Free oscillations of the sun and the giant planets

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Usp. Fiz. Nauk **134**, 675–710 (August 1981)

The current state of research on the free oscillations of the sun is reviewed. Observational data on oscillations with periods from 5 to 160 min published through the end of 1980 are described. The interpretation of these data in connection with research on the internal structure of the sun is discussed. A theory of the free oscillations in the linear adiabatic approximation is described; differential rotation is taken into account. The principles for classifying the theoretical normal modes are discussed. The problem of the excitation of the solar oscillations is outlined. The theoretical normal-mode spectra of Jupiter and Saturn are discussed.

PACS numbers: 96.60.Cp, 96.30.Kf, 96.30.Mh

CONTENTS

1. INTRODUCTION

Just a short time ago the problem of the internal structure of the sun was considered basically solved. Detailed evolutionary models of the internal structure had been constructed from laboratory and theoretical results on the cross sections for nuclear reactions and the average-mass radiation absorption coefficients (opacities) and from assumptions regarding the structure of the convection zone (mixing-length theory). These "standard solar models" were constructed under the assumption that the chemical composition was homogeneous during the formation of the sun and that the matter below the convection zone did not become involved in the mixing during the evolution; in other words, the products of the fusion reactions were assumed to accumulate where they were formed. Calculations were carried out for the evolution according to these models, and by varying the original helium content it was found possible to match the modern luminosity of the sun with an age of $4.7 \cdot 10^9$ yr. By varying a free parameter (the ratio of the length of the mixing path to the pressure scale height) in the theory for the convection zone it was found possible to match the radius of the sun. It was believed that the slight uncertainties associated with the errors in the physical properties used could be eliminated without any radical revision of the original assumptions. Research in the past few years, however, has not only failed to justify this optimism but has in fact spurred a reexamination of the original assumptions regarding the structure and evolution of the sun.

The solar-neutrino problem remains unresolved.[1] The standard models predict a flux density of high-energy neutrinos 2.6–4.0 times the observed values, if it is assumed that the neutrinos produced in the solar interior reach the earth without undergoing any changes.[27]

There is a second problem regarding paleoclimatology. The standard models predict a significant increase in the luminosity of the sun during the earth's history. A billion years ago the luminosity should have been 10% lower than the modern value, and four billion years ago it should have been 30–40% lower. The early quantitative models for the earth's climate showed that a decrease of only 2% in the solar luminosity would have led to global glaciation on the earth, if the atmosphere had the same composition as it has today. Even if the luminosity subsequently increased to the modern value the earth would have remained covered with snow because of the high albedo. On the other hand, the earliest life forms are estimated to go back at least three billion years. Some ancient rocks have a morphology which would require liquid water. Isotope analysis indicates that the climate in the past was in fact somewhat warmer than the modern climate. Furthermore, certain features of the Martian channels indicate that there was liquid water on Mars in the past, so that the climate was warmer; this conclusion is again difficult to reconcile with a low solar luminosity. There are still a large number of unresolved problems in the effort to construct a theoretical model for the climate over a large time scale, and the reliability of such conclusions should not be over-estimated. A low luminosity could be offset to some extent by an increase in the greenhouse effect because of a different chemical composition of the atmosphere in the past. At the moment, therefore, all we can say is that the theoretical modelling runs into difficulties in attempts to relate clima-

 0038-5670/81/080697-20$01.10

tological data with the predictions of the standard models of a low solar luminosity in the past. The state of the climate-history problem is described in books by Budyko[104] and Monin and Shishkov[2] and in a review by Pollack.[3]

The third problem, to which the present review is devoted, arose in connection with the discovery of free oscillations of the sun. This research basically began with the discovery of the 5-min oscillations of the sun by Leighton, Noyes, and Simon.[4] Hill, Stebbins, and Brown[5] later reported detecting a large number of periods in the fluctuations of the solar diameter, ranging from a few minutes to an hour. Severny, Kotov, and Tsap[6] and, independently, Brooks, Isaak, and van der Raay[7] reported the detection of oscillations with a period of 160 min.

These oscillations have now been detected by six groups of investigators.[6-10,116] The 160-min oscillation is difficult to interpret in the standard model. There are also some difficulties in interpreting some recent data obtained by the Hill group.[11] We will return to this question later on in this review.

Aside from these problems there are still many puzzling phenomena associated with the solar activity. As Parker[13] has written, the sun is the only star about which we know enough to sense just how little we actually know. The problem of the internal structure of the sun has left the category of apparently solved problems to become one of the most critical and urgent problems of astrophysics. Research on free oscillations is rapidly developing into a promising new field of solar physics.

The free oscillations incorporate quantitative information on the internal structure of the sun which cannot be obtained by any other method. Since the periods of the oscillation modes are determined by regions of different depths in the sun, it will become possible at some time in the future, as experimental data are accumulated, to take up the inverse problem of constructing a solar model. Aside from the periods of the oscillations, the oscillation amplitudes and the time evolution of these amplitudes potentially carry much information about the internal structure. The precession of the surface displacement pattern contains information on the differential rotation of the solar interior. A solid theoretical basis has been developed for the normal-mode method, and much experience has been gained in the use of this method to study the internal structure of the earth, for which more than 1000 observed frequencies have been identified with the theoretical spectrum.

We will also discuss the theoretical normal-mode spectra of Jupiter and Saturn. These planets are attracting considerable interest because of the first flybys of space vehicles. It is presently believed that both of these planets are in a gaseous–liquid convective state.[14] So far, the detailed models for the internal structure have been based on extremely scanty data, specifically the average density, the first two gravitational moments, and the boundary conditions on the pressure and temperature in the atmosphere.[15] It is thus an extremely urgent problem to acquire more experimental data bearing information on the interiors of Jupiter and Saturn.

The problem of the free oscillations of the sun has much in common with the corresponding problem for the giant planets. In both cases, rotation is important. A comparative analysis of the theoretical spectra will make it possible to refine the principles for classifying the oscillations, with direct application to the interpretation of the observed oscillations of the sun.

2. THEORY

In this section we will outline the normal-mode theory as adapted to the interpretation of the observed solar oscillations and to calculations of the theoretical normal-mode spectra of the giant planets. We will discuss only some of the most important studies which bear directly on these problems. In describing the general aspects of the normal-mode theory for the sun and the gaseous–liquid giant planets, we will refer to the object in question as simply a "star." To a reader who is interested in the development of the theory in a broader scope we suggest the reviews by Ledoux and Valraven[16] and Ledoux[17] and the recent book by Cox.[105] Some recent advances in the normal-mode theory for stars are summarized in a review by Cox.[18]

a) Equations for small adiabatic oscillations of a differentially rotating star

The initial equations in the inertial coordinate system are as follows:

1) the equation of motion,

$$\mathbf{a} = -\rho^{-1}\nabla p - \nabla\psi, \tag{2.1}$$

where $\mathbf{a}$ is the acceleration of an element of mass, and ρ, p, and ψ are the density, pressure, and gravitational potential;

2) the Poisson equation,

$$\nabla^2\psi = 4\pi G\rho; \tag{2.2}$$

3) the continuity equation,

$$\rho' = -\nabla\cdot(\rho_0\mathbf{u}), \tag{2.3}$$

where u is the displacement of the element of mass from its position in the unperturbed star, ρ' is the Euler perturbation of the density, and ρ_0 is the unperturbed density;

4) an equation relating the Lagrange pulsations of the pressure and the density,

$$\Delta p = K\rho_0^{-1}\Delta\rho, \tag{2.4}$$

which would be written in Euler coordinates as

$$p' + \mathbf{u}\cdot\nabla p_0 = K\rho_0^{-1}(\rho' + \mathbf{u}\cdot\nabla\rho_0). \tag{2.5}$$

In the calculation of the adiabatic compression modulus

$$K = p\left(\frac{d\ln p}{d\ln\rho}\right)_{ad} \equiv p\Gamma_1 \tag{2.6}$$

the solar matter will be treated as a mixture of a gas of particles and radiation. Under the physical conditions prevailing in the solar interior, the gas of particles can be described well by the ideal-gas equation of state, while the radiation can be described well by the law of blackbody emission:

$$p_G = \frac{R}{\mu}\rho T, \quad p_F = \frac{4\sigma}{3c}T^4, \quad p = p_G + p_F, \tag{2.7}$$

where σ is the Stefan–Boltzmann constant. For this mixture we have[19]

$$\Gamma_1 = \beta + \frac{(4-3\beta)^2(\gamma-1)}{\beta + 12(\gamma-1)(1-\beta)}, \quad \beta = \frac{p_G}{p}, \quad \gamma = \frac{5}{3}. \tag{2.8}$$

If $p_G \gg p_F$ (this is a good approximation), we would have $\Gamma_1 = 5/3$.

In the adiabatic model of the gaseous–liquid planets Jupiter and Saturn the compression modulus is calculated from

$$K = \rho\frac{dp}{dr}\left(\frac{d\rho}{dr}\right)^{-1}. \tag{2.9}$$

We will assume that the unperturbed velocity field of the differentially rotating star is a stationary field and that the velocity does not depend on the azimuthal angle φ in the spherical coordinate system (r, θ, φ). In this case the velocity field can be written

$$v_0(r, \theta) = r_c\,\Omega(r, \theta)\,\hat{\boldsymbol{\varphi}}, \tag{2.10}$$

where r_c is the distance from the rotation axis (this distance is the radius in the cylindrical coordinate system with z axis along the rotation axis), Ω is the angular rotation frequency, and $\hat{\varphi}$ is the unit vector in the direction of φ. We will determine the dependence of the displacement u on the angle φ during the oscillations by the factor $\exp(im\varphi)$, where m is an integer. This procedure is legitimate since the φ dependence must be periodic, and in the linear theory any oscillation can be treated as a superposition of elementary waves of this type with different values of m.

After linearization, the equations of the small adiabatic oscillations take the following form in the inertial coordinate system:

$$\begin{aligned}\rho_0\,[-\omega^2\mathbf{u} + \Omega\,(2i\omega\hat{\mathbf{z}}\times\mathbf{u} - 2m\omega\mathbf{u}) + \Omega^2\,(2im\hat{\mathbf{z}}\times\mathbf{u} - m^2\mathbf{u})]\\ = \nabla\,(K\nabla\cdot\mathbf{u}) - \nabla\,[\mathbf{u}\cdot\rho_0\,(\nabla\psi_0 - \mathbf{r}_c\Omega^2)]\\ - \rho'\,(\nabla\psi_0 - \mathbf{r}_c\Omega^2) - \rho_0\nabla\psi' - \rho_0\mathbf{r}_c\,(\mathbf{u}\cdot\nabla\Omega^2),\end{aligned} \tag{2.11}$$

$$\nabla^2\psi' = 4\pi G\rho', \tag{2.12}$$

$$\rho' = -\nabla\cdot(\rho_0\mathbf{u}). \tag{2.13}$$

Here $\mathbf{z}$ is a unit vector along the rotation axis. The time dependence is singled out in the factor $\exp(i\omega t)$, where ω is the angular frequency of the oscillation. The quantity in brackets on the left side of (2.11) is the Lagrange pulsation of the acceleration.[20,21] In the derivation of (2.11) we used the equation for the pressure pulsations, (2.5), and the continuity equation, (2.3). The solutions of (2.11)–(2.13) must satisfy the free boundary conditions at the deformed surface of the star (these conditions are that there are no stresses and that the gravitational potential and its gradient are continuous).

Equations (2.11)–(2.13) with the appropriate boundary conditions constitute an eigenvalue boundary-value problem with partial differential equations. This problem is solved by perturbation theory, which yields the corrections for a sufficiently slow rotation to the solution of the simpler problem of a nonrotating star.

b) Perturbation theory

Two different approaches are taken to the derivation of a perturbation theory. The first approach—a variational approach—can be outlined as follows: One finds the frequencies and eigenfunctions (the shapes) of the modes in the absence of rotation. The mode frequency is then written as a functional of the parameters of the model and of the eigenfunctions of the corresponding mode. This can be done by integrating the scalar product of the vector oscillation equation and the vector displacement field over the volume of the star. By varying the rotational frequency from zero to a given value in this functional, the correction to the mode frequency can be found. The justification for this method is that the frequency found in this manner is, in a first approximation, stationary with respect to variations of the eigenfunctions. This approach has the advantage of simplicity, but it yields only a first correction to the frequency, and it does not yield corrections to the eigenfunctions. The effect of a slow differential rotation on the frequencies of the normal modes has been studied by the variational method by Hansen, Cox, and Van Horn[21] (their results have been put in a form more convenient for numerical calculations by Cuypers[106]).

A second and more general method is the perturbation theory for Hermitian operators which is used in quantum mechanics. We will briefly outline the application of this method to the calculation of the normal modes of a rotating star.[22-24]

The differential rotation of the star is conveniently described in dimensionless form by singling out the average angular frequency of the rotation, Ω_0:

$$\Omega(r, \theta) = \Omega_0\Omega_d(r, \theta). \tag{2.14}$$

The equations of small adiabatic oscillations are written in operator form:

$$\begin{aligned}-\omega^2\mathbf{u} + \Omega\,(2i\omega\hat{\mathbf{z}}\times\mathbf{u} - 2m\omega\mathbf{u}) + \Omega^2\,(2im\hat{\mathbf{z}}\times\mathbf{u} - m^2\mathbf{u})\\ = -\mathbf{H}_0\mathbf{u} - \Omega_0^2\,(\mathbf{\Psi} + \mathbf{E})\,\mathbf{u}.\end{aligned} \tag{2.15}$$

The operator $\mathbf{H}_0$ in (2.15) corresponds to the problem without rotation:

$$\omega_0^2\mathbf{u}_0 = \mathbf{H}_0\mathbf{u}_0. \tag{2.16}$$

The form of this integrodifferential operator is determined by Eq. (2.11), with ψ' and ρ' from (2.12) and (2.13). The operator $\mathbf{\Psi} + \mathbf{E}$ is then determined from the condition that Eqs. (2.15) and (2.11) be equivalent. On the left side of (2.15) there are operators which represent Coriolis forces. The operator $\mathbf{\Psi}$ describes the effect of centrifugal forces, while $\mathbf{E}$ describes the effect of the eccentricity (the deformation of the star caused by the rotation).

The small parameter

$$\lambda = \frac{\Omega_0}{\omega_0} \tag{2.17}$$

is also introduced, and solutions are sought in the form

$$\omega = \omega_0 (1 + \sigma_1\lambda + \sigma_2\lambda^2 + \ldots), \tag{2.18}$$

$$\mathbf{u} = \mathbf{u}_0 + \lambda\mathbf{u}_1 + \lambda^2\mathbf{u}_2 + \ldots . \tag{2.19}$$

Substituting expansions (2.18) and (2.19) into operator equation (2.15), and equating terms with identical powers of λ, we find the system of perturbation-theory equations:

$$[\mathbf{I} - \omega_0^{-2}\mathbf{H}_0]\,\mathbf{u}_0 = 0, \tag{2.20}$$

$$[\mathbf{I} - \omega_0^{-2}\mathbf{H}_0]\,\mathbf{u}_1 = -2\sigma_1\mathbf{u}_0 - \Omega_d\,[2m - 2i\hat{\mathbf{z}}\times]\,\mathbf{u}_0, \tag{2.21}$$

$$\begin{aligned}[\mathbf{I} - \omega_0^{-2}\mathbf{H}_0]\,\mathbf{u}_2 = {} & -2\sigma_1\mathbf{u}_1 - (\sigma_1^2 + 2\sigma_2)\,\mathbf{u}_0 - \Omega_d\,[2m - 2i\hat{\mathbf{z}}\times]\,\mathbf{u}_1 \\ & -\sigma_1\Omega_d[2m - 2i\hat{\mathbf{z}}\times]\,\mathbf{u}_0 - \Omega_d^2[m^2 - 2im\hat{\mathbf{z}}\times]\,\mathbf{u}_0 + [\mathbf{\Psi} + \mathbf{E}]\,\mathbf{u}_0, \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots ;\end{aligned} \tag{2.22}$$

where I is the unit operator. Equations (2.20)–(2.22) are solved in succession. Equation (2.20) corresponds to the problem without rotation (the zeroth order approximation), Eq. (2.21) determines the first-order perturbation theory, Eq. (2.22) determines the second-order, and so forth.

c) The problem without rotation. Classification of oscillations

The zeroth order approximation is the foundation for a scheme for classifying the oscillations and for studying the properties of the various normal modes. The effects of rotation and other possible perturbations, such as the magnetic field and tides, and also the problems of the stability and excitation of the oscillations are usually studied on the basis of the zeroth-order approximation solutions. In the zeroth-order approximation the nonrotating star has a spherical shape.

The normal modes of a nonrotating, self-gravitating elastic sphere fall into two groups: spheroidal and torsional modes. For the spheroidal modes the vector displacement field is

$$\mathbf{u} = \hat{\mathbf{r}}U(r)\,Y_{lm}(\theta, \varphi) + V(r)\,\nabla_1 Y_{lm}(\theta, \varphi), \tag{2.23}$$

and that for the torsional modes is

$$\mathbf{u} = -W(r)\,\hat{\mathbf{r}} \times \nabla_1 Y_{lm}(\theta, \varphi); \tag{2.24}$$

where Y_{lm} is the spherical harmonic of indices l and m, ∇_1 is the angular part of the gradient operator, $\nabla_1 = \hat{\theta}\partial/\partial\theta + \hat{\varphi}\sin^{-1}\theta\partial/\partial\varphi$, and $\hat{\mathbf{r}}$, $\hat{\theta}$, $\hat{\varphi}$ are the unit vectors along r, θ, φ. The index m takes on the $2l+1$ values $m = -l, \ldots, 0, \ldots, l$. The eigenfunctions of the spheroidal and torsional modes form a complete set: Any free motion of a gravitating sphere can be expanded in these eigenfunctions. For the spheroidal mode, the radial component of the curl (or rotor) of the displacement is zero. For the torsional modes there are no radial displacements, and the divergence of the displacement is zero. A particular case of the spheroidal modes with $l = 0$ is represented by radial oscillations with displacement directed along the radius. The oscillations with $l = 1$ are dipole oscillations; those with $l = 2$ are quadrupole oscillations; etc. In the absence of rotation, a gaseous or liquid sphere would not be subject to torsional oscillations (there are no restoring forces in the absence of shear stresses), and all the corresponding frequencies would be zero. These modes cannot, however, be ignored in the analysis: Aside from the fact that the vector fields of the torosional modes complement the vector field of the spheroidal mode in forming the complete set of eigenfunctions of the elastic gravitating sphere,[25] their frequencies are nonzero when a rotation or a magnetic field is introduced.

Because of the spherical symmetry, a separation of variables is possible by substituting (2.23) for a spheroidal vector field into the vector oscillation equations (2.11)–(2.13) for $\Omega = 0$. This step reduces the problem to a linear, homogeneous system of four first-order ordinary differential equations[16,17,37]:

$$\begin{aligned} \frac{dy_1}{dr} &= -\frac{2}{r}y_1 + \frac{1}{K}y_2 + \frac{l(l+1)}{r}y_0, \\ \frac{dy_2}{dr} &= -\left(\omega^2\rho_0 + \frac{4\rho_0 g_0}{r}\right)y_1 + \frac{l(l+1)\rho_0 g_0}{r}y_0 - \rho_0 y_4, \\ \frac{dy_3}{dr} &= 4\pi G\rho_0 y_1 + y_4, \\ \frac{dy_4}{dr} &= -\frac{4\pi G\rho_0 l(l+1)}{r}y_0 + \frac{l(l+1)}{r^2}y_3 - \frac{2}{r}y_4, \\ y_0 &= \frac{1}{r\omega^2}\left(g_0 y_1 - \frac{1}{\rho_0}y_2 - y_3\right). \end{aligned} \tag{2.25}$$

Here g_0 is the gravitational acceleration, and the following variables have been introduced:

$$\begin{aligned} & y_1 = U, \quad y_2 = KX, \quad y_3 = P, \quad y_4 = \frac{dP}{dr} - 4\pi G\rho_0 U, \quad y_0 = V, \\ & \psi' = -P(r)\,Y_{lm}(\theta, \varphi), \quad \nabla\cdot\mathbf{u} = X(r)\,Y_{lm}(\theta, \varphi). \end{aligned} \tag{2.26}$$

These variables have a simple physical meaning: They are the radial factors in the radial displacement component (y_1), in the pressure pulsation (y_2), in the perturbation of the gravitational potential (y_3), and in the horizontal component of the displacement (y_0). The boundary conditions are that all quantities are regular at the origin, that there are no stresses at the deformed surface of the star, and that the gravitational potential and its gradient are continuous at this surface:

$$y_2 = 0, \quad y_4 + \frac{l(l+1)}{R}y_3 = 0 \quad (r = R); \tag{2.27}$$

where R is the radius of the star. Inside the star all the functions y_i must be continuous, except that y_0—the tangential displacement—may have discontinuities in the star where discontinuities are caused in the properties of the medium by changes in the chemical composition or by phase transitions. Since the density and the pressure both fall off continuously to zero in the direction toward the surface for the sun and the giant planets, the boundary conditions in (2.27) are imposed at some sufficiently large $r_b < R$. This approach is equivalent to calculating the modes of a model by discarding the light outer shells of the star and replacing them by a constant pressure equal to the pressure which the discarded outer shells exerted. The validity of this procedure is checked in the course of the numerical calculations by a direct variation of r_b; the procedure works because the outermost shells have only a minor effect on the periods and shapes of the oscillations. Equations (2.25) with boundary conditions (2.27) are solved numerically by iterations in ω. The different papers on the normal modes use different choices of the functions y_i to describe the modes, so that Eqs. (2.25) may take different forms in the different papers.[16] The choice of the function in the form in (2.26) corresponds to the choice adopted in the normal-mode theory for the earth.[26] In addition, the mode index (the first index on the spherical harmonic) is denoted by the letter "n" in some papers. To avoid confusion we have adopted for this review the indexing scheme used in the overwhelming majority of the papers cited.

At a fixed value of l all the modes with $m = -l, \dots, l$ have the same frequency; in other words, in the absence of rotation the frequencies are degenerate with respect to the azimuthal index of the spherical harmonic. This $(2l+1)$-fold degeneracy is a consequence of the spherical symmetry of the problem.

Furthermore, for each value of l there exists an infinite set of solutions with discrete values of ω, whose eigenfunctions have (a) different numbers of nodes and (b) amplitudes with different radial profiles. A description of these eigenfunctions requires a classification scheme which reflects the differences in the properties of these modes. Below we will discuss the principles for such a scheme for the particular cases of the theoretical normal-mode spectra of polytropic models, a standard solar model, and models for the giant planets. We will see that the mode spectrum of stars may be extremely complicated, so that the classification of the modes is an important question in practice.

For a preliminary study of the basis for the classification of the theoretical normal-mode spectrum, the problem can be simplified considerably by using the Cowling approximation,[28] i.e., to ignore the perturbations of the gravitational potential which result from the mass redistribution in the star during its oscillations. If the mass is significantly concentrated toward the center of the star, where the displacements in the oscillations are basically horizontal, this approximation yields good quantitative results.[29] In this case the Poisson equation drops out of the problem, and the problem becomes one of solving a system of second-order ordinary differential equations, which may be written[17,30]

$$\frac{dv}{dr} = \left[\frac{l(l+1)}{\omega^2} - \frac{\rho_0 r^2}{\Gamma_1 p_0}\right]\frac{p_0{}^{2/\Gamma_1}}{\rho_0} w,$$
$$\frac{dw}{dr} = \frac{1}{r^2}\left(\omega^2 + A g_0\right)\frac{\rho_0}{p_0{}^{2/\Gamma_1}} v, \tag{2.28}$$

where $v = r^2 U p_0^{1/\Gamma_1}$, $w = p'/p_0^{1/\Gamma_1}$, p' is the Euler perturbation of the pressure, $\Gamma_1 = 5/3$ (we are ignoring the elasticity of the radiation), and

$$A = \frac{d\ln\rho_0}{dr} - \frac{1}{\Gamma_1}\frac{d\ln p_0}{dr}. \tag{2.29}$$

The boundary condition at the surface is $p' + U(dp_0/dr) = 0$ $(r = R)$.

Let us examine the nature of the solutions of this problem for a polytropic gaseous sphere whose structure is determined by the following relationship between the pressure and the density: $p_0 = \text{const}\cdot\rho_0{}^{(n+1)/n}$, where n, the index of the polytrope, may lie between 0 and 5. The polytrope of index 0 makes the model one of constant density, while the polytrope of index 1.5 corresponds to an adiabatic gaseous mode. The concentration of mass toward the center intensifies with inreasing index of the polytrope, tending toward infinity at $n = 5$. The polytropic law is a major convenience in describing the internal structure of stars.

The solutions of differential equations (2.28) may be exponential or oscillatory functions of the radius, depending on whether the coefficients on the right sides of these equations have the same or different signs. For a given model these coefficients are functions of the radius and the frequency. The coefficient in the first equation is zero under the condition $\omega^2 = [l(l+1)/r^2]\Gamma_1 p_0/\rho_0$, and that in the second equation is zero under the condition $\omega^2 + Ag_0 = 0$, i.e., under the condition $\omega^2 = N^2$, where N is the Brunt–Väisälä frequency. The coordinate plane $(x = r/R, \omega^2)$ thus breaks up into four regions, in two of which the solutions are oscillatory in x, while in the other two the solutions are exponential in x. These regions are the regions bounded by the solid curves in Fig. 1 for the polytrope of index 3. The regions G and A are those for which the solutions are oscillatory in x (Ref. 30).

The horizontal lines in Fig. 1 show the oscillation frequencies calculated through a numerical integration of Eq. (2.28) for the polytropic model of index 3. For each frequency the positions of the nodes in the eigenfunctions of the radial displacements are marked by circles. The nodes occur in regions in which the solutions are oscillatory in x. We see that there is one mode without nodes, and this mode is labelled the "fundamental mode," f. The modes of higher frequencies are concentrated toward the outer part of the star and correspond to acoustic or p modes; the index of a p mode is equal to the number of its nodes. The lower-frequency modes, which are concentrated closer to the center of the star, correspond to gravity or g modes, and the index of a g mode is again equal to the number of nodes in the corresponding eigenfunctions. This classification of modes as p, f, and g modes was introduced by Cowling.[28]

The classification scheme becomes slightly more complicated for the polytropic modes of higher index, as can be seen from Fig. 2 for an index of 4. Regions

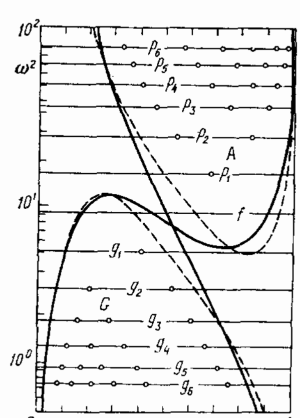

FIG. 1. The polytrope of index 3. The solid curves bound regions A and G. The horizontal lines show the frequencies of several modes ($l = 2$), and the circles show the positons of the nodes. The frequency is given in units of GM/R^3. The dashed curves show the limiting frequencies for plane waves.[30]

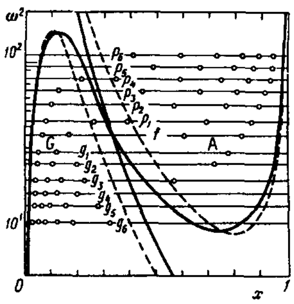

FIG. 2. The same as in Fig. 1, but for the polytrope of index 4.

G and A overlap in frequency, and as a result the oscillations in a certain intermediate frequency interval are of a mixed nature, having some of their nodes in the gravity or G region and some in the acoustic or A region. For each oscillation, the additional nodes enter in pairs as the index of the polytrope is increased, and the number of additional nodes in regions G and A is the same. Formally, the classification scheme is slightly modified: The fundamental mode f is taken to be that for which the number of nodes in region G is equal to that in region A. The index of a p mode is determined by the difference between the numbers of nodes in regions A and G; analogously, the index of a g mode is equal to the number of nodes in region G minus the number of nodes in region A. As before, the frequencies of all the p modes are higher than those of the g modes, and the frequency of the f mode lies between the two cases. In this case the oscillations of the first p and g modes are mixed in nature.

The physical nature of the oscillations is conveniently demonstrated by using an analogy from the theory of plane waves.[30,31]

The two regions in which the solutions are oscillatory in the radius according to calculations based on this analogy are shown by the dashed curves in Figs. 1 and 2. We see a good correspondence between the regions of the acoustic and gravity waves, on the one hand, and the A and G regions on the other. The mixed oscillations (Fig. 2) exist because at these frequencies parts of the star at different depths are undergoing oscillations which differ in physical nature. In other words, in a certain frequency interval gravity waves may be propagating in the interior of a star, while acoustic waves are simultaneously propagating in the outer part.

The classification of the theoretical spectrum in terms of the p, f, and g modes can also be carried out formally by using phase diagrams for the eigenfunctions, e.g., plots of the radial displacement vs. the pressure pulsation.[30]

The use of this classification is illustrated in Fig. 3, which shows the eigenfunctions of the radial displacements $U(x)$ for quadrupole oscillations for three polytropic models of a star having the mass and radius of the sun.[32] Plotted along the abscissa in each part of the figure is the dimensionless radius (the center of the star is at the left, and the surface is at the right). The classification in terms of the p, f, and g modes is specified for each oscillation, along with the period in minutes. The oscillations are arranged in order of increasing period from bottom to top. Where necessary, the radial positions of the nodes are indicated by line segments, while the positions of antinodes are marked by arrows. The circles show the relative amplitudes of the tangential displacements V at the surface. For the model with a polytrope index of 3.7 we can see the appearance of a pair of additional nodes in the p_1, f, g_1 modes, reflecting the mixed nature of these oscillations. These results were obtained through a numerical solution of the complete problem, without the use of Cowling's approximation (the oscillations of polytropic models have been studied by many workers; see Ref. 29, for example).

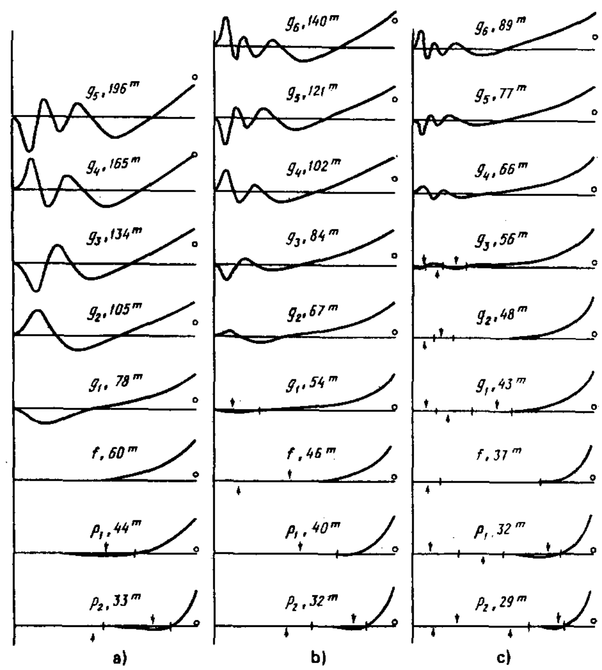

FIG. 3. Eigenfunctions of the quadrupole oscillations of the polytropic models of the sun.[32] The dimensionless radius $x = r/R$ of subscript ⊙ is plotted along the abscissa. The center of the sun is at the left, and the surface is at the right. The normalized amplitude of the radial displacement, $U(x)$, is plotted along the ordinate. Where necessary, the positions of the nodes are marked by line segments, while the antinodes are marked by arrows. The relative amplitudes of the tangential displacement V at the surface are shown by the circles. The p, f, g classification is shown for each oscillation, along with its period in minutes. a—Polytrope of index 3; b—polytrope of index 3.4; c—polytrope of index 3.7.

Figure 4 shows the dependence of the periods of the quadrupole oscillations of the polytropic models on the index of the polytrope.[32] The dashed curve shows the density at the center of the mode. As the polytrope index is reduced to 1.5 (corresponding to the adiabatic model), the periods of all the g modes increase without bound. To illustrate the simple physical meaning of this result, we consider an arbitrary mass element in the interior of the star which is displaced upward by some perturbation, while it simultaneously undergoes an adiabatic expansion. In the adiabatic model the density of this mass element becomes equal to the density of

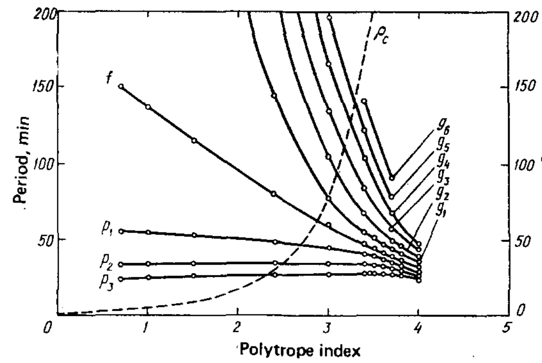

FIG. 4. Periods of the quadrupole (l = 2) oscillations of polytropic solar models, plotted as a function of the index of the polytrope. The dashed curve shows the density of the center of the sun.[32]

the surrounding shells as the mass element rises, so that there is no restoring force, and the periods of the g modes determined by perturbations of this type are infinitely large. For a model with a polytrope index less than 1.5 the adiabatic expansion of the mass element during its ascent leads to a density lower than that of the surroundings. Forces arise which tend to thrust the mass element along the direction of the displacement, so that the model becomes unstable. This instability is analogous to the Rayleigh–Taylor instability for an incompressible fluid with a density which increases in the upward direction. Such unstable g modes have a negative square of the frequency, i.e., an exponential time dependence. A physical manifestation of unstable g modes in stars is convection. If the polytrope index is below the critical value, the motions of the convective type rapidly lead to a redistribution of the mass in the star (over a time of the order of an hour for a star with the mass and radius of the sun), and the polytrope index reaches a value very nearly equal to the critical value, even if the convection is continuously sustained by a heat flux from the interior.

The critical value of the polytrope index is equal to 1.5 only for an ideal gas with $\Gamma_1 = 5/3$. In general, the critical condition for the onset of convection is determined by the vanishing of the parameter A in (2.29). If the star has a homogeneous chemical composition, the condition for the onset of convection is determined by the extent to which the temperature gradient exceeds the adiabatic gradient (the Schwarzschild criterion). If, on the other hand, the average molecular weight increases toward the center of the star, convection might not arise even if there is a superadiabatic temperature gradient. In a model in which the effective polytrope index varies along the radius, the unstable g modes and convection are limited to the corresponding regions of the star.

The oscillations with $l \neq 2$ are classified by analogy with the quadrupole oscillations, with the single exception of the radial oscillations (l = 0), whose spectrum contains no gravity modes, which are by nature nonradial.

We turn now to the oscillation spectra of the detailed modern models of the sun and the giant planets, again from the standpoint of classification of these oscillations. By establishing the physical nature of the oscillations, the classification of the theoretical spectrum leads to conclusions regarding the probabilities for the excitation and detection of the various periods. The theoretical results are usually reported as a spectrum of periods classified in accordance with some scheme or other. When the p,f,g scheme is used it is assumed that the p and f modes are the most likely to be observed, since they have the maximum amplitudes at the surface, in contrast with the g modes, which are concentrated in the interior. This conclusion must be approached cautiously, however, since the formal application of the p,f,g classification may in certain cases mask the actual nature of the individual oscillations.

Figure 5 shows a theoretical spectrum of free oscillations of a modern standard solar model for l = 2, l = 4, and l = 6. The notation is the same as in Fig. 3. The calculations of Ref. 32 used the internal-structure model constructed by Abraham and Iben[33] (model I; the relative abundance of heavy elements is 0.0149; the initial helium abundance is 0.253; the central temperature is $15.38 \cdot 10^6$ K; and the central density is 161.3 g/cm³) and the model of a convection zone with a depth of 198 000 km, calculated by Spruit.[34] This model agrees well with both the internal-structure model and the HSRA model for the solar atmosphere.[35]

Distinctive among the gravity modes with l = 6 is that

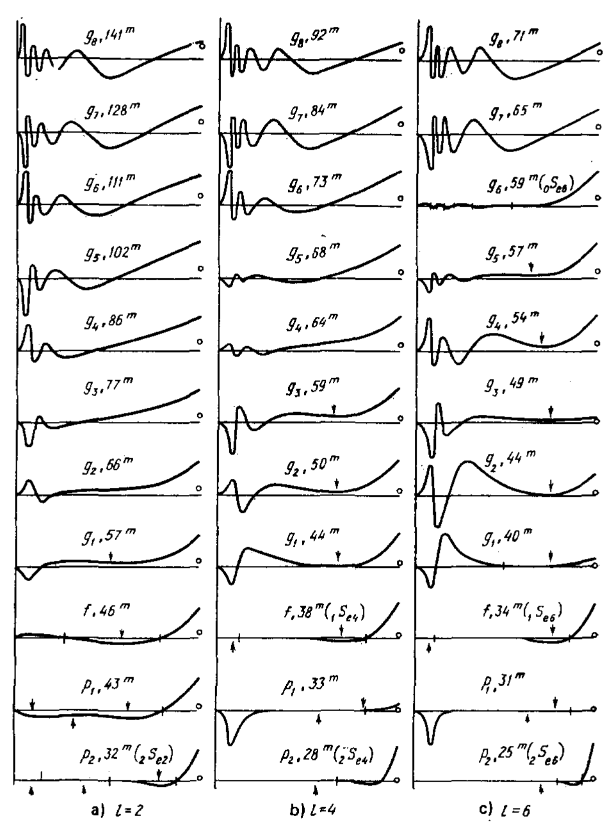

FIG. 5. Eigenfunctions of the various nonradial (l=2, 4, 6) oscillations according to the standard solar model.[32] The notation is the same as in Fig. 3. Shown in parentheses for some of the modes is their classification according to the auxiliary scheme. a—l = 2; b—l = 4; c—l = 6.

with a period of 59 min, whose structure is markedly different from that of other g modes with similar periods. This is actually an oscillation of the outer part of the sun, with very small amplitudes in the interior. The detection of this oscillation would be more probable than that of other g modes of similar period for $l = 6$, since for a given energy this oscillation has an incomparably higher amplitude at the surface. Here we see an important disadvantage of the formal p,f,g classification: It ignores the existence of a distinctive mode of this sort among gravity modes of other periods. Furthermore, the 31-min oscillation with $l = 6$ is formally classified as a p_1 mode, while the amplitudes at the surface are comparatively small.

With reference to these disadvantages, the classification scheme could be refined by taking two different approaches.

The first approach is to introduce an auxiliary scheme in which the oscillations are classified on the basis of their spatial nature.[32] This auxiliary classification is shown in parentheses in Fig. 5 for the oscillations of the outer regions with $l = 6$. Here ${}_0S_{e6}$ denotes the fundamental spheroidal mode (S) of the external region (e) with $l = 6$; ${}_1S_{e6}$ is the second harmonic; etc. The existence of a fundamental mode for the external region has a simple physical meaning. Returning to the incompressible sphere of homogeneous density, we find that all the acoustic modes, whose frequencies become infinite, disappear, as do all the internal g modes, whose frequencies vanish. For each $l = 0$, a single oscillation (a Kelvin mode)[36] remains; in physical nature, this remaining mode corresponds to gravity waves at the surface of an incompressible fluid. This mode is the limiting case of the fundamental mode of the external region in the transition to this simple model. We wish to emphasize that this oscillation is definitely of a gravity nature in a compressible model also, despite the fact that it refers to the external regions. The higher harmonics of the external regions are determined by elastic forces. The classification of these harmonics in the auxiliary scheme is the same as their classification in terms of p modes, or at least it becomes the same at a certain harmonic index. All the other modes corresponding to a given value of l are gravity modes and refer to oscillations of the internal regions. For the standard model with $l = 6$ these other modes are easily classified: The oscillation with a period of 31 min is the fundamental mode of the internal regions, that with 40 min is the second harmonic, etc. The indices of the higher harmonics correspond to the index of the g mode. For $l = 6$, the spectrum therefore breaks up into two parts: one corresponding to the internal regions and one corresponding to the external regions. The two parts of the spectrum overlap somewhat along the period scale. This overlap, which determines the possibility of mixed oscillations, has essentially already been discussed in the p,f,g classification, in which oscillations of a mixed nature—acoustic in the external region and gravitational in the internal region—appear for a polytrope of high index. These discussions of modes are of a local nature, however, and cannot incorporate a distinctive fundamental mode of the external region in the g-mode spectrum.

For $l = 4$ the fundamental mode of the external regions is not unambiguously singled out; it could with equal justification be identified as the 64-min mode (g_4) or the 68-min mode (g_5). In the case $l = 2$, this fundamental mode is not singled out at all; it is mixed with a large number of internal gravity modes, and it gives them substantial amplitudes at the surface. The reason is that with decreasing l the oscillations of the outer regions penetrate deeper toward the center of the star. This mixed nature makes it difficult to apply the auxiliary classification scheme outlined above; if the fundamental harmonic of the outer regions is not distinctive, the p,f,g classification is free of this shortcoming.

A second method for refining the classification scheme is to retain the p,f,g scheme, but to be always alert to the possible existence of a distinct oscillation in the g-mode spectrum, concentrated in the external parts of the star and having maximum amplitudes at the surface.

We turn now to the oscillation spectra of Jupiter and Saturn.[23,37] Models for these planets will be described briefly in Sec. 4 below. From the standpoint of the classification of oscillations, these models have two important characteristic features: an adiabatic structure throughout their volume and the presence of two discontinuities in the radial profile of the material properties. The first density discontinuity results from a phase transition of hydrogen to a metallic state at high pressures; the deeper second discontinuity corresponds to the boundary of the inner core and results from a change in chemical composition.

Figure 6 shows the eigenfunctions for the radial displacements for the oscillations of Jupiter with $l = 0$, $l = 2$, and $l = 8$. All the radial oscillations [Fig. 6(a)] are acoustic in nature. The fundamental mode, ${}_0S_0$, has the longest period. The periods decrease monotonically with increasing harmonic index, and the index of a harmonic is equal to the number of nodes in the eigenfunctions.

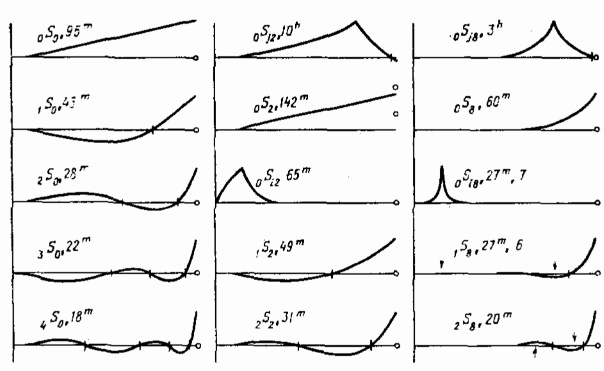

FIG. 6. Eigenfunctions of the oscillations of Jupiter. The notation is the same as in Figs. 3 and 5. A classification scheme based on the spatial nature of the oscillations has been used. The maximum amplitudes of the interior gravity modes correspond to the radial positions of the density discontinuities. (These are the results of Ref. 37, with some additions).

The spectrum of quadrupole oscillations [Fig. 6(b)] contains, in addition to the fundamental ${}_0S_2$ and the acoustic higher harmonics ${}_1S_2$, ${}_2S_2$, etc., two gravity modes, ${}_0S_{j2}$ and ${}_0S_{i2}$. The mode ${}_0S_{i2}$ is the fundamental mode of the oscillations of the inner core and is related to the density discontinuity at the boundary of this core. The ${}_0S_{j2}$ mode is similar in nature and is related to the density discontinuity at the boundary of the region of metallic hydrogen. These two modes do not have corresponding higher harmonics, because of the adiabatic structure of this model. In summary, these two modes exhaust the list of gravity oscillations, and the mode ${}_0S_{j2}$ has the longest period in the spectrum of quadrupole oscillations.

The spectra of the oscillations of higher values of l [see Fig. 6(c) for $l = 8$] have the same structure as the quadrupole oscillations. It can be seen from the comparison of the $l = 8$ and $l = 2$ oscillations that with increasing l the oscillations concentrate near the physical boundaries at which they have their maximum amplitudes. For example, the oscillations of the external regions are displaced toward the surface, while for the gravity oscillations the displacements are concentrated closer to the density discontinuities, which determine these oscillations. The oscillation spectrum for the model of Saturn has a similar structure.

The p,f,g classification scheme cannot be used for the oscillations of the giant planets. The auxiliary scheme discussed above, in which the oscillations are classified as modes corresponding to external and internal regions, is more versatile. It is essentially this scheme which is used to classify the oscillations of Jupiter. The spectrum shown for Jupiter clearly demonstrates the fact that the period of the fundamental mode for the external region may lie among the periods of the internal gravity modes.

If the structure of the inner core in the Jupiter model were not adiabatic, then the fundamental core mode would be supplemented by core harmonics with increasing periods. On the other hand, there would be only a slight change in the nature of the oscillation at the fundamental mode of the external region, since this change in the model would not penetrate into the inner core. The fundamental mode would have a period lying among those of the higher harmonics of the core (g modes in the Cowling scheme); having nodes in the inner core, by virtue of the interaction with these higher core harmonics, this mode would be classified formally as a g mode. This situation also illustrates the distinction of an "anomalous" g mode in the gravity mode spectrum.

The theoretical normal-mode spectrum of a star with a continuous density profile has been studied in detail by Christensen-Dalsgaard.[38] The interaction of the modes has also been studied by Gabriel.[107,108] The spectrum of modes for a star with density discontinuities has been discussed by Gabriel and Scuflaire.[109,110] The properties of oscillations with a large number of nodes have been studied analytically by Wolff.[39] References 38, 39, and 107–110 also contain a more extensive bibliography of the oscillation-classification problem.

d) Effect of rotation

The effect of rotation is found by solving the perturbation-theory equations, (2.20)–(2.22), with the help of the frequencies and eigenfunctions calculated beforehand in zeroth-order.

We denote the volume of the star by V, and we define the following scalar product for any pair of vector functions specified in V:

$$(\mathbf{u}_1, \mathbf{u}_2) = \int\limits_V \rho_0 \mathbf{u}_1^* \cdot \mathbf{u}_2 \, dv. \tag{2.30}$$

The asterisk denotes the complex conjugate. It is not difficult to see that the operators H_0 and $i\hat{\mathbf{z}}\times$, defined in the space of piecewise-continuously differentiable functions which satisfy the free boundary conditions, are Hermitian. Since H_0 is Hermitian, those eigenfunctions $\mathbf{u}_0$ of the zeroth order, (2.20), which correspond to different frequencies are orthogonal. Since H_0 is Hermitian, the stationary perturbation theory for Hermitian operators developed in quantum mechanics can be used.

The degeneracy of the oscillation frequencies with respect to the index m is lifted by the rotation. The first-order correction for the frequency is determined by

$$2\sigma_1 = -(\mathbf{u}_0, \Omega_d [2m - 2i\hat{\mathbf{z}} \times] \mathbf{u}_0), \tag{2.31}$$

which is found after taking the scalar product of (2.21) and $\mathbf{u}_0$ and making use of the Hermitian nature of H_0. The first correction to the eigenfunctions is found in the form of an expansion in the zeroth-order eigenfunctions:

$$\mathbf{u}_1 \equiv \mathbf{u}_{l,m,n,1} = \sum_{(l',m',n') \neq (l,m,n)} a_{l,m,n,1}^{l',m',n'} \mathbf{u}_{l',m',n',0} + \mathbf{u}^T_{l,m,n,1}. \tag{2.32}$$

Here we are using the l,m,n classification of oscillations. The index n determines the classification for fixed values of l and m, as discussed in the preceding section, i.e., the classification in terms of the set of radial eigenfunctions. The last term in (2.32) gives the increment of the torsional or toroidal type. It is determined by the projection of both sides of Eqs. (2.21) onto the space of eigenfunctions of the torsional oscillations, (2.24):

$$\mathbf{u}^T_{l,m,n,1} = \{-\Omega_d [2m - 2i\hat{\mathbf{z}} \times] \mathbf{u}_{l,m,n,0}\}_T. \tag{2.33}$$

The index "T" denotes the projection. We have used the orthogonality of the vector fields of the spheroidal and torsional types, the Hermitian nature of H_0, and the vanishing of the frequencies of the torsional oscillations in the unperturbed problem. The coefficients in the expansion in spheroidal oscillations are determined by taking the scalar product of both sides of Eq. (2.21) with $u_{l',m',n',0}$:

$$\frac{\omega^2_{l,m,n,0} - \omega^2_{l',m',n',0}}{\omega^2_{l,m,n,0}} a^{l',m',n'}_{l,m,n,1} = -(\mathbf{u}_{l',m',n',0}, \Omega_d [2m - 2i\hat{\mathbf{z}} \times] \mathbf{u}_{l,m,n,0}). \tag{2.34}$$

Then, by analogously multiplying (2.22) by $\mathbf{u}_{l,m,n,0} \equiv \mathbf{u}_0$, we find the second-order correction to the frequency:

$$2\sigma_2 - \sigma_1^2 = -(\Omega_d [2m - 2i\hat{\mathbf{z}} \times] \mathbf{u}_0, \mathbf{u}_1) - m(\mathbf{u}_0, \Omega_d^2 [m - 2i\hat{\mathbf{z}} \times] \mathbf{u}_0) + (\mathbf{u}_0, [\mathbf{\Psi} + \mathbf{E}] \mathbf{u}_0). \tag{2.35}$$

The problem thus reduces to one of evaluating volume integrals determined by the zeroth-order functions. For numerical calculations, the angular dependence is singled out and reduced to one-dimensional integrals. The differential-rotation law is specified through an expansion in spherical harmonics:

$$\Omega_d(r,\theta)=\sum_{l=0}^{\infty}\Omega_{dl}(r)\,Y_{l0}(\theta,\varphi). \tag{2.36}$$

For practical calculations, a finite number of terms in (2.36) is sufficient, so that each integral can be evaluated as a finite sum. If the model contains discontinuities in the radial profile of the material properties (as, for example, in the models of Jupiter and Saturn), the surfaces of these discontinuities are deformed when the rotation is introduced. This deformation leads to additional terms in those integrals which determine the effect of the eccentricity of the star. To calculate these terms is a separate additional problem; the method used is that developed in the theory of normal modes of the earth by Woodhouse.[40]

The rapid rotation of the giant planets has strong effects on the normal modes, changing the frequencies of the fundamental modes by some tenths. The second-order correction to the frequency is extremely important in this case; the spectra given for the giant planets at the end of this review were calculated on the basis of the entire second-order perturbation theory. In the calculations for the normal modes of the sun, the rotation of the sun is so slow that first-order perturbation theory is sufficient. The first-order calculations become particularly simple and have a graphic interpretation if the differential rotation is ignored and some average angular frequency $\Omega_\odot$ is adopted for the rotation.

In the first-order for rigid-body rotation, in which only the Coriolis forces are considered, the frequencies of the normal modes are determined by a simple expression which corresponds to the result first derived some time ago by Cowling and Newing[41] and Ledoux[42]:

$$\omega=\omega_0+m\,(1-\tau)\,\Omega_\odot, \tag{2.37}$$

$$\tau=\frac{\int_0^R \rho_0 r^2\,(2UV+V^2)\,dr}{\int_0^R \rho_0 r^2\,[U^2+l\,(l+1)\,V^2]\,dr}, \tag{2.38}$$

where m is the longitudinal index of the spherical harmonic, and the functions U and V are defined in (2.23). Since τ does not depend on m, the multiplet which is degenerate in m is split symmetrically. The normal modes with m of identical magnitude and opposite sign are two waves which are travelling along the equator, one in the direction of the solar rotation and one in the opposite direction. In the absence of rotation, and with equal amplitudes, these waves combine to form a standing wave with antinodes in fixed positions with respect to the observer. When rotation is introduced, the first wave is slowed down slightly, and the second is slightly accelerated. As a result, the spatial pattern of displacements of the resultant oscillation precesses around the rotation axis of the sun. The angular frequency of this precession for an observer in an inertial coordinate system can be found directly from (2.37):

$$\Omega_{\rm prec}=(1-\tau)\,\Omega_\odot. \tag{2.39}$$

Here τ can take on values from 0 to 1. The rotation of the spatial picture of the oscillations with respect to the observer may therefore lag considerably behind the solar rotation. This effect is important for experimental observations made over long time intervals. If the precession frequency can be determined from observations, the results would be extremely informative.

In addition to the rotation effects, it would be interesting to evaluate the effect of the large-scale magnetic fields of the sun and the giant planets on the normal modes. For the giant planets the magnetic field has an effect because of an internal conducting region of metallic hydrogen. The correction to the mode frequency for the magnetic field, $\Delta\omega_{\rm magn}$, depends on the field H, its configuration, and the nature of the mode under consideration. Theoretical studies have been carried out for some special cases,[43,44] and some general estimates of the magnitude of the effect have been made.[16] Simple estimates, on the other hand, are sufficient for finding the comparative magnitude of the effect of the magnetic field for the lower-order modes.[21] In order of magnitude we have $|\Delta\omega_{\rm magn}|\approx|\omega_0|(E_{\rm magn}/E_{\rm grav})$, where $E_{\rm magn}$ is the total magnetic energy, of order H^2R^3, and $E_{\rm grav}$ is the total gravitational energy of the star, of the order of GM^2/R (M is the mass of the star). Taking the correction to the frequency for the rotation, $\Delta\omega_{\rm rot}$, to be of the order of Ω [see (2.37)], we can easily show that the condition $|\Delta\omega_{\rm magn}|\ll|\Delta\omega_{\rm rot}|$ is equivalent to the condition $E_{\rm magn}\ll\sqrt{E_{\rm rot}E_{\rm grav}}$, where $E_{\rm rot}$ is the total rotational energy of the star, of the order of $MR^2\Omega^2$. The latter condition holds for both the sun and the giant planets at $H\ll10^6$ G. The effect of the magnetic field can thus be completely ignored in comparison with the rotation.

e) Excitation of oscillations

The normal-mode problem, covered in the preceding sections, can tell us only the periods and shapes of the normal modes. For a theoretical prediction of the mode amplitudes it becomes necessary to study the excitation and damping mechanisms at a quantitative level. Calculations of this sort run into serious difficulties, and the results reported to date have been largely confined to the study of the linear stability of the oscillations.

The mechanisms which are primarily responsible for the excitation and damping of the solar oscillations stem from perturbations of the region in which fusion energy is released, from dissipation as a result of radiative heat transfer, from perturbations of the radiation flux toward the surface because of perturbations in the opacity, from convective motions, from dissipation because of turbulent friction, and from the emission of acoustic waves into the corona.

The simplest studies of the linear stability of the oscillations are based on the quasiadiabatic approximation, which uses the frequencies and eigenfunctions of the adiabatic problem and treats the deviations from this problem as small perturbations.[16,17] The sign of the imaginary correction to the frequency calculated in this manner determines the stability of the oscillation. The stability of the solar p modes was studied by this approach by Scuflaire *et al.*[45] The stability of g modes during the solar evolution has been studied by Christensen-Dalsgaard *et al.*[46] Since the deviations from the adiabatic case may be quite large in certain regions (particularly in the outer, optically thin shells of the sun) and thus cannot be treated as small perturbations, some more general, completely nonadiabatic, methods have been developed for calculating the complex frequencies of the linear oscillations.[47-49]

A study of the stability of the solar g modes by Saio[111] confirmed the results found in the quasiadiabatic approximation: Some of the lower-order g modes turn out to be unstable during the early stages of the solar evolution. The g_2 mode for $l=1$ (with a period of about 80 min) turns out to be unstable even for a model with an age of $4.5\cdot10^9$ yr. It is not clear, however, whether the amplitudes grow fast enough to mix the matter in the solar interior (this mixing provides one of the ways out of the neutrino problem), since the characteristic time for the growth of the oscillations is very long ($\gtrsim10^6$ yr).

One of the most detailed studies of the linear stability of the acoustic oscillations of the sun in the nonadiabatic approach was carried out by Goldreich and Keeley.[50] They restricted their calculations to radial oscillations. The damping by the turbulence in the convection zone was parametrized by introducing a scalar turbulent-viscosity coefficient. If the damping by this turbulent viscosity is ignored, all the oscillations with periods longer than 6 min turn out to be unstable. The oscillations with shorter periods are stabilized by the radiative damping in the atmosphere. The primary excitation mechanism is an anomalous behavior of the opacity because of the ionization of hydrogen near the upper boundary of the convection zone (the "k mechanism," which has been studied thoroughly in connection with the pulsations of the cepheids[51]). When damping is taken into account, the turbulent friction makes all the modes stable, but the stability margin is small. The large uncertainty regarding the choice of the turbulent-viscosity coefficient leads us to the conclusion that the theory in its present state is not capable of unambiguously solving the problem of the linear stability of acoustic oscillations of the sun. The primary obstacle is the lack of a reliable theory for turbulent convection.

Similar methods have been used by Ando and Osaki[48] for calculations of the linear stability of nonradial oscillations. Those calculations, however, ignored the interaction of the oscillations with the convection. The damping rates found for the nonradial p modes for the low values of l agree with the results found by Goldreich and Keeley for radial modes of similar period, provided that the damping due to turbulent viscosity is ignored. At the same time, there are discrepancies in the estimated damping of the short-period oscillations in the atmosphere.

Studying the possibility that the solar p modes were nevertheless stabilized by turbulent viscosity, Goldreich and Keeley estimated the amplitudes of these modes, working from a stochastic excitation of these oscillations by turbulent convection.[52] The predicted amplitudes range from 10^{-2} cm/s for the fundamental radial mode to 0.6 cm/s for the higher radial harmonics with periods of about 5 min. Such low amplitudes are difficult to reconcile with the observational results reported by the Hill group, but we cannot rule out the possibility that some of the p modes may actually be linearly unstable, despite the turbulent damping. Calculations of the amplitudes of these modes must take into account their nonlinear interaction with other modes, but this complication presents serious theoretical difficulties.

In summary, whether information on the amplitudes of the observed oscillations can be used to study the internal structure of the sun in the near future remains an open question. While the periods of the free oscillations of the sun can be determined quite accurately with a single radial density profile, so that there is the hope that the normal-mode method can in fact be used to refine the model, the solution of the excitation problems requires a detailed physical model of the sun. Even with the detailed model presently available, such studies run into serious difficulties. At present it is not possible to resolve even the question of the linear stability of the oscillations, primarily because we lack a reliable theory for turbulent convection.

3) FREE OSCILLATIONS OF THE SUN

a) 5-min oscillations

Since Leighton, Noyes, and Simon's discovery[4] of the 5-min oscillations of the sun two decades ago, these oscillations have been studied intensely by many research groups. Rhodes *et al.*[53] have published a comparative analysis of some of the most detailed observation methods. A bibliography of the experimental and theoretical work has been published by Dubov[54]; see also Refs. 55–57. In the present review we will describe only some of the recent results.

In the observations the period of the 5-min oscillations has been found subject to random fluctuations over the approximate range 3–7 min. These apparent fluctuations in the period are actually the result of an interference of a large number of oscillations of different frequencies ω, with different horizontal wave numbers k_H, and with different amplitudes. Observations with a high spatial and temporal resolution have yielded the power spectrum of the periodic signal in a plot of k_H vs ω in the form of clearly defined bands.[12,53,58,59] Figure 7 shows one such two-dimensional power spectrum, obtained by Deubner *et al.*[12]

These oscillations penetrate only a short distance into the interior of the sun, so that they may be treated theoretically as a superposition of oblique acoustic waves which are captured in the outer part of the sun[61,60] or global acoustic waves with high values of l {the horizontal wave number is related to l by[62] $k_H^2=[l(l+1)]/R^2$}.[62] Numerical calculations based on both

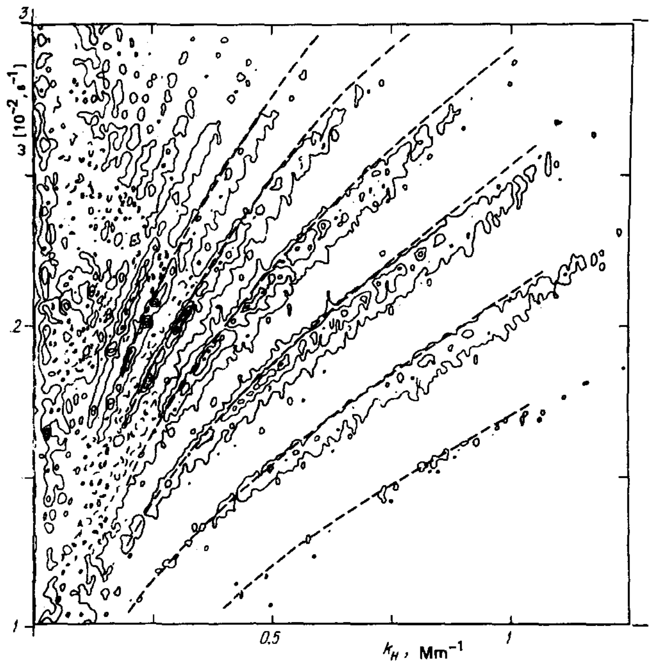

FIG. 7. Power spectrum of the 5-min oscillations on the k_H, ω plane. The contours show the relative power levels; a quadratic scale is used for the power. The lower level corresponds to 2.8% of the maximum power. For convenience, the values of the power were multiplied by $\sqrt{k_H}$ before being plotted. The dashed curves show the theoretical frequencies of the acoustic modes for the model of a convection zone with $\alpha = 2$ (from Ref. 12).

these approaches are in good agreement for similar models.

The observed oscillations involve only the outer shells of the convection zone, but they potentially carry information about the structure of the sun down to the lower boundary of the convection zone, which is determined by the condition for convective stability. The model for the structure of the convection zone based on the mixing-length theory can be parametrized in terms of the free parameter α, which is the ratio of the mixing length to the pressure scale height. Ulrich and Rhodes[61] have calculated theoretical spectra of the acoustic oscillations for various values of α; the dashed curves in Fig. 7 show the theoretical frequencies for the case $\alpha = 2$. The close agreement with the experimental data leaves no doubt that the 5-min oscillations are solar p modes with high values of l.

The resolution which has been achieved is such that we can identify some slight systematic discrepancies with the theoretical frequencies; the theoretical frequencies are slightly higher than the experimental frequencies everywhere on the k_H, ω diagram (except for the lowest-frequency branch). The discrepancies decrease with increasing α, in accordance with an increase in the depth of the convection zone.[61,63] This increase, however, is accompanied by a transition of the base of the convection zone to higher temperatures, and this effect is limited by the observed abundance of lithium. Estimates of the rate at which lithium is burnt up, combined with the present abundance of lithium, impose a lower limit of about $0.62R_\odot$ on the radius of the base of the convection zone. These calculations left a small systematic discrepancy between the theoretical and experimental frequencies,[53] but the discrepancy was not fundamental, and it has now been eliminated by improving the accuracy of the calculations.[64,112] If the calculations are correct (a question which remains open is the correct formulation of the boundary conditions in the atmosphere[65,113]), the convection zone should have a depth of about 200 000 km (Ref. 64).

The structure of the sun below the convection zone does not directly affect the oscillations with which we are concerned here, but we can derive some indirect information about the model from the depth of this zone. From this standpoint, the data on the 5-min oscillations constitute a serious argument against those solar models which have low abundances of heavy elements in the interior (such models have been proposed in an attempt to solve the neutrino problem[66,67]), since such models have a thin convection zone.

Improvements in the quality of the experimental data on the 5-min oscillations have made it possible to use these oscillations to study the solar rotation. For oscillations of such short periods, the rotation has only a small effect, and it is sufficient to use first-order perturbation theory, which incorporates only the Coriolis forces. According to the first-order results, the spatial pattern of the resultant oscillation formed by the superposition of waves travelling in opposite directions is rotating slowly with respect to the observer at an angular frequency $\Omega_{\mathrm{prec}} = (1 - \tau)\Omega_\odot$. For the oscillations under consideration here ($l > 100$) the parameter τ in (2.37) is small, and for a preliminary study it is quite legitimate to ignore the Coriolis forces altogether, setting $\tau = 0$. If a rigid-body rotation is then assumed, the spatial pattern of the oscillations is tied rigidly to the solar rotation; by measuring the velocity at which this pattern drifts across the field of view one can determine the rate at which the solar matter is rotating (the drift velocity is measured by measuring the observable difference between the frequencies of waves travelling toward the east and toward the west, whose superposition is the given oscillation). Although such measurements have been carried out,[68] it has been difficult to determine the absolute value of the rotation velocity accurately because of imperfections of the telescope tracking system.

An even more unusual possibility is that of determining the relative change in the rotation velocity with increasing depth. This possibility arises because of the different depths to which the p modes penetrate into the convection zone. For each p mode the theory furnishes an effective depth at which, in the case of solar rotation differential with respect to the radius, the motion of the spatial pattern of the oscillations is frozen in the velocity field.[69] The drift velocities observed experimentally for the various p modes therefore determine the rotation velocity at the corresponding effective depths.

The results which have been found[12] are shown in Fig. 8. Despite the large scatter in the data there is a visible tendency for the rotation velocity to increase with

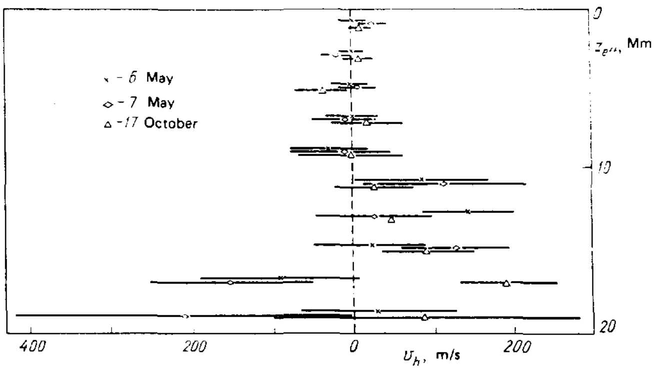

FIG. 8. Relative rotation velocity at the equator, plotted as a function of the depth below the photosphere. The average rotation velocity is normalized to zero for the four upper levels. A positive velocity corresponds to a rotation more rapid than in the photosphere. These results were obtained from the data of three days of observations in 1977 (Ref. 12).

increasing depth. Such an increase in the velocity, of about 80 m/s down to a depth of about 11 000–15 000 km, is in agreement (within the measurement error) with the observed difference in the rotation periods of sunspots and of the photospheric plasma (the most recent results on the solar rotation are reviewed by Howard[70]). Below 15 000 km the situation is less clear, although a tendency toward a decrease in the velocity is not ruled out. There is the hope that this unique method for measuring the differential velocity field in the convection zone will be significantly improved in the future.

Doppler measurements of the velocities in the integrated light from the entire solar disk (without spatial resolution) have been published in the last two years, and these results also show a large number of oscillations with periods near 5 min (Refs. 71 and 114–116). The most detailed data have been obtained in observations from the south pole.[116] The frequencies of the peaks in the power spectrum of the Doppler signal are spaced approximately uniformly with an average separation of 68.0 μHz.

Such oscillations cannot be explained by the interpretation described above, since the high-l oscillations with a large number of waves over the solar disk cannot be observed in a spatially averaged signal. It has now been established beyond doubt that these oscillations are higher acoustic harmonics with low values of l. In the asymptotic limit of high harmonics (when the index of the harmonic is considerably higher than l), the frequencies in the theoretical mode spectrum are also spaced in an equidistant manner.[72,117] At a fixed value of l the difference between the frequencies is given by

$$\Delta\nu \approx \left(2\int_0^R c^{-1}dr\right)^{-1},$$

where c is the adiabatic sound velocity. The frequencies for different even values of l are nearly equal, and the same is true of the odd values of l, for which the frequencies lie roughly halfway between the even-l frequencies. The theoretical values of $\Delta\nu$ are slightly sensitive to the choice of model and are roughly twice the observed value of 68.0 μHz. It follows that oscillations with different values of l are excited. Analysis of the alternation of peaks in the power spectrum and a comparison of the theoretical predictions[116] have made it possible to identify modes with $l = 0, 1, 2, 3$, and, possibly, 4. The corresponding harmonic indexes run from about 15 to 35. Such oscillations penetrate deep into the solar interior and bear information not accessible to the high-l oscillations. The results from the first comparisons with the theoretical spectrum seemed to argue in favor of models with a reduced abundance of heavy elements,[71,73] but this conclusion has been refuted by more detailed calculations incorporating the effect of the solar atmosphere. The model of the standard chemical composition has proved to agree well with the observed frequency distribution.[118]

b) Oscillations with periods of 7-70 min

Several research groups have obtained data which may indicate free oscillations of the sun in this range of periods. Kaufman[74] detected a period of about 41 min in recordings of the solar microwave emission. Kobrin and Korshunov[75] have observed fluctuations with a period of about 50 min in the difference between the intensities of the solar radio emission at two adjacent frequencies; in a study of longer recordings, this period split into two other periods, about 57 and 33 min. Fosset and Ricort[76] detected oscillations with a period of about 40 min in the average velocity field in the photosphere. Brooks *et al.*[7] reported observing oscillations with periods of 58 and 40 min in the Doppler shift of a solar absorption line (in addition to the 160-min oscillations, which will be discussed in the following section). Gal'per *et al.*[77] have observed variations in the γ flux density in the upper atmosphere of the earth with periods of 11.7 ± 0.1, 12.7 ± 0.1, 7 ± 0.1, 15.8 ± 0.2, 23.2 ± 0.2, and 33 ± 1 min.

The most detailed results have been obtained by Hill and his colleagues in observations carried out at the Santa Catalina Laboratory for Experimental Relativity by Astrometry (SCLERA). In measurements of the oblateness of the solar disk which were undertaken as an experimental test of the general theory of relativity, a large number of periods were detected in the fluctuations of the solar diameter.[5,78,79] Repeated observations have confirmed the reality of these oscillations.[78,79,119] Some of the periods are listed in Table I. The observed amplitudes are of the order of $10^{-6}R_\odot$. The actual displacement amplitudes may be much smaller than the observed amplitudes if the resultant signal is determined primarily by brightness pulsations at the limb of the

TABLE I. Periods (in minutes) of the pulsations in the solar diameter according to the average data from 11 days of observations in 1975 (Ref. 80). The periods marked with asterisks might have been detected as the result of anomalously large peaks in the power spectrum in the data from only one or two observation days.

66.16	28.7	13.3	9.9	7.6
44.66	24.8	12.1	9.3	6.9
39.0	21.0*	11.4	8.5*	6.7
32.1	19.5	10.7	7.8	6.5

disk, rather than by surface displacements.

At present we have essentially no information on the spatial pattern of the oscillations, i.e., on the probable values of l, which may correspond to the observed periods. The method used to observe the pulsations of the solar diameter is capable of detecting oscillations with l up to several tens,[11,80] so there is no possibility of an unambiguous identification with the theoretical spectrum, in which the range of 7–70 min is densely packed with the periods of radial and nonradial acoustic oscillations. Table II shows the theoretical oscillation periods of the standard solar model with a relative heavy-element abundance $Z = 0.02$ according to calculations by Iben and Mahaffy.[81] Taking into account the errors in the experimental and theoretical periods (which are at least 1–2%), we conclude from a comparison of Tables I and II that the identification is extremely ambiguous. Furthermore, the theoretical periods depend on the choice of model (for example, with $Z = 0.01$ and $Z = 0.03$, the maximum periods of the radial oscillations are 56.22 and 66.14 min, respectively).[81] It is thus not surprising that the periods which have been detected can be identified with the theoretical spectra of some wildly different models.[81-83]

A detailed analysis of observations of the solar diameter[11] has, however, yielded indications that the oscillations with periods of about 66 and 45 min probably have high values of l, in the range $20 < l < 40$. If this is the case, then these two oscillations could not be acoustic modes, whose periods at these values of l are much shorter. At the same time, an attempt to make an identification with gravity modes in the standard model also runs into serious difficulties. According to calculations by Dziembowski and Pamjatnykh,[84] the amplitudes of the g modes with such high values of l are concentrated in the interior of the sun and decrease by a factor of at least 10^5 in the transition through the convection zone to the surface. It is doubtful that such oscillations could be observed, because the amplitudes in the interior would have to be exceedingly high. This difficulty does not arise for models with a low value of Z in the interior, whose thin convection zones correspond to a slighter damping of the amplitudes as the modes propagate to the surface.[85] Such models, however, do not agree with the data on the 5-min oscillations, in addition to having other difficulties.[85]

TABLE II. Mode periods according to the standard solar model with a relative heavy-element abundance of $Z = 0.02$ according to the calculations of Iben and Mahaffy.[31]

Mode	Period, min					Mode	Period, min			
	$l=0$	$l=1$	$l=2$	$l=3$	$l=4$		$l=1$	$l=2$	$l=3$	$l=4$
$p1$	62.29	57.25	42.50	39.53	37.58	f		45.90	40.97	38.82
$p2$	40.94	36.98	32.19	29.42	27.62	$g1$	61.58	55.05	47.94	44.18
$p3$	30.93	27.88	25.09	23.21	21.92	$g2$	84.4	63.03	54.88	49.59
$p4$	24.49	22.30	20.52	19.26	18.31	$g3$	105.8	72.58	61.88	57.73
$p5$	20.19	18.68	17.39	16.44	15.72	$g4$	127.3	83.49	67.76	61.11
$p6$	17.17	16.04	15.10	14.38	13.81	$g5$	149.2	95.38	74.9	64.89
$p7$	14.93	14.08	13.35	12.77	12.32	$g6$	171.1	107.7	83.1	70.30
$p8$	13.21	12.55	11.97	11.51	11.14	$g7$		120.2	91.8	76.83
$p9$	11.86	11.34	10.87	10.49	10.18	$g8$		132.9	100.7	83.62
$p10$	10.78	10.35	9.97	9.65	9.39	$g9$		145.9	109.7	90.56
$p11$	9.90	9.54	9.21	8.94	8.71	$g10$		158.9	118.9	97.62
$p12$	9.15	8.84	8.56	8.32	8.11	$g11$		172.1	128.1	104.5
$p13$	8.50	8.23	7.99	7.78	7.60	$g12$			137.6	111.7
$p14$	7.94	7.71	7.49	7.31	7.15	$g13$			147.0	118.9
$p15$	7.45	7.25	7.06	6.89	6.75	$g14$			156.5	126.5
$p16$	7.02	6.84	6.67	6.52	6.39	$g15$			166.7	133.3
$p17$	6.64	6.47	6.32	6.18	6.06	$g16$			175.9	141.5
$p18$	6.29	6.14	6.00	5.87	5.77	$g17$				148.6
$p19$	5.98	5.84	5.71	5.60	5.50	$g18$				156.4
$p20$	5.69	5.56	5.45	5.34	5.25	$g19$				164.0
						$g20$				171.1

Another way to eliminate the difficulties presented by the oscillations at 45 and 66 min is related to the distinctive or anomalous g modes in the theoretical spectrum, with large amplitudes at the surface. For the standard models at high values of l, however, these oscillations have periods which are too low.

Boury *et al.*[122,123] have studied the oscillations of a solar model with small and zero initial abundances of hydrogen in a small central region (including 3% of the solar mass). The density discontinuity in such a model leads to the appearance of distinct modes in the theoretical spectrum which are associated with the discontinuity. This model does not, however, simplify the problem of identifying the oscillations: The theoretical spectrum turns out to be very dense in the range 7–70 min.

The reality of the observed oscillations still raises some doubt. The oscillations found from the observations of the solar diameter could not be confirmed in measurements of the velocities[86,87] or brightness pulsations.[88-90] Hill and Caudell[11] and Knapp *et al.*[120] have reviewed the continuing discussion of the possible reasons for the discrepancies. An exceptionally important argument for identifying the observed periods with free oscillations of the sun is the phase coherence of the observed oscillations[11,119,121] (in preliminary estimates,[11] the statistical significance of the observed coherence was greatly overestimated[91]). If future experiments confirm the reality of these oscillations, then more accurate and more reliable information on the spatial structure of the oscillations will be required for an unambiguous identification with the theoretical spectrum and for the use of this information to refine the model for the internal structure of the sun.

c) 160-min oscillations

The detection of these oscillations in the Doppler shift of absorption spectral lines was first reported by Severny, Kotov, and Tsap[6] and, independently, by Brooks, Isaak, and van der Raay.[7] In observations at the Crimean Astrophysical Observatory,[6,92] the difference in the velocities along the line of sight from the central part of the solar disk, with a radius of $0.66R_\odot$, and from the limb was measured. Observations by a Birmingham group[7] were carried out in the integrated light from the entire solar disk. Oscillations of the same period have been detected in observations by Snider *et al.*[8] The 160-min period has also been found in variations of the geomagnetic field[9] which are possibly of solar origin.

The extensive observations at the Crimean Astrophysical Observatory over the years 1974–1978 demonstrated a high phase coherence of the oscillations and made it possible to refine the value of the period[93,94]: 160.01 min. Similar observations carried out at Stanford

University[10] yielded the same period and were also in good agreement in terms of the phase. These results have been confirmed by observations in 1979 at both observatories.[124] The research at the Crimean observatory also showed that there are changes in the brightness, the general magnetic field of the sun[92,95] and its radial emission[96] which are synchronized with the velocity oscillations (whose amplitude is about 1 m/s). A detailed study of the power spectrum of the oscillations has also revealed several similar periods: 134.398, 148.359, 171.099, 175.061 min (Ref. 92). The amplitudes corresponding to these additional periods are much smaller. The detection of these periods is less reliable and may to some extent be an artifact of the analysis of the observations.

The reality of the 160-min oscillations has been questioned on several points. It has been suggested that the observed 160-min period may be caused by the passage of solar supergranules across the field of view.[97] This interpretation has turned out to be incompatible with the phase coherence of the signal, which has been observed over several years. Some doubt was raised by later observations by the Birmingham group, which failed to confirm the presence of the 160-min oscillations.[125] Since the period of 160 min is precisely 1/9 of a terrestrial day, and the long series of observations, carried out only in daytime, has been modulated by the earth's rotation period, serious doubt has been raised regarding the solar nature of the oscillations. In particular, it has been suggested that these oscillations could be a consequence of fluctuations in the transparency of the earth's atmosphere.[98] Long-term observations in the Crimean and at Stanford have shown, however, that the exact value of the period is slightly higher than 160 min and is thus not precisely 1/9 of the terrestrial day. Possible consequences of certain terrestrial effects have also been the subject of a special study,[126-128] and the results speak in favor of a solar nature of the oscillations.

Some unique observations, free of the effects of the alternation of day and night, were carried out at the south pole in late December 1979 and January 1980 (Ref. 116). Doppler measurements of the velocities were carried out by a method of resonant spectroscopy in the integrated light from the entire solar disk. Oscillations with a period of 160 min were extracted in an analysis by the superimposed-epoch method (this method has been used at the Crimean and Stanford observatories) of the data from 5 days of continuous observations. The oscillations found are in good agreement with the Crimean and Stanford data in terms of amplitude and phase.

The observation method which has been used has so far not yielded an unambiguous determination of the mode index l, but it is most probably $l = 2$ (the quadrupole mode). It is difficult to detect oscillations with higher values of l because the signal is averaged over large regions on the solar disk. Radial oscillations could not have such a long period (the only exception to this statement is for the solar model with a homogeneous density). The detection of dipole oscillations ($l = 1$) would be probable only if there were a significant difference between the amplitudes of the radial and tangential displacements (U and V) at the surface: If these amplitudes are approximately equal, the dipole oscillations correspond to translational motions of the surface as a whole, and it would be difficult to detect these motions in measurements of the velocity difference.

Attempts to interpret the 160-min oscillations in the standard solar model run into serious difficulties. In the theoretical normal-mode spectrum this period corresponds to high-index gravity modes. At a formal level, there is no difficulty in identifying this oscillation with one of these modes (for example, g_{10} for $l = 2$); the periods can be matched exactly for example by slightly adjusting the value of Z in the model. Gravity modes with such periods are linearly stable in the standard model. The excitation of turbulent convection could not lead to the observed amplitudes.[129] It is also difficult to explain why this particular g mode has been excited, while no others have been, over a time of several years. The possibility is not ruled out that this selective excitation is of a resonant nature and is a consequence of an interaction with other oscillations, because of (for example) an approximately equal beat period of two radial modes.[81] It is difficult to find a more convincing interpretation in the standard model. No help comes from the possibility, discussed above, of singling out in the theoretical spectrum an anomalous g mode (the fundamental mode for oscillations of the outer regions). If the mixed nature of the oscillations did not prevent it, this fundamental mode in the case $l = 2$ could have a rather long period, as has been shown by calculations carried out by a numerical algorithm with a formal suppression of the interior gravity oscillations.[99] The actual interaction of the oscillations of the outer and inner regions at low values of l, however, is so pronounced that the fundamental mode of the outer regions is not singled out as an isolated mode in the theoretical spectrum: All the g modes have essentially the same amplitude (Fig. 5). Anomalous g modes are clearly distinguished at $l \leq 6$, but their periods are too short.

The identification of the 160-min oscillations with the theoretical normal-mode spectrum based on the standard solar model thus requires a problematic mechanism which leads to the selective excitation of only one of several modes of similar properties and similar periods. These difficulties, as well at the neutrino problem, may mean that the actual structure of the sun is quite different from that predicted by the standard evolutionary model. A question of interest for future research, therefore, is that of which change in the model might lead to a distinct, isolated oscillation of this particular period in the theoretical spectrum. A circumstance which would be of assistance here is that, from the standpoint of adiabatic oscillations, the model can be specified quite accurately by specifying simply the radial density profile. One likely possibility is a density discontinuity. Such a discontinuity could make the oscillations of the outer and inner regions largely independent of each other, and it could also determine oscillations of a new type, with maximum amplitudes at the discontinuity (the spectrum of Jupiter in Fig. 6 illustrates one such core oscillation). The density discontinuity could arise because of an inhomogeneity of the chemical composition. A model of this type, with an anomalously high central abundance of heavy elements and with a convective core, has been proposed by Hoyle[100] as a possible solution of the neutrino problem. In this model a solar core with a mass of $(0.3-0.5)M_{\odot}$ has a low initial abundance of helium, $Y\approx 0.04$ (so that the present value is $Y\approx 0.15$), and a high abundance of iron-group metals ($Z = 0.075-0.15$ in the cases considered). The high opacity caused by these metals makes the core convective. The low emission of high-energy neutrinos results from a high concentration of hydrogen, $X\approx 0.7$, which is maintained at the center of the sun and also from the rapid convective mixing of ^7Be, which retards the reaction $^7Be(p,\gamma)^8B$. Hoyle's model predicts that the solar luminosity remains roughly constant over the history of the earth. It would be interesting to study models of this type to see the implications for the theoretical normal-mode spectrum, in which a "core" mode can be singled out for each value of $l\neq 0$. This mode might be excited as a result of convective motions in the core. A slightly different possibility was considered by Zatsepin *et al.*[101]: Working from their calculations for multilayer polytrope models of the sun, they suggested identifying the 160-min period with one of the lowest-order quadrupole g modes of a core in a nearly convective state. A periodic mixing of the matter in such a core would also reduce the neutrino production.

Observations of the 160-min oscillations have revealed regular variations in their amplitude with a period equal to or approximately equal to the 27-day solar rotation period.[93,94] Such a modulation could be explained directly by a precession of the spatial picture of the oscillations caused by Coriolis forces.[102] If the oscillations are quadrupole oscillations (if they have two diametrically opposite antinodes which are in phase), then the observation period would correspond to a rotational-splitting parameter $\tau\approx 1/2$. Once the 160-min oscillations have been unambiguously identified with a theoretical mode spectrum, measurements of the period of this modulation could yield information about the rotation in the solar interior. The observed amplitude modulation is a further argument in favor of a solar nature for the observed oscillations.

4. THEORETICAL NORMAL-MODE SPECTRA OF JUPITER AND SATURN

a) Models for the internal structure

Modern detailed models for the internal structure of Jupiter and Saturn have been used for calculations of the free oscillations of these planets.[15] These models are adiabatic, having two shells with different chemical compositions. Each shell consists of a homogeneous mixture of hydrogen and helium with some admixture of heavy components. The core consists exclusively of heavy components, water, methane, ammonia, silicates, and iron in their solar proportions. The models have been constructed from accurate equations of state and are consistent with the existing observational data: the mass, the radius, the gravitational moments J_2 and J_4, and the boundary conditions on the pressure and temperature in the atmosphere.

b) Normal-mode spectra

The basic characteristic properties of the models, which determine the structure of the mode spectrum, are the adiabatic structure throughout and the presence of two discontinuities in the radial density profile. The inner discontinuity corresponds to a change in the chemical composition at the core boundary, and the second discontinuity corresponds to a phase transition of hydrogen to a metallic state at high pressures. The general structure of the theoretical spectrum in the absence of rotation (in zeroth-order perturbation theory) was discussed above in connection with the classification of the oscillations (Fig. 6). The existence of two density discontinuities determines two modes with maximum amplitudes at these discontinuities for each oscillation index $l\neq 0$. Since the model has an adiabatic structure, these two modes constitute all the interior gravity oscillations. These oscillations are sensitive to the detailed structure of the core and to a possible radial blurring of the density discontinuity associated with the phase transition of hydrogen. In this regard the present ideas regarding the internal structure of the giant planets are extremely tentative. Such interior oscillations have, furthermore, relatively small amplitudes at the surface. The first step to be taken, therefore, is to study the fundamental modes with the maximum amplitudes at the surface and the acoustic higher harmonics.

The theoretical mode spectra have been calculated in second-order perturbation theory incorporating all the rotation effects: the Coriolis forces, the centrifugal forces, and the shape deformation of the planet. Both rigid-body and differential rotations have been considered.[23,24] In the differential-rotation case, some simple model distributions of cylindrically symmetric differential rotations have been used; the angular velocity of the rotation, Ω, is a function of the distance from the rotation axis, and the function is chosen to be quadratic:

$$\Omega=\Omega_0\left[1+0.01\left(\frac{r}{R}\sin\theta\right)^2\right]\text{— for Jupiter,} \tag{4.1}$$

$$\Omega=\Omega_0\left[1+0.1\left(\frac{r}{R}\sin\theta\right)^2\right]\text{— for Saturn.} \tag{4.2}$$

On the average, these model distributions reflect the observed increase in the rotation velocity toward the equator. The value of Ω_0 corresponds approximately to a 10-h rotation period.

Figures 9 and 10 show the mode spectra which have been derived for Jupiter and Saturn. For a clear picture of the rotation effects, the results are shown in a coordinate system which is rotating at the angular frequency Ω_0. The planet is at rest in this system in the case of a rigid-body rotation, and if rotation effects are ignored the oscillation periods are degenerate in the azimuthal index m. These degenerate periods correspond to the zeroth-order problem[37] and are shown by the circles. For each value of l, the fundamental

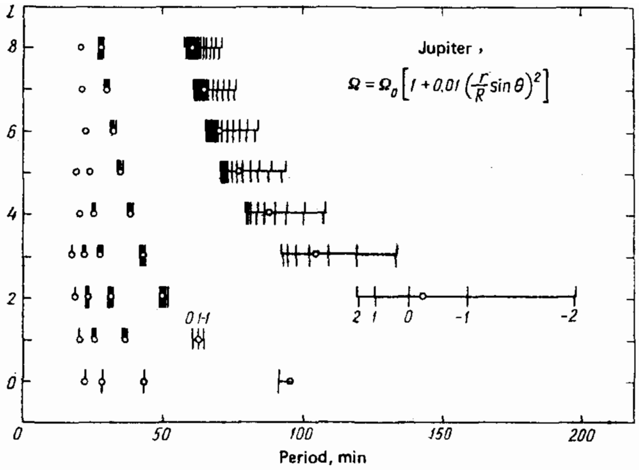

FIG. 9. Normal-mode spectrum of Jupiter. The circles show the mode periods in the zeroth order. Shown above the circles is the splitting of the multiplets for the case of a rigid-body rotation; the corresponding splitting for the case of a differential rotation is shown below the circles. The values of the index m are shown near two of the multiplets. The results are given in a coordinate system which is rotating at the angular velocity Ω_0 (Refs. 23 and 24).

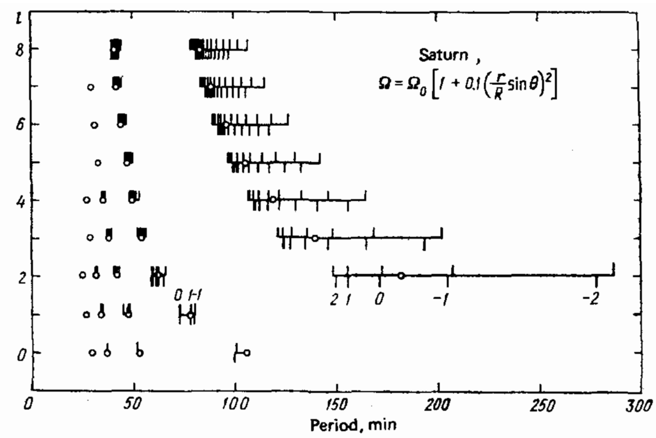

FIG. 10. Normal-mode spectrum of Saturn. The notation is the same as in Fig. 9.

modes have the longest periods. In the short-period range there are branches of higher acoustic harmonics; the periods of the interior gravity modes are not shown. The vertical line segments show the splitting of the multiplets caused by the rotation. The results shown above the horizontal lines correspond to rigid-body rotation, and those below these lines correspond to the differential rotation. The sequential order of the mode periods in terms of the index m is shown near two of the multiplets; this order is characteristic of the entire spectrum.

Beginning with the results for the rigid-body rotation, we see that multiplets of the fundamental modes exhibit the greatest splitting. At low values of l the rotation has its effects primarily through the Coriolis forces. These forces cause waves which are travelling in the direction of the rotation ($m < 0$) to be slowed, with the result that the period is increased, while the waves travelling in the opposite direction ($m > 0$) are accelerated. We can also see an asymmetry in the splitting, which becomes more pronounced with increasing mode index l. At high values of l the mode periods are much shorter than the rotation period, and the splitting is caused primarily by the eccentricity of the shape of the planet. In the limit $l \to \infty$, the oscillations are displaced toward the surface and become analogous to surface waves. In the case $m = 0$, these oscillations are formed by waves which are travelling along the meridian; in the case $|m| = l$ these oscillations correspond to waves which are propagating along the equator. Because of the eccentric shape, the distance around the planet along the equator is longer than that along a meridian, so that the periods of the modes with $|m| = l$ are larger than those with $l = 0$. The eccentricity thus causes the wings of the multiplets to acquire periods longer than those of the $m = 0$ central line. For the higher acoustic harmonics there is a generally smaller splitting because of the shorter periods.

The differential nature of the rotation has greater effects in the case of Saturn (Fig. 10). It affects the fundamental modes most. In general, it makes the splitting of the multiplets somewhat less than in the case of the rigid-body rotation. In the case of the differential rotation, the outer equatorial regions of the planet are rotating slightly faster than the rotating coordinate system. Because of this effect, waves travelling in the direction of the rotation ($m < 0$) receive an additional acceleration in the direction of the motion, so that their periods decrease. In the case $m > 0$, the effect is the opposite. With increasing l, the effect of the differential rotation intensifies, because the oscillations are displaced toward the surface, where the inhomogeneity of the rotation is at maximum.

c) Interaction of modes

It can be seen from Figs. 9 and 10 that the effect of the rotation on the normal-mode spectra of the giant planets is quite pronounced. The small parameter of the perturbation theory—the ratio of the zeroth-order oscillation frequency to the rotation frequency—reaches a value of 1/4 for the fundamental quadrupole mode of Jupiter and a value of nearly 1/3 for Saturn. The multiplets split by the rotation overlap along the period scale, raising the question of just how much we can trust the perturbation-theory results.

The actual mode frequencies may be quite different from the perturbation-theory results because of the interaction between different modes. The eigenfunctions (the shapes of the oscillations) of a rotating planet are mixed in nature and consist of a superposition of the eigenfunctions of certain of the zeroth-order modes. If the frequencies of the interacting modes are approximately equal, and the rotation is sufficiently rapid, then the corresponding corrections to the zeroth-order eigenfunctions may be substantial.

At the same time, the accuracy with which the mode frequencies are calculated by perturbation theory is determined by the accuracy with which the eigenfunctions are calculated. Consequently, if the correction to the eigenfunctions for the substantial interaction with adjacent modes is ignored (this correction may arise only in the higher orders of the perturbation theory), the accuracy of all the results may suffer severely.

The perturbation theory has been modified for a quantitative account of the mode-interaction effects.[22,24] The case of a rigid-body rotation has been studied. The modification follows the method used in quantum-mechanical perturbation theory for Hermitian operators in the case of a quasidegeneracy.[103] In the initial equations, (2.15), the operator H_0 of the zeroth-approximation problem ($\omega_0^2 \mathbf{u}_0 = \mathbf{H}_0 \mathbf{u}_0$) is replaced by the operator

$$\mathbf{H}_p = \mathbf{H}_0 + (\omega_0^2 - \omega_{01}^2)\,\mathbf{P}^1 + (\omega_0^2 - \omega_{02}^2)\,\mathbf{P}^2; \qquad (4.3)$$

where ω_{01} and ω_{02} are the zeroth-order frequencies of the two adjacent modes with which the mode in question, of frequency ω_0, is interacting; and $\mathbf{P}^1$ and $\mathbf{P}^2$ are the operators which project onto the zeroth-order eigenfunctions, $\mathbf{u}_{01}$ and $\mathbf{u}_{02}$, respectively, of these adjacent modes ($\mathbf{P}^1\mathbf{u}_{01} \equiv \mathbf{u}_{01}$; $\mathbf{P}^2\mathbf{u}_{02} \equiv \mathbf{u}_{02}$). This replacement of the unperturbed operator reduces the interacting modes to a common frequency, the frequency of the zeroth approximation, ω_0. A perturbation theory is constructed for this problem with an artificial degeneracy, and a conversion is made from the results to the solutions of the original problem. This algorithm evaluates the correction to the eigenfunctions for the interaction of adjacent modes and determines how this correction affects the oscillation frequency.

The symmetry of the problem dictates the selection rules for the interacting modes. Only modes with identical values of m can interact; a mode with index l interacts with modes with l and $l \pm 2$.

Numerical calculations have been carried out on the basis of this modified perturbation theory. The interaction of the fundamental mode l with the fundamental $l+2$ and $l-2$ modes has been studied. The results are illustrated by Fig. 11, which shows the multiplet of the fundamental $l = 4$ mode in expanded scale. The results shown above the horizontal line are the results of the ordinary perturbation theory, while those shown below the linear are the results of the perturbation theory modified to incorporate the mode interaction. The slight differences in the results reflect this interaction. The resulting corrections to the periods, however, are much smaller than the total splitting. This result confirms the applicability of the ordinary perturbation theory, although the splitting of the multiplets is greater than the separation of the unperturbed periods. This unusually broad applicability of perturbation theory for the normal-mode problem of a rotating planet results

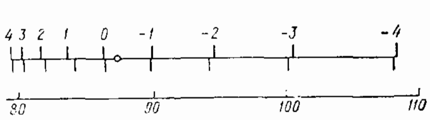

FIG. 11. Multiplet of the fundamental $l = 4$ mode of Jupiter. The circle shows the unperturbed period. The results shown above the line are those of the ordinary perturbation theory, while the results shown below the line are those of the perturbation theory modified to incorporate the mode interaction.[22,24]

from the stringent selection rules, which place a severe limit on the number of interacting modes.

We wish to thank A.B. Severnyĭ, V.A. Kotov, and T.T. Tsap for useful discussions.

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Translated by Dave Parsons