral diverges. The interesting case of gravitational fields with Newtonian asymptotic behavior belongs to the class (2.1), so that there are no physical reasons for relaxing the conditions (2.1).

In their recent preprint of Ref. 35, Denisov and Logunov use a coordinate transformation with asymptotic behavior

$$\eta^\mu(x) - x^\mu = O\left(\frac{1}{r^{1/2}}\right), \quad \partial_\mu\eta^\nu = \delta^\nu_\mu + O\left(\frac{1}{r^{3/2}}\right), \tag{2.17}$$

which carries the metric from the class (2.1) into the class (2.16) with $\alpha = \frac{1}{2}$. With respect to such transformations, the definition of the energy given below is not invariant. From our point of view, such coordinate transformations are inadmissible. For example, the action (2.12) is not invariant with respect to them. More rigorous arguments must be based on a detailed discussion of the compactification of the manifold corresponding to the asymptotically flat space-time. The coordinate transformations (2.17) are singular on this manifold. However, a detailed discussion goes beyond the framework of the naive but perspicuous definition of admissible coordinates adopted in this review.

Let us conclude with a few words on general covariance. Of course, as in any theory formulated in space-time, arbitrary coordinates can be used in the theory of gravitation. However, the concept of asymptotically flat space-time, which has an objective physical meaning, can be formulated simply and clearly in the coordinates that we have used that satisfy the asymptotic condition (2.1). In particular, the important concept of displacement in time is given by the simple formula (2.9). In some papers,[21,22] criticism has been advanced of the definition of energy in Einstein's theory of gravitation based on the use of homogeneous coordinate transformations which are not Lorentz transformations. It is clear that the paradoxes which then arise are associated with such a violation of the asymptotic conditions.

3. GENERALIZED HAMILTONIAN FORMULATION FOR THE GRAVITATIONAL FIELD

Just as we treated electrodynamics in Sec. 1, we shall now describe the Hamiltonian dynamics for a gravitational field interacting with a matter field, using initial data on the surface $x^0 = 0$, which we shall assume is spacelike. As the variables of the gravitational field, we take $g_{\mu\nu}$ and $\Gamma^{\sigma}_{\mu\nu}$. These data induce on the initial surface a metric g_{ik} (the first quadratic form), which is positive, and the second quadratic form Γ^0_{ik}.

It is convenient to use tensor densities instead of these tensors. Let

$$h^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu} \tag{3.1}$$

be the contravariant tensor density of the four-dimensional metric tensor. We denote

$$q^{ik} = h^{0i}h^{0k} - h^{00}h^{ik}, \quad \Pi_{ik} = \frac{1}{h^{00}}\Gamma^0_{ik}. \tag{3.2}$$

The matrix q^{ik} is the contravariant density of weight 2 of the metric g_{ik}, $q^{il}g_{lk} = \delta^i_k\gamma$, $\gamma = \det\|g_{ik}\|$. The matrix Π_{ik} is a covariant density of weight -1. For details, see Appendix I, in which it is shown in detail that after elimination of the unimportant variables in the action (2.12) the Lagrange function of the gravitational field and the matter fields, which we denote by φ_α, π^α, takes the form

$$L = \int \left(\Pi_{ik} \frac{d}{dt} q^{ik} + \pi^\alpha \frac{d}{dt}\varphi_\alpha - \lambda^0 C_0 - \lambda^k C_k - H \right) d^3x, \tag{3.3}$$

where

$$C_0 = q^{ik}q^{mn}(\Pi_{ik}\Pi_{mn} - \Pi_{im}\Pi_{kn}) + \gamma R_3 - T_{00}, \tag{3.4}$$

$$C_k = 2\nabla_k (q^{il}\Pi_{il}) - 2\nabla_l (q^{il}\Pi_{ik}) - T_{0k}, \tag{3.5}$$

$$H = -C_0 - \partial_i\partial_k q^{ik} \tag{3.6}$$

and

$$\lambda^0 = \frac{1}{h^{00}} + 1, \quad \lambda^k = \frac{h^{0k}}{h^{00}}, \tag{3.7}$$

∇_i is the covariant derivative with respect to the metric g_{ik}, and R_3 is its scalar curvature. Further, T_{00} and T_{0k} are the energy and momentum densities of the matter field, which depend on the canonical variables of the matter field and the three-dimensional metric g_{ik}.

For example, for the massive scalar field

$$T_{00} = \frac{1}{2}(\pi^2 + q^{ik}\partial_i\varphi\partial_k\varphi + \gamma m^2\varphi^2), \tag{3.8}$$

$$T_{0k} = -\pi\partial_k\varphi, \tag{3.9}$$

and it can be assumed that φ is a scalar and π a scalar density of weight 1. We shall assume that the energy–momentum tensor of the matter field satisfies the positivity condition

$$T_{00} > |T_{0k}|, \quad |T_{0k}|^2 = q^{ik}T_{0i}T_{0k}, \tag{3.10}$$

which holds for the example (3.8)–(3.9) and all normal Lagrangians of a matter field.

We note that the scalar curvature contains the second derivatives of the metric tensor linearly, so that $Q = -\gamma R_3 - \partial_i\partial_k q^{ik}$ is a quadratic form of the first derivatives of the metric with respect to the spatial variables:

$$Q = -\frac{1}{2} q^{ik}\partial_i \ln\gamma\partial_k \ln\gamma - \frac{1}{2} q_{ik}\partial_l q^{im}\partial_m q^{kl} - \frac{1}{4} q^{ik}\partial_i q_{lm}\partial_k q^{lm}. \tag{3.11}$$

In particular, $H(x)$ has the asymptotic behavior

$$H(x) = O\left(\frac{1}{r^4}\right), \tag{3.12}$$

and the integral in (3.3) converges.

The Poisson brackets

$$\begin{aligned} &\{\Pi_{ik}(x), q^{lm}(y)\} = \frac{1}{2}(\delta^l_i\delta^m_k + \delta^m_i\delta^l_k)\,\delta^{(3)}(x-y), \\ &\{\pi^\alpha(x), \varphi_\beta(y)\} = \delta^\alpha_\beta\delta^{(3)}(x-y), \end{aligned} \tag{3.13}$$

induced by the Lagrangian (3.3) are consistent with the natural interpretation of the δ function as a (bi)scalar density of weight 1. To express the commutation relations of the constraints C_0 and C_k, it is convenient to introduce functionals of the vector $X^k(x)$ and the scalar density $f(x)$ of weight -1:

$$C(X) = \int C_k(x) X^k(x)\, d^3x, \quad C_0(f) = \int C_0(x) f(x)\, d^3x, \tag{3.14}$$

$$[C(X_1), C(X_2)] = C([X_1, X_2]), \tag{3.15}$$

$$[C(X), C_0(f)] = C_0(Xf), \tag{3.16}$$

$$[C_0(f_1), C_0(f_2)] = C([f_1, f_2]), \tag{3.17}$$

where $[X_1, X_2]$ is the vector field with the components

$$X_1^l\partial_l X_2^k - X_2^l\partial_l X_1^k, \tag{3.18}$$

Xf is the scalar density of weight -1 of the form

$$Xf = X^l\partial_l f - f\partial_l X^l, \tag{3.19}$$

and $[f_1, f_2]$ is the vector field with components

$$q^{ik}(f_1\partial_i f_2 - f_2\partial_i f_1). \tag{3.20}$$

The action of the constraints $C(X)$ and $C_0(f)$ on the canonical variables reveals their geometrical significance: $C(X)$ are the generators of three-dimensional coordinate transformations, and $C_0(f)$ corresponds to the transformation of the first and second quadratic forms of the surface when it is deformed.

The equations of motion that follow from the Lagrangian (3.3) are identical to the Einstein–Hilbert equations. One can therefore say that this Lagrangian gives the generalized Hamiltonian formulation of the theory of gravitation. In particular, the generalized Hamiltonian is given by the expression

$$H = \int H(x)\, d^3x = \int [q^{ik}q^{lm}(\Pi_{il}\Pi_{km} - \Pi_{ik}\Pi_{lm}) + Q + T_{00}]\, d^3x, \tag{3.21}$$

and the numerical values of the energy are equal to the possible values of this functional on the constraint surface:

$$C_k(x) = 0, \quad C_0(x) = 0. \tag{3.22}$$

We emphasize that the Hamiltonian (3.21) has the usual structure for relativistic field theory, i.e., it is a quadratic form in the momenta plus a quadratic form of the first derivatives of the generalized coordinates.

Comparing (3.4) and (3.6), we see that the Hamiltonian density $H(x)$ differs from the constraint $C_0(x)$ by an expression of divergence type. Thus, the numerical values of the energy can be calculated as the limit of an integral over a closed two-dimensional surface S which is "inflated" to infinity:

$$E=\lim_{S\to\infty}\int_{(S)}(-\partial_l q^{lk})\,\mathrm{d}S_k. \tag{3.23}$$

It is such an expression for the energy that is given in monographs on the theory of gravitation; see Refs. 2–5. From our point of view, (3.23) gives only the numerical value of the energy, and the Hamiltonian and the generator of a displacement in time are given by the expression (3.21).

Formula (3.23), which expresses an observable quantity—the energy—in terms of the asymptotic behavior of the field, is not a characteristic feature of the theory of gravitation. In electrodynamics, the constraint equation makes it possible to express the total charge in terms of the asymptotic behavior of the electric field:

$$Q=\int\rho(x)\,\mathrm{d}^3x=\lim_{S\to\infty}\int E_k\,\mathrm{d}S_k. \tag{3.24}$$

Formulas (3.23) and (3.24) are similar in that in them the field sources—the charge in electrodynamics and mass–energy in the theory of gravitation—are expressed in terms of the asymptotic behavior of the field. An important difference, however, is that in electrodynamics the charge has two signs, and the vanishing of Q does not entail vanishing of the field, whereas in the theory of gravitation mass is always positive, and vanishing of the total mass leads to an absence of matter and gravitational field, i.e., to flat space-time. This assertion will be proved formally in the following section.

Unfortunately, in contrast to electrodynamics, the expression (3.21) for the total energy is not manifestly positive. Therefore, the question of the positivity of the energy of the gravitational field cannot be readily solved. We shall consider here the comparatively simple case of weak gravitational waves. The general case of a strong gravitational field interacting with matter will be discussed in the following section.

Thus, suppose

$$q^{ik}=\delta^{ik}+2\chi^{ik}, \tag{3.25}$$

where χ^{ik} and Π_{ik} are small and matter fields are absent. The constraint equations can then be linearized:

$$\partial_i\partial_k\chi^{ik}=0,\quad \partial_i\Pi_{ik}=\partial_k\Pi_{ii}, \tag{3.26}$$

and we do not distinguish subscripts and superscripts, since they are raised and lowered by means of the metric tensor δ^{ik}. The energy (3.21) in the first nonvanishing order is given by the quadratic form

$$H=\int\left(\Pi_{ik}\Pi_{ik}-\Pi_{ii}\Pi_{kk}+\partial_i\chi^{kl}\partial_i\chi^{kl}-2\partial_i\chi^{kl}\partial_k\chi^{il}-\frac{1}{2}\partial_i\chi^{kk}\partial_i\chi^{ll}\right)\mathrm{d}^3x, \tag{3.27}$$

which is still not positive. However, it becomes positive when we take into account the constraint (3.26). Indeed, we use the orthogonal expansions

$$\begin{aligned}\chi_{ik}&=\chi^{\mathrm{T}}_{ik}+\frac{1}{2}(\partial_i\chi_k+\partial_k\chi_i)-\delta_{ik}\partial_l\chi_l+\partial_i\partial_k\chi,\\ \Pi_{ik}&=\Pi^{\mathrm{T}}_{ik}+\frac{1}{2}(\partial_i\Pi_k+\partial_k\Pi_i)-\delta_{ik}\partial_l\Pi_l+\partial_i\partial_k\Pi,\end{aligned} \tag{3.28}$$

where

$$\Pi^{\mathrm{T}}_{ii}=0,\quad \partial_i\Pi^{\mathrm{T}}_{ik}=0,\quad \chi^{\mathrm{T}}_{ii}=0,\quad \partial_i\chi^{\mathrm{T}}_{ik}=0. \tag{3.29}$$

The tensor χ^T_{ik} is parametrized by two functions, so that χ^T_{ik} together with the three functions χ_i and the one function χ parametrize the arbitrary symmetric tensor χ_{ik}. The same applies to Π_{ik}. The constraints (3.26) lead to the equations

$$\nabla^4\chi=0,\quad \nabla^2\Pi_i+3\partial_i\partial_k\Pi_k=0, \tag{3.30}$$

from which, using the boundary conditions, we obtain

$$\chi=0,\quad \Pi_i=0. \tag{3.31}$$

Further, it can be shown that χ_i and Π disappear from the expression (3.27). This is natural, since they are canonically conjugate to the constraints (3.31). As a result, (3.27) is reduced to the following manifestly positive expression

$$H=\int[(\partial_l\chi^{\mathrm{T}}_{ki})^2+(\Pi^{\mathrm{T}}_{ik})^2]\,\mathrm{d}^3x, \tag{3.32}$$

which contains the wave energy of the transverse gravitational waves.

In the presence of matter fields, the total energy also includes the instantaneous Newtonian energy of attraction, which arises when χ and Π_i are eliminated by means of the constraint equations, these being modified by the presence of the components T_{00} and T_{0i} of the energy–momentum tensor of the matter on the right-hand sides of (3.26).

We emphasize that the expression (3.32) is quadratic in the deviation of the metric q^{ik} from flatness. In Ref. 22, it is incorrectly asserted that the energy of gravitational fields vanishes on the basis of the fact that the energy vanishes in the first approximation.

Like the energy, we can introduce the total momentum P_k as the generator of the coordinate displacement

$$x^k\to x^k+a^k,\quad x^0\to x^0, \tag{3.33}$$

so that

$$P_k=\int(-\Pi_{lm}\partial_k q^{lm}+T_{0k})\,\mathrm{d}^3x. \tag{3.34}$$

Note that the integrand of $P_k(x)$ in (3.34) differs by a divergence from the constraint $C_k(x)$. To see this, we take into account the weights of the densities Π_{ik} and q^{ik} and write $C_k(x)$ in the form

$$C_k(x)=2\partial_k(q^{lm}\Pi_{lm})-2\partial_l(q^{lm}\Pi_{mk})+\Pi_{lm}\partial_k q^{lm}-T_{0k}. \tag{3.35}$$

Therefore, the numerical value of the momentum is given by a formula analogous to (3.23):

$$P_k=\lim_{S\to\infty}\int 2(q^{lm}\Pi_{lm}\delta^i_k-q^{im}\Pi_{mk})\,\mathrm{d}S_i, \tag{3.36}$$

and in (3.36) we can replace q^{ik} by its asymptotic value.

The other generators of the Poincaré group can be introduced similarly. Here, we shall not make the corresponding calculations, since they are not important for the discussion of the energy problem. We merely point out that the final expressions can be cast into the customary form for relativistic field theory by using the asymptotically Cartesian coordinates and taking $H(x)$ as T_{00} and $P_k(x)$ as $T_{0k}(x)$.

To conclude this section, we give a brief history of the results and discuss a number of typical objections. The constraints $C_k(x)$ and $C_0(x)$ and their Poisson brackets were first given by Dirac in Refs. 9 and 16. Their derivation from the Hilbert–Palatini Lagrangian and a detailed discussion were given in a series of papers by Arnowitt, Deser, and Misner[11] and Schwinger.[12] The differences between the formulas in the quoted papers are explained by the choice of the weights and variants of the first and second quadratic forms. In the present paper, we follow Schwinger's choice. The clearest proof of the necessity of subtracting the divergence in (3.6) is given in Ref. 14 by Regge and Teitelboim.

The expression (3.23) for the energy is identical in asymptotically flat space-time with the expression for it in terms of the so-called energy–momentum pseudotensor already given in the first papers of Einstein.[1] We shall now consider some typical critical objections to such a definition of the energy of the gravitational field.

a) The expressions (3.21) and (3.23) are not generally covariant.

As a rule, this criticism is leveled at the noncovariant energy–momentum pseudotensor $\tau_{\mu\nu}$, whose component τ_{00} is (in Fock's formulation) equal to the density in the integral (3.21). The objections of Lorentz, Levi-Civita, and Schrödinger mentioned in the Introduction refer precisely to this concept. In the papers of Logunov *et al.*,[23] this criticism is taken to extremes: "In Einstein's theory, the energy–momentum pseudotensors are not physical characteristics of the gravitational field and have no meaning."

I agree that the procedure of introducing the energy–momentum pseudotensor in Einstein's papers and in the fundamental monographs Refs. 2–5 has a formal heuristic nature. (Actually, Fock does not even use such an expression.) Therefore, in the present review I have used the Hamiltonian approach to the definition of the energy as the generator of displacement in time. However, the agreement between the results of the Hamiltonian approach and the approach based on application of the energy–momentum pseudotensor to definition of the total energy in asymptotically flat space-time shows that Einstein's definition of the total energy was correct.

With regard to the critical comment concerning general covariance, the answer to it was already given at the end of Sec. 2. To give a clear physical description of localized masses and fields, it is convenient to use the asymptotically flat coordinates (2.1). In these coordinates, the energy is given by the expression (3.23). To calculate it in arbitrary coordinates (for example, in spherical coordinates, which were first used by Bauer in Ref. 24 in a criticism that was then repeated in Refs. 23, 8 and other papers) it is necessary to make a conversion using the appropriate Lamé coefficients, etc. (see, for example, Ref. 36).

b) The expression (3.23) gives a value identically equal to zero for the energy.

This assertion is formulated in its clearest form in Ref. 23. The proof is based on the following argument: For fields concentrated in a compact spatial region $q^{ik} = \delta^{ik}$ and $\partial_i q^{ik} = 0$ identically outside this region. Therefore, the integral (3.23) gives a vanishing expression for the energy.

The natural and correct way out of this problem is to note that $\partial_i q^{ik}$ has a nonvanishing term $O(1/r^2)$ in the asymptotic behavior as $r \to \infty$, namely, the Newtonian tail. The physically obvious but mathematically nontrivial fact is that a gravitational field that decreases too rapidly in spatial directions is identically flat.

4. PROOF OF POSITIVITY OF THE ENERGY

The property of positivity of the energy has fundamental significance and is associated with stability of the system. In relativistic field theory, the expression for the energy of matter fields deduced from the energy–momentum tensor or from the Hamiltonian formulation is manifestly positive. We have demonstrated this once more in Sec. 1 for the example of electrodynamics. However, the expression (3.21) obtained in Sec. 3 for the energy of the gravitational field is not manifestly positive. Even less can be said about the numerical value of the total energy of the gravitational field and the matter fields given by formula (3.23). The example of a weak field considered in Sec. 3 shows that the proof of positivity must be based on solution of the constraint equations, which in the general case form a complicated nonlinear system of partial differential equations.

The question of the positivity of the energy of the gravitational field was not discussed seriously during the classical period of development of this theory. Its active history has lasted about 20 years. Examples of special strong fields considered by Araki[25] and Brill[26] in 1959 showed that the actual formulation of the problem of positivity of the energy is meaningful. The hypothesis of positivity was supported by the papers of Brill, Deser, and the present author,[27,28] though our variational arguments were far from mathematically rigorous. Throughout the seventies, the positivity problem attracted the attention of many specialists in mathematical physics,[29-31] and it was finally solved by Schoen and Yau.[32,33] Their work is based on complicated mathematical methods, and we cannot present it here. Fortunately, a remarkable paper has recently been written by Witten,[6] and this gives a new and formally simple proof of the positivity. We shall give this proof in a form somewhat different from the original in Ref. 6.[1)]

[1)]In the journal version[37] of Witten's preprint, which was published after the present review had been sent to press, Witten notes that his original arguments contain an error. In preparing this review, we found this error (or rather, two errors that cancelled each other), and it is not contained in our text.

Witten's main result consists of the following assertion: The total energy of the gravitational field and matter fields with positive energy-momentum tensor can be represented as a manifestly positive quadratic form in the solution of an auxiliary linear equation in which the gravitational field plays the part of an external field.

The linear equation used by Witten is a Dirac equation restricted to the initial surface $x^0 = 0$. As is well known, to give expression to the Dirac equation it is necessary to use the orthogonal frame formalism, in which the gravitational field is described by a set of four orthogonal vectors e^μ_a and connection coefficients $\omega_{\mu,ab}$. The local index a (the number of the vector) is raised and lowered by means of the tensor η^{ab} (2.3). The connection with the variables $g_{\mu\nu}$ and $\Gamma^\sigma_{\mu\nu}$ is given by

$$g^{\mu\nu} = e^\mu_a \eta^{ab} e^\nu_b, \quad \omega_{\mu,ab} = e^\nu_a \partial_\mu e_{\nu b} - \Gamma^\sigma_{\mu\nu} e_{a\sigma} e^\nu_b, \tag{4.1}$$

where e^a_μ is the matrix that is the inverse of e^μ_a:

$$e^a_\mu e^\nu_a = \delta^\nu_\mu, \quad e^a_\mu e^\mu_b = \delta^a_b. \tag{4.2}$$

The connection coefficients $\omega_{\mu,ab}$ can be expressed directly in terms of e^μ_a as follows:

$$\omega_{\mu,ab} = \frac{1}{2} e^c_\mu (\Omega_{cab} - \Omega_{abc} - \Omega_{bca}), \tag{4.3}$$

$$\Omega_{cab} = e_{\nu c} (e^\mu_a \partial_\mu e^\nu_b - e^\mu_b \partial_\mu e^\nu_a). \tag{4.4}$$

We shall make the further calculations in a synchronous coordinate system, imposing the conditions

$$e^0_0 = 1, \quad e^{00} = e_{00} = -1, \quad e^\alpha_0 = e^i_0 = 0, \tag{4.5}$$

which are compatible with the asymptotic conditions (2.1).

We define the three-dimensional Dirac operator D by the formula

$$D = e^i_\alpha \gamma^\alpha \nabla_i, \quad \nabla_\mu = \partial_\mu + \frac{1}{8} \omega_{\mu,ab} [\gamma^a, \gamma^b], \tag{4.6}$$

where γ^a, $a = 1, 2, 3, 0$, are the ordinary constant Dirac matrices satisfying

$$\gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab}. \tag{4.7}$$

Consider the solution of the equation

$$D\psi = 0, \tag{4.8}$$

for which the spinor $\psi(x)$ at large r has the asymptotic behavior

$$\psi = \psi_0 + O\left(\frac{1}{r}\right), \quad \partial_\mu \psi = C\left(\frac{1}{r^{1+\alpha}}\right), \quad \alpha > \frac{1}{2}, \tag{4.9}$$

and ψ_0 is a constant spinor. Witten showed that for an asymptotically flat gravitational field satisfying the Einstein-Hilbert equation such a solution exists and is unique (see also Appendix II).

Elementary but lengthy calculations, which we give in Appendix II, lead to the identity

$$\int \sqrt{\gamma} \left[\frac{1}{2} \psi^* (G_{00} + e^k_\alpha G_{0k} \gamma^0 \gamma^\alpha) \psi + g^{ik} (\nabla_i \psi)^* \nabla_k \psi \right] d^3x = \frac{1}{4} (E (\psi_0^* \psi_0) + P_\alpha (\psi_0^* \gamma^0 \gamma^\alpha \psi_0)), \tag{4.10}$$

where $G_{0\mu}$ are components of the Einstein-Hilbert tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{4.11}$$

and E and P_α are the energy and momentum defined by (3.23) and (3.36) in Sec. 3. The positivity of the energy follows directly from this identity.

Indeed, if the Einstein-Hilbert equations are satisfied,

$$G_{\mu\nu} = T_{\mu\nu}, \tag{4.12}$$

then the left-hand side in (4.10) is positive. For the second term this is obvious, and for the first it follows from the fact that the matrix $a_\alpha \gamma^0 \gamma^\alpha$ has the eigenvalues $\pm |a|$, where $|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$, and from the inequality (3.10). Taking now ψ_0 to be an eigenvector of the matrix $P_\alpha \gamma^0 \gamma^\alpha$ with eigenvalue $-|P|$, we obtain from (4.10)

$$E \geqslant |P|, \tag{4.13}$$

so that the vector $P_a = (E, P_\alpha)$ is timelike.

We now show that the energy E can vanish only if the matter fields are absent and the metric $g_{\mu\nu}$ is flat, i.e., there are also no gravitational fields. Indeed, for $E = 0$ it also follows from (4.13) that $P_\alpha = 0$, and then from (4.10) we find that

$$\nabla_i \psi = 0, \tag{4.14}$$

$$\psi^* (T_{00} + e^k_\alpha T_{0k} \gamma^0 \gamma^\alpha) \psi = 0 \tag{4.15}$$

for any solution ψ of Eq. (4.8). A covariantly constant spinor which does not vanish at infinity does not vanish at all x on the surface $x^0 = 0$. Considering different asymptotic values ψ_0 for ψ, we can construct four linearly independent spinors ψ_s, $s = 1, 2, 3, 4$, for which (4.14) is satisfied, and also

$$[\nabla_i, \nabla_k] \psi_s = \frac{1}{8} R_{ik,ab} [\gamma^a, \gamma^b] \psi_s = 0. \tag{4.16}$$

By virtue of the linear independence of the ψ_s, we obtain from this

$$R_{ik\ ab} = 0, \tag{4.17}$$

i.e., the curvature tensor restricted to the initial surface vanishes.

Further, choosing $\psi_0 s$ in such a way that ψ_s for given x are eigenvectors of the matrix $e^k_\alpha T_{0k} \gamma^0 \gamma^\alpha$, we find from (4.15) that $T_{00} = \pm T_{0k}$, whence

$$T_{00} = 0. \tag{4.18}$$

This last equation leads to the vanishing of the matter field by virtue of the positivity of its energy density. Thus, $T_{\mu\nu} = 0$, and from Eqs. (4.12)

$$R_{\mu\nu} = 0. \tag{4.19}$$

It is easy to show that (4.19) in conjunction with (4.17) leads to the equation

$$R_{0k,0\alpha} = 0, \tag{4.20}$$

from which we find that on the initial surface $x^0 = 0$ the total curvature tensor vanishes:

$$R_{\mu\nu,ab} = 0. \tag{4.21}$$

We now note that the energy is an integral of the motion and that, thus, we can repeat our arguments for any surface $x^0 = a^0$. Thus, the curvature tensor vanishes identically in the entire space-time and the metric $g_{\mu\nu}$ is flat. This completes the proof of the positivity of the energy for any nontrivial configuration of the gravitational field and matter fields.

CONCLUSIONS

Using the systematic Hamiltonian approach to Einstein's theory of gravitation, we have shown that in the case of asymptotically flat space-time this theory admits the fundamental integrals of the motion of relativistic theory, including the energy and the momentum. The total energy of the gravitational field and matter fields is positive and vanishes only in the absence of matter and gravitational waves. We have also considered a number of criticisms that have been leveled against this result and have shown that they are without substance.

Thus, the generally accepted theory of gravitation is completely self-consistent and corresponds to the main physical requirements. The positive solution to the energy problem removes all doubts with regard to this question and shows once more that Einstein's theory is the most natural and beautiful variant of the theories of gravitation.

APPENDIX I

Here, we reduce the general Lagrangian (2.11) to the form (3.3). The only specific features of the derivation are associated with the gravitational field, so that for simplicity we shall assume that there is no matter field.

The Lagrangian density

$$L = \Gamma^{\sigma}_{\nu\sigma}\partial_{\mu}h^{\mu\nu} - \Gamma^{\sigma}_{\mu\nu}\partial_{\sigma}h^{\mu\nu} + h^{\mu\nu}(\Gamma^{\rho}_{\mu\nu}\Gamma^{\sigma}_{\rho\sigma} - \Gamma^{\sigma}_{\mu\rho}\Gamma^{\rho}_{\sigma\nu}) \tag{A1}$$

contains the 10 variables $h^{\mu\nu}$ and 40 variables $\Gamma^{\sigma}_{\mu\nu}$. Of the 50 equations of motion, 30 do not contain time derivatives, and we can use them to eliminate the nondynamical variables, in the same way that the magnetic field is eliminated in electrodynamics.

We write these equations in the three-dimensional form

$$\partial_l h^{ik} + \Gamma^{i}_{l\sigma}h^{\sigma k} + \Gamma^{k}_{l\sigma}h^{\sigma i} - h^{ik}\Gamma^{\sigma}_{l\sigma} = 0, \tag{A2}$$

$$\partial_l h^{0k} + \Gamma^{0}_{lm}h^{mk} + \Gamma^{k}_{lm}h^{0m} + \Gamma^{k}_{l0}h^{00} - h^{0k}\Gamma^{m}_{lm} = 0, \tag{A3}$$

$$\partial_l h^{00} + 2\Gamma^{0}_{lm}h^{m0} + \Gamma^{0}_{l0}h^{00} - h^{00}\Gamma^{m}_{lm} = 0. \tag{A4}$$

We denote

$$q^{ik} = h^{0l}h^{0k} - h^{00}h^{ik} \tag{A5}$$

and show that

$$q^{ik} = \gamma\gamma^{ik}, \tag{A6}$$

where γ^{ik} is the contravariant three-dimensional metric corresponding to the restriction g_{ik} of the metric $g_{\mu\nu}$ to the surface $x^0 = 0$, and γ is the metric determinant. Indeed, by definition, $\gamma^{il}g_{lk} = \delta^i_k$, so that

$$\gamma^{ik} = g^{ik} - \frac{g^{0i}g^{0k}}{g^{00}} = \frac{1}{gg^{00}}q^{ik}, \tag{A7}$$

where g is the determinant of the metric $g_{\mu\nu}$. On the other hand, by the definition of the inverse matrix $g^{00} = \gamma g^{-1}$, so that

$$gg^{00} = \gamma. \tag{A8}$$

and (A6) is proved. Combining Eqs. (A2), (A3), and (A4) in an obvious manner, we arrive at the relation

$$\partial_l q^{ik} + \Gamma^{i}_{lm}q^{mk} + \Gamma^{k}_{lm}q^{ml} - 2\Gamma^{m}_{lm}q^{ik} + \Gamma^{0}_{lm}h^{mk}h^{0l} + \Gamma^{0}_{lm}h^{ml}h^{0k} - 2\Gamma^{0}_{lm}h^{ik}h^{0m} = 0. \tag{A9}$$

In the first row, we have an expression whose vanishing is the definition of the Christoffel symbols γ^l_{ik} of the metric g_{ik}. As a result, solving (A9) for Γ^l_{ik}, we obtain

$$\Gamma^{l}_{ik} = \gamma^{l}_{ik} + \frac{h^{0l}}{h^{00}}\Gamma^{0}_{ik}. \tag{A10}$$

Further, from Eqs. (A3) and (A4) we obtain

$$\Gamma^{k}_{i0} = -\frac{1}{h^{00}}(\nabla_i h^{0k} + \Gamma^{0}_{im}h^{mk}), \tag{A11}$$

$$\Gamma^{0}_{i0} = -\frac{1}{h^{00}}(\nabla_i h^{00} + \Gamma^{0}_{im}h^{0m}), \tag{A12}$$

where ∇_i is the covariant derivative with respect to the metric g_{ik}. Here, it is borne in mind that h^{00} and h^{0k} are scalar and vector densities of weight 1, respectively. Indeed, from (A8) we have

$$\sqrt{-g} = \frac{\sqrt{-\gamma}}{g^{00}}, \tag{A13}$$

so that

$$h^{00} = \sqrt{-g}\,g^{00} = -\sqrt{-\gamma g^{00}}, \quad h^{0k} = \sqrt{\gamma}\frac{g^{0k}}{\sqrt{-g^{00}}}, \tag{A14}$$

and g^{00} and g^{0k} are to be regarded as a three-dimensional scalar and three-dimensional vector, respectively.

We now note that Γ^{μ}_{00} occurs in the Lagrangian (A1) linearly, and the corresponding coefficient is a linear combination of the equations (A3) and (A4). Thus, Γ^{μ}_{00} disappear from the Lagrangian once these equations are used.

As a result, if we substitute (A10), (A11), and (A12) in (A1), the new Lagrangian can be expressed in terms of the 10 variables $h^{\mu\nu}$ and the six variables Γ^0_{ik}. Let us make this substitution. We begin with the terms with time derivatives; after substitution of (A10)–(A12) and elementary manipulations we obtain

$$\Pi_{ik}\partial_0 q^{ik} + \partial_l \ln h^{00}\partial_0 h^{0l} - \partial_0 \ln h^{00}\partial_l h^{0l}, \tag{A15}$$

where

$$\Pi_{ik} = \frac{\Gamma^{0}_{ik}}{h^{00}} \tag{A16}$$

is a symmetric tensor density of weight –1.

The substitution of (A10)–(A12) in the remaining part of the Lagrangian involves a more lengthy calculation. The terms quadratic in Π can be collected together into the expression

$$\frac{1}{h^{00}}q^{ik}q^{ml}(\Pi_{kl}\Pi_{im} - \Pi_{ik}\Pi_{ml}). \tag{A17}$$

The terms linear in Π are

$$2\nabla_k\left(\frac{h^{0k}}{h^{00}}\right)g^{il}\Pi_{il}-2\nabla_l\left(\frac{h^{0k}}{h^{00}}\right)q^{il}\Pi_{ik}. \tag{A18}$$

Finally, the terms without Π are

$$\frac{1}{h^{00}}\Big[\partial_l q^{lk}\gamma^l_{ik}-\partial_k q^{kl}\gamma^m_{lm}+q^{ik}(\gamma^m_{kl}\gamma^l_{mi}-\gamma^l_{ik}\gamma^m_{lm}) + \partial_k h^{l0}\partial_l h^{0k}-\partial_l h^{0l}\partial_k h^{0k}+\partial_l h^{00}\partial_k h^{lk} + \partial_l h^{00}\partial_k h^{00}\frac{h^{lk}}{h^{00}}-2h^{0k}\frac{\partial_k h^{00}}{h^{00}}\partial_l h^{0l}\Big]. \tag{A19}$$

We transform the last expression. We have

$$\partial_l q^{ik}\gamma^l_{ik}-\partial_k q^{kl}\gamma^m_{lm}=\partial_l(q^{ik}\gamma^l_{ik}-q^{kl}\gamma^m_{km})+q^{ik}(\partial_i\gamma^m_{km}-\partial_m\gamma^m_{ik}). \tag{A20}$$

The second term on the right-hand side of (A20) can be combined with the second term in the first row of (A19) to make the expression

$$-\frac{1}{h^{00}}q^{ik}R^{(3)}_{ik}=-\frac{1}{h^{00}}\gamma R_3, \tag{A21}$$

and the first term on the right-hand side of (A20) can be written in the form $-(1/h^{00})\partial_i\partial_k q^{ik}$ in accordance with the definition of the Christoffel symbols. After this, it can be combined with the second and third row in (A19) into an expression of the form

$$-\partial_i\left(\frac{1}{h^{00}}\partial_k q^{ik}\right)+\partial_l\left(\frac{h^{0k}}{h^{00}}\right)\partial_k h^{0l}-\partial_l\left(\frac{h^{0l}}{h^{00}}\right)\partial_k h^{0k}. \tag{A22}$$

The last two terms in (A22) are the divergence

$$\partial_l\left(\frac{h^{0k}}{h^{00}}\partial_k h^{0l}-\frac{h^{0l}}{h^{00}}\partial_k h^{0k}\right), \tag{A23}$$

which vanishes after integration over the whole of space, since $h^{0k}\partial_k h^{0l}$ and $h^{0l}\partial_k h^{0k}$ have the asymptotic behavior $O(1/r^3)$ as $r\to\infty$. The first term in (A22) can be rewritten in the form

$$\partial_i\partial_k q^{ik}-\partial_i\left[\left(\frac{1}{h^{00}}+1\right)\partial_k q^{ik}\right], \tag{A24}$$

and the second term again vanishes after integration.

One further vanishing divergence can be separated in (A18) by rewriting it in the form

$$2\frac{h^{0k}}{h^{00}}[\nabla_l(q^{il}\Pi_{ik})-\nabla_k(q^{il}\Pi_{il})] +2\partial_k\left(\frac{h^{0k}}{h^{00}}q^{il}\Pi_{il}-\frac{h^{0l}}{h^{00}}q^{ik}\Pi_{il}\right). \tag{A25}$$

Here, in the second row we have replaced the covariant derivative by the ordinary derivative because the terms in the brackets are vector densities of weight +1. As a result, the second row in (A25) again vanishes after integration.

Finally, the last two terms in (A15) can be written in the form

$$\partial_0(h^{0k}\partial_k\ln h^{00})-\partial_k(h^{0k}\partial_0\ln h^{00}). \tag{A26}$$

The second term is a vanishing divergence, and the first makes an uninteresting contribution to the Lagrange function of the type of a total derivative with respect to the time.

Collecting together the terms which do not vanish after integration and ignoring the derivative with respect to the time, we obtain the final expression for the Lagrange function:

$$L=\int\Big\{\Pi_{ik}\frac{d}{dt}q^{ik}+\frac{1}{h^{00}}[q^{ik}q^{mn}(\Pi_{kl}\Pi_{im}-\Pi_{ik}\Pi_{ml})-\gamma R_3] +2\frac{h^{0k}}{h^{00}}[\nabla_l(q^{il}\Pi_{ik})-\nabla_k(q^{il}\Pi_{il})]+\partial_i\partial_k q^{ik}\Big\}d^3x, \tag{A27}$$

which can be rewritten in the form (3.3) after the identification

$$\lambda_0=\frac{1}{h^{00}}+1,\quad \lambda_k=\frac{h^{0k}}{h^{00}} \tag{A28}$$

and the addition of the Lagrangian of the matter fields.

APPENDIX II

We here derive the identity (4.10). Let

$$D=e^i_\alpha\gamma^\alpha\nabla_i,\quad \nabla_i=\partial_i+\Gamma_i, \tag{A29}$$

be the Dirac operator restricted to the surface $x^0=0$. For the two arbitrary spinors ψ_1 and ψ_2, consider the expression

$$\Phi(\psi_1,\psi_2)=\psi_1^*D^2\psi_2+g^{ik}(\nabla_i\psi_1)^*\nabla_k\psi_2-\frac{1}{\sqrt{\gamma}}\partial_i\left(\sqrt{\gamma}g^{ik}\psi_1^*\nabla_k\psi_2\right). \tag{A30}$$

Using the property of the γ matrices

$$\gamma^a\gamma^b=\eta^{ab}+S^{ab},\quad S^{ab}=\frac{1}{2}(\gamma^a\gamma^b-\gamma^b\gamma^a), \tag{A31}$$

we obtain for Φ the expression

$$\Phi=\psi_1^*A^k\nabla_k\psi_2+\psi_1^*B\psi_2, \tag{A32}$$

where

$$A^k=g^{ik}(\Gamma_i+\Gamma_i^*)+e^i_\alpha e^k_\beta\gamma^\alpha[\Gamma_i,\gamma^\beta]+e^i_\alpha\partial_i e^k_\beta(\delta^{\alpha\beta}+S^{\alpha\beta})-\frac{1}{\sqrt{\gamma}}\partial_i(\sqrt{\gamma}g^{ik}), \tag{A33}$$

$$B=\frac{1}{2}e^i_\alpha e^k_\beta S^{\alpha\beta}\hat{R}_{ik}, \tag{A34}$$

and

$$\hat{R}_{ik}=\partial_i\Gamma_k-\partial_k\Gamma_i+[\Gamma_i,\Gamma_k] \tag{A35}$$

is the spinor curvature tensor restricted to the initial surface.

We show that the matrices A^k drop out. They contain three matrix structures: I, $\gamma^0\gamma^\alpha$, and $S^{\alpha\beta}$. We collect together the coefficients of each of them. From the commutation relation

$$[S^{ab},\gamma^c]=2\eta^{bc}\gamma^a-2\eta^{ac}\gamma^b \tag{A36}$$

and the definition (4.6) of the matrix Γ_i, we obtain

$$[\Gamma_i,\gamma^\beta]=\frac{1}{4}\omega_{i,ab}[S^{ab},\gamma^\beta]=\frac{1}{2}\omega_{i,ab}(\eta^{b\beta}\gamma^a-\eta^{a\beta}\gamma^b)=\omega^{\ \beta}_{i,\ a}\gamma^a, \tag{A37}$$

so that

$$\gamma^\alpha[\Gamma_i,\gamma^\beta]=\omega_i^{\ \alpha\beta}+\omega^{\ \beta}_{i,\ \gamma}S^{\alpha\gamma}+\omega^{\ \beta}_{i,\ 0}\gamma^\alpha\gamma^0. \tag{A38}$$

Further,

$$\Gamma_i+\Gamma_i^*=2\cdot\frac{1}{4}(\omega_{i,\alpha0}\gamma^\alpha\gamma^0+\omega_{i,0\alpha}\gamma^0\gamma^\alpha)=\omega_{i,\alpha0}\gamma^\alpha\gamma^0. \tag{A39}$$

Thus, we must prove the equations

$$e^i_\alpha e^k_\beta\omega^{\ \beta}_{i,\ 0}+g^{ik}\omega_{i,\alpha0}=0, \tag{A40}$$

$$e^i_\alpha e^k_\beta\omega^{\ \beta}_{i,\ \gamma}+e^i_\alpha\partial_i e^k_\gamma=0, \tag{A41}$$

$$e^i_\alpha e^k_\beta\omega_i^{\ \alpha\beta}+e^i_\alpha\partial_i e^{\alpha k}=\frac{1}{e}\partial_i(eg^{ik}), \tag{A42}$$

where we have denoted

$$e=\sqrt{\gamma}=\det\|e_{ak}\|. \tag{A43}$$

Note that in our system of coordinates (4.5) the following symmetry relation holds by virtue of (4.1):

$$\omega_{\alpha\beta0}=\omega_{\beta\alpha0},\quad \omega_{abc}=e^\mu_a\omega_{\mu,bc}, \tag{A44}$$

as a result of which (A40) is also satisfied once we have rewritten it in the form

$$e^{\beta k}\,(\omega_{\alpha 0\beta}+\omega_{\beta\alpha 0})=0. \tag{A45}$$

The relations (A41) and (A42) follow from the definition (4.3), which expresses $\omega_{\mu,ab}$ in terms of e^{μ}_{a}, if we bear in mind that

$$\partial_i e=e^{k}_{\alpha}\partial_i e^{\alpha}_{k}=-e_{k\alpha}\partial_i e^{k\alpha}. \tag{A46}$$

We now transform B. For this, we note that

$$\hat{R}_{ik}=\frac{1}{4}\,R_{ik,ab}S^{ab}, \tag{A47}$$

where $R_{\mu\nu,ab}$ is the total curvature tensor. We use an obvious identity for the γ matrices:

$$[\gamma^{\alpha},\gamma^{\beta}]\,[\gamma^{a},\gamma^{b}]=2\varepsilon^{\alpha\beta ab}\gamma_5+4\eta^{\alpha b}\eta^{\beta a}+2\,(\eta^{\alpha b}\,[\gamma^{\beta},\gamma^{a}]+\eta^{\beta a}\,[\gamma^{\alpha},\gamma^{b}])-(a\leftrightarrow b). \tag{A48}$$

The first term in (A48) gives a vanishing contribution to B by virtue of the Bianchi identity

$$R_{ab,cd}+R_{ac,db}+R_{ad,bc}=0. \tag{A49}$$

The second term in (A48) gives

$$\frac{1}{4}\,\eta^{ab}\eta^{\beta a}e^{i}_{\alpha}e^{k}_{\beta}R_{ik,ab}=-\frac{1}{4}\,R^{ik}_{ik}=-\frac{1}{2}\left(R_{00}-\frac{1}{2}\,g_{00}R\right). \tag{A50}$$

Finally, in the last term it follows from the symmetry

$$R_{ab,cd}=R_{cd,ab} \tag{A51}$$

that the only term to "survive" is

$$\frac{1}{4}\,R_{\alpha\beta,}{}^{\beta}{}_{0}\,[\gamma^{\alpha},\gamma^{0}]=\frac{1}{2}\,R_{\alpha 0}\gamma^{\alpha}\gamma^{0}. \tag{A52}$$

Thus, we obtain the final result

$$B=-\frac{1}{2}\,(G_{00}+e^{k}_{\alpha}G_{0k}\gamma^{0}\gamma^{\alpha}). \tag{A53}$$

We now consider a solution ψ of the Dirac equation

$$D\psi=0 \tag{A54}$$

that has asymptotic behavior at infinity as $r\to\infty$

$$\psi=\psi_0+O\left(\frac{1}{r}\right),\quad \partial\psi=O\left(\frac{1}{r^{1+\alpha}}\right),\quad \alpha>\frac{1}{2}, \tag{A55}$$

where ψ_0 is a constant spinor. The standard methods of scattering theory show that such a solution exists and is unique if Eq. (A54) does not have nontrivial solutions with $\psi_0=0$. We shall show that there are indeed no such solutions if the gravitational field satisfies Eqs. (4.12). We integrate the identity $\Phi(\psi,\psi)=\psi^{*}B\psi$ over the whole space. If $\psi_0=0$, then the integral of the divergence vanishes, and as a result we obtain the equation

$$\int e\left[\frac{1}{2}\,\psi^{*}\,(T_{00}+T_{0k}e^{k}_{\alpha}\gamma^{0}\gamma^{\alpha})\,\psi+(\nabla_i\psi)^{*}\,(\nabla^{i}\psi)\right]\mathrm{d}^3x=0. \tag{A56}$$

As was already shown in Sec. 4, both terms are here negative. Thus, we obtain

$$\nabla_i\psi=0, \tag{A57}$$

from which it follows that ψ vanishes, since $\psi\to 0$ as $r\to\infty$. Thus, a solution of Eq. (A54) with the asymptotic behavior (A55) exists.

To derive the identity (4.10), we again integrate the identity $\Phi(\psi,\psi)=\psi^{*}B\psi$ over the whole space. This immediately leads to the left-hand side of (4.10), and for the complete derivation it remains to transform the surface integral

$$\int\limits_{(S)} e g^{ik}\psi^{*}\nabla_k\psi\cdot\mathrm{d}S_i. \tag{A58}$$

As S, we take a sphere of radius R. It is clear that in the limit $R\to\infty$ only the asymptotic part $O(1/r^2)$ of the integrand contributes to the integral (A58). We show that this asymptotic behavior can be expressed in terms of ψ_0 and the asymptotic behavior of the gravitational field. Indeed, multiplying Eq. (A54) by γ^{β}, we obtain

$$e^{i\beta}\partial_i\psi+e^{i}_{\alpha}S^{\beta\alpha}\partial_i\psi=-\gamma^{\beta}e^{i}_{\alpha}\gamma^{\alpha}\Gamma_i\psi \tag{A59}$$

or, after multiplication by e^{k}_{β},

$$g^{ik}\partial_k\psi=e^{i}_{\beta}e^{k}_{\alpha}S^{\alpha\beta}\partial_k\psi-e^{i}_{\alpha}e^{k}_{\beta}\gamma^{\alpha}\gamma^{\beta}\Gamma_k\psi. \tag{A60}$$

Returning to the integral (A58), we see that the contribution from the first term on the right-hand side of (A60) can, by virtue of the asymptotic behavior (A55) and the asymptotic behavior

$$e^{i}_{\alpha}=\delta^{i}_{\alpha}+O\left(\frac{1}{r}\right), \tag{A61}$$

which can be assumed without loss of generality, be cast in the form

$$\psi_0^{*}\int\limits_{r=R}[\gamma^{\alpha},\gamma^{\beta}]\,\partial_{\beta}\psi\,\mathrm{d}S_{\alpha}+O(R^{-\alpha})=\psi_0^{*}\int\limits_{r<R}[\gamma^{\alpha},\gamma^{\beta}]\,\partial_{\alpha}\partial_{\beta}\psi+O(R^{-\alpha}) \tag{A62}$$

which vanishes as $R\to\infty$. Thus, the entire integral (A58) in the limit $R\to\infty$ takes the form

$$\int\limits_{r=R} e\psi_0^{*}\,(g^{ik}\Gamma_k-e^{i}_{\alpha}e^{k}_{\beta}\gamma^{\alpha}\gamma^{\beta}\Gamma_k)\,\psi_0\,\mathrm{d}S_i=\psi_0^{*}\left(-\int\limits_{r=R} ee^{i}_{\alpha}e^{k}_{\beta}S^{\alpha\beta}\Gamma_k\,\mathrm{d}S_i\right)\psi_0. \tag{A63}$$

We now recall the definition (4.6) and once more use (A48). As a result, the integrand in (A63) is transformed as follows:

$$\frac{1}{4}\,e^{i}_{\alpha}e^{k}_{\beta}\omega_{k,cd}\,S^{\alpha\beta}S^{cd}=\frac{1}{4}\,\varepsilon^{\alpha\beta cd}e^{i}_{\alpha}\omega_{\beta cd}+\frac{1}{2}\,e^{i}_{\alpha}e^{k}_{\beta}\omega_{k,cd}\eta^{\alpha d}\eta^{\beta c}$$
$$+\frac{1}{4}\,e^{i}_{\alpha}e^{k}_{\beta}\omega_{k,cd}\,(\eta^{\alpha d}\,[\gamma^{\beta},\gamma^{c}]+\eta^{\beta c}\,[\gamma^{\alpha},\gamma^{d}]). \tag{A64}$$

The first term on the right-hand side of (A64) vanishes. Indeed, one of the indices c or d must be a time index, and the complete term disappears because of the symmetry of (A44). Further, using (4.1) and (4.3), we can rewrite the coefficient of $\gamma^{\alpha}\gamma^{\beta}$ in the last term on the right-hand side of (A64) in the form

$$\frac{1}{4}\,ee^{i}_{\alpha}\,(\omega_{\beta}{}^{\alpha}{}_{\gamma}\,[\gamma^{\beta},\gamma^{\gamma}]+\omega_{\beta}{}^{\beta}{}_{\gamma}\,[\gamma^{\alpha},\gamma^{\gamma}])=\frac{1}{4}\,\{-ee^{i\alpha}\Gamma_{\beta\gamma\alpha}\,[\gamma^{\beta},\gamma^{\gamma}]+\partial_k\,(ee^{i}_{\alpha}e^{k}_{\beta})\,[\gamma^{\alpha},\gamma^{\beta}]\}. \tag{A65}$$

Here, the first term does not contribute because of the symmetry of $\Gamma_{\beta\gamma\alpha}$ with respect to the first two indices. The integral of the second can be rewritten in the form

$$\int\partial_k\,(ee^{i}_{\alpha}e^{k}_{\beta})\,[\gamma^{\alpha},\gamma^{\beta}]\,\mathrm{d}S_i=\int\partial_i\partial_k\,(ee^{i}_{\alpha}e^{k}_{\beta})\,[\gamma^{\alpha},\gamma^{\beta}]\,\mathrm{d}^3x=0 \tag{A66}$$

using the symmetry.

We collect together the remaining nontrivial contribution to the right-hand side of (A64). From (A42), we obtain

$$\frac{1}{2}\,ee^{i}_{\alpha}e^{k}_{\beta}\omega^{\beta\alpha}_{k}=\frac{1}{2}\,[\partial_k\,(eg^{ik})-ee^{k}_{\alpha}\partial_k e^{i\alpha}]=\frac{1}{4e}\,\partial_k g^{ik}+\frac{e}{2}\left(\frac{1}{2}\,\partial_k g^{ik}-e^{k}_{\alpha}\partial_k e^{i\alpha}\right). \tag{A67}$$

Further, using (4.1) and (4.3) and the definition of the Christoffel symbols, we have

$$\frac{1}{4} e e^i_\alpha e^k_\beta (\omega_{k,0\alpha} [\gamma^\beta, \gamma^0] + \omega_{k,\beta 0} [\gamma^\alpha, \gamma^0]) = \frac{1}{2e} (g^{ik}\Gamma^0_{km} e^m_\alpha - g^{km}\Gamma^0_{km} e^i_\alpha)\, \gamma^\alpha \gamma^0. \tag{A68}$$

In integrating over the asymptotic region, we can assume that $e \approx 1$ and $e^\alpha_i = \delta^\alpha_i$. Therefore, recalling Eqs. (3.23) and (3.36), we find finally that the integral (A58) reduces to

$$\frac{1}{4} (E\psi_0^* \psi_0 + P_\alpha \psi_0^* \gamma^0 \gamma^\alpha \psi_0). \tag{A69}$$

This concludes the transformation of the surface integral (A58), and with it the proof of the identity (4.10).

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