Solutions of the instanton type in chiral models

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A review of the solutions of the instanton type in chiral modes of the field theory is given.

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CONTENTS

INTRODUCTION

The remarkable discovery by Gardner, Green, Kruskal, and Miura[1] of a new method for integrating nonlinear differential equations, the so-called method of the inverse scattering problem, together with the algebraic formulation of this method given by Lax,[2] has led to the possibility of studying a broad class of nonlinear equations which could not be dealt with by older methods.

Detailed expositions of this method can be found, for example in a recent monograph,[3] and one of a number of review articles.[4] Here we merely point out that in the method of the inverse scattering problem the solution of a given nonlinear equation is reduced to the solution of linear problems, namely to finding the scattering data for a given potential and the inverse problem of finding the potential from the scattering.

For the simplest integrable nonlinear equation, that of Korteweg and de Vries,

$$\frac{\partial u}{\partial t}=6u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3},\qquad u=u(x,t)$$

the solution consists of three stages.

1) From the potential $u(x,t)$ given at the initial time $t=0$, $u_0(x)=u(x,0)$, one finds the scattering coefficient $r_0(k)$ for the Schrödinger equation

$$-\frac{\partial^2\psi}{\partial x^2}+u_0(x)\,\psi=k^2\psi.$$

2) The scattering coefficient $r(k,t)$ at an arbitrary time t can be found from the formula

$$r(k,t)=\exp(i8k^3t)\,r_0(k).$$

The last step in the procedure is to find the potential $u(x,t)$ from the scattering coefficient $r(k,t)$, i.e., to solve the inverse scattering problem for the Schrödinger equation.

In its degree of effectiveness the inverse scattering method is analogous to the Fourier transformation method for linear equations with constant coefficients and has found wide application in all branches of theoretical and mathematical physics.

The use of this method in field theory has led to striking progress in our understanding of the structure of both classical and quantum nonlinear field theories in two-dimensional space-time, outside the framework of perturbation theory.

In recent years many papers have been devoted to the study of the solutions of the equations of classical field theory. Among these we point out a number[5-92] in which more detailed information and references to the literature can be found.

As a result of these researches a new type of solutions has been found, which play an important part in physics but cannot be derived in the framework of the ordinary theory.

These include, for example, the so-called soliton solutions, or simply solitons, i.e., localized solutions of wave equations which preserve their shape and size for arbitrarily long times. Such solutions were first discovered in the last century in the equations of hydrodynamics (cf. Refs. 3,4).

The existence of such solutions in nonabelian gauge field theories used in the theory of elementary particles was discovered by 't Hooft[5] and Polyakov.[6] The solutions turned out to be stable; they cannot smear out and disappear, for topological reasons; that is, in just the same way that a knot tied in a rope cannot be removed without cutting the rope. Another important property is that the particles corresponding to these solutions can be monopoles, i.e., they can carry an isolated magnetic charge (the properties of such solutions have been examined in detail in Ref. 7).

Another type of solutions of nonlinear equations is the so-called instantons, i.e., solutions localized in a small region in both space and time. Instantons cannot be interpreted as real objects, but rather as processes, not as particles, but rather as quantum-mechanical transitions between different states of particles.

The existence of instantons, like that of solitons, is primarily due to topological causes and is not associated with the specific form of the Lagrangian of the theory in question, but with the geometric structure of the fields involved in the Lagrangian.

Soliton and instanton solutions have been found for many equations of field theory: In the Yang-Mills gauge theory,[8-17] (see the reviews in Refs. 15—17), in the Einstein gravitation theory,[18-25] and in the simplest essentially nonlinear field-theory models, the so-called chiral models.[27-92] An elementary treatment of this type of solutions is given in Ref. 26.

The chiral models are field-theory models in which the interaction is introduced not by adding to the free-field Lagrangian an interaction Lagrangian, but in a purely geometric way.

Namely, in such models the Lagrangian is kept unchanged from the free-field case, but restrictions are placed on the field itself, so that now the field φ takes on values in a certain nonlinear manifold Φ. (In the simplest version, the three real fields forming a vector n are subjected to the condition $\mathrm{n}^2=1$, so that the manifold Φ is a two-dimensional sphere S^2 in three-dimensional space.) This model already describes interacting fields.

Furthermore, chiral models in two-dimensional space-time, or, for short, two-dimensional chiral models, are particularly interesting. They are in many respects analogous to four-dimensional gauge field theories, which at present are regarded as most probably suitable for describing the strong interactions. (This analogy has been pointed out repeatedly by a number of authors, and was particularly emphasized by Polyakov.)

This is true, for example, for such properties as asymptotic freedom, nontrivial topological structure, the existence of instantons, and the presence of a high hidden symmetry, which in the two-dimensional case leads to an infinite number of conservation laws.

A field-theory model is characterized, as usual, by a Lagrangian $\mathscr{L}(\varphi)$ and its integral S over space-time, the action, a functional of the field $\varphi(x)$: $S=S[\varphi]$. The equations which serve as equations of motion are obtained as usual by requiring that the action be an extremal: $\delta S=0$.

The instanton solutions are obtained by going over to a pure imaginary time, or, in other words, by changing from the φ pseudoeuclidean metric dt^2-dx^2 to the Euclidean metric $dx_1^2+dx_2^2$.

In going to the quantum case it is convenient to use the method of the functional integral (or integral over paths) introduced by Feynman. In this method the probabilities of various processes are described by functional integrals involving the action S.

In the calculation of these integrals in the quasiclassical approximation instantons correspond to extremal values of the action. They involve a transition in the functional integral to a pure imaginary time and describe tunnelling processes in the quasiclassical approximation. Expansion of the functional integral near the instanton configuration and calculation of the resulting Gaussian integrals gives the first nontrivial quantum corrections (this is analogous to the calculation of the one-loop Feynman diagrams). We emphasize that it is in principle impossible to obtain these corrections in the framework of ordinary perturbation theory. Owing to limited space, however, we cannot consider here the role of instantons in the quantum case.

The purpose of the present paper is to give a survey of the properties of the instanton solutions in two-dimensional chiral models of general type. Such solutions have been found in Refs. 28, 38–42 for ordinary chiral models and in Refs. 85, 86, 88, 89 for supersymmetric generalizations of these models.[1)] The more complicated case of instantons for the Yang-Mills field and the Einstein gravitational field[18-24] will not be considered here.

It must be especially emphasized that in the study of chiral models (and still more so for the Yang-Mills field and the gravitational field) topological concepts and methods, Lie group theory, and also differential and algebraic geometry are very important. In the present review we shall explain the necessary ideas, appealing to physical intuition and referring the reader interested in a more rigorous formulation to the appropriate literature.

The article consists of four sections. In the first section we consider, from various points of view, the simplest chiral model, the so-called n-field model, for which instanton solutions were first found in a paper by Belavin and Polyakov.[28] In this model the field $\mathrm{n}(x)$ is a three-dimensional vector, $\mathrm{n}=(n_1,n_2,n_3)$, on which the restriction $\mathrm{n}^2=1$ is imposed. The field $\mathrm{n}(x)$ is defined on the two-dimensional plane $x=(x_1,x_2)$. The action $S[\mathrm{n}]$ is determined by the metric on the sphere $(\mathrm{n}^2=1)$, which is invariant under rotations in three dimensional space

$$S=\frac{1}{2}\int d^2x\,\partial_\mu \mathrm{n}\cdot\partial_\mu \mathrm{n},\quad \partial_\mu=\frac{\partial}{\partial x_\mu},\quad \mu=1,\,2.$$

The resulting "equations of motion" are

$$\partial_\mu\partial_\mu \mathrm{n}+(\partial_\mu \mathrm{n}\cdot\partial_\mu \mathrm{n})\,\mathrm{n}=0.$$

We shall be concerned with solutions of these equations that have finite action; therefore we require that the field $\mathrm{n}(x)$ have a definite limit

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1) Ordinary chiral models describe boson fields; supersymmetric chiral models describe both boson and fermion fields in such a way that there is a definite symmetry between them.

$$n(x) \to n_0 \quad \text{for} \quad |x| \to \infty.$$

Such a field can already be considered on an extended plane of the variable x, to which is associated an infinitely distant point $\{\infty\}$. By means of stereographic projection, this plane can be mapped with the well known formulas

$$x_1 + ix_2 = \frac{\nu_1 + i\nu_2}{1-\nu_3},$$

onto the unit sphere in the three-dimensional space of vectors ν with $\nu^2 = 1$.

Accordingly, the field $n(x)$ determines a mapping of a sphere in ν space onto a sphere in n space. But, as is well known from topology (cf. e.g., Ref. 96), two such mappings [corresponding to two fields $n(x)$] cannot always be deformed into each other. More exactly, to a field $n(x)$ one can assign an integer Q, called the topological charge

$$Q = \frac{1}{8\pi}\int d^2x\varepsilon_{\mu\nu}(\mathbf{n}[\partial_\mu \mathbf{n}, \partial_\nu \mathbf{n}]),$$

and two fields $n_1(x)$ and $n_2(x)$ can be deformed into each other only when their corresponding topological charges Q_1 and Q_2 are equal.

It then turns out that the action S has a lower bound given by the topological charge

$$S \geqslant 4\pi |Q|.$$

Furthermore the equality sign holds here only when one or the other of the equations

$$\partial_\mu \mathbf{n} = \pm\varepsilon_{\mu\nu}[\mathbf{n}, \partial_\nu \mathbf{n}],$$

which are called "duality equations," is satisfied.

It is not hard to see that any solution of these equations is also a solution of the equations of motion.

The equations of duality are nonlinear, and to solve them it is convenient to go over from the variable n to a new variable w, again using stereographic projection

$$w = w_1 + iw_2 = \frac{n_1 + in_2}{1-n_3}, \quad \mathbf{n}^2 = 1.$$

The duality equations are thus simplified; they become linear, namely they go over into the Cauchy-Riemann equations

$$\frac{\partial w}{\partial \bar{z}} = 0 \quad \text{or} \quad \frac{\partial w}{\partial z} = 0, \quad z = x_1 + ix_2.$$

Using the boundary condition $n(x) = n_0$ for $|x| \to \infty$, we find from this that the function $w(z)$ [or $w(\bar{z})$] must be rational:

$$w(z) = c\prod_j^{k_1}(z-a_j)\left[\prod_l^{k_2}(z-b_l)\right]^{-1}.$$

Furthermore, owing to the invariance of our equation under rotations in the n space we can take as the limiting value w_0 any number, for example $w_0 = 1$. Then the conditions $c = 1$ and $k_1 = k_2 = k$ must hold.

The topological charge corresponding to this solution is equal to k.

Accordingly, in the case of the model of the n field the instanton solutions have been completely described. It turns out that analogous results hold for a broad class of chiral field-theory models.

The remaining three sections of this paper are devoted to chiral models of this class.

In Sec. 2 we examine in detail chiral models of general form. It turns out that the results of Sec. 1 can be transferred to the case of chiral models in which there are N complex fields $w^\alpha(x)$ and the action is of the form

$$S = \int h_{\alpha\bar{\beta}}(w, \bar{w})\,\partial_\mu w^\alpha \partial_\mu \bar{w}^\beta d^2x, \quad \mu = 1, 2,$$

and is invariant under a group of transformations which is large enough so that any point w^α can be taken into any point w^β. Besides this it is required that the metric $h_{\alpha\bar{\beta}}$ be of a special form; it must be a Kähler metric, i.e., it must satisfy the condition

$$\frac{\partial h_{\alpha\bar{\beta}}}{\partial w^\gamma} = \frac{\partial h_{\gamma\bar{\beta}}}{\partial w^\alpha}.$$

In this case, just as in that of the n field, the duality equations can be reduced to the Cauchy-Riemann equations, and the instanton solutions are given by rational functions.[40] In this section we must use a number of concepts of topology, Lie group theory, and differential and algebraic geometry which are unfamiliar to a physicist, and which we have tried to explain with very simple examples.

A reader who is not interested in chiral models of general type can omit this section and go on at once to the next one, where a model is considered in which the field takes on values in an n dimensional complex projective space, usually denoted as CP^n. We note that the usual n field can be regarded as a CP^1 model. However, the situation for the CP^n model becomes different for $n > 1$. For $n \geqslant 2$ the instanton solutions do not exhaust all the possibilities for solutions with finite action. Such solutions for CP^n models have recently been completely described.[76,77]

The last section of the paper is devoted to a consideration of the so-called Grassmann chiral model, in which the field $\varphi(x)$ takes on values in a so-called complex Grassmann manifold, which can be regarded as a definite set of matrices.

Accordingly, the n field model, the CP^n model, and the Grassmann model can be regarded as models of a complex scalar field, a complex vector field, and a complex matrix field.

In concluding this section we would like to thank L. B. Okun', who examined the manuscript of this paper, for a number of helpful comments.

1. THE SIMPLEST CHIRAL MODEL, THAT OF THE n-FIELD

We shall begin with the simplest case, the so-called n field model, whose instanton solutions were found in a paper by Belavin and Polyakov.[28] In view of further generalizations, we consider this model from various points of view.

1. In this model [often called the SO(3)-invariant σ model] we deal with a field

$$n(x,t)=(n^1,n^2,n^3) \quad \text{or} \quad n^\alpha(x,t),\ \alpha=1,2,3, \tag{1.1}$$

with a Lagrangian which is formally identical with that for the free field

$$\mathscr{L}=\frac{1}{2}\partial_\mu \mathbf{n}\cdot\partial_\mu \mathbf{n}, \quad \mu=0,1, \quad x_0=t, \quad x_1=x. \tag{1.2}$$

In order to get a theory of interacting fields, we shall not proceed in the usual way, adding an interaction to the Lagrangian, but impose a simple quadratic restriction on the n field:

$$\mathbf{n}^2=1. \tag{1.3}$$

Accordingly, the field n now takes on values in a nonlinear manifold Φ, a two-dimensional sphere S^2 defined in a three-dimensional space by the condition (1.3).

In this model the interaction is due to the intrinsic curvature of the manifold (in this case a sphere) and is purely geometrical in origin. This enables us to study this kind of theory in considerable detail.

The instanton solutions are solutions that describe a tunnelling process, and the convenient and usual procedure for finding them is to change from the ordinary time t to a purely imaginary time it, i.e., from the pseudoeuclidean case to the Euclidean case. Therefore we shall suppose from the very beginning that the field n is defined on a two-dimensional Euclidean plane:

$$\mathbf{n}=\mathbf{n}(x), \quad x=(x_1,x_2).$$

Let us introduce a coordinate system of some kind on the sphere, so that a point on the sphere is determined by coordinates u^γ, $\gamma=1,2$. Such a model is then determined by giving the action (energy) functional:

$$S=\frac{1}{2}\int d^2x g_{\gamma\delta}\partial_\mu u^\gamma\partial_\mu u^\delta, \quad \partial_\mu=\frac{\partial}{\partial x_\mu}, \tag{1.4}$$

where $g_{\gamma\delta}$ is the metric tensor

$$ds^2=g_{\gamma\delta}\,du^\gamma\,du^\delta. \tag{1.5}$$

The equations of motion follow as usual from the condition that the action be an extremal:

$$\delta S=0\to\partial_\mu\partial_\mu u^\gamma+\Gamma^\gamma_{\alpha\beta}\partial_\mu u^\alpha\partial_\mu u^\beta=0; \tag{1.6}$$

here

$$\Gamma^\gamma_{\alpha\beta}=\frac{1}{2}g^{\gamma\delta}\left(\frac{\partial g_{\delta\alpha}}{\partial u^\beta}+\frac{\partial g_{\delta\beta}}{\partial u^\alpha}-\frac{\partial g_{\alpha\beta}}{\partial u^\delta}\right) \tag{1.7}$$

are the Christoffel symbols.

These equations are a generalization of the well-known equations for geodesics

$$\ddot{u}^\gamma+\Gamma^\gamma_{\alpha\beta}\dot{u}^\alpha\dot{u}^\beta=0. \tag{1.8}$$

We note that the sphere is a uniform manifold; it is invariant under rotations of the three-dimensional space. Therefore it is natural to take for $g_{\alpha\beta}$ a metric which is rotation invariant:

$$ds^2=d\mathbf{n}\cdot d\mathbf{n}. \tag{1.9}$$

From this we get

$$S=\frac{1}{2}\int d^2x\partial_\mu\mathbf{n}\cdot\partial_\mu\mathbf{n}, \quad \partial_\mu=\frac{\partial}{\partial x_\mu}, \quad \mu=1,2, \tag{1.10}$$

and the "equation of motion"

$$\delta S=0\to\partial_\mu\partial_\mu\mathbf{n}+(\partial_\mu\mathbf{n}\cdot\partial_\mu\mathbf{n})\,\mathbf{n}=0. \tag{1.11}$$

We note that it follows from Eq. (1.11) that

$$\partial_\mu[\mathbf{n},\partial_\mu\mathbf{n}]=0. \tag{1.12}$$

We point out that this model can also be regarded as a model of an n field in (2+1) dimensional space-time, with the condition that we concern ourselves only with solutions that do not depend on the time.

We shall consider only solutions with a finite action S, and therefore require that the field $n(x)$ have a definite limit

$$\mathbf{n}(x)\to\mathbf{n}_0 \quad \text{for} \quad |x|\to\infty. \tag{1.13}$$

In this case we can deal with the field $n(x)$ as given on an extended plane of the variables $x=(x_1,x_2)$ with an infinitely distant point $\{\infty\}$ associated with it.

To do this it is convenient to use stereographic projection to map this extended plane onto a two-dimensional sphere $S^2=\{\nu:\nu^2=1\}$; the mapping is

$$x_1+ix_2=\frac{\nu_1+i\nu_2}{1-\nu_3}, \quad \nu^2=1. \tag{1.14}$$

It can be seen from this that the extended plane is topologically equivalent to the two-dimensional sphere.

Accordingly, we can consider that the field n is given on a two-dimensional sphere: $n=n(\nu)$, and this field determines a continuous mapping

$$\boldsymbol{\nu}\to\mathbf{n}(\boldsymbol{\nu})$$

of the two-dimensional sphere $S^2=\{\nu:\nu^2=1\}$ onto the two-dimensional sphere $S^2=\{n:n^2=1\}$.

A characteristic peculiarity of such a field is that in general it (unlike a field given on an ordinary plane) cannot be deformed continuously into a field independent of x: $n=n_0=\text{const}$. Namely, to each field $n(x)$ that satisfies the condition (1.13) one can assign a whole number Q (usually called the topological charge) which does not change in a continuous deformation of the sphere. Then two fields that have the same topological charge can be continuously deformed into each other.[2)]

In the present case we can write an integral representation for the topological charge

$$Q=\frac{1}{8\pi}\int d^2x\varepsilon_{\mu\nu}(\mathbf{n}[\partial_\mu\mathbf{n},\partial_\nu\mathbf{n}]), \tag{1.15}$$

where $\varepsilon_{\mu\nu}$ is the antisymmetric tensor and $\varepsilon_{12}=1$. That Q is an integer follows from the fact that in this case the density of topological charge is nothing other than the Jacobian of the mapping $x=n(x)$. In fact, using the usual parametrization

$$\mathbf{n}=(\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta), \tag{1.16}$$

we get

$$Q=\frac{1}{4\pi}\int\sin\theta(x)\,d\theta(x)\,d\varphi(x). \tag{1.17}$$

Then, from the obvious inequality

$$(\partial_\mu\mathbf{n}\mp\varepsilon_{\mu\nu}[\mathbf{n},\partial_\nu\mathbf{n}])^2\geqslant 0 \tag{1.18}$$

it follows that the action is bounded below by the topological charge

[2)] This property of fields $\mathbf{n}(\nu)$ is well known in topology (see, for example, a recent monograph[99]).

$$S \geqslant 4\pi \mid Q \mid. \tag{1.19}$$

Furthermore the equals sign in (1.19) is reached when one or the other of the conditions holds

$$\partial_\mu \mathbf{n} = \pm \varepsilon_{\mu\nu} [\mathbf{n}, \partial_\nu \mathbf{n}]. \tag{1.20}$$

The equations (1.20) play an important part in the theory and are called the "equations of duality." Unlike the "equations of motion" (1.11), the equations (1.20) are of the first order. It is not hard to show, however, that any solution of the "equations of duality" is also a solution of the "equations of motion" (1.11). Solutions of the "equations of duality" are also called instanton solutions.

2. Although Eqs. (1.20) are first-order equations, they are nonlinear, and to solve them it is convenient to use a different parametrization. Namely, by stereographic projection from the north pole of the sphere we change to new variables

$$w = w_1 + iw_2 = \frac{n_1 + in_2}{1 - n_3} = \operatorname{ctg} \frac{\theta}{2} e^{i\varphi}, \quad z = x_1 + ix_2. \tag{1.21}$$

In these variables the expression (1.10) for the action S takes the form

$$S = 2 \int \frac{d^2x}{(1 + |w|^2)^2} \partial_\mu w \cdot \partial_\mu \bar{w}, \tag{1.22}$$

or

$$S = 4 \int \frac{d^2x}{(1 + |w|^2)^2} \left(\frac{\partial w}{\partial z} \frac{\partial \bar{w}}{\partial \bar{z}} + \frac{\partial w}{\partial \bar{z}} \frac{\partial \bar{w}}{\partial z} \right),$$

and the topological charge is given by

$$Q = \frac{1}{\pi} \int \frac{d^2x}{(1 + |w|^2)^2} \left(\frac{\partial w}{\partial z} \frac{\partial \bar{w}}{\partial \bar{z}} - \frac{\partial w}{\partial \bar{z}} \frac{\partial \bar{w}}{\partial z} \right). \tag{1.23}$$

From this it follows at once that the topological charge has the lower-bound property (1.19), and also, which is more important, the equations of duality (antiduality) take the simple form

$$\frac{\partial w}{\partial \bar{z}} = 0 \quad \text{or} \quad \frac{\partial w}{\partial z} = 0, \tag{1.24}$$

i.e., the duality equations reduce in this case simply to the Cauchy-Riemann equations.

At first glance it seems that the general solution of the equations of duality is given by the formula

$$w = f(z) \quad \text{or} \quad w = f(\bar{z}),$$

where f is an arbitrary analytic function. But we must also require that this function have a definite limit w_0 (including $w_0 = \infty$) for $|z| \to \infty$. It follows that the function f must be a rational function.

Because of the homogeneity of our space $\Phi = S^2$ under the group SU(2) we can take for the limiting value w_0 any number, for example $w_0 = 1$. Then the solution of the equations of duality takes the form

$$w = \prod_{j=1}^{k} \frac{z - a_j}{z - b_j} \quad \text{or} \quad w = \prod_{j=1}^{k} \frac{\bar{z} - a_j}{\bar{z} - b_j}. \tag{1.25}$$

It is not hard to calculate that for this solution the topological charge is

$$Q = k \quad \text{or} \quad Q = -k, \quad k > 0, \tag{1.26}$$

and it is natural to call such a solution a k-instanton (or k-anti-instanton) solution.

We see that a k-instanton solution is characterized by $4k$ parameters: $2k$ complex numbers a_j and b_j. This is natural, since one instanton can be characterized by four parameters; they are its position b, its extent $a - b$, and one more quantity, its angle (phase) $\theta = \arg(a - b)$.

The analogous function $w = f(\bar{z})$ gives a solution with $k < 0$, which we can call a solution with $|k|$ antiinstantons.

We also point out that with parametrization by means of w the equations of motion (1.11) take the form

$$\frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} w = \frac{2\bar{w}}{1 + |w|^2} \frac{\partial w}{\partial z} \frac{\partial w}{\partial \bar{z}}. \tag{1.27}$$

It follows from them that the function

$$\frac{(\partial w / \partial z)\, \partial \bar{w} / \partial z}{(1 + |w|^2)^2} = f(z) \tag{1.28}$$

does not depend on $\bar{z}$.

Similarly, the function

$$(1 + |w|^2)^{-2} \frac{\partial w}{\partial \bar{z}} \frac{\partial \bar{w}}{\partial \bar{z}} = f(\bar{z}) \tag{1.29}$$

does not depend on z.

It can also be shown[32] that many-instanton and anti-instanton functions give the only solutions of the equations of motion (1.11) with finite action.

3. The connection with group theory becomes clearer if we examine a different formulation of the n field model. Namely, we go from the vector n to a two-row square matrix φ according to the formula

$$\varphi(x) = \sigma \mathbf{n}(x), \tag{1.30}$$

where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices.

We can now regard the field $\varphi(x)$ as an element of the Lie algebra $\mathscr{G}$ of the group SO(3), the rotation group of three-dimensional space, or of the group $G = \mathrm{SU}(2)$ which is locally isomorphic to it; this is the group of unitary second-rank matrices with determinant unity. The restriction (1.3) then takes the form

$$\operatorname{tr}(\varphi^2) = 1. \tag{1.31}$$

The group $G = \{g\} = \mathrm{SU}(2)$ acts naturally on the Lie algebra $\mathscr{G}$ of the group SU(2), or, what is the same thing, in the space of fields φ:

$$g\colon \varphi \to g\varphi g^+, \tag{1.32}$$

where g^+ is the matrix Hermitean adjoint to the matrix g. [This type of action is called the adjoined representation of the group SU(2).]

Choosing some element φ_0 of $\mathscr{G}$ and acting on it with all elements of the group G, we obtain the orbit Ω of the adjoined φ representation of the group SU(2), which, as can be easily seen, is the two-dimensional sphere S^2 given by Eq. (1.31).

Accordingly, in this case the space Φ is the orbit of the adjoined representation of the group SU(2).

The expressions for the action and the topological charge in terms of φ are

$$S=\frac{1}{2}\int \mathrm{tr}\,(\partial_\mu\varphi\partial_\mu\varphi)\,d^2x, \quad (1.33)$$

$$Q=c\int \mathrm{tr}\,(\varphi\,[\partial_\mu\varphi,\,\partial_\nu\varphi])\,\varepsilon_{\mu\nu}\,d^2x, \quad (1.34)$$

and, as we shall see in the next section, these formulas can be naturally extended to the case in which Φ is the orbit of the adjoined representation of an arbitrary compact Lie group.

4. We shall give another formulation of the theory of the n field. Let us consider a two-component spinor field $\psi(x)=\binom{\psi_1(x)}{\psi_2(x)}$, $\psi^+(x)=[\bar\psi_1(x),\bar\psi_2(x)]$ and use it to form a vector field

$$\mathbf{n}(x)=(\psi^+(x),\,\sigma\psi(x)), \quad (1.35)$$

where $\sigma_1,\sigma_2,\sigma_3$ are the Pauli matrices. Then the condition (1.3) will be satisfied if we impose the restriction (normalization condition)

$$\psi^+\psi=\bar\psi_1\psi_1+\bar\psi_2\psi_2=1. \quad (1.36)$$

It must, however, be kept in mind that the two spinors $\psi(x)$ and $\exp[i\alpha(x)]\psi(x)$ give the same vector $\mathbf{n}(x)$, and therefore such spinors must be regarded as equivalent.

Accordingly, we are dealing with normalized spinors $\psi(x)$ defined up to a phase factor $\exp[i\alpha(x)]$. It is therefore natural to change to a new space such that the quantities (ψ_1,ψ_2) and $(\lambda\psi_1,\lambda\psi_2)$ define one point in it. Such a space is well known in mathematics. It is called a one-dimensional complex projective space and is denoted by CP^1.[3)] It is also well known that such a space is topologically equivalent to a two-dimensional sphere: $CP^1\sim S^2$.

Substituting $\mathbf{n}=\psi^+\sigma\psi$ in the expression (1.10) for the action and using Eq. (1.36), we get

$$S=\int d^2x\,\{(\partial_\mu\psi^+,\partial_\mu\psi)-(\psi^+,\partial_\mu\psi)(\partial_\mu\psi^+,\psi)\}. \quad (1.37)$$

It is not hard to see that this action determines a metric on $\Phi=CP^1$ which is invariant under the action of the group SU(2). Similarly for the topological charge Q we have

$$Q=c\int d^2x\varepsilon_{\mu\nu}\,(\partial_\mu\psi^+,\,\partial_\nu\psi). \quad (1.38)$$

We note that the expression (1.37) for the action is gauge invariant under the action of the group $U(1)$, i.e., under the replacement $\psi(x)\to\exp[i\alpha(x)]\psi(x)$.

5. One more formulation of the n field model; this is a formulation in terms of an Abelian gauge field $A_\mu(x)$ and a complex field $B_\mu(x)$.[4)] This formulation is most convenient for tracing the analogy between the two-dimensional chiral model and four-dimensional non-abelian gauge field theories.

We here expound a version of the formulation following Ref. 48.

We consider our space Φ as a unit sphere imbedded in a three-dimensional space: $\Phi=S^2=\{\mathbf{n}:\mathbf{n}^2=1\}$. We introduce a movable orthogonal coordinate system, so that at each point x, the set $\mathbf{e}_1(x)$, $\mathbf{e}_2(x)$, and $\mathbf{e}_3(x)=\mathbf{n}(x)$ are a triad of mutually orthogonal unit vectors

$$\mathbf{e}_j\cdot\mathbf{e}_k=\delta_{jk}. \quad (1.39)$$

In other words, we consider that at each point x there is given an element $g(x)$ of the three-dimensional rotation group, which takes the standard reference frame into the frame $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$. As x varies, the movable system undergoes a rotation. Therefore

$$\partial_\mu\mathbf{n}=-B^j_\mu\mathbf{e}_j,\quad \partial_\mu\mathbf{e}_j=A_\mu\varepsilon_{jk}\mathbf{e}_k+B^j_\mu\mathbf{n},\quad j,\ k=1,\ 2. \quad (1.40)$$

From this we get

$$B^j_\mu=(\mathbf{n},\,\partial_\mu\mathbf{e}_j),\quad A_\mu=(\mathbf{e}_2,\,\partial_\mu\mathbf{e}_1). \quad (1.41)$$

We note that the transition to a different movable orthogonal coordinate system is given by the formulas

$$\begin{aligned}\mathbf{e}'_1&=\cos\alpha(x)\cdot\mathbf{e}_1+\sin\alpha(x)\cdot\mathbf{e}_2,\\ \mathbf{e}'_2&=-\sin\alpha(x)\cdot\mathbf{e}_1+\cos\alpha(x)\cdot\mathbf{e}_2,\end{aligned} \quad (1.42)$$

where $\alpha(x)$ is a gauge function. In going from $\mathbf{e}_1,\mathbf{e}_2$ to $\mathbf{e}'_1,\mathbf{e}'_2$ the vectors A_μ and B^j_μ are also changed:

$$\left.\begin{aligned}B^{1'}_\mu&=\cos\alpha\cdot B^1_\mu+\sin\alpha\cdot B^2_\mu,\\ B^{2'}_\mu&=-\sin\alpha\cdot B^1_\mu+\cos\alpha\cdot B^2_\mu,\\ A'_\mu&=A_\mu+\partial_\mu\alpha.\end{aligned}\right\} \quad (1.43)$$

From the fields A_μ and $B_\mu=B^1_\mu+B^2_\mu$ we can recover the movable reference system $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3=\mathbf{n})$ by means of Eq. (1.40), provided that the conditions

$$D^\mu\tilde B_\mu=0,\quad \varepsilon_{\mu\nu}\partial_\mu A_\nu=\frac{1}{2}\bar B^\mu\tilde B_\mu, \quad (1.44)$$

are satisfied; here $\partial_\mu+iA_\mu=D_\mu$ is covariant differentiation, $\bar B_\mu$ is the field complex conjugate to B_μ, and $\tilde B_\mu=i\varepsilon_{\mu\nu}B^\nu$ is the field dual to B_μ.

In terms of the fields A_μ and B_μ the action and the equations of motion take the forms

$$S=\frac{1}{2}\int d^2x\bar B^\mu B_\mu, \quad (1.45)$$

$$D^\mu B_\mu=0, \quad (1.46)$$

and the equations of duality (antiduality) can now be written

$$B_\mu\pm\tilde B_\mu=0. \quad (1.47)$$

Accordingly, the n field model is equivalent to the theory of an Abelian gauge field A_μ and a charged vector field B_μ which have to satisfy the conditions (1.44).

We note that the quantity $\bar B_\mu\tilde B$ is proportional to the density of topological charge $\varepsilon_{\mu\nu}(\mathbf{n}[\partial_\mu\mathbf{n},\partial_\nu\mathbf{n}])$, so that, according to Eq. (1.44), we have

$$Q=\frac{1}{4\pi}\int d^2x\varepsilon^{\mu\nu}\partial_\mu A_\nu. \quad (1.48)$$

This formula is analogous to the corresponding one for the Yang-Mills field [cf. Ref. 16, Eq. (4.3)]

$$Q=\frac{1}{8\pi^2}\int d^4x\varepsilon^{\mu\nu\rho\sigma}\partial_\mu A_{\nu\rho\sigma},\quad \mu,\ \nu,\ \rho,\ \sigma=1,\ 2,\ 3,\ 4, \quad (1.49)$$

where the field $A_{\nu\rho\sigma}$ is given by the expression

$$A_{\nu\rho\sigma}=\mathrm{tr}\left\{A_\nu F_{\rho\sigma}-\frac{2}{3}A_\nu A_\rho A_\sigma\right\}, \quad (1.50)$$

and $A_\mu=(2i)^{-1}A^j_\mu\sigma_j$ is the Yang-Mills field,

[3)] The n-dimensional complex projective space CP^n is defined analogously; a point in this space is a nonzero $(n+1)$-dimensional complex vector, defined up to a common factor.

[4)] This sort of formulation for the general case was given in a paper by Semenov-Tyan-Shanskii and Fadeev.[33]

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

We note that the field equations (1.44) and (1.46) have the obvious symmetry (dual symmetry)

$$B'_\mu \pm \tilde{B}'_\mu = e^{\pm i\beta}(B_\mu \pm \tilde{B}_\mu), \quad A'_\mu = A_\mu, \tag{1.51}$$

where β is a constant parameter.

This symmetry transformation was first discovered by Pohlmeyer[29] and was called by him R^β: It has a very simple form in precisely this formalism. As was shown in Ref. 36, there are associated with this symmetry an infinite set of nonlocal integrals of the motion, which in the quantum case lead to a strong restriction on the dynamics of scattering processes: No particles are created. Regarding the dual symmetry for general chiral models see Ref. 49.

The instanton solutions are invariant under the dual symmetry, which in this case reduces to a global gauge transformation. Conversely, any field that is invariant under the transformation of dual symmetry must be either self-dual or antiselfdual.

This indicates that the existence of instantons is closely connected with the existence of the dual symmetry.

2. CHIRAL MODELS OF GENERAL FORM

In the general case we have a field $\varphi(x)$ in a d-dimensional Euclidean space ($x \in \mathbf{R}^d$), which takes on values in a nonlinear manifold Φ. Let there be introduced in Φ local coordinates u^α and a Riemannian metric

$$ds^2 = g_{\alpha\beta}\, du^\alpha\, du^\beta. \tag{2.1}$$

Then the chiral model is determined by prescribing the action

$$S = \frac{1}{2} \int dx_1 \ldots dx_d g_{\alpha\beta}(u)\, \partial_\mu u^\alpha \partial_\mu u^\beta. \tag{2.2}$$

The condition $\delta S = 0$ leads to the "equations of motion"

$$\partial_\mu \partial_\mu u^\alpha + \Gamma^\alpha_{\beta\gamma} \partial_\mu u^\beta \partial_\mu u^\gamma = 0, \tag{2.3}$$

where $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(\frac{\partial g_{\delta\beta}}{\partial u^\gamma} + \frac{\partial g_{\delta\gamma}}{\partial u^\beta} - \frac{\partial g_{\beta\gamma}}{\partial u^\delta} \right). \tag{2.4}$$

We shall concern ourselves with smooth fields $\varphi(x)$ that have a definite limit φ_0 for $|x| \to \infty$. In this case we can adjoin to the Euclidean space $\mathbf{R}^d$ an infinitely remote point $\{\infty\}$ and regard this space as a d-dimensional sphere

$$S^d = \mathbf{R}^d \cup \{\infty\}. \tag{2.5}$$

Accordingly, each point of the sphere S^d is put into correspondence with a point of the space Φ, or, in other words, the field $\varphi(x)$ defines a mapping of the sphere S^d into the space Φ:

$$\varphi\colon S^d \to \Phi. \tag{2.6}$$

We recall that two such mappings φ_1 and φ_2 are called homotopic[5)] to each other if they can be continuously deformed into each other. We unite all mappings that are homotopic to each other into one class. Then, as is well known,[96] in the set of such classes we can introduce an operation of multiplication relative to which they form a group; this is the d-dimensional homotopic group of the manifold Φ, or $\pi_d(\Phi)$. Accordingly, as a topological characteristic of a field $\varphi(x)$ we can take its homotopic class, or, in other words, an element of the homotopic group $\pi_d(\Phi)$. Therefore it is natural to regard a chiral theory as topologically nontrivial if $\pi_d(\Phi) \neq 0$.

a) Chiral models with nontrivial topology

We confine ourselves to consideration of only the most fully studied two-dimensional Euclidean chiral models. Readers who are interested in the pseudoeuclidean case are referred to Refs. 29 and 35, and those interested in the many-dimensional case, to the fundamental work in Ref. 74.

Following Refs. 38 and 40, we suppose that the field $\varphi(x)$, $x = (x_1, x_2) \in \mathbf{R}^2$, takes on values in a compact manifold Φ, which is topologically nontrivial: $\pi_2(\Phi) \neq 0$. We further assume that Φ is a homogeneous space such that it can be acted on transitively by a compact semisimple Lie group $G = \{g\}$.[6)] In other words, for any two elements φ_1 and φ_2 there exists a transformation of the group G that takes φ_1 into φ_2. We take some fixed element φ_0 of the space Φ and denote by $H = \{h\}$ the set of elements of the group G that leave it fixed: $h\varphi_0 = \varphi_0$. It is not hard to see that this set is a subgroup of the group G, the stationary subgroup of the element φ_0.

We now divide all the elements of the group G into so-called adjacency classes; we regard elements g_1 and g_2 as belonging to the same class if there exists an element $h \in H$ such that $g_1 = g_2 h$.

It is not hard to see that our space Φ is isomorphic to a factor space, the space of adjacency classes of the group G with respect to the subgroup H, denoted ordinarily by

$$\Phi = G/H. \tag{2.7}$$

We remark that any homogeneous space can be represented in this way. Examples:

1. The space $\Phi = S^2$ considered in the preceding section can be represented in the form

$$S^2 = \mathrm{SU}(2)/\mathrm{U}(1) \quad \text{or} \quad S^2 = \mathrm{SO}(3)/\mathrm{SO}(2), \tag{2.8}$$

2. the n-dimensional sphere S^n can be written in the form

$$S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n). \tag{2.9}$$

3. For the complex n-dimensional projective space $\mathbb{C}P^n$

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5) We shall not give here the rigorous definitions of the topological concepts. They can be found, for example, in Refs. 96, 97, and 99.

6) The required information about the theory of Lie groups can be found, for example, in a book by Kirillov.[100] We mention that a semisimple Lie group is the direct product of simple Lie groups, and a simple Lie group is a group that has no invariant subgroup.

$$\mathbb{C}P^n = \mathrm{SU}(n+1)/\mathrm{SU}(n)\times\mathrm{U}(1). \tag{2.10}$$

Let us further assume that the space Φ is connected and simply connected, i.e., any two of its points can be connected with a curve, and any closed curve can be deformed within Φ into a point. In this case, to find the second homotopic group of the space Φ we may use a formula well known in topology[96,97]:

$$\pi_2(\Phi) = \pi_2(G/H) = \pi_1(H), \tag{2.11}$$

where $\pi_1(H)$ is the first homotopic group of the space H, i.e., the group of classes of mutually homotopic closed curves passing through some fixed point of the space H.

Thus a chiral theory will be topologically nontrivial if the first homotopic group of the stationary subgroup H of some point of the space Φ is nontrivial, or in other words, if the subgroup H is not simply connected. The calculation of the group $\pi_1(H)$ in our case presents no difficulty, since the structure of the subgroup H is well known. Namely, if G is a simple compact Lie group of rank l, i.e., if the maximal subgroup of G of the form $\mathrm{U}(1)\times\mathrm{U}(1)\times\ldots\times\mathrm{U}(1)$ consists of l factors, then H is of the form

$$H = (\mathrm{U}(1)\times\ldots\times\mathrm{U}(1)\times H_0)/K, \tag{2.12}$$

where the factor U(1) occurs $k\leq l$ times, and the group H_0 is simply connected: $\pi_1(H_0)=0$, and the group K is finite. Since $\pi_1(X\times Y)=\pi_1(X)+\pi_1(Y)$, it follows from Eq. (2.12) that

$$\pi_2(\Phi)=\pi_2(G/H)=\pi_1(H)=\underbrace{\mathbb{Z}+\ldots+\mathbb{Z}}_{k\ \text{times}}+K, \tag{2.13}$$

where $\mathbb{Z}=\pi_1[\mathrm{U}(1)]$ is the additive group of integers. Accordingly, up to the finite group K

$$\pi_2(\Phi)=\underbrace{\mathbb{Z}+\ldots+\mathbb{Z}}_{k\ \text{times}},\quad k\leqslant l, \tag{2.14}$$

or, in other words, k whole numbers can be assigned to the field $\varphi(x)$, its topological charges.

Examples:

1. From Eqs. (2.8)–(2.10) it follows at once that

$$\pi_2(S^2)=\mathbb{Z},\quad \pi_2(S^n)=0,\quad\text{for}\quad n\geqslant 3,\quad \pi_2(\mathbb{C}P^n)=\mathbb{Z}. \tag{2.15}$$

We note that a SO(n)-invariant chiral theory with $n\geq 4$ is topologically trivial, and owing to this has no instanton solutions. At the same time there is a broad class of topologically nontrivial homogeneous spaces. The ones most thoroughly studied are the so-called orbits of adjoined representations of simple compact Lie groups, which we shall consider in greater detail.

Then let G be a compact simple Lie group, and let $\mathcal{G}$ be its Lie algebra. The group G acts on the algebra $\mathcal{G}$ naturally, but nontransitively, i.e., it cannot take an element of $\mathcal{G}$ into an arbitrary element of $\mathcal{G}$. Accordingly, under the action of G the algebra $\mathcal{G}$ breaks up into orbits. An orbit Ω is the set of elements obtained by applying all the transformations of the group G to some one element of $\mathcal{G}$.

Examples:

1) Let $G=\mathrm{SO}(3)$, the group of rotations of three-dimensional space, i.e., the group $\{g\}$ of three-rowed matrices obeying the condition $gg'=1$, where the prime denotes transposition.

Then $\mathcal{G}$ is the algebra of real skew-symmetric matrices:

$$\mathcal{G}=\{x\},\quad x=\begin{pmatrix}0 & x_3 & -x_2\\ -x_3 & 0 & x_1\\ x_2 & -x_1 & 0\end{pmatrix},\quad x'=-x.$$

The adjoined representation of the group G is given by the formula $x=gxg'$. If we prescribe an element of $\mathcal{G}$ with the three-dimensional vector $\mathbf{x}=(x_1,x_2,x_3)$, the orbits of G will be a sphere in three-dimensional space with its center at the origin: $\Omega=\mathrm{SO}(3)/\mathrm{SO}(2)$, $\pi_2(\Omega)=\mathbb{Z}$, including a degenerate orbit, the origin of coordinates itself.

2) $G=\mathrm{SU}(3)=\{g\}$, where g is a unitary three-row matrix with determinant equal to unity: $gg^+=I$, $\det g=1$; $\mathcal{G}=\{x\}$, the algebra (eight-dimensional) of anti-Hermitean three-row matrices: $x^+=-x$. The action of the adjoined representation is given by $x=gxg^+$. In this case there are two types of orbits besides the origin of coordinates:

a) All the eigenvalues of the matrix x are different. Then the orbit Ω is six-dimensional: $\pi_2(\Omega)=\mathbb{Z}+\mathbb{Z}$, $\Omega=\mathrm{SU}(3)/\mathrm{U}(1)\times\mathrm{U}(1)$.

b) Two eigenvalues of the matrix x coincide. The orbit is four-dimensional: $\Omega=SU(3)/SU(2)\times U(1)$, $\pi_2(\Omega)=\mathbb{Z}$. In this case an orbit Ω is isomorphic to the two-dimensional complex projective space: $\Omega=\mathbb{C}P^2$.

3) $G=\mathrm{SU}(n)$, $\mathcal{G}=\{x\}$, $x^+=-x$, $(n-1)$ eigenvalues of the matrix x coincide. Orbit $\Omega=\mathrm{SU}(n)/\mathrm{SU}(n-1)\mathrm{U}(1)$, $\pi_2(\Omega)=\mathbb{Z}$, and Ω is isomorphic to the space $\mathbb{C}P^{n-1}$.

It is well known that the orbit of the adjoined representation is a connected and singly connected manifold: $\pi_1(\Omega)=0$. Therefore the reader versed in topology can use the theorem of Gurevich[96]:

If Φ is connected and singly connected: $\pi_1(\Phi)=0$ and $\pi_j(\Phi)=0, j=2,\ldots,(k-1), \pi_k(\Phi)\neq 0$, then $H_j(\Phi)=0, j=2,\ldots,(k-1), H_k(\Phi)=\pi_k(\Phi)$; here $H_j(\Phi)$ is the j-th homology group. Consequently, $H^j(\Phi)\sim 0, j=2,\ldots,(k-1)$, where $H^j(\Phi)$ is the j-th cohomology group, and the sign ~ means equality up to a finite group.

But in virtue of De Rham's theorem $H^j(\Phi)$ is the factor space of closed external j-forms with respect to the subspace of exact j-forms. On the other hand, it is known that there exists on Φ a closed 2-form which is not exact, the so-called Kirillov form (cf. Ref. 100). Consequently, for the orbit of the adjoined representation $H^2(\Phi)\neq 0$, and by Gurevich's theorem:

$$\pi_2(\Phi) = H^2(\Phi). \tag{2.16}$$

We shall explain what we have said for readers not acquainted with the homology and cohomology groups. The fact that the group $\pi_2(\Phi)$ is nontrivial means that in the space Φ there are closed two-dimensional manifolds or two-dimensional cycles, deformed two-dimensional spheres, which cannot be deformed to a point. Over them one can integrate closed 2-forms, i.e., expressions that are in the form of a total derivative.

A quantity so derived remains unchanged by deformations of the 2-cycle over which the integration is performed, and therefore is topologically invariant.

It follows from Gurevich's theorem that all the topological charges can be obtained in this way as integrals of 2-forms, i.e., as twofold integrals. Symbolically this is written

$$Q = c^{-1} \int \tilde{\omega}. \tag{2.17}$$

Here $\tilde{\omega}$ is a closed 2-form on $S^2 = \mathbf{R}^2 \cup \{\infty\}$ which is not exact.

The form $\tilde{\omega}$ can be obtained in the following way. The mapping φ: $S^2 \to \Phi$ induces a corresponding mapping in the space of cohomologies, φ^*: $H^2(\Phi) \to H^2(S^2)$, and

$$\tilde{\omega} = \varphi^* \omega. \tag{2.18}$$

If we regard the solutions of interest to us as solutions independent of the time in a chiral theory with d space dimensions and one time dimension, the assertion just arrived at can be formulated differently: A topological charge can be represented as the integral of the zeroth component of a topological current $I_a = (I_0, I_\mu)$:

$$Q = c^{-1} \int I_0 \, dx_1 \ldots dx_d, \tag{2.17'}$$

where I_0 is a complete divergence

$$I_0 = \partial_\mu K_\mu. \tag{2.19}$$

We give examples of such currents.

1) The case of the so-called principal chiral field: $\Phi = G, d = 3$. In this case the topological charge is given by the formula[67]

$$Q = c^{-1} \int d^3x \varepsilon_{\mu\nu\lambda} \operatorname{tr} ([L_\mu, L_\nu] L_\lambda), \tag{2.20}$$

where the chiral field φ is a field on the group G: $\varphi(x) = g(x), L_\mu = \partial_\mu g \cdot g^{-1}$ is an element of the Lie algebra $\mathcal{G}$, taken in the adjoined representation, and c is a normalization constant. Here the topological charge is

$$I_a = \varepsilon_{abcd} \operatorname{tr} ([L_b, L_c] L_d) \tag{2.21}$$

and is conserved:

$$\partial_a I_a = 0.$$

Here Greek indices run through the values 1,2,3 and Latin ones through 0,1,2,3; $\varepsilon_{\mu\nu\lambda}$ and ε_{abcd} are completely antisymmetric tensors. Conservation of the current I_a follows from the identity (condition of zero curvature)

$$\partial_a I_b - \partial_b I_a + [I_a, I_b] = 0 \tag{2.22}$$

and the Jacobi identity for commutators. With a suitable normalization the topological charge takes on integer values.

2) The n field, $d = 2$:

$$I_a = \varepsilon_{abc} \varepsilon_{\alpha\beta\gamma} n^\alpha \partial_b n^\beta \partial_c n^\gamma, \quad \alpha, \beta, \gamma = 1, 2, 3; \quad a, b, c, d = 0, 1, 2, \tag{2.23}$$

from which we have

$$I_0 = \varepsilon_{\mu\nu} \varepsilon_{\alpha\beta\gamma} n^\alpha \partial_\mu n^\beta \partial_\nu n^\alpha, \quad \mu, \nu = 1, 2, \tag{2.24}$$

and arrive at the expression (1.15) for the topological charge. Conservation of the current I_a follows from the linear dependence of the vectors $\partial_\mu \mathbf{n}$.

3) The n field, $d = 3$. In this case a current analogous to that in the foregoing case cannot be introduced, but another topological invariant can be associated with the n field.[67] In analogy with Eq. (2.23) we introduce the vector

$$H_\mu = \varepsilon_{\mu\nu\lambda} \varepsilon_{\alpha\beta\gamma} n^\alpha \partial_\nu n^\beta \partial_\lambda n^\gamma. \tag{2.25}$$

Then owing to the fact that $\operatorname{div} \mathbf{H} = 0$ we can represent $\mathbf{H}$ in the form

$$\mathbf{H} = \operatorname{rot} \mathbf{A}. \tag{2.26}$$

For the n field without singularities the integral

$$\frac{1}{4\pi} \int \mathbf{A} \operatorname{rot} \mathbf{A} \, d^3x \tag{2.27}$$

exists and takes on integer values. This topological invariant is called the Hopf invariant.[101]

For the reader familiar with topology we point out that for the vector A there is a corresponding 1-form $\tilde{\omega}$ and for the vector rot A, a 2-form $d\tilde{\omega}$, and the expression (2.27) can be rewritten in the invariant form

$$(4\pi)^{-1} \int \tilde{\omega} \wedge d\tilde{\omega}, \tag{2.28}$$

where the sign $\wedge$ denotes outer multiplication.

In the general case, when $\Phi = \Omega$, the orbit of the adjoined representation of the group G, we can make use of the fact that on the orbit there exists a universal closed 2-form ω, the Kirillov form[100]

$$\omega = \omega(X, Y) = (\varphi, [X, Y]). \tag{2.29}$$

Here X and Y are vectors tangent to the orbit at the point φ, and (φ, ψ) is the invariant scalar product in the Lie algebra $\mathcal{G}$. In the simplest case $\mathcal{G} = SU(N)$, φ and ψ are N-rowed antihermitean matrices, and $(\varphi, \psi) = -\operatorname{tr}(\varphi\psi)$.

The topological charge corresponding to the form (2.29) is

$$Q = c^{-1} \int d^2x \varepsilon_{\mu\nu} (\varphi [\partial_\mu \varphi, \partial_\nu \varphi]), \quad \mu = 1, 2. \tag{2.30}$$

Here $\partial_1 \varphi$ and $\partial_2 \varphi$ are two vectors tangent to the orbit, which in the case $\mathcal{G} = SU(N)$ is defined by the conditions $\operatorname{tr}(\varphi^k) = c_k$. This formula is a generalization of the formula (1.15) for the topological charge of the n field.

b) Instanton solutions in chiral models

Up to this point the specific form of the action functional has not been of essential concern to us. We now choose it so that it is bounded below by the topological charge Q. This approach has already been used in Refs. 27, 28, and 67. Here, following Ref. 8, we consider the general case, in which Q is given by Eq. (2.30).

We denote by ψ_μ the quantity $\varepsilon_{\mu\nu}[\varphi, \partial_\nu \varphi]$, where $\varphi \in \mathcal{G}$ is an element of the Lie algebra $\mathcal{G}$, and by (φ, ψ) the positive-definite invariant scalar product in $\mathcal{G}$. We have the obvious inequality

$$(\partial_\mu \varphi \mp \psi_\mu, \partial_\mu \varphi \mp \psi_\mu) \geqslant 0. \tag{2.31}$$

Therefore, if we take as the action S

$$S=\frac{1}{2}\int d^2x\{(\partial_\mu\varphi,\ \partial_\mu\varphi)+([\varphi,\ \partial_\nu\varphi],\ [\varphi,\ \partial_\nu\varphi])\},\tag{2.32}$$

we have the obvious bound

$$S\geqslant c\mid Q\mid.\tag{2.33}$$

This inequality becomes an equation if the so-called equations of duality

$$\partial_\mu\varphi=\pm\varepsilon_{\mu\nu}[\varphi,\ \partial_\nu\varphi]\tag{2.34}$$

are satisfied.

Accordingly, the situation in the general case is reminiscent of that for the n field (Sec. 1), but the expression for the action is now more complicated. At first glance the expression (2.32) for S reminds one of Skyrme's expression,[27] but actually it is quite different, since it contains the derivatives $\partial_\mu\varphi$ only quadratically, while Skyrme used an expression of the fourth degree in $\partial_\mu\varphi$.

The equations of duality (2.34) are simplified somewhat if we go over to the complex variable $z=x_1+ix_2$. They become

$$\partial\varphi=\pm i[\varphi,\ \partial\varphi],\quad \bar\partial\varphi=\mp i[\varphi,\ \bar\partial\varphi],\tag{2.35}$$

where

$$\partial=\frac{\partial}{\partial z}=\frac{1}{2}(\partial_1-i\partial_2),\quad \bar\partial=\frac{\partial}{\partial\bar z}=\frac{1}{2}(\partial_1+i\partial_2).$$

We note that from the equations of duality we have

$$(\partial_1\varphi,\ \partial_2\varphi)=0,\quad (\varphi,\ \partial_1\varphi)=(\varphi,\ \partial_2\varphi)=0,\quad (\partial_1\varphi,\ \partial_1\varphi)=(\partial_2\varphi,\ \partial_2\varphi).\tag{2.36}$$

This means that in the case when the duality conditions hold the mapping φ: $S^2\to\Phi$ is conformal.

We now return to the treatment of chiral models of general form and assume further that the manifold Φ is a complex manifold.[7)] Let w^α be local coordinates in the neighborhood of a point $w^\alpha=0$, and let $h_{\alpha\bar\beta}(w^\gamma)$ be a Hermitean metric

$$ds^2=h_{\alpha\bar\beta}\,dw^\alpha\,d\bar w^\beta.\tag{2.37}$$

Then as the action functional we can take the functional

$$S=\frac{1}{2}\int d^2x h_{\alpha\bar\beta}\partial_\mu w^\alpha\partial_\mu\bar w^\beta.\tag{2.38}$$

Substituting in Eq. (2.38) the quantity $\partial_\mu w^\alpha\pm i\varepsilon_{\mu\nu}\partial_\nu w^\alpha$ instead of $\partial_\mu w^\alpha$, we get the inequality

$$S\geqslant c\mid Q\mid,\quad Q=\frac{i}{2}\int\varepsilon_{\mu\nu}h_{\alpha\bar\beta}\partial_\mu w^\alpha\partial_\nu\bar w^\beta\,d^2x.\tag{2.39}$$

This inequality becomes an equation only for fields which satisfy the duality equations

$$\partial_\mu w^\alpha\pm i\varepsilon_{\mu\nu}\partial_\nu w^\alpha=0,\tag{2.40}$$

or, after changing to the complex coordinate $z=x_1+ix_2$,

$$\bar\partial w^\alpha=0\quad(\text{or}\quad \partial w^\alpha=0).\tag{2.41}$$

The local solution of these equations is $w^\alpha=f^\alpha(z)$ [or $w^\alpha=f^\alpha(\bar z)$]. Accordingly, the equation $S=cQ$ (or, respectively, $S=-cQ$) can be obtained only for holomorphic (or antiholomorphic) mappings φ of the compactified z-plane z: $\mathbf{R}^2\cup\{\infty\}$ (which can be regarded as a one-dimensional complex projective space $\mathbf{C}P^1$, into the manifold Φ.

Unfortunately, Q is in general not a topological invariant and changes with deformations of $w^\alpha(x)$. Therefore in the general case it cannot be asserted that a solution of the duality equations is also a solution of the Euler equations that follow from the condition $\delta S=0$.

However, as was pointed out in Ref. 40, there exists a broad class of complex compact manifolds for which this difficulty does not arise. Namely, let Φ be a Kähler manifold, i.e., a complex manifold on which there exists a Hermitean metric $h_{\alpha\bar\beta}$, whose imaginary part $\omega=(1/2)h_{\alpha\bar\beta}dw^\alpha\wedge d\bar w^\beta$ is a closed nondegenerate 2-form. We note that the condition that this form be closed is equivalent to the conditions

$$\frac{\partial h_{\alpha\bar\beta}}{\partial w^\gamma}=\frac{\partial h_{\gamma\bar\beta}}{\partial w^\alpha},\quad \frac{\partial h_{\alpha\bar\beta}}{\partial\bar w^\gamma}=\frac{\partial h_{\alpha\bar\gamma}}{\partial\bar w^\beta}.\tag{2.42}$$

The mapping φ: $S^2\to\Phi$ defines a two-dimensional cycle in Φ. In this case the quantity cQ is the integral of ω over this cycle, and since the form ω is closed, this quantity depends only on the class of homologies to which this cycle belongs. Consequently, in a given class the quantity Q is constant, and the action integral S is equal to its minimal value if the conditions (2.41) are satisfied, i.e., for holomorphic mappings φ: $S^2=\mathbf{C}P^1\to\Phi$. These mappings, if they exist, give solutions of the duality equations (2.41), so-called instanton solutions.

We note that if the manifold Φ is algebraic, i.e., an analytic submanifold without singularities in the complex projective space $\mathbf{C}P^N$ for some N, then there exists a Kähler metric (the so-called Hodge metric) such that Q will always be an integer.

We further point out that if we go from the variables x_1 and x_2 to the complex variable $z=x_1+ix_2$, the expressions for the action and the topological charge take the form

$$S=\int h_{\alpha\bar\beta}(\partial w^\alpha\,\bar\partial\bar w^\beta+\bar\partial w^\alpha\,\partial\bar w^\beta)\,d^2x,\tag{2.43}$$

$$Q=c^{-1}\int h_{\alpha\bar\beta}(\partial w^\alpha\,\bar\partial\bar w^\beta-\bar\partial w^\alpha\,\partial\bar w^\beta)\,d^2x.\tag{2.44}$$

It is now obvious that S coincides with Q for holomorphic and antiholomorphic fields.

The "equations of motion" are obtained as usual from the condition $\delta S=0$. Recalling the condition (2.24), we get

$$h_{\alpha\bar\beta}(\partial\,\bar\partial w^\alpha)+\frac{\partial h_{\alpha\bar\beta}}{\partial w^\gamma}\,\bar\partial w^\alpha\,\partial w^\gamma=0\tag{2.45}$$

and the equation complex conjugate to this. We now multiply the left side of Eq. (2.45) by $\partial\bar w^\beta$, sum over β, and add to it the left side of the conjugate equation, similarly multiplied and summed. Using the relations (2.42) we can transform this expression into

$$\bar\partial(h_{\alpha\bar\beta}\,\partial w^\alpha\,\partial\bar w^\beta)=0.\tag{2.46}$$

From this we get

$$h_{\alpha\bar\beta}\,\partial w^\alpha\,\partial\bar w^\beta=f(z)\tag{2.47}$$

and similarly

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7) This means, roughly speaking, that points of the manifold can be parametrized with complex coordinates; for the rigorous definition, see Refs. 98, 99.

$$h_{\alpha\bar{\beta}}\bar{\partial}w^{\alpha}\bar{\partial}\bar{w}^{\beta}=f(\bar{z}). \tag{2.48}$$

Let us now consider the solutions of the equations of duality (2.41). These equations are of the form of the Cauchy Riemann equations, but owing to the compactness of the manifold Φ there is by no means always a global solution or, what is the same thing, a holomorphic mapping φ: $\mathbb{C}P^1\to\Phi$. For example, if Φ is a two-dimensional compact manifold of type g (for $g=0$, $\Phi=S^2=\mathbb{C}P^1$; for $g=1, \Phi=T^2$, a two-dimensional torus), i.e., a Riemann surface, such a mapping exists only if $\Phi=\mathbb{C}P^1$. (This case is considered in a paper by Belavin and Polyakov.[28]) Here a mapping with topological charge depends on $4k$ parameters.

Such a mapping exists, however, if $\Phi=\mathbb{C}P^n$, a complex projective space (this case is considered in detail in the next section).

Let us now consider the important class of Kähler manifolds for which holomorphic mappings $\mathbb{C}P^1\to\Phi$ exist. This case includes singly connected compact Kähler manifolds. It follows from Ref. 102 that they all have the form G/H, where G is a compact connected semisimple Lie group with trivial center, and H is the centralizer of some torus in G. It is not hard to see that these spaces can be regarded as orbits of adjoined representations of compact semisimple Lie groups G in their respective Lie algebras $\mathcal{G}$. These spaces are not only algebraic, but also rational.[105,8)]

In this case it is also known that not only the real group G, but also the corresponding complex group G^c act transitively on Φ. Therefore the manifold Φ can also be represented in the form $\Phi=G/H=G^c/P$, where P is the parabolic subgroup, i.e., the subgroup of G^c that contains the maximal connected solvable subgroup. Here $H=P\cap G$.

It is known[104] that any such subgroup can be constructed in a canonical way in terms of the subsystem I of the simple roots of the Lie algebra of the group G^c.

Let R_I be the subsystem of positive roots, consisting of linear combinations of elements of I. Let G_I be the subgroup of G generated by H and the subgroups

$$N_\gamma=\{\exp(tE_\gamma)\colon\ t\in\mathbb{C}\}\quad\text{for}\quad \gamma\in R_I\cup\{-R_I\},\quad P_I=G_I\cdot N_I.$$

As is well known, any parabolic subgroup is conjugate in G^c to one of these subgroups.

We present here the construction of invariant Kähler metrics given by Bore.[102] It is constructed by means of left-invariant forms, so-called Maurer-Cartan forms.[98]

Let us consider the simplest case $\Phi=G/T$, where T is the maximal torus: $T=S^1\times\ldots\times S^1$ (r factors, r being the rank of the group G). Let w^α be left-invariant Maurer-Cartan forms, which induce on the Lie algebra a basis dual to the basis of vector fields X_α, and which are orthogonal to H^c. Using the Maurer-Cartan equations and the well known properties of structure constants, one can show that the form

$$\omega=\frac{i}{2}\sum_{\alpha\in R_+}c_\alpha\omega^\alpha\Lambda\omega^{-\alpha} \tag{2.49}$$

is closed provided only that

$$c_\alpha+c_\beta=c_\gamma\quad\text{for}\quad\alpha+\beta=\gamma. \tag{2.50}$$

Therefore the form ω is determined by the constants c_α for the simple roots α, which can be regarded as arbitrary: $c_\alpha=(h,\alpha)$. Let us consider the restriction of this form to the group G. We thus get a form left-invariant with respect to G and right-invariant with respect to T. Since on complex conjugation the form w^α goes over into the form $w^{-\alpha}$, the form ω is a form of the type (1.1) on G/T. This form is real for real values of the constants c_α, and it can be shown that the class of real cohomologies that corresponds to it is obtained from the element $h\in H^{(1)}(T)$ for which $(\alpha,h)=c_\alpha$ (the α are simple roots) by transgression. If h belongs to the interior of the positive Weyl cell, all the numbers c_α are positive, and the metric

$$ds^2=\sum_{\alpha\in R_+}c_\alpha\omega^\alpha\bar{\omega}^\alpha \tag{2.51}$$

is a Kähler metric on G/T.

If, besides this, $h\in H^{(1)}(T;\mathbf{Z})$, then the corresponding orbit is an integral one and the image of the element h in transgression gives an integral class of cohomologies. Then the corresponding metric is a Hodge metric, and by Kodaira's theorem the manifold G/T is algebraic. We note also that, as is shown in Ref. 105, any orbit of the adjoined representation is a rationale algebraic manifold.

This fact is especially important for us, since in this case there exist homomorphic mappings $\mathbb{C}P^1\to\Phi=G/H$ which are different from the constant one,[9)] or, equivalently, are nontrivial instanton solutions of the corresponding chiral theories.

3. AN SU(N)-INVARIANT CHIRAL MODEL. THE CASE OF THE COMPLEX PROJECTIVE SPACE $\Phi=\mathbb{C}P^n$[10)]

In this section we study a special case of the chiral models considered in the preceding section. Namely, we examine the case in which Φ is the most degenerate orbit (the orbit of smallest dimensionality) of the adjoined representation of the group SU(N). It is well known that this orbit is a homogeneous space $\Phi=G/H=\mathbb{C}P^n$ (i.e., a complex projective space of n dimensions) where the stationary subgroup is $H=\mathrm{SU}(n)\times\mathrm{U}(1)$, $n=N-1$. We recall that since the group G is connected and simply connected, $\pi_2(G/H)=\pi_1(H)$, and consequently

8) A manifold M^n is said to be rational if the field of meromorphic functions on it is isomorphic to the field of meromorphic functions of n complex variables. For example, the two-dimensional sphere $S^2\sim\mathbb{C}P^1$ is a rational manifold, while the two-dimensional torus is a Kähler manifold, but not a rational manifold.

9) Yu. I. Manin has pointed out that this follows from certain theorems of algebraic geometry.

10) The special case $n=1$ was studied by Belavin and Polyakov[28]; cf. Sec. 1. The pseudoeuclidean case is treated in a paper by Zakharov and Mikhaĭlov.[25] The Euclidean case is considered in Refs. 39, 41-43. Here we follow Ref. 39.

$$\pi_2(\Phi) = \pi_2(\mathbb{C}P^n) = \mathbf{Z}, \tag{3.1}$$

where $\mathbf{Z}$ is the additive group of integers. Consequently, the field $\varphi(x)$ can be assigned a topological charge if it is such that its limit exists for $|x| \to \infty$. As was already pointed out in Sec. 2, we can consider that $\varphi \in \mathcal{G}$, i.e., in this case $\varphi(x)$ is a Hermitean $(n+1)\times(n+1)$ matrix with zero trace and the eigenvalues

$$\lambda_1 = \lambda_2 = \ldots = \lambda_n = \frac{\lambda}{n+1}, \quad \lambda_{n+1} = \frac{\lambda n}{n+1}, \tag{3.2}$$

where without loss of generality we can set $\lambda = 1$. It is not hard to see that such a $\varphi(x)$ can be written in the form

$$\varphi(x) = (n+1)^{-1} I - u \otimes \bar{u}, \tag{3.3}$$

where I is the unit matrix and

$$(u \otimes \bar{u})_{\alpha\beta} = u_\alpha \bar{u}_\beta, \quad \alpha, \beta = 1, \ldots, (n+1),$$
$$|u|^2 = (\bar{u}u) = \sum_\alpha \bar{u}_\alpha u_\alpha = 1.$$

Consequently, the field $\varphi(x)$ is determined [up to multiplication by a phase factor $\exp(i(\alpha(x)))$] by a single complex $(n+1)$-dimensional vector u. Now, using (3.3) we can readily show that

$$\frac{1}{2}\operatorname{tr}(\partial_\mu\varphi\partial_\mu\varphi) = (\partial_\mu\bar{u}, \partial_\mu u) - (\bar{u}, \partial_\mu u)(u, \partial_\mu\bar{u}) \tag{3.4}$$

and

$$\operatorname{tr}([\varphi, \partial_\mu\varphi], [\varphi, \partial_\mu\varphi]) = -\operatorname{tr}(\partial_\mu\varphi, \partial_\mu\varphi). \tag{3.5}$$

Therefore we can take the action S in the form

$$S = \frac{1}{2}\int d^2x \operatorname{tr}(\partial_\mu\varphi, \partial_\mu\varphi) = \int d^2x \{(\partial_\mu\bar{u}, \partial_\mu u) - (\bar{u}, \partial_\mu u)(u, \partial_\mu\bar{u})\}. \tag{3.6}$$

We see that the action given by Eq. (3.6) differs from that for the ordinary n field, Eq. (1.3), by the presence of an additional term. We note that this term is necessary in order for the action (3.6) to be gauge invariant, remaining unchanged for $u \to \exp[i\alpha(x)]u$.

The gauge invariance becomes obvious if we change in Eq. (3.6) from the ordinary derivative ∂_μ to the covariant derivative

$$D_\mu = \partial_\mu - A_\mu = \partial_\mu - \bar{u}\partial_\mu u, \quad A_\mu = \bar{u}\partial_\mu u = -\bar{A}_\mu. \tag{3.7}$$

Then the expression (3.6) for the action can be rewritten in the form

$$S = \int d^2x \mathcal{L} = \int d^2x (\overline{D_\mu u}, D_\mu u). \tag{3.8}$$

The equations of motion that follow from the condition $\delta S = 0$ are

$$D_\mu D_\mu u + (\overline{D_\mu u}, D_\mu u) u = 0. \tag{3.9}$$

Changing, as before, to new variables

$$z = x_1 + ix_2, \quad \bar{z} = x_1 - ix_2,$$
$$\partial = \frac{\partial}{\partial z} = \frac{1}{2}(\partial_1 - i\partial_2), \quad \bar{\partial} = \frac{\partial}{\partial\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2),$$
$$A_z = \bar{u}\,\partial u = \frac{1}{2}(A_1 - iA_2), \quad A_{\bar{z}} = \bar{u}\,\bar{\partial}u = \frac{1}{2}(A_1 + iA_2),$$
$$D_z = \partial - A_z, \quad D_{\bar{z}} = \bar{\partial} - A_{\bar{z}}, \tag{3.10}$$

we get

$$\mathcal{L} = 2[(\overline{D_z u}, D_z u) + (\overline{D_{\bar{z}} u}, D_{\bar{z}} u)], \tag{3.11}$$

$$(D_z D_{\bar{z}} + D_{\bar{z}} D_z) u + \frac{1}{2}\mathcal{L} u = 0. \tag{3.12}$$

Another important difference between the CP^n model and that of the n field in N-dimensional space is that the model now considered is always topologically nontrivial and always has instanton solutions, whereas the $SO(N)$-invariant model of the n field is topologically nontrivial only for $N = 3$.

For the density of topological charge it is not hard to derive the expression

$$q = \operatorname{tr}(\varphi[\partial_1\varphi, \partial_2\varphi]) = (\partial_1 u, \partial_2 u) - (\partial_2 u, \partial_1 u) = (\overline{D_1 u}, D_2 u) - (\overline{D_2 u}, D_1 u) \tag{3.13}$$

or

$$\frac{1}{2} q = [D_z, D_{\bar{z}}] = (\overline{D_z u}, D_z u) - (\overline{D_{\bar{z}} u}, D_{\bar{z}} u). \tag{3.14}$$

Consequently,

$$Q = c^{-1}\int \varepsilon_{\mu\nu}(\partial_\mu\bar{u}, \partial_\nu u)\, d^2x = c^{-1}\int \varepsilon_{\mu\nu}(\overline{D_\mu u}, D_\nu u)\, d^2x. \tag{3.15}$$

We note that by means of Eq. (3.14) the "equations of motion" (3.12) can be written in the form

$$D_z D_{\bar{z}} u + \mathcal{L}_{\bar{z}} u = 0, \tag{3.12'}$$

or

$$D_{\bar{z}} D_z u + \mathcal{L}_z u = 0, \tag{3.12''}$$

where

$$\mathcal{L}_z = \frac{1}{4}(\mathcal{L} + q) = (\overline{D_z u}, D_z u), \quad \mathcal{L}_{\bar{z}} = \frac{1}{4}(\mathcal{L} - q) = (\overline{D_{\bar{z}} u}, D_{\bar{z}} u).$$

We now form the Hermitean matrices

$$\Psi_\mu^\pm = \partial_\mu\varphi \mp i\varepsilon_{\mu\nu}[\varphi, \partial_\nu\varphi]. \tag{3.16}$$

By considering the obvious inequalities

$$\frac{1}{2}\operatorname{tr}(\Psi_\mu^\pm, \Psi_\mu^\pm) = \frac{1}{2}\operatorname{tr}(\partial_\mu\varphi\partial_\mu\varphi) - \frac{1}{2}\operatorname{tr}([\varphi, \partial_\mu\varphi][\varphi, \partial_\mu\varphi]) \mp i\operatorname{tr}(\partial_\mu\varphi[\varphi, \partial_\nu\varphi])\varepsilon_{\mu\nu} \geqslant 0, \tag{3.17}$$

we get

$$S \geqslant c|Q|. \tag{3.18}$$

The inequalities (3.17) become equations only for fields that satisfy the "equations of duality"

$$\partial_\mu\varphi = \pm i\varepsilon_{\mu\nu}[\varphi, \partial_\nu\varphi], \tag{3.19}$$

which in the variables u take the form

$$\partial u = \mp(u, \partial\bar{u})\, u, \quad \bar{\partial}u = \pm(\bar{u}, \bar{\partial}u)\, u \tag{3.20}$$

or

$$D_z u = 0, \quad D_{\bar{z}} u = 0. \tag{3.21}$$

In order to find the solutions of the equations (3.20), we go from the variables $(u_1, u_2, \ldots, u_n, u_{n+1})$ to new variables $(w_1, w_2, \ldots, w_n, u_{n+1})$ according to the formula[11)]

$$(u_1, \ldots, u_n, u_{n+1}) = (u_{n+1}w_1, \ldots, u_{n+1}w_n, u_{n+1}). \tag{3.22}$$

From this we have

$$|u_0|^2 = (1 + |w|^2) = 1.$$

Consequently, the last of the equations (3.20) can be written in the form

$$\partial u_{n+1} = (\bar{u}\,\partial u)\, u_{n+1}. \tag{3.23}$$

[11)] The coordinates w_α are indeed those in which the invariant metric on CP^n takes the Kähler form.

By means of this equation we can rewrite the remaining equations in the form

$$\partial w_\gamma = 0 \quad (\text{or} \quad \bar{\partial} w_\gamma = 0), \quad \gamma = 1, \ldots, n. \tag{3.24}$$

Accordingly, the equations of duality have been reduced to the form of the Cauchy-Riemann equations; this was first done for the special case $\Phi = S^2 = \mathbb{C}P^1$ by Belavin and Polyakov.[28] Consequently, the solution of the equations has the form $w_\gamma = f_\gamma(z)$. But we must also satisfy the condition $\varphi(x) \to \varphi_0$ for $|x| \to \infty$. Since the space $\Phi = \mathbb{C}P^n$ is homogeneous, we can take as φ_0 any point of $\mathbb{C}P^n$, for example the point with the coordinates $w_\gamma = 1$. In this special case our condition takes the form $f_\gamma \to 1$ for $|z| \to \infty$. Consequently, we can consider that the $f_\gamma(z)$ are rational functions, and on reducing them to a common denominator we get

$$w_\gamma = f_\gamma(z) = \frac{Q_k^\gamma(z)}{P_k(z)} = \left[\prod_{j=1}^{k}(z - a_j^0)\right]^{-1} \prod_{j=1}^{k}(z - a_j^\gamma). \tag{3.25}$$

Accordingly, an instanton solution in an SU(N)-invariant chiral model depnds on $2Nk$ real parameters, and corresponds to the topological charge

$$Q = k. \tag{3.26}$$

Consequently, each instanton is characterized by $2N$ parameters.

We give the expressions for the action and the topological charge in the coordinates w_γ:

$$S = \int d^2x\, [\delta_{\beta\alpha}(1 + |w|^2) - \bar{w}_\alpha w_\beta] \frac{\partial w_\alpha \bar{\partial}\bar{w}_\beta + \bar{\partial} w_\alpha \partial \bar{w}_\beta}{(1 + |w|^2)^2}, \tag{3.27}$$

$$Q = c^{-1} \int d^2x\, [\delta_{\beta\alpha}(1 + |w|^2) - \bar{w}_\alpha w_\beta] \frac{\partial w_\alpha \bar{\partial}\bar{w}_\beta - \bar{\partial} w_\alpha \partial \bar{w}_\beta}{(1 + |w|^2)^2}. \tag{3.28}$$

We point out that both of these expressions are determined by the Kähler structure for $\Phi = \mathbb{C}P^n$.

Returning from the variables w_γ to the variables u_α ($\alpha = 1, \ldots, N$), we come to the conclusion that the most general instanton solution is of the form

$$u_\alpha = \frac{P_\alpha(z)}{|P_\alpha|}, \quad \alpha = 1, \ldots, N, \tag{3.29}$$

where $P_\alpha(z)$ are polynomials in z that have no common roots, and

$$|P_\alpha| = \sqrt{\sum_{\alpha=1}^{N} |P_\alpha|^2}. \tag{3.30}$$

The instanton solutions (3.29) are solutions of the "equations of motion" (3.9) [or (3.12), (3.12′), (3.12″)] with the finite action (3.8).

The question naturally arises of describing all the solutions of the "equations of motion" that have finite action.

For the ordinary n field model ($\Phi = \mathbb{C}P^1$) it is not hard to show that the instanton and antiinstanton solutions exhaust all solutions with finite action. For the case of the $\mathbb{C}P^n$ model with $n \geq 2$ this is not true; solutions exist with finite action which do not reduce to instanton or antiinstanton ones.[75] Namely, it was remarked in Refs. 75 and 76 that such solutions can be obtained from known solutions of the SO(N)-invariant model of a real n field. For finding solutions of the n field model in Ref. 79, a mathematical method previously developed in Ref. 80 was used.

It must be pointed out, however, that in the model of the real n field the situation is more complicated in comparison with the $\mathbb{C}P^n$ model, since in the n field model a number of supplementary conditions must be satisfied, and therefore in Ref. 79 only a recurrence procedure was indicated for finding the general procedure in the SO(N)-invariant model (with odd N).

Using these techniques[80,79] explicit expressions were obtained[76-78] for the general solution of the $\mathbb{C}P^n$ model with finite action, without any sort of supplementary conditions.

We shall present these results, following Refs. 76 and 77.

The explicit form of the general solution of the $\mathbb{C}P^n$ model with finite action

Let $u(z, \bar{z})$ be a solution of the "equations of motion" of the $\mathbb{C}P^n$ model, Eq. (3.12), with finite action. For instanton (or antiinstanton) solutions the vector u satisfies the equation $D_{\bar{z}}u = 0$ (or $D_z u = 0$). Therefore it is natural to consider two sequences of vectors

$$D_{\bar{z}}u,\ D_{\bar{z}}^2 u,\ \ldots \tag{3.31}$$

and

$$D_z u,\ D_z^2 u,\ \ldots .$$

The vectors of these two sequences turn out to be orthogonal to each other. Namely,

$$A_{j,k}^m = (\overline{D_{\bar{z}}^j u},\ D_z^k u) = 0 \quad \text{for} \quad m = j + k \geq 1. \tag{3.32}$$

It is obvious that $A_{0,1}^1 = A_{1,0}^1$, and not hard to show that $A_{1,1}^2 = 0$.

Let us now assume that the quantities $A_{j,k}^1, A_{j,k}^2, \ldots, A_{j,k}^m$ are equal to zero. Then, using the identity

$$\partial(\bar{a}, b) = (D_{\bar{z}} a, b) + (\bar{a}, D_z b), \tag{3.33}$$

we get

$$A_{i+1,j}^{m+1} = (\overline{D_{\bar{z}}^{i+1}u},\ D_z^j u) = \partial(\overline{D_{\bar{z}}^i u},\ D_z^j u) - (\overline{D_{\bar{z}}^i u},\ D_z^{j+1} u) = -A_{i,j+1}^{m+1}. \tag{3.34}$$

Therefore it suffices to verify that

$$A_{0,m+1}^{m+1} = 0. \tag{3.35}$$

The proof of this fact reduces to proving that the quantities $A_{0,m+1}^{m+1}$ are analytic and using a variant of Liouville's theorem together with the inequality

$$|A_{0,m+1}^{m+1}|^2 \leq |D_z^{m+1} u|^2. \tag{3.36}$$

Now, let $u(z, \bar{z})$ be a solution of the equations of motion (3.12) with finite action. We denote by H_k and H'_m the subspaces spanned by the vectors $D_{\bar{z}}u, D_{\bar{z}}^2 u, \ldots$ and $D_z u, D_z^2 u, \ldots$, respectively; here k and m are the dimensions of the spaces H_k and H'_m. These spaces are orthogonal to each other ($H_k \perp H'_m$) and to the vector u.

It is obvious that

$$k + m \leq n \tag{3.37}$$

and for a solution of general type the equals sign is attained here (with the possible exception of individual points).

It can also be shown that the first k vectors $D_z u, D_z^2 u, \dots, D_z^k u$ form a basis in the space H_k, and correspondingly the vectors $D_z u, D_z^2 u, \dots, D_z^m u$ form a basis in the space H'_m.

Let $u(z,\bar{z})$ be a solution of the equations (3.12) which is not an instanton solution. Then $k \geq 1$. It turns out that in the space $\hat{H}_k = \{u, H_k\}$ there exists a holomorphic vector $f = (f_1, \dots, f_n)$, i.e., a vector satisfying the equation $\bar{\partial} f = 0$, and that it can be chosen so that is satisfies the condition

$$(\bar{f}, D_z^j u) = \omega \delta_{jk}, \quad j = 0, 1, \dots, k, \tag{3.38}$$

where ω is some function. We shall not give the proof of this assertion; it can be found in Ref. 77. It can also be shown that the components of the vector f are rational functions of z and that the points of the z plane at which f is singular correspond to the case $k + m < n$.

It can also be shown that

$$(\overline{\partial^l f}, D_{\bar{z}}^j u) = (-1)^l \omega \delta^{j+l, k}, \quad 0 \leqslant j + l \leqslant k. \tag{3.39}$$

Accordingly, the vectors $f, \partial f, \dots, \partial^k f$ form a basis in the space H_k which is dual to the basis $u, D_{\bar{z}} u, \dots, D_{\bar{z}}^k u$.

In Ref.76 an explicit expression was found for the expansion of the vector u in the basis $f, \partial f, \dots, \partial^k f$. It is of the form

$$u = \frac{v}{|v|}, \tag{3.40}$$

where

$$v = (-1)^k \left[\partial^k f - \sum_{j=0}^{k-1} \left(\sum_{l=0}^{k-1} (M^{-1})_{jl}\, \partial M_{l,\,k-1}\right) \partial^j f\right], \tag{3.41}$$

and M_{jl} is a k-row matrix

$$M_{lj} = (\overline{\partial^l f}, \partial^j f), \quad j, l = 0, \dots, k-1. \tag{3.42}$$

This matrix is positive definite and invertible.

We note that furthermore

$$\omega = |v|, \tag{3.43}$$

and that the vector space H'_m is spanned by the vectors $f, \partial f, \dots \partial^m f$.

It can also be shown[76] that for an arbitrary rational analytic vector $f = f(z)$ in $(n+1)$-dimensional complex space such that the vectors $f, \partial f, \dots, \partial^n f$ are linearly independent (with the exception of individual points, including the infinitely remote point), and for any integer k such that $0 \leq k \leq n$, the vector defined by Eqs. (3.40)–(3.42) is actually a solution of the equations (3.42), and the action S for this solution is finite.

Let us consider the simplest nontrivial case, that of the $\mathbf{C}P^2$ model.

Here the solutions with $k = 0, m = 2$ (or $k = 2, m = 0$) are instanton (antiinstanton) solutions. The case $k = 1$, $m = 1$ corresponds to solutions that are neither of instanton nor of antiinstanton type. In this case

$$f = (f_1(z), f_2(z), f_3(z)), \qquad u = \frac{v}{|v|}, \quad v = -(\partial f - |f|^{-2} (\bar{f}, \partial f) \cdot f). \tag{3.44}$$

Let

$$f = (1, z, z^2). \tag{3.45}$$

Then

$$u = \frac{(\bar{z}(1 + 2|z|^2),\ |z|^4 - 1,\ -z(2 + |z|^2))}{(1 + 4|z|^2 + 6|z|^4 + 5|z|^6 + |z|^8)^{1/2}}. \tag{3.46}$$

For $|z| \to \infty, u \to (0,1,0)$, and consequently this solution corresponds to topological charge $Q = 0$.

4. THE SU$(n+m)$-INVARIANT CHIRAL MODEL. THE CASE OF A COMPLEX GRASSMANN MANIFOLD: $\Phi = G_{m,n}$[12)]

In this section we consider another SU(N)-invariant model namely the case in which the field $\varphi(x)$ takes on values in the complex Grassmann manifold $G_{m,n}$.

The space $G_{m,n}$ can be regarded as a space of m-dimensional (or n-dimensional) hyperplanes in the $(m+n)$-dimensional complex space $\mathbf{C}^{m+n}$. These spaces are a natural generalization of the complex projective space $\mathbf{C}P^n$ considered in the preceding section; the space $\mathbf{C}P^n$ corresponds to the case $m = 1$.

A point of this space can be regarded as a set of m vectors $\{u_\alpha\}$ in an $(n+m)$-dimensional complex space, under the condition that two such sets $\{u_\alpha\}$ and $\{\tilde{u}_\beta\}$ are counted as equivalent if they are connected by a transformation from the group $U(m)$; $\tilde{u}_\alpha = g_{\alpha\beta} u_\beta$, $g \in U(m)$.

In other words, we can consider a point of this space as a rectangular matrix u with m columns and $(m+n)$ rows, this matrix being required to satisfy the condition

$$u^+ u = I^{(m)} \quad \text{or} \quad u_\alpha^j \bar{u}_\alpha^k = \delta_{jk}, \qquad \alpha = 1, \dots, m+n; \quad j, k = 1, \dots, m, \tag{4.1}$$

where $I^{(m)}$ is the unit matrix of order m.

On the other hand, the field $\varphi(x)$ can be regarded as a field which takes on values in the Lie algebra of the group G, that is, we regard $\varphi(x)$ as a Hermitean matrix of order $(n+m)$ with zero trace and the eigenvalues

$$\lambda_1 = \lambda_2 = \dots = \lambda_n = \frac{\lambda m}{m+n}, \quad \lambda_{n+1} = \dots = \lambda_{n+m} = -\frac{\lambda n}{m+n}. \tag{4.2}$$

Here without loss of generality we can assume that $\lambda = 1$.

It is not hard to show that $\varphi(x)$ is of the form

$$\varphi(x) = \frac{m}{m+n} I - uu^+, \quad \varphi_{\alpha\beta} = \frac{m}{m+n} \delta_{\alpha\beta} - u_\alpha^j \bar{u}_\beta^j. \tag{4.3}$$

We note that the matrix

$$P = uu^+ \tag{4.4}$$

satisfies the condition

$$P^2 = P \tag{4.5}$$

(i.e., this matrix is a projection operator), and the matrix

$$g = I - 2P, \quad g^+ = g \tag{4.6}$$

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12) For the pseudoeuclidean case this sort of model was first considered in Ref. 35. For the Euclidean case see Refs. 39, 40, 42.

satisfies the condition

$$g^+g = g^2 = I^{(m+n)}. \tag{4.7}$$

Therefore we can regard the matrix g as an element of the group $G = SU(m+n)$. Accordingly, in the Grassmann case we can regard the chiral field as an element of the group G which satisfies, first, the condition

$$g^+ = g \tag{4.8}$$

and second, the condition that n of the eigenvalues of the matrix g are equal to $+1$, and the other eigenvalues are equal to -1. Consequently, an arbitrary element of our space can be obtained from the element

$$g_0 = \mathrm{diag}\,(1, \ldots, 1, -1, \ldots, -1) \tag{4.9}$$

by using the action of the group[13] G.

We remark that in the space $G_{m,n}$ under consideration there exists a unique (up to a constant factor) $SU(m+n)$-invariant metric

$$ds^2 = \mathrm{tr}\,(d\varphi \cdot d\varphi), \tag{4.10}$$

by means of which we obtain an expression for the action

$$S^2 = \frac{1}{2} \int d^2x\, \mathrm{tr}\,(\partial_\mu \varphi \cdot \partial_\mu \varphi). \tag{4.11}$$

Substituting in this formula the expression of φ in terms of u [Eq. (4.3)], we get

$$S = \int d^2x\, \mathrm{tr}\, \{(\partial_\mu u^+ \partial_\mu u) - (u^+ \partial_\mu u \cdot \partial_\mu u^+ u)\}. \tag{4.12}$$

We now give a different formulation of this model. We introduce Hermitian matrices A_μ ($\mu = 1, 2$) of order m:

$$A_\mu = \frac{i}{2}\,[(u^+\partial_\mu u) - (\partial_\mu u^+ u)] = i u^+ \partial_\mu u. \tag{4.13}$$

It is not hard to see that under the transformation

$$u \to u' = u \cdot \exp\,(i\alpha\,(x)), \tag{4.14}$$

where α is a Hermitian matrix of order m, the field A_μ transforms as a gauge field does relative to the group $U(m)$:

$$A_\mu \to A'_\mu - \partial_\mu \alpha, \quad A'_\mu = \exp\,(-i\alpha)\, A_\mu \exp\,(i\alpha). \tag{4.15}$$

It is convenient to introduce the covariant derivative

$$D_\mu = \partial_\mu + iA_\mu. \tag{4.16}$$

Then the expression (4.12) can be rewritten in the form

$$S = \int d^2x\, \mathrm{tr}\, ((D_\mu u)^+ D_\mu u), \tag{4.17}$$

where

$$\begin{aligned} D_\mu u &= \partial_\mu u + iuA_\mu, \\ (D_\mu u)^+ &= \partial_\mu u^+ - iA_\mu u^+. \end{aligned} \tag{4.18}$$

We note that the second term in Eq. (4.12) is necessary in order that the action S be invariant under the group $U(m)$. Accordingly, the theory in question is globally $SU(m+n)$-invariant and invariant relative to $SU(m)$ with respect to gauge.

[13] An analogous statement also holds for a number of other homogeneous spaces (namely for so-called symmetric spaces[93]). For details see Refs. 40, 50, 70.

The $G_{m,n}$ model, like the CP^n model, is topologically nontrivial; the second homotopic group $\pi_2(\Phi) = \pi_2(G_{mn}) = \mathbf{Z}$. Therefore the field $\varphi(x)$ can be characterized by a single integer, the topological charge.

According to the general formula (2.30) we have the integral representation

$$Q = c^{-1} \int d^2x\, (\varphi,\ [\partial_\mu \varphi,\ \partial_\nu \varphi])\, \varepsilon_{\mu,\nu} \tag{4.19}$$

for the topological charge. Substituting here the expression (4.3) for φ, we get

$$Q = \frac{1}{2\pi} \int d^2x\, \{\mathrm{tr}\,(\partial_\mu u \partial_\nu u^+)\, \varepsilon_{\mu\nu}\}, \quad \nu,\ \mu = 1,\ 2 \tag{4.20}$$

or

$$Q = \frac{1}{2\pi} \int d^2x\, \varepsilon_{\mu\nu}\, \mathrm{tr}\,(\partial_\mu A_\nu). \tag{4.21}$$

We now proceed to find the instanton solutions. We form the Hermitean matrices of order $(m+n)$ given by

$$\psi_\mu^\pm = \partial_\mu \varphi \mp i\varepsilon_{\mu\nu}\, [\varphi,\ \partial_\nu \varphi]. \tag{4.22}$$

From the obvious identity

$$\mathrm{tr}\,(\psi^\pm \psi^\pm) \geqslant 0 \tag{4.23}$$

it follows that

$$S \geqslant c\, |\, Q\, |. \tag{4.24}$$

This inequality becomes an equation only for fields which satisfy the equations of duality

$$\partial_\mu \varphi = \pm i\varepsilon_{\mu\nu}\, [\varphi,\ \partial_\nu \varphi], \tag{4.25}$$

which can be written in a different form

$$D_\mu u = \pm\, \varepsilon_{\mu\nu} D_\nu u \tag{4.26}$$

or

$$\partial u = \pm u\, (u^+ \partial u), \quad \bar{\partial} u = \mp u\, (u^+ \bar{\partial} u), \tag{4.27}$$

where

$$\partial = \frac{\partial}{\partial z} = \frac{1}{2}\,(\partial_1 - i\partial_2), \quad \bar{\partial} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\,(\partial_1 + i\partial_2).$$

To find solutions of the equations (4.27) we carry out a change of variables. We go from the variables u_α^j [$j = 1, \ldots, m$; $\alpha = 1, 2, \ldots, (m+n)$] to the new variables w_a^j ($j = 1, \ldots, m$; $a = 1, \ldots, n$):[14]

$$w_a^j = u_a^k\, (u^{(0)-1})_k^j, \tag{4.28}$$

where $u^{(0)}$ is a matrix of order m

$$(u^{(0)})_k^j = u_{n+k}^j, \quad j,\ k = 1, \ldots, m. \tag{4.29}$$

From Eq. (4.27) it follows that

$$\partial w_a^j = 0 \quad (\text{or} \quad \bar{\partial} w_a^j = 0), \quad j = 1, \ldots, m, \ a = 1, \ldots, n. \tag{4.30}$$

Therefore the quantities w_a^j are rational functions of the variable z (or $\bar{z}$), and besides this we can consider (just as in the preceding section) that for $|z| \to \infty$, $w_a^j(z) \to 1$. Reducing the expressions for w_a^j to a common denominator, we get

$$\begin{gathered} w_a^j = f_a^j(z) = \frac{Q_k^{j,a}(z)}{P_k(z)} = \prod_{l=1}^{k} (z - a_l^{ja}) \Big[\prod_{l=1}^{k} (z - a_l)\Big]^{-1}, \\ j = 1, \ldots, m, \quad a = 1, \ldots, n. \end{gathered} \tag{4.31}$$

[14] The coordinates ω^j are coordinates in which the invariant metric on $G_{m,n}$ is a Kähler metric.

Accordingly, an instanton (anti-instanton) solution for the $SU(n+m)$-invariant Grassmann chiral model depends on $2(mn+1)k$ real parameters. The topological charge corresponding to this solution is

$$Q = k. \tag{4.32}$$

Consequently, each instanton (anti-instanton) is characterized by $2(mn+1)$ real parameters.

We further give expressions for the action and the topological charge in the coordinates w_a^j:

$$S = \int d^2x\,(h_{a\bar{b}}^{j\bar{k}} \partial_\mu w_a^j \partial_\mu \overline{w}_b^k), \tag{4.33}$$

$$Q = c^{-1} \int d^2x\,(h_{a\bar{b}}^{j\bar{k}} \partial_\mu w_a^j \partial_\nu \overline{w}_b^k)\,\varepsilon_{\mu\nu}, \tag{4.34}$$

where

$$\begin{gathered} |w|^2 = \sum_{j,a} |w_a^j|^2 = \mathrm{tr}\,(ww^+), \\ h_{a\bar{b}}^{j\bar{k}} = \frac{\partial^2 F}{\partial w_a^j\, \partial \overline{w}_j^k}, \quad F = \ln \det (I + ww^+). \end{gathered} \tag{4.35}$$

In concluding the section we point out that in the Grassmann model considered, as also in the CP^n model, besides the instanton and antiinstanton solutions there exist also other solutions with finite action. Their description is a more complicated problem than for the CP^n model, and we shall not deal with it here (on this matter see Ref. 78).

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