METHODOLOGICAL NOTES

Ambiguity in the definition of the magnetic permeability of material media

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Usp. Fiz. Nauk **24**, 171–174 (September 1981)

It is shown that the magnetic permeability of material media taking into account both temporal and spatial dispersion is not defined uniquely. The requirement that experimentally measured quantities be obtained in the quasistatic limit does not significantly change this arbitrariness.

PACS numbers: 75.30.Cr

1. In textbooks on electrodynamics of material media, in order to describe the response of a spatially uniform and stationary medium to an external monochromatic field depending on time and coordinates as

$$A(\omega, \mathbf{k}) \exp(-i\omega t + i\mathbf{k}\mathbf{r}) \quad (1)$$

(It is assumed in what follows that all electrodynamic quantities, including external field sources ρ_0 and $\mathbf{j}_0$ are of this form), two independent tensor quantities $\varepsilon_{ij}(\omega,\mathbf{k})$ and $\mu_{ij}(\omega,\mathbf{k})$, called the dielectric permittivity and magnetic permeability tensors, respectively, are introduced. In so doing, the field equations in the medium are written with the use of two additional fields, the electric displacement **D** and the magnetic field **H**:*

$$\begin{gathered} \mathbf{k}\mathbf{D} = -4\pi i\rho_0, \quad [\mathbf{k}\mathbf{E}] = \frac{\omega}{c}\mathbf{B}, \\ \mathbf{k}\mathbf{B} = 0, \quad [\mathbf{k}\mathbf{H}] + \frac{\omega}{c}\mathbf{D} = -\frac{4\pi i}{c}\mathbf{j}_0, \\ D_i = \varepsilon_{ij}E_j, \quad B_i = \mu_{ij}H_j. \end{gathered} \quad (2)$$

On the other hand, it is well known[1] that the response of the medium to an external field can be expressed by a single relation $\mathbf{j}=\mathbf{j}(\mathbf{E})$, since the magnetic induction **B**, according to Maxwell's equations for nonstationary processes, can always be expressed in terms of the electric field **E**. For this reason, in linear electrodynamics of spatially uniform and stationary media, all effects of the field on the medium must be described only by a single conductivity tensor $\sigma_{ij}(\omega,\mathbf{k})$. The charge density ρ is expressed with the help of the continuity equation in terms of the current density **j** and for the field equations in a medium we have[1)]

$$\left.\begin{gathered} \mathbf{k}\mathbf{E} = -4\pi i(\rho+\rho_0), \quad [\mathbf{k}\mathbf{E}] = \frac{\omega}{c}\mathbf{B}, \\ \mathbf{k}\mathbf{B} = 0, \quad [\mathbf{k}\mathbf{B}] + \frac{\omega}{c}\mathbf{E} = -\frac{4\pi i}{c}(\mathbf{j}+\mathbf{j}_0), \\ j_i = \sigma_{ij}E_j, \quad \omega\rho = \mathbf{k}\mathbf{j}. \end{gathered}\right\} \quad (3)$$

*Translator's note: $[\cdots] \equiv$ vector product.

[1)] We note that for stationary fields, the material equations $\rho=\rho(\mathbf{E}\mathbf{B})$ and $\mathbf{j}=\mathbf{j}(\mathbf{E},\mathbf{B})$ can be obtained from the equations $j_i=\sigma_{ij}E_j$ or from $D_i=\varepsilon_{ij}E_j$ and $B_i=\mu_{ij}H_j$, respectively, by passing to the limit $\omega\to 0$.

Here, the equations are written for the electric field **E** and the magnetic induction **B**, which solely have direct physical meaning, determining the Lorentz force acting on a macroscopic charge q, moving in a medium with velocity **v**:

$$\mathbf{F} = q\left\{\mathbf{E} + \frac{1}{c}[\mathbf{v}\mathbf{B}]\right\}. \quad (4)$$

In connection with writing the field equations in the form (3), there arises the problem as to the independence of the tensors ε_{ij} and μ_{ij} and their relation to the tensor σ_{ij} for arbitrary ω and **k**.

2. In Ref. 1, for an isotropic medium, when

$$\sigma_{ij}(\omega,\mathbf{k}) = \frac{k_ik_j}{k^2}\sigma^{l}(\omega,\mathbf{k}) + \left(\delta_{ij} - \frac{k_ik_j}{k^2}\right)\sigma^{tr}(\omega,\mathbf{k}), \quad (5)$$

it is assumed that $\varepsilon_{ij}=\delta_{ij}\varepsilon$ and $\mu_{ij}=\delta_{ij}\mu$. Then, from the equivalence of systems (2) and (3), it is easy to show that

$$\begin{gathered} \varepsilon = 1 + \frac{4\pi i}{\omega}\sigma^{l} \equiv \varepsilon^{l}(\omega, k), \\ \mu^{-1} = 1 - \frac{4\pi i\omega}{c^2k^2}(\sigma^{tr} - \sigma^{l}) \equiv 1 - \frac{\omega^2}{c^2k^2}[\varepsilon^{tr}(\omega,k) - \varepsilon^{l}(\omega,k)]. \end{gathered} \quad (6)$$

This choice, however, even in the case of an isotropic medium is arbitrary, since the tensor ε_{ij} should be chosen in the form (5), i.e. consisting of two scalar functions. Due to the transverse nature of the induction vector **B**, the tensor μ_{ij} in an isotropic medium is always proportional to the unit tensor. In this case, it would no longer be possible to establish a relation of the type (6).

On the other hand, given a tensor μ_{ij} for an arbitrary anisotropic medium, it is possible to express the tensor ε_{ij} in terms of μ_{ij} and σ_{ij}. Thus, if it is assumed that

$$\varepsilon_{ij} = \delta_{ij} + \frac{4\pi i}{\omega}\sigma_{ij} + \frac{c^2}{\omega^2}e_{ikl}k_k(\delta_{lm} - \mu^{-1}_{lm})e_{mnj}k_n, \quad (7)$$

where e_{ijk} is the unit completely antisymmetric tensor ($e_{123}=1, e_{213}=-1$), then Eqs. (2) and (3) turn out to be equivalent. On the other hand, assuming that the tensor ε_{ij} is arbitrary, it is no longer possible to express μ_{ij} in terms of ε_{ij} and σ_{ij}, since in the form adopted

0038-5670/81/090795-03$01.10

for the material equations (1), the vector **H** does not depend on the potential part of the electric field **E** (which cannot be expressed in terms of **B**). We note that in this approach the tensor μ_{ij} can be chosen in a completely arbitrary manner.

For an isotropic medium, when σ_{ij} is represented in the form (5) $\mu_{ij}=\delta_{ij}\mu$, relation (7) takes the form

$$\varepsilon_{ij}=\frac{k_ik_j}{k^2}\varepsilon^l+\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\left[\varepsilon^{tr}-\frac{c^2k^2}{\omega^2}\left(1-\frac{1}{\mu}\right)\right]. \qquad (8)$$

From here, in particular, the inconsistency of the choice $\varepsilon_{ij}=\delta_{ij}\varepsilon$ is evident. The tensor ε_{ij} in an isotropic medium, just as σ_{ij}, consists of two functions and is represented in the form (5).

Thus, the procedure for defining the tensors ε_{ij} and μ_{ij} in terms of σ_{ij}, and at the same time defining the vectors **D** and **H**, is always ambiguous. This arbitrariness can be demonstrated in general form by transforming to new quantities **D′** and **H′**:

$$\mathbf{D}'=\mathbf{D}-[\mathbf{kf}],\qquad \mathbf{H}'=\mathbf{H}+\frac{\omega}{c}\mathbf{f}, \qquad (9)$$

where **f** is an arbitrary vector depending on **B**. It can be shown that the fields **D′** and **H′** satisfy the same equations (1) with appropriate changes in ε_{ij} and μ_{ij}. This precisely indicates the ambiguity in defining the tensors ε_{ij} and μ_{ij} without any additional conditions. In other words, the current **j** induced in the medium cannot be separated, without some additional assumptions, into a curl (rotor) of the magnetization and the derivative of the polarization with respect to time, i.e. having written the current in the form $\mathbf{j}=-i(\omega/c)\mathbf{P}+ic[\mathbf{kM}]$, the vector **P** will be determined to within an arbitrary transverse vector.

3. Let us examine several possible definitions of **H** and **D**. First, however, we note that it is often assumed in the electrodynamics of material media that $\mu_{ij}=\delta_{ij}$. In this case, according to (7)

$$\varepsilon_{ij}=\delta_{ij}+\frac{4\pi i}{\omega}\sigma_{ij}. \qquad (10)$$

It should be emphasized that this does not at all mean that magnetic properties of the media are neglected: all the magnetic properties of the medium are included in the tensor ε_{ij}. This form of writing the material equations, however, is not very convenient for studying the magnetic properties of media in the quasistatic limit. The tensor μ_{ij} should be constructed so that in the static limit ($\omega/k\to 0, k\to 0$) the expression usually used for the quasistatic magnetic permeability is obtained:

$$\mu_{ij}^{-1}=\frac{\partial}{\partial B_i}(B_j-4\pi M_j). \qquad (11)$$

The magnetization of the medium **M**, generally speaking, can depend not only on the magnetic induction, but also on the intensity of the potential electric field $\mathbf{E}^l$. In the static limit, two different conditions can be imposed on the electric field: either $\mathbf{E}^l$=const or $\mathbf{D}^l$=const. For this reason, it makes sense to examine two different magnetic permeability tensors.

Transforming to arbitrary frequencies and wave vectors, we separate the total current induced in the medium into longitudinal and transverse parts $\mathbf{j}=\mathbf{j}^l+\mathbf{j}^{tr}$, where

$$\mathbf{j}^l=\frac{\mathbf{k}(\mathbf{k}\cdot\mathbf{j})}{k^2},\quad \mathbf{j}^{tr}=\mathbf{j}-\mathbf{j}^l. \qquad (12)$$

The transverse part of the current $\mathbf{j}^{tr}$ can be represented in the form $\mathbf{j}^{tr}=ic[\mathbf{kM}]$. As a result, we have

$$M_i=\frac{i}{ck^2}[\mathbf{kj}^{tr}]_i=\frac{i}{ck^2}e_{ijk}k_j\sigma_{kl}E_l. \qquad (13)$$

Further, representing **E** as $\mathbf{E}=\mathbf{E}^l+\mathbf{E}^{tr}$ and taking into account the fact that $\mathbf{E}^{tr}=-(\omega/ck^2)[\mathbf{kB}]$, we define μ_{ij}^{-1} with $\mathbf{E}^l$=const as

$$\mu_{ij}^{-1}=\delta_{ij}+\frac{4\pi i}{c^2k^4}e_{ikl}k_k\sigma_{ln}e_{nmj}k_m. \qquad (14)$$

The condition $\mathbf{E}^l$=const means that the potentials of external sources are fixed. In this case, according to (7)

$$\varepsilon_{ij}=\delta_{ij}+\frac{4\pi i}{\omega}\left[\sigma_{ij}-\left(\delta_{il}-\frac{k_ik_l}{k^2}\right)\sigma_{lm}\left(\delta_{mj}-\frac{k_mk_j}{k^2}\right)\right]. \qquad (15)$$

In an isotropic medium, Eqs. (14) and (15) take the form

$$\mu_{ij}^{-1}=\delta_{ij}-\frac{4\pi i\omega}{c^2k^2}\sigma^{tr}\delta_{ij}=\delta_{ij}\left[1-\frac{\omega^2}{c^2k^2}(\varepsilon^{tr}-1)\right],$$
$$\varepsilon_{ij}=\delta_{ij}+\frac{4\pi i}{\omega}\frac{k_ik_j}{k^2}\sigma^l=\delta_{ij}+\frac{k_ik_j}{k^2}(\varepsilon^l-1). \qquad (16)$$

Here, we took into account the fact that μ_{ij}^{-1} always acts on the transverse vector **B**.

We now determine μ_{ij}^{-1}, assuming that $\mathbf{D}^l$=const, i.e. with external charges fixed. In this case, a potential field $\mathbf{E}^l$, depending on the magnetic induction **B**, arises in the medium. Further, determining the longitudinal field $\mathbf{E}^l$ from the equation $\mathbf{kD}^l=-4\pi i\rho_0$ and using (13), we easily find

$$\mu_{ij}^{-1}=\delta_{ij}+\frac{4\pi i\omega}{c^2k^4}e_{ikl}k_k\sigma_{ln}e_{nmj}k_m-\frac{i}{c^2k^6\varepsilon^l}e_{ikl}k_k\sigma_{lm}k_mk_n\sigma_{ns}e_{sjt}k_t, \qquad (17)$$

$\varepsilon^l=1+(4\pi i/\omega)k_i\sigma_{ij}k_j/k^2$ is the longitudinal dielectric permittivity of an anisotropic medium. Now, following (17), we find the tensor ε_{ij}, which we will not write out here. Evidently, in an isotropic medium, expressions (17) and (14) coincide, so that relations (16) are valid for $\mathbf{D}^l$=const as well.

Although the choice of one form or another of the tensor μ_{ij} is to a large extent a matter of taste, we feel that the most natural choice of the tensor μ_{ij} is given by Eq. (17). For a wave vector **k** different from zero, the only condition realized in practice is the constancy of external charges, i.e. $\mathbf{D}^l$=const,[2] and the tensor μ_{ij}, given by Eq. (17), satisfies the causality condition and the Kramers–Kronig relations. The tensor (16) can be viewed as a response function only for **k**=0.

Thus, the equations of electrodynamics of material media can be formulated with the use of the tensors ε_{ij} and μ_{ij}, taking into account both frequency and spatial dispersion. However, the choice of one formulation or another is to a large extent arbitrary and the definition of the tensors ε_{ij} and μ_{ij} is not unique. The requirement that definite quantities measured experimentally are obtained in the static limit does not remove the arbitrariness. In particular, according to (9), for μ_{ij} to be unique in the static limit $\omega/k\to 0$ and $k\to 0$, it is enough to require that $\mathbf{f}\to 0$. At the same time, we wanted to emphasize that definition (17) is better grounded from the physical point of view as corresponding to the real formulation of the problem.

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Translated by M. E. Alferieff