A short guide to modern geometry for physicists

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Some concepts of differential geometry and topology which have applications in the theory of classical gauge fields are explained by means of simple examples. A guide to the mathematical literature is given.

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CONTENTS

The most modern methods of differential geometry and topology are entering into physics. An example of this is provided by the reviews of Refs. 1–4, which are devoted to the application of these branches of mathematics in field theory and in solid-state physics. However, the absence of sufficiently "democratic" books aimed at physicists makes it difficult to read the corresponding literature not to mention the practical application of the new techniques. It is hoped that the present guide will help to partially fill this gap. It should be regarded as a mathematical supplement to the review of Ref. 1. The structure of the guide is as follows:

1) A list of symbols commonly used in the mathematical literature; 2) an index of terms with corresponding references to the literature; 3) four short introductory articles (see Sec. 3), which, albeit self-contained in character, should be read consecutively; 4) a guide to the literature.

We stress again that the articles are intended for physicists and do not pretend to either rigor or completeness. It would be helpful for the reader to acquaint himself with the corresponding mathematical literature by following the bibliographical guide.

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1. LIST OF SYMBOLS

$a \in A$ -The element a belongs to the set A
$\forall a \exists c$-For every a there exists c
$A \approx B$ -The sets A and B are isomorphic
$A \cap B$-The intersection of the sets A and B
$\alpha \wedge \beta$ -The exterior product of the forms α and β
$\mathbf{R}^n$ -The real n-dimensional space
$\mathbf{C}^n$ -The complex n-dimensional space
S^n -The n-dimensional sphere
$\mathbf{R}P^n$ -The n-dimensional real projective space
$\mathbf{C}P^n$ -The n-dimensional complex projective space
$\mathbf{Z}$ -The additive group of integers
$\mathbf{Z}_p$ -The cyclic group of p elements
φ^{-1} -The mapping inverse to the mapping φ

2. INDEX OF MATHEMATICAL TERMS

The index is compiled according to the following principle. Whenever a term is explained in the review of Ref. 1, we indicate the corresponding section of that review. In other cases, we give references to books published in Russian:

Homotopy groups: Ref. 7, p. 573.
Homotopy: Ref. 7, p. 499; Ref. 22, p. 60.
Cohomology groups: Ref. 16; Ref. 15, p. 214 of Russian translation.
The action of a group is free: Ref. 1, Sec. 1.A.
Diffeomorphism: Ref. 7, p. 194; Ref. 22, p. 139.
Compactness: Ref. 22, p. 31; Ref. 24, p. 222.
Differentiable manifold: Ref. 1, Sec. 2.A.1.
Module: Ref. 17, p. 285.
Lift of a vector field: Ref. 1, Sec. 2.E.
Vector field: Ref. 1, Sec. 2.A.1.
Fundamental vector field: Ref. 1, Sec. 2.C.
Exact sequence: Ref. 19, p. 38; Ref. 22, p. 372.
Obstruction: Ref. 21, p. 168; Ref. 20, p. 177.
Projection: Ref. 1, Sec. 1.A.
Exterior product (of forms): Ref. 1, Sec. 2.A.2.
Inner product: Ref. 1, Sec. 2.A.4.
Exterior derivative: Ref. 1, Sec. 2.A.3.
Associated fiber bundle: Ref. 1, Sec. 1.E.
Principle fiber bundle: Ref. 1, Sec. 1.A.
Principal coordinate bundle: Ref. 1, Sec. 1.C.3.
Trivial bundle-direct product: Ref. 1, Sec. 1.D.
Retract: Ref. 20, p. 5; Ref. 22, p. 22.
Cross section: Ref. 1, Sec. 1.C.1.
Postnikov system: Ref. 23, p. 440.
Fiber: Ref. 1, Sec. 1.A.
Sobolev extension: Ref. 24.
Transpose: Ref. 1, Sec. 2.A.5.
Induced form: Ref. 1, Sec. 2.A.2.
Maurer–Cartan form: Ref. 11, p. 91.
Adjoint form: Ref. 1, Sec. 2.A.4.
Homotopic equivalence: Ref. 7, p. 537; Ref. 22, p. 62.

3. BASIC CONCEPTS

a) Differentiable manifolds

The abstract concept of a differentiable manifold is equivalent to the usual definition of an m-dimensional hypersurface in the n-dimensional Euclidean space $\mathbf{R}^n$. This surface can be specified by means of a system of equations

$$f_j(x_1, \ldots, x_n) = 0 \quad (j = m + 1, \ldots, n), \tag{1}$$

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where f_j are infinitely differentiable functions. Manifolds can also be specified parametrically:

$$x_k = x_k(u_1, \ldots, u_m) \quad (k = 1, \ldots, n). \tag{2}$$

The parameters $(u_1, \ldots, u_m)$ may cover not the entire space $\mathbf{R}^n$, but may be, for example, periodic. The form (2) for the description of a differentiable manifold is preferable to the form (1), since it indicates explicitly the required number of parameters.

However, the second method of describing a differentiable manifold, like the first, cannot be regarded as satisfactory. As an example, let us consider the sphere S^2. Its parameters—the spherical coordinates θ and φ—do not enable us to define all the points of the sphere uniquely: in the neighborhood of the poles $(\theta = 0, \pi)$, the angle φ is not defined.

In addition, the two descriptions (1) and (2) have a common defect. They give explicitly an embedding of a differentiable manifold in the Euclidean space $\mathbf{R}^n$. But one and the same differentiable manifold can be embedded in $\mathbf{R}^n$ in different ways; in other words, it can be described by different functions $f_j(1)$ or $x_j(2)$. Therefore, if we want an invariant definition of a differentiable manifold, we must dispense with attempts to describe a differentiable manifold globally by means of a single set of functions $f_j(1)$ or $x_j(2)$.

For a sphere, the way out of the difficulty is well known. The coordinates of any point on the sphere are defined by an atlas consisting of some set of overlapping maps. The number of such maps can be arbitrary, provided that with some overlap they cover the entire sphere. For example, we can confine ourselves to two maps defined by means of stereographic projections:

$$U_1 = \left\{z = \tan\frac{\theta}{2}\, e^{i\varphi},\ \pi - \varepsilon \geqslant \theta \geqslant 0,\ \varepsilon > 0,\ 0 \leqslant \varphi < 2\pi\right\},$$
$$U_2 = \left\{w = \cot\frac{\theta}{2}\, e^{i\varphi},\ \pi \geqslant \theta \geqslant \varepsilon,\ \varepsilon > 0,\ 0 \leqslant \varphi < 2\pi\right\}.$$

Direct verification shows that the transition from one map to another, i.e., the conversion of the z coordinates to the w coordinates and vice versa, is accomplished by means of infinitely differentiable functions. In other words, the maps are mutually compatible. Each map establishes a one-to-one correspondence between some region of the sphere and a region of the plane $\mathbf{R}^2$. We note that this description of a sphere by means of an atlas is the only correct one in a number of physical problems, for example, in the study of a monopole (see Ref. 26).

Generalizing this construction, we can give the abstract definition of a differentiable manifold. Suppose that we are given a set M. It is a differentiable manifold if the following conditions are satisfied: 1) for some subjects $U_\alpha \subset M$ we are given a one-to-one mapping φ_α into a region of the m-dimensional Euclidean space $\mathbf{R}^m$, i.e., we are given a set of *maps* $(\{U_\alpha\})$; 2) any set of maps $\{U_\alpha\}$ should form an *atlas*. By this we mean: a) each point $x \in M$ belongs to at least one map U_α; b) overlapping maps U_α and U_β are *compatible*, i.e., mappings $\varphi_\alpha\varphi_\beta^{-1}$ and $\varphi_\beta\varphi_\beta^{-1}$ of regions of the space $\mathbf{R}^m$ must be one-to-one and infinitely differentiable.

Thus, the atlas $\{U_\alpha, \varphi_\alpha\}$ specifies the structure of the differentiable manifold in the original set M. Locally, the differentiable manifold has the structure of the sphere $\sum_{j=1}^m z_j^2 < 1$ of the Euclidean space $\mathbf{R}^m$. The number m is called the dimension of the differentiable manifold. We stress that in general any differentiable manifold requires for its description more than one map in the atlas. If a differentiable manifold is topologically equivalent to a Euclidean space, one map is sufficient. For example, the upper sheet of the two-sheeted hyperboloid $H^2 = \{x_0^2 - x_1^2 - x_2^2 = 1,\ x_0 > 0\}$ can be mapped in a one-to-one manner onto the plane $\mathbf{R}^2 = \{x_1, x_2\}$, thereby giving an atlas containing only one map, i.e., H^2 and $\mathbf{R}^2$ are topologically equivalent. In the general case, one map is insufficient, i.e., it is insufficient to specify a single mapping (2).

Local coordinates make it possible to employ the usual concepts of analysis on a differentiable manifold: to consider functions on a differentiable manifold, to construct tangent vectors, and so forth. In particular, the maps $\{U_\alpha\}$ occur in the definition of fiber bundles over a differentiable manifold.

Let f be a mapping of one differentiable manifold onto another $(f: M \to M_1)$, If this mapping defines a smooth deformation of the differentiable manifold, it is called a *diffeomorphism*.

More precisely, let the point x belong to the map $U_\alpha \subset M$ and its image $f(x)$ belong to the map $V_\alpha \subset M_1$. If φ_α is a mapping of the map U_α into $\mathbf{R}^n$, and h_α is a mapping of the map V_α into $\mathbf{R}^m$, then the mapping f is a diffeomorphism if compositions of mappings of $\mathbf{R}^m$ into $\mathbf{R}^m$ of the form $h_\alpha f \varphi_\alpha^{-1}$ and $\varphi_\alpha f^{-1} h_\alpha^{-1}$ are infinitely differentiable. Thus, for example, we have an obvious diffeomorphism of the cylinder $\{x_1^2 + x_2^2 = 1, x_0 \text{ arbitrary}\}$ and the single-sheeted hyperboloid $\{x_0^2 - x_1^2 - x_2^2 = -1\}$.

Nontrivial examples of differentiable manifolds are provided by Lie groups. Thus, the group SU(2)—the set of complex matrices of the form $\left(\begin{smallmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{smallmatrix}\right) = g$ with determinant 1—is diffeomorphic to the sphere S^3. In fact, the condition $\det g = |\alpha|^2 + |\beta|^2$ defines the sphere S^3 in the four-dimensional space $\mathbf{R}^4$.

By means of a stereographic projection, it is possible to construct an atlas of the sphere S^3 in the same way that this was done for the two-dimensional sphere S^2. It is well known that the group SU(2) covers the group SO(3) twice. Let us see what the group SO(3) represents as a differentiable manifold. Every rotation $g \in \mathrm{SO}(3)$ can be specified by a vector directed along the axis of rotation, the length of the vector being equal to the angle of rotation. These vectors fill a sphere of radius π, the antipodal points on the surface of the sphere being identified, since rotations through angles π and $-\pi$ are identical. The resulting manifold is diffeomorphic to the three-dimensional projective space $\mathbf{R}P^3$. The points of $\mathbf{R}P^3$ are the lines of the space $\mathbf{R}^4$ passing through the origin. We shall not prove the diffeomorphism of the manifolds SO(3) and $\mathbf{R}P^3$, but shall indicate the structure of an atlas for $\mathbf{R}P^3$.

It follows from the definition of $\mathbf{R}P^3$ that its points can be specified by so-called homogeneous coordinates

(x_1, x_2, x_3, x_4), where the numbers x_j are determined with accuracy up to a common factor. We form an atlas of four maps

$$U_\alpha = \{(x_1, x_2, x_3, x_4), \ |x_\alpha| \geqslant \varepsilon > 0\},$$
$$\varphi_\alpha: \quad U_\alpha \to \mathbb{R}^3 (y_1, y_2, y_3) \quad (\alpha = 1, \ldots, 4),$$

where the functions φ_α are defined as follows:

$$\varphi_\alpha: \ y_\beta^{(\alpha)} = \frac{x_\beta}{x_\alpha} \quad \alpha \neq \beta.$$

An arbitrary real projective space $\mathbf{R}P^n$ is defined as the set of lines passing through the origin in the space $\mathbf{R}^{n+1}$. It can be shown that $\mathbf{R}P^n$ is a differentiable manifold of dimension n. We shall indicate another realization of $\mathbf{R}P^n$. We construct the unit sphere S^n in the space $\mathbf{R}^{n+1}$. Then each line in $\mathbf{R}^{n+1}$ corresponds to two antipodal points on the sphere S. Thus, the sphere S^n with identified antipodal points is another model of the space $\mathbf{R}P^n$. Similarly, the set of complex lines in the space $\mathbb{C}^{n+1}$ forms a complex projective space, which is a differentiable manifold of dimension $2n$. Points on the unit sphere S^{2n+1} in the space $\mathbb{C}^{n+1}$ which differ by a phase factor $\exp(i\varphi)$ correspond to a single complex line.

b) Equivalence classes

Various types of sets can be divided into nonoverlapping classes according to certain criteria. For example, the plane can be represented as the set of concentric circles with radii varying from zero to infinity, and the set of vector potentials can be represented as the totality of classes of gauge-equivalent potentials. In the first case, points in the plane fall into a single class if they are equidistant from a fixed center. In the second case, equivalent potentials are related by gauge transformations.

Another example of a partition into equivalence classes is the partition of continuous mappings into classes of homotopically equivalent mappings (see Sec. 3d on homotopy).

It is easy to see that in the first two examples an equivalence relation is established by means of the action of a group of transformations. In the first case, points of the plane fall into a single class if there exists a rotation which carries one point into the other, or, in other words, the points belong to a single *orbit* of the group of rotations. In the second case, equivalent potentials lie on a single orbit of the group of gauge transformations.

This construction also holds in the general situation. Suppose that the group G acts as a group of transformations of the manifold P. Then P is partitioned into a set of equivalence classes P/G with respect to the action of the group G. Each equivalence class is an orbit O of the group G. The set of equivalence classes P/G is called the factor space, and the correspondence π relating to each point $x \in P$ containing its orbit $O \in P/G$ $(\pi: P \to P/G)$ is called the projection.

We shall give two examples.

1) Let P be the sphere S^n, and let $G = \mathbf{Z}_2$ be the group which takes the point x into $-x$. Then each orbit O of the group $\mathbf{Z}_2$ consists of two points, and the factor space $S^n/\mathbf{Z}_2$ is the real projective space $\mathbf{R}P^n$.

2) If P is the sphere S^{2n+1} in the space $\mathbb{C}^{n+1}: \{|z_0|^2 + |z_1|^2 + |z_n|^2 = 1\}$ and G is the group U(1), which takes points of the sphere into themselves, $z_k \to \exp(i\varphi) z_k$, then $S^{2n+1}/\mathrm{U}(1) = \mathbb{C}P^n$.

An important special case of this construction occurs when the original space P is itself a group. Suppose, for example, that P is the group SO(3) and that $G = \mathrm{SO}(2)$ and suppose that the action of G on P is multiplication to the right by elements $g \in \mathrm{SO}(2)$. Thus, the factor space P/G is the set of elements of the form $\{pG\}$. Elements p_1 and p_2 belonging to $P = \mathrm{SO}(3)$ fall in a single class if $p_1^{-1} p_2 \in \mathrm{SO}(2)$. We shall show that P/G is the two-dimensional sphere S^2. Let x_0 be a fixed point of the sphere. Then any other point x_1 of the sphere can be obtained by means of a rotation from the point x_0: $x_1 = p x_0$. The subgroup G of rotations around the axis passing through x_0 and the origin is SO(2). Thus, if $g \in \mathrm{SO}(2)$, then $x_0 = g x_0$. Clearly, the point x_1 also satisfies the representation $x_1 = p_1 x_0$, where $p_1 = pg$ and g is an arbitrary element of SO(2). Therefore the set of points of the sphere S^2 coincides with the set $\{pG\}$, i.e., with the factor space P/G.

The sphere is a *homogeneous space* with respect to the action of the group SO(3). This means that all its points are related by a rotation from the group SO(3). The subgroup SO(2) is the *stationary* (little) *subgroup* of the point x_0. In the general case, a partition of a group P into equivalence classes with respect to the action of a subgroup G leads to a homogeneous space P/G with the stationary subgroup G. Other examples of homogeneous spaces are the spheres $S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$, the upper sheet of the hyperboloid $H = \{x_0^2 - x_1^2 - x_2^2 - x_3^2 = 1, x_0 > 0\} = \mathrm{SO}(3,1)/\mathrm{SO}(3)$, where SO(3) preserves the point $(1, 0, 0, 0)$, and the spheres $S^{2n+1} = \mathrm{SU}(n+1)/\mathrm{SU}(n)$ of odd dimension. It follows from the last example that the complex space $\mathbb{C}P^n$ can be regarded as the homogeneous space $\mathrm{SU}(n+1)/\mathrm{U}(1) \otimes \mathrm{SU}(n)$.

c) Exact sequences

The formalism of exact sequences is convenient for the calculation of homotopy groups (see the next subsection).

Let f be a homomorphism of the group X_1 into the group X_2, i.e., a mapping such that $f(ab) = f(a) f(b)$ and $f(e_1) = e_2$, where e_1 (e_2) is the unit element of the group X_1 (X_2). The set of elements of X_1 which are mapped into e_2 is called the *kernel* of the homomorphism f and is denoted by $\mathrm{Ker} f$. The image of the group X_1 in the group X_2 under the homomorphism f is denoted by $\mathrm{Im} f (\mathrm{Im} f = f(X_1))$. Thus, $\mathrm{Ker} f \subset X_1$ and $\mathrm{Im} f \subset X_2$.

Suppose, for example, that X_1 and X_2 are two vector spaces, regarded as Abelian groups, and let f be a matrix which maps X_1 into X_2. The matrix f determines a homomorphous (mapping) of the Abelian group X_1 into the Abelian group X_2. Then $\mathrm{Ker} f$ is the set of solutions of the homogeneous equation $fx = 0$, and $\mathrm{Im} f$ consists of the vectors of X_2 which admit the representation $y = f \cdot x$ (the space X_2 itself may be larger and

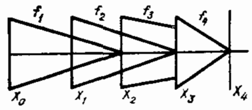

FIG. 1. *The unit elements of the groups X_j are plotted along the horizontal axis.*

may contain other vectors). We note that a nontrivial kernel occurs when $\det f = 0$.

Consider a sequence of groups and homomorphisms $f_j\colon X_{j-1} \to X_j$. It is said to be *exact* in the term X_j if

$$\operatorname{Im} f_j = \operatorname{Ker} f_{j+1}.$$

A graphical illustration of a sequence which is exact in the terms X_1 and X_2 but not in X_3 is given in Fig. 1.

We shall consider some interesting special cases.

1) Suppose that we have the exact sequence

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} e.$$

Then by definition $\operatorname{Im} f_1 = \operatorname{Ker} f_2$. But $\operatorname{Ker} f_2 = X_1$. Therefore $\operatorname{Im} f_1 = X_1$, i.e., the entire group X_1 is the image of the group X_0.

2) We shall show that exactness of the sequence

$$e \xrightarrow{f_1} X_1 \xrightarrow{f_2} X \xrightarrow{f_3} e$$

leads to the isomorphism $X_1 \simeq X_2$. Indeed, $\operatorname{Ker} f_3 = X_2$, and exactness in the term X_2 leads to the equality $\operatorname{Im} f_2 = X_2$. Exactness in the term X_1 implies the equality $\operatorname{Ker} f_2 = e$. Thus, the homomorphism f_2 maps the group X_1 onto the entire group X_2 with a trivial kernel, i.e., the groups X_1 and X_2 are isomorphic.

3) Consider the exact sequence of groups

$$e \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} X_3 \xrightarrow{f_4} e.$$

Since $\operatorname{Ker} f_2 = e$, the image $\operatorname{Im} f_2$ is isomorphic to X_1 ($\operatorname{im} f_2 \approx X_1$). But $\operatorname{Ker} f_3 = \operatorname{Im} f_2$. Therefore $\operatorname{Ker} f_3 \simeq X_1$. On the other hand, $\operatorname{Im} f_3 = X_3$. Consequently, if we consider in the group X_2 the set of orbits (see Sec. 3b on equivalence classes) of its subgroup $\operatorname{Im} f_2 \simeq X_1$, the resulting factor space $X_2/\operatorname{Im} f_2$ is isomorphic to X_3, since the orbit passing through the unit element is mapped onto the unit element of the group X_3.

We shall give examples which illustrate the last case.

1) Consider the exact sequence of Abelian groups

$$0 \xrightarrow{f_1} \mathbb{Z} \xrightarrow{f_2} \mathbb{R} \xrightarrow{f_3} S^1 \xrightarrow{f_4} 1.$$

Here f_1 is the embedding of zero into the group of integers, f_2 is the embedding of the group of integers into the group of real numbers, f_3 is the exponential mapping $f_3(t) = \exp(2\pi i t)$, and f_4 is the mapping into unity. Thus, the circle S^1 is obtained with the factorization $S^1 = \mathbf{R}/\mathbf{Z}$, and the mapping f_3 is the projection π (see Sec. 3b on equivalence classes).

2) There exists an exact sequence

$$e_1 \xrightarrow{f_1} \mathbb{Z}_2 \xrightarrow{f_2} \mathrm{SU}(2) \xrightarrow{f_3} \mathrm{SO}(3) \xrightarrow{f_4} e_2,$$

where

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbb{Z}_2 = \{e_1, -e_1\},$$

$$e_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathrm{SU}(2) = \left\{ \begin{matrix} \alpha & \beta \\ -\bar{\beta} & \alpha \end{matrix} \right\}.$$

The homomorphism f_3 is accomplished by means of a stereographic projection of the bilinear transformation of the complex plane $z \to (\alpha z - \bar{\beta})/(\beta z + \bar{\alpha})$ into the group of rotations of the sphere, SO(3). Thus, $\mathrm{SO}(3) = \mathrm{SU}(2)/\mathbf{Z}_2$, i.e., an element $g \in \mathrm{SO}(3)$ has two inverse images in SU(2) under the homomorphism f_3: u and $-u$. This means that the group SU(2) covers the group of rotations twice.

3) The exact sequence of unitary groups

$$e \xrightarrow{f_1} \mathrm{SU}(n) \xrightarrow{f_2} \mathrm{U}(n) \xrightarrow{f_3} \mathrm{U}(1) \to e.$$

As usual, f_2 is the embedding of the group $\mathrm{SU}(n)$ into $\mathrm{U}(n)$. The mapping f_3 has the form $f_3(g) = \det g$ for $g \in \mathrm{U}(n)$. In other words, we have $\mathrm{U}(1) = \mathrm{U}(n)/\mathrm{SU}(n)$.

d) Homotopy group

Homotopy groups constitute one of the possible variants of the algebraic description of the topological properties of manifolds.

First of all, we introduce the concept of homotopy of two continuous mappings of one manifold into another. Let φ and g be two such mappings of X into Y. They are said to be *homotopic* if there exists a parameteric family of mappings $f_t\colon X \to Y$, continuous in $x \in X$ and $t \in [0, 1]$, such that $f_0 = \varphi$ and $f_1 = g$. In other words, homotopy implies the possibility of continuous deformation of one mapping into the other. All continuous mappings of X into Y can be partitioned into classes of mutually homotopic mappings.

Let us see what form the partition into classes takes in the case of a continuous mapping of a circle into a circle ($X = S^1 \to S^1 = Y$). Let the circle be given by the equation $|z| = 1$. Then the mapping $\exp(i\theta) = \exp(i\varphi(\theta))$ is continuous under the condition $\varphi(2\pi) = \varphi(0) + 2k\pi$, where k is an integer. The number k is called the *degree of the mapping* and indicates the number of rotations under the mapping φ. It can be shown that mappings having the same degree are homotopic; at the same time, if mappings have different degrees, then they belong to different homotopy classes. Thus, every mapping is homotopic to a canonical mapping $z \to z^k$.

One of the most important problems of topology is the problem of homotopic classification of the mappings of the n-dimensional sphere S^n into a given manifold X. It turns out that it is possible to define an operation of multiplication of homotopy classes of mappings and to convert the set of classes into a group.

We shall first consider continuous mappings of the circle S^1 into a manifold X. We fix a point x_0 in X and consider only mappings $\gamma(t)\colon S^1 \to X$ (t is a parameter on the circle) such that $\gamma(0) = x_0$. The set of such mappings $\{\gamma(t)\} = \Omega(X, x_0)$, which represents all possible closed contours in X passing through the point x_0, is called the *space of contours* (*loops*). Contours from $\Omega(X, x_0)$ can be multiplied together as follows. By the product of two contours $\gamma_1 * \gamma_2(t)$, we mean the contour $\gamma(t)$ which begins at the point x_0, passes along the contour γ_1, and then along the contour γ_2. However, this product does not convert the space of loops $\Omega(X, x_0)$ into a group. The point is that this multiplication is not associative, owing to the difference between the parameterizations: $\gamma_1 * (\gamma_2 * \gamma_3) \neq (\gamma_1 * \gamma_2) * \gamma_3$. Although the contours are not identical, they are mutually homotopic: $\gamma_1 * (\gamma_2 * \gamma_3) \approx (\gamma_1 * \gamma_2) * \gamma_3$. Therefore we shall identify homotopic contours in the space $\Omega(X, x_0)$. Thus, if $\gamma_1 \approx \sigma_1$ and $\gamma_2 \approx \sigma_2$, then their products are homotopic: $\gamma_1 * \gamma_2(t) = \sigma_1 * \sigma_2(t)$. The multiplication which we have introduced converts the classes of homotopic contours into a group. The unit element of the group is the class of contours homotopic to the constant mapping $\gamma(t) = x_0$. In other words, this is the class of contours which can be contracted continuously into a point. The group which we have constructed is called the *fundamental group* of the manifold X and is denoted by $\pi_1(X, x_0)$.

By definition, the group $\pi_1(X, x_0)$ depends on the choice of the point x_0. However, if the manifold X is linearly connected, i.e., if any of its points can be joined by a path, then the groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ constructed for different points are isomorphic. Therefore we make no reference to x_0 in these cases.

A manifold X is said to be *simply connected* if its fundamental group is trivial, i.e., if any contour in X can be contracted into a point. Examples of simply connected spaces are the Euclidean spaces $\mathbf{R}^n$, spherical regions, and the spheres S^n for $n \geq 2$. However, the circle S^1 is not simply connected. We have in fact already calculated its fundamental group: $\pi_1(S^1) = \mathbf{Z}$. It is easy to see that on the torus T^2 there are two independent noncontractable contours, which generate its fundamental group. Therefore $\pi_1(T^2) = \mathbf{Z} \oplus \mathbf{Z}$. The torus is a direct sum of two circles: $T^2 = S^1 \oplus S^1$. This example is a particular case of a general assertion: $\pi_1(X \oplus Y) = \pi_1(X) \oplus \pi_1(Y)$.

In Sec. 3a on differentiable manifolds, it was shown that the group SU(2) represents the sphere S^3, and it is therefore a simply connected manifold. At the same time, in the group SO(3) there exists a closed contour, not contractable into a point, joining the rotations around a given axis through the angles π and $-\pi$. But the product of this contour with itself is contractable into a point. This means that $\pi_1(\mathrm{SO}(3)) = \mathbf{Z}_2$. The groups SU(2) and SO(3) are locally isomorphic, and the fact that their fundamental groups are different indicates precisely that the group SU(2) covers the group SO(3) twice. We shall give other examples in what follows. But there we note that although the fundamental groups were Abelian in all the foregoing examples, this is not the general rule. For example, the fundamental knot group is non-Abelian. However, the fundamental groups of Lie groups are Abelian.

Higher homotopy groups are defined by analogy with the fundamental group. For this, it is necessary to define multiplication of homotopy classes of mappings of the n-dimensional sphere S^n into a manifold X. We shall consider the mappings $\alpha: S^n \to X$ which carry a fixed point of the sphere $s_0 \in S^n$ into a fixed point $x_0 = \alpha(s_0) \in X$. The product $\alpha * \beta$ of the mappings α and β is defined as the mapping of S^n into X which is identical to α on one hemisphere and identical to β on the other. Then the equator S^{n-1} of the sphere S^n, which separates the hemispheres and contains the point s_0, is mapped entirely into the point x_0. If the mappings $\alpha \approx \alpha_1$ and $\beta \approx \beta_1$ are homotopic, then their product is homotopic: $\alpha * \beta \approx \alpha_1 * \beta_1$. Therefore multiplication carries over to homotopy classes of mappings of S^n into X. This operation determines the homotopy group $\pi_n(X, x_0)$ of order n. It is easy to show that for $n > 1$ these groups are Abelian. The relation between the groups $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$ is the same as for the fundamental groups constructed for different points. We note also that the set $\pi_0(X, x_0)$ does not have a group structure. It indicates the number of components of connectivity of the manifold X. This follows from the fact that the sphere S^0 is a pair of points. But if X is a Lie group, we can introduce a group operation in the set of components of connectivity, and $\pi_0(X, x_0)$ is converted into a group. In addition, it can be shown that multiplication in the group $\pi_n(X)$ is specified by means of multiplication in the group X.

The next assertion enables us to understand the importance of homotopy groups in topology. If f is a continuous mapping of a manifold X into Y, there exists a homomorphism f_* of topological groups:

$$f_*: \pi_n(X) \to \pi_n(Y). \tag{3}$$

If the manifolds are topologically equivalent, their homotopy groups are identical. In general, the converse is not true.

The calculation of homotopy groups of manifolds in the general case is a complicated problem. In certain cases, the answers are intuitively obvious. For example, the homotopy groups of a Euclidean space are trivial. Similarly, $\pi_i(S^n) = 0$ for $i < n$ and $n > 1$, and $\pi_n(S^n) = \mathbf{Z}$.

We shall now indicate a method which makes it possible to express unknown homotopy groups in terms of known ones. Consider a manifold P, on which we are given an equivalence relation by the action of a group G (see Sec. 3b on equivalence classes). We can construct two mappings: 1) the embedding i of the orbit O of the group G into the manifold P,

$$i: O \to P;$$

2) the projection of P onto the space of orbits $M = P/G$:

$$\pi: P \to M.$$

According to (3), there exists a sequence of homomorphisms of homotopy groups:

$$\pi_n(O) \xrightarrow{i_*} \pi_n(P) \xrightarrow{\pi_*} \pi_n(M).$$

Moreover, there exists a relation between the homotopy groups of the space of orbits M and the group G. It is given by the so-called boundary homomorphism:

$$\pi_n(M) \xrightarrow{d} \pi_{n-1}(O), \quad n \geq 1$$

We shall not construct it in the general case, but shall give it in the examples which follow. It turns out that the sequence continued by means of the boundary homomorphism becomes exact (see Sec. 3c on exact sequences):

$$\xrightarrow{d} \pi_n(O) \xrightarrow{i_*} \pi_n(P) \xrightarrow{\pi_*} \pi_n(M) \xrightarrow{d} \pi_{n-1}(O) \xrightarrow{i_*}.$$

For the examples given in Secs. 1b and 1c, we shall show how this exact sequence can be used to calculate homotopy groups.

1) The factorization $S^1 = \mathbf{R}/\mathbf{Z}$ leads to the exact sequence

$$0 = \pi_n(\mathbb{R}) \xrightarrow{\pi_*} \pi_n(S^1) \xrightarrow{d} \pi_{n-1}(\mathbb{Z}) \xrightarrow{i_*} \pi_{n-1}(\mathbb{R}) = 0.$$

Since $\pi_0(\mathbf{Z}) = \mathbf{Z}$ and $\pi_n(\mathbf{Z}) = 0$ for $n > 0$, we have $\pi_n(S^1) = 0 (n > 1)$ and $\pi^1(S^1) = \mathbf{Z}$.

2) Consider the sphere S^m as a homogeneous space $S^m = \mathrm{SO}(m+1)/\mathrm{SO}(m)$. Since for $m > 2$ we have $\pi_2(S^m) = 0$ and $\pi_1(S^m) = 0$, we obtain the exact sequence

$$0 = \pi_2(S^m) \xrightarrow{d} \pi_1(\mathrm{SO}(m)) \xrightarrow{i_*} \pi_1(\mathrm{SO}(m+1)) \xrightarrow{\pi_*} \pi_1(S^m) = 0.$$

It follows from this that for $m \geqslant 3$ we have

$$\pi_1(\mathrm{SO}(m)) = \pi_1(\mathrm{SO}(m+1)).$$

But we have proved that $\pi_1(\mathrm{SO}(3)) = \mathbf{Z}_2$. Consequently, $\pi_1(\mathrm{SO}(m)) = \mathbf{Z}_2$. This means that the group $\mathrm{SO}(m)$ is not simply connected and covers the group $\mathrm{Spin}(m)$ twice; for $m = 3$, the latter is identical to the group $\mathrm{SU}(2)$. Constructing the exact sequence for the higher homotopy groups, we obtain

$$\pi_k(\mathrm{SO}(m+1)) = \pi_k(\mathrm{SO}(m)) \quad \text{for} \quad k < m - 1.$$

3) The sphere of odd dimension, $S^{2n-1} = \mathrm{SU}(n)/\mathrm{SU}(n-1)$ $(n \geqslant 2)$, leads to the exact sequence

$$0 = \pi_2(S^{2n-1}) \xrightarrow{d} \pi_1(\mathrm{SU}(n-1)) \xrightarrow{i_*} \pi_1(\mathrm{SU}(n)) \xrightarrow{\pi_*} \pi_1(S^{2n-1}) = 0.$$

It follows from this that the group $\mathrm{SU}(n)$ is simply connected: $\pi_1(\mathrm{SU}(n)) = 0$.

4) Let us calculate the fundamental group of the group $\mathrm{U}(n)$. From the relation $\mathrm{U}(1) = \mathrm{U}(n)/\mathrm{SU}(n)$, we have the exact sequence

$$0 = \pi_1(\mathrm{SU}(n)) \xrightarrow{i_*} \pi_1(\mathrm{U}(n)) \xrightarrow{\pi_*} \pi_1(\mathrm{U}(1)) \xrightarrow{d} \pi_0(\mathrm{SU}(n)).$$

Since $\pi_1(\mathrm{U}(1)) = \pi_1(S^1) = \mathbf{Z}$ and $\pi_0(\mathrm{SU}(n)) = 0$, we have $\pi_1(\mathrm{U}(n)) = \mathbf{Z}$.

For arbitrary $k > 1$, we have the exact sequence

$$0 = \pi_{k+1}(\mathrm{U}(1)) \xrightarrow{d} \pi_k(\mathrm{SU}(n)) \xrightarrow{i_*} \pi_k(\mathrm{U}(n)) \xrightarrow{\pi_*} \pi_k(U(1)) = 0.$$

Consequently,

$$\pi_k(\mathrm{SU}(n)) = \pi_k(\mathrm{U}(n)).$$

5) Consider the real projective space $\mathbf{R}P^n = S^n/\mathbf{Z}_2$. For $n \geqslant 2$, the corresponding exact sequence has the form

$$0 = \pi_1(S^n) \xrightarrow{\pi_*} \pi_1(\mathbb{R}P^n) \xrightarrow{d} \pi_0(\mathbb{Z}_2) \xrightarrow{i^*} \pi_0(S^n) = 0.$$

Since $\pi_0(\mathbf{Z}_2) = \mathbf{Z}_2$, we have $\pi_1(\mathbf{R}P^n) = \mathbf{Z}_2$. By considering this same sequence for groups of order $k \leqslant n$, we obtain $\pi_k(\mathbf{Z}P^n) = 0$ $(k < n)$ and $\pi_n(\mathbf{R}P^n) = \mathbf{Z}$. In particular, $\pi_2(\mathrm{SO}(3)) = 0$. It follows from example 2 that for all groups $\mathrm{SO}(n)$ the group π_2 is trivial. It can be shown that it is trivial for any Lie group.

6) For the complex projective space $\mathbf{C}P^n = S^{2n+1}/\mathrm{U}(1)$, we have $\pi_1(\mathbf{C}P^n) = 0$, which is a consequence of the fact that the sphere S^{2n+1} is connected and simply connected. At the same time, it follows from the exact sequence

$$0 = \pi_2(S^{2n+1}) \xrightarrow{\pi_*} \pi_2(\mathbb{C}P^n) \xrightarrow{d} \underset{\substack{\| \\ \mathbb{Z}}}{\pi_1(\mathrm{U}(1))} \xrightarrow{i_*} \pi_1(S^{2n+1}) = 0$$

that $\pi_2(\mathbf{C}P^n) = \mathbf{Z}$. Thus, there exist nontrivial homotopy classes of mappings $S^2 \to \mathbb{C}P^n$. In the $\mathbb{C}P^n$ model, they correspond to instanton solutions, whose charge is determined by the homotopy class of the solution.[27]

7) Consider again a sphere of odd dimension as a factor space: $S^{2n-1} = \mathrm{SU}(n)/\mathrm{SU}(n-1)$. For $n \geqslant 3$, we have the exact sequence

$$0 = \pi_4(S^{2n-1}) \xrightarrow{d} \pi_3(\mathrm{SU}(n-1)) \xrightarrow{i_*} \pi_3(\mathrm{SU}(n)) \xrightarrow{\pi_*} \pi_3(S^{2n-1}) = 0.$$

Hence $\pi_3(\mathrm{SU}(n)) = \pi_3(\mathrm{SU}(n-1))$. Since $\pi_3(\mathrm{SU}(2)) = \pi_3(S^3) = \mathbf{Z}$, we have $\pi_3(\mathrm{SU}(n)) = \mathbf{Z}$. This fact is related to the classification of instantons in the Yang-Mills theory.[1] There is a more general assertion: for any simple Lie group, its third homotopy group is isomorphic to the group $\mathbf{Z}$.

8) Finally, we give the homotopy classification of the mappings of the sphere S^3 into the sphere S^2; we shall calculate the group $\pi_3(S^2)$. Since $S^2 = \mathrm{SO}(3)/\mathrm{SO}(2)$, by making use of the results obtained in the preceding examples we have the exact sequence

$$0 = \pi_3(\mathrm{SO}(2)) \xrightarrow{i_*} \underset{\substack{\| \\ \mathbb{Z}}}{\pi_3(\mathrm{SO}(3))} \xrightarrow{\pi_*} \pi_3(S^2) \xrightarrow{d} \pi_2(\mathrm{SO}(2)) = 0,$$

from which it follows that $\pi_3(S^2) = \mathbf{Z}$.

We summarize the foregoing information in Table I.

4. GUIDE TO THE LITERATURE

First of all, we shall give a guide to the geometry of classical gauge fields. A lucid and rather detailed exposition (and one of the earliest ones) of the geometrical approach to the theory of gauge fields was given in the book of Konopleva and Popov.[5] The paper of Popov[6] is also useful.

An exposition of the methods of differential geometry used in gauge theories which is acceptable to physicists can be found in the book of Dubrovin, Novikov, and Fomenko.[7] Daniel and Viallet,[1] who introduced the mathematical concepts, made essential use of the two-volume monograph of Kobayashi and Nomizu,[8] whose

TABLE I.

Differentiable manifold	π_1	π_2	π_3
S^1	$\mathbb{Z}$	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$
S^3	0	0	$\mathbb{Z}$
$T^2 = S^1 \oplus S^1$	$\mathbb{Z} \oplus \mathbb{Z}$	0	0
$\mathrm{SO}(n)$ $(n > 2)$	$\mathbb{Z}_2$	0	$\mathbb{Z}$
$\mathrm{SU}(n)$	0	0	$\mathbb{Z}$
Simple Lie groups	—	0	$\mathbb{Z}$
$U(n)$	$\mathbb{Z}$	0	$\mathbb{Z}$
$\mathbb{R}P^n$	$\mathbb{Z}_2$ $(n > 1)$	$\begin{cases} 0 & n > 2 \\ \mathbb{Z} & n = 2 \end{cases}$	$\begin{cases} 0 & n > 3 \\ \mathbb{Z} & n = 3 \end{cases}$
$\mathbb{C}P^n$	0	$\mathbb{Z}$	—

Russian translation was published in 1981. There is a closely related book by Nomizu.[9] Of the mathematical books, in addition to the foregoing we can recommend the books of Steenrod,[21] Bishop and Crittenden,[10] and Chern.[11] In particular, the last of these contains the theory of characteristic classes. This theory is the main subject of the paper by Chern[12] and the book of Milnor and Stasheff.[13] The theory of differential forms is outlined in the books by Efimov,[14] Cartan,[15] and de Rham.[16] The last book contains an exposition of cohomology theory. For the geometry of Lie groups, see the collective monograph of Ref. 17. An exposition of certain problems of topology addressed to physicists is given in the lectures of Schwarz[18] and Shapiro and Ol'shanetskiĭ.[19]

Drinfel'd and Manin[25] have given a geometrical description of instantons and a new approach to the geometry of instanton configurations.

Finally, we give a brief characterization of the above-mentioned reviews.[2-4] The review of Ref. 2 gives an account of the application of homotopy theory to various field-theoretic models. It also contains considerable factual material on homotopy groups. Mermin's review[3] contains the application of the same theory to certain problems of solid-state physics. It explains in detail the basic concepts of homotopy theory. The review of Ref. 4 is devoted to the application of the methods of algebraic topology and differential geometry in the Yang–Mills theory and in the theory of gravitation.

We turn now to the introductory papers in the literature. First of all, we note that the definition of differentiable manifolds is contained in most of the mathematical books mentioned in the bibliography. The reader can best acquaint himself with homotopy theory from the lectures of Schwarz,[18] the book of Dubrovin, Novikov, and Fomenko,[7] or the monograph of Hu,[20] which is intended for the reader with previous preparation. Exact sequences are considered in the lectures of Shapiro and Ol'shanetskiĭ.[19] The reader can acquaint himself with the material contained in Sec. 3c from the books of Dubrovin, Novikov, and Fomenko[7] and Nomizu.[9]

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