The pion-pion interaction (review of experimental data)

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Experimental methods for studying the $\pi\pi$ interaction are reviewed. Problems involved in the phase-shift analysis of reactions of the type $\pi N \to \pi\pi N$, which are the primary source of information on $\pi\pi$ scattering, are discussed. The experimental characteristics of the $\pi\pi$ interaction are summarized.

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1. INTRODUCTION

It has now been more than 20 years since the introduction of the first concepts regarding the $\pi\pi$ interaction and more than 15 years since theoreticians and experimentalists began their vigorous study of it. This interaction nevertheless remains one of the most popular and interesting problems in the physics of elementary particles, apparently because of a combination of three important circumstances: the fact that this interaction is so widespread ("omnipresent"), the relative simplicity of the theoretical description, and the novelty of the method by which experimental information is obtained.

Pion–pion scattering is a particular case of the strong interaction, so it is a governing factor in a broad range of effects. A classic example of its importance was in the analysis, some years ago, of the electromagnetic form factors of the nucleons, which led to the prediction of vector pion resonances. Vector mesons were discovered soon thereafter, as we know, first in strong interactions and then in electromagnetic interactions. With the passage of time this field has grown into an extensive spectroscopy of pion resonances. Among more recent examples of the importance of pion–pion scattering are the significant developments in research on NN interactions, which became possible as a result of our improved understanding of the $\pi\pi$ amplitudes and the use of the concept of dipion exchange, and the concept of a new form of the pion–pion interaction: pion condensation.

The $\pi\pi$ interaction cannot be studied by direct experiments because intersecting pion beams are not available. Everything which has been learned experimentally about this interaction has thus come indirectly: from analysis of reactions in which pions are produced in the final state (K decays, the reactions $\pi N \to \pi\pi N$, the reactions $e^+e^- \to 2\pi$ and $\bar{p}p \to 2\pi$, etc.). To a large extent this circumstance determined the nature of the early theoretical work on $\pi\pi$ scattering. Foremost among the early developments are the Ansel'm–Gribov method, which in principle makes it possible to determine certain characteristics of pion–pion scattering by studying various processes near the threshold for three-particle production in the final state,[1-4] and the fundamental work by Chew and Low,[5] who offered a specific method for extracting the cross section for $\pi\pi$ scattering from the cross section for the reaction $\pi p \to \pi\pi N$ (and similar reactions), which can be measured directly. The methods for extracting information on the $\pi\pi$ interaction will be discussed in more detail in Section 2. Here we would simply like to emphasize the important contributions of several Soviet groups, particularly Anisovich and his colleagues (on the determination of the $\pi\pi$-scattering phase shifts from $K \to 3\pi$ decays[4-6,7]), Aref'ev and his colleagues (on the determination of the scattering lengths[8]), and Boreskov, Kaĭdalov, and Ponomarev (on the reggeization of π exchange and a description on the basis of it of reactions in which one and two pions are produced[9-12]). Various aspects of this problem have been discussed elsewhere (see, for example, Refs. 13–17). The state of the problem as of January 1969 is covered well and in detail in the review[1)] by Leksin.[18]

One reason for the considerable interest in the $\pi\pi$ interaction is its unique nature (the scattering of a quantum by a quantum). However, the interaction is more important than simply as an end in itself. Since this interaction is so common, information on it is required

[1)] We hope that the reader is familiar with this review and with the terminology used in it.

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for testing many theoretical models. Since the pion has a small mass and no spin, and since $\pi\pi$ scattering has complete crossing symmetry, reactions of the type $\pi\pi \to \pi\pi$ are extremely sensitive to theoretical assumptions. The reactions $\pi\pi \to \pi\pi$ and $\pi\pi \to \pi\omega$ play a special role in the "dual" models. The duality principle may be described in a very simplified manner by saying that the high-energy behavior of amplitudes may be used to predict resonant states at low energies, and in a certain sense the s-channel reasonances are constructed from Regge poles in the t channel. An s, t amplitude of this type, symmetric in the Mandel'stam variables, was introduced by Veneziano,[19] for example. The Veneziano model, like several other dual models, predicts new states. For a study of these states—for the spectroscopy of resonances—the study of the $\pi\pi$ interaction is extremely important. A detailed phase-shift analysis of $\pi\pi$ scattering is one of the powerful tools available for finding new structures, for studying their characteristics, and for comparing the results with the dual-model predictions.

For a long time now there has been interest in the possibility of constructing models from the general principles of analyticity, unitarity, and crossing symmetry. The "bootstrap" hypothesis reflects the early stage of this approach. In it, the properties of hadron dynamics are built up under the assumption that all the particles are bound states of each other and that long-range forces are dominant. The $\pi\pi$ interactions played a central role in this approach. The subsequent development of $SU(3)$ symmetry and the discovery of several structures for which the bootstrap hypothesis did not work caused this hypothesis to be abandoned, but the effort to construct models from the principles of analyticity, unitarity, and crossing symmetry continued. The most promising candidate for the application of these principles is the $(\pi\pi)$ system, which has complete crossing symmetry. Particular progress has been made with the help of the Roy equations, which will be discussed in Section 7. The uncertainties in the $\pi\pi$ amplitudes at low energies can be reduced by using these equations. The threshold region is of special interest because here a comparison can be made with the predictions of current algebra, of models with broken chiral symmetry, and the hypothesis of partial conservation of axial current (PCAC). For all these models it has been $\pi\pi$ scattering which has yielded the decisive experimental data. Information on $\pi\pi$ interactions which is important for constructing realistic multiperipheral models is presently being obtained. The field of research has recently attracted even more interest because of the development of the theory of the π condensate and the anomalous state of nuclear matter.[20,21] It turns out that information on $\pi\pi$ scattering is required for exact calculations from these models.

Many years of research on the $\pi\pi$ interaction failed to clarify the picture. More recently, several experiments have been of substantial help. Since the 1970 publication of Leksin's review, mentioned above, few papers have been published in the Soviet literature on the overall $\pi\pi$-interaction problem. We therefore felt it was worthwhile to reanalyze the approaches and methods of the research on the $\pi\pi$ interaction and to review the experimental situation as it exists today. In Section 2 we will briefly describe the primary sources of information on $\pi\pi$ scattering, and we will compare the advantages and disadvantages of these sources. In Sections 3 and 4 we will analyze the research method which is presently proving the most successful: study of reactions of the type $\pi N \to \pi\pi N$ and $\pi N \to \pi\pi\Delta$. We will briefly review the existing models, discuss the extrapolation of experimental data to the pion pole, and discuss the ambiguity of the concept of a resonance. In Sections 5–7 we will review the phase-shift analysis method and the particular features of research on $\pi\pi$ interactions at low energies. In Section 8 we will see the experimental results on the scattering phase shifts and scattering lengths, and in the Conclusion we will summarize the progress, point out the unresolved questions, and discuss some possible approaches for future research.

This review covers, albeit not exhaustively, work published up to November 1979.

2. METHODS FOR STUDYING $\pi\pi$ INTERACTIONS

What can we learn about the $\pi\pi$ interaction, and how can we get this information? The most interesting bits of information are an integral characteristic—the cross section $\sigma_{\pi\pi}$—and a more detailed characteristic—the partial-wave amplitudes, i.e., the phase shifts and elasticity parameters. These characteristics are usually studied as functions of the dipion mass $m_{\pi\pi}$. For convenience, the range of dipion masses may be considered to consist of three subranges, with distinct problems and research methods. The first subrange is the low-energy subrange, near the threshold. The most important unresolved questions here are the values of the pion–pion scattering lengths. Next comes the subrange of intermediate energies, $500 \leq m_{\pi\pi} < 1800$ MeV. In this "resonance" region the characteristics of the pion resonances are studied. The third subrange is that of high energies, $m_{\pi\pi} > 1800$ MeV. We will have very little to say about this third subrange in the present review, since essentially no experimental information is available for such masses, and the questions of joining the results of phase-shift analysis in the intermediate subrange with theoretical calculations for high energies (e.g., calculations from the theory of reggeized exchange) go beyond the scope of this review.

In the absence of intersecting pion beams and meson targets, indirect methods must be used to study the $\pi\pi$ interaction. In principle, any process which includes a $\pi\pi$ interaction might be a source of information. So far, the most important sources have been reactions in which several pions are produced in the bombardment of a simple nuclear target (a proton or deuteron) by a π beam (for example, $\pi N \to \pi\pi N$ and $\pi N \to \pi\pi\Delta$). The one-pion-exchange diagram is predominant in these reactions at a low 4-momentum transfer t. With a sufficiently large statistical base it becomes possible to single out the contribution of this diagram and, by studying the scattering of a pion by a virtual pion, to move on to a description of the process at the pion pole, i.e., to the scattering of real pions. Then the problem reduces to

TABLE I.

Reaction	Waves studied	Possible isospin	References
1. $\pi^-p \to \pi^-\pi^+n$	S, P, D, F	0, 1, 2	22–42
2. $\pi^\pm p \to \pi^\pm\pi^0p$	S, P, D, F	1, 2	43–45
3. $\pi^+p \to \pi^+\pi^+n$	S, D	2	45–49
4. $\pi^-p \to \pi^0\pi^0n$	S, D	0, 2	50–52
5. $\pi^+p \to \pi^0\pi^0\Delta^{++}$	S, D	0, 2	53–54
6. $\pi^+p \to \pi^+\pi^-\Delta^{++}$	S, P, D, F	0, 1, 2	55–56
7. $\pi^-p \to \pi^-\pi^-\Delta^{++}$	S, D	2	57–59
8. $\pi^+d \to \pi^+\pi^-pps$	S, P, D, F	0, 1, 2	60
9. $\pi^-d \to \pi^-\pi^-pps$	S, D	2	61–63
10. $\pi^+d \to \pi^0\pi^0pps$	S, D	0, 2	64

ordinary phase-shift analysis in its simplest form (the scattering of a spin-zero particle by a spin-zero scattering center). A more detailed justification for the hypothesis of one-pion exchange and a more detailed description of the procedure for extrapolating the experimental data to the pole will be given in Section 3.

Table I lists the basic reactions under consideration here, the isospins of the dipion system, and the partial waves whose amplitudes can be studied in each of these reactions at intermediate energies. We see that a study of these reactions in principle yields everything we wish to know about all possible states of the $\pi\pi$ system. Reactions 1–10 in this table are usually studied at high energies of the incident pion, sufficient for the manifestation of waves with $l \leq 3$. It is also extremely useful, however, to study these reactions at small values of $m_{\pi\pi}$, near the threshold. Study of reactions 1–10 has substantially broadened our understanding of the $\pi\pi$ interaction. Within certain uncertainties, which will be discussed below, it can be said that the behavior of the $\pi\pi$ scattering phase shifts is known up to $m_{\pi\pi} \simeq 1.8$ GeV. Many new resonant states (S^*, h, etc.) have been discovered and studied.

An understanding of the details of the resonance phenomenology requires a study of such reaction channels as $\pi K \to \pi K$ and $\pi\pi \to K\bar{K}$. The latter reaction must also be studied for a final resolution of some uncertainties which remain in the S_0 wave for $\pi\pi$ scattering. An important source of detailed information on the $\pi\pi$ interactions at large values of $m_{\pi\pi}$ is the annihilation reaction $\bar{p}p \to \pi^+\pi^-$. This reaction has been the subject of considerable recent interest.[65–67]

There are certain other ways to obtain information about the $\pi\pi$ interaction. The most useful of these other ways is to analyze K_{e4} decay, which in principle permits a completely model-independent determination of the phase shifts in the region $2\mu < m_{\pi\pi} < m_K$, and to analyze the reaction $e^+e^- \to \pi^+\pi^-$, which determines the P_1 wave. It is difficult to study K_{e4} decay because of the small partial width of this channel ($\sim 4 \cdot 10^{-5}$), so that the accuracy of the results cannot be improved very much. Nevertheless, this method has yielded important data on the scattering lengths and phase shift for the S_0 wave near the threshold, as will be discussed in Section 7.

The characteristics of the ρ^0 resonance were obtained some time ago in a study[68] of the reaction $e^+e^- \to \pi^+\pi^-$ at Novosibirsk, but we believe that this reaction is not being used adequately for a study of the $\pi\pi$ interaction. This method might, for example, resolve the question of the value of the P_1-wave scattering length.

Most of the other potential sources of information on $\pi\pi$ interactions are used less frequently, since—with certain exceptions—these other sources are either highly model-dependent (for example, $K \to 3\pi$) or require an accuracy unattainable in practice ($K_S \to 2\pi$). Elastic pion–nucleon interactions are studied in order to obtain information on the S-wave $\pi\pi$ interaction near the threshold.[69] Analysis of data on πN scattering in the physical region and the crossing-symmetry properties can yield information on the $\pi\pi$ phase shifts. Reactions in which nucleons are annihilated, and a large number of pions are formed in the final state, are sometimes also used, but more frequently the $\pi\pi$ scattering phase shifts are introduced here to study other aspects of the reactions.

We see that at present the primary source of information on the $\pi\pi$ interaction is the study of reactions 1–10 in Table I. This method will accordingly be discussed in detail in the sections which follow, while other methods will be touched on from time to time.

3. REACTIONS OF THE TYPE $\pi N \to \pi\pi N$ AND $\pi N \to \pi\pi\Delta$

Study of reactions 1–10, which are presently the primary sources of information about $\pi\pi$ scattering, is based on the hypothesis that one-pion exchange is predominant (the OPE model). It is assumed in this model that the interacting particles are exchanged in the t channel by a single π meson (Fig. 1a). This model arose because at energies above 1 GeV essentially all inelastic reactions are peripheral in nature, i.e., have very anisotropic c.m. angular distributions and occur primarily with a small 4-momentum transfer t. In πN and NN collisions in which one or two pions are produced, most of the cross section is concentrated at $|t| < 0.5$ $(\text{GeV}/c)^2$.

It can be seen from geometric considerations that small values of t correspond to a large impact parameter (thus the term "peripheral interaction") and to a small mass of the exchange particle. Since the π mass is small, the pion pole is near the physical region, so that π exchange becomes predominant. This circumstance can be seen easily from the expression for the amplitude for one-particle exchange:

$$T = \frac{f}{m^2_{\text{ex}} - t}, \tag{1}$$

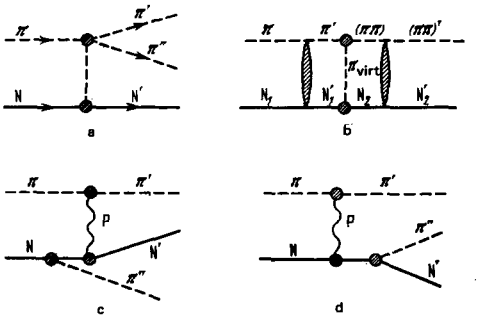

FIG. 1. Feynman diagrams for various reactions. a—One-pion exchange (OPE); b—one-pion exchange with absorption (OPEA); c, d—diffractive dissociation.

where m_{ex} is the mass of the exchange particle, and f is an unknown function. If the values of f for the various particles are of approximately the same order of magnitude and also independent of t, that is, if the departure of the exchange pion from the mass shell is ignored, then everything is determined by the propagator, and π exchange is predominant at small values of t (this is the OPE pole approximation).

One of the most direct pieces of evidence that OPE is important is the agreement of the behavior of the πN cross sections[71] and the angular distributions[55] found experimentally on the mass shell with the results calculated by the Chew–Low method. The results of these and many other experiments have shown that OPE plays a leading role in reactions 1–10, in which we are interested here.

In the pure pole OPE model, however, only a qualitative description of the experimental data can be found. Three methods are presently being used to improve the agreement between theory and experiment:

1) Empirical form factors are being introduced.

2) Absorption is taken into account in the initial and final states (the absorption models).

3) Models with exchange of a reggeized pion are being developed.

The meaing of the form factors and the auxiliary functions $\Phi(t)$, which allow for the t dependence at the vertices, is that their introduction allows for the departure of the exchange pion from the mass shell. Corrections of this sort cannot be calculated exactly, so that the form factor has an arbitrary parametrization under certain restrictions (the conditions of positiveness and analyticity and a normalization condition). The function $\Phi(t)$ is a universal function independent of the charge reaction channel, and it is normalized in such a manner that the condition $\Phi(\mu^2)=1$ holds at the pole. A rather large number of distinct form factors have been introduced at one time or another; here we will mention only those which have been used most frequently in work on the reactions of interest here.

Ferrari and Selleri[72] were the first to introduce a form factor for the OPE model; it had a structure similar to that of the electromagnetic form factors of the nucleon:

$$\Phi(t)=0.28+\frac{0.72}{1-[(t-1)/4.73]}, \tag{2}$$

where t is in units of μ^2, the square mass of the π meson. This form factor, like the Amaldi–Selleri form factor,[73] has been widely used by experimentalists in the past, but it has now essentially been abandoned. The Dürr–Pilkuhn form factor,[74] found from arguments based on the spirit of potential-scattering theory, is used more frequently. The interaction vertex has a spatial region of radius R. For each partial wave with angular momentum l one determines the penetration factor v_l, which is the probability that the particle will enter the target particle. Then the meson vertex may be written

$$V(t)=\sqrt{\frac{v_l(t,\ R_l)}{v_l(\mu^2,\ R_l)}}\,V(\mu^2), \tag{3}$$

where

$$v_l(t,\ R_l)=\frac{1}{[q(t)\,R_l]^2}\,[j_l^2(qR_l)+\eta_l(qR_l)]^{-1},$$

q is the momentum of the incident pion in the rest frame of the dipion, R_l is an adjustable parameter, and j_l and η_l are the spherical Bessel and Neumann functions. For the lowest orbital angular momenta the specific expressions for the Dürr–Pilkuhn form factors are

$$\left.\begin{aligned}
&\Phi_{l=0}(t)=1 && \text{for the S-wave},\\
&\Phi_{l=1}(t)=\sqrt{\frac{1+R_D^2q^2(\mu^2)}{1+R_D^2q^2(t)}} && \text{for the P-wave},\\
&\Phi_{l=2}(t)=\sqrt{\frac{9+3R_D^2q^2(\mu^2)+R_D^4q^4(\mu^2)}{9+3R_D^2q^2(t)+R_D^4q^4(t)}} && \text{for the D-wave}.
\end{aligned}\right\} \tag{4}$$

Form factors are also introduced at the second vertex, in a similar way. For practial use of the Dürr–Pilkuhn kinematic form factors or the related Benecke–Dürr form factors,[75] two questions must be answered: 1. What are the values of R_l? 2. What contribution is made by each partial wave? The values of R_l have been studied in detail by Wolf,[76] who selected values for R_l from an extensive body of experimental data. With regard to the contribution of each partial wave, the situation is different: Although an *a priori* answer to this question is not known, it turns out that the effect of the form factor is not very sensitive to small changes in the contributions of the individual waves. It is thus possible to use the form factor even with rather crude estimates of the partial waves.

Form factors may be used both for work in the physical region and to improve the extrapolation process, as discussed above. As shown in Ref. 77, for example, the Dürr–Pilkuhn form factor for π^-p scattering found from the reaction $pp\to n\Delta^{++}$ gives satisfactory results in both cases.

The model of scattering by an absorbing disk (the optical model) works well for high-energy elastic scattering. The strong absorption by the target particle is responsible for the sharp diffraction peak in the differential cross section. The same peak is observed in quasi-two-particle reactions of the type $\pi^-p\to\rho^0n$ and $\pi^+p\to\rho^0\Delta^{++}$. This fact suggests that peripheral interactions can be studied by considering absorption. In each partial-wave amplitude it is necessary to introduce an absorption coefficient proportional to the total cross section in other possible channels. Taking into account absorption effects, which are strongest for small-l waves, improves agreement with experiment, although calculations of corrections of this sort are model-dependent.

Various versions of the OPE model with absorption (OPEA) have been proposed. One of the most popular is the OPEA formalism proposed in Refs. 78 and 79, which is based on the diagram in Fig. 1b. Here the hatched features denote the initial- and final-state interactions. The OPEA models are successful in a rather narrow energy range, and not for all reactions, but they are satisfactory in the important case of quasi-two-particle reactions, and they can be used to describe experimental data on differential cross sections and angular distributions and to calculate the correct values for the spin density matrix elements. Many studies of

the $\pi\pi$ interaction have been based on the OPEA model. Equations convenient for calculations in this model may be found, for example, in Ref. 80.

From the theoretical standpoint, the most consistent approach seems to be to introduce reggeization in the one-pion exchange. Models of this type arose as a result of developments in the theory of complex angular momentum. The model of the exchange of a reggeized pion (the OPER model) is the same as the OPE model at the pole, but in the modified model all possible states with the quantum numbers of the pion can be exchanged in the t channel. The OPER model gives a common basis for studying a broad range of reactions, and it correctly predicts the anisotropy with respect to the Treiman–Yang angle (this cannot be done by introducing form factors), the energy dependence of the cross section, and the elements of the density matrix. The OPER model was applied to various reactions in Refs. 9–11; this work is reviewed in Ref. 12.

Turning to the procedure for extracting experimental information on the $\pi\pi$ interaction, we first note that none of these theoretical models can claim general acceptance among experimentalists. The pattern is for each experimental paper to give some theoretical background and introduce a new set of changes and additions to the models and a new set of adjustable parameters.

The simplest way to extract information on $\pi\pi$ scattering is to study reactions 1–10 (Table I) in the physical region. As mentioned above, at sufficiently small values of t it can be hoped that one-pion exchange will be governing and that the effects of other mechanisms will be small. These conditions justify the use of this method. In practice, analysis of the experimental data reduces to taking their average over some interval of t and approximating them with some model (OPEA, OPER, etc.). The partial-wave characteristics—the phase shifts and elasticities—are treated as adjustable parameters and found by fitting the experimental data.

One of the most common methods is to extrapolate the experimental data to the pion pole, to the point corresponding to scattering of a real pion by a real pion, and then employ phase-shift analysis.

We will return to the extrapolation problem below; at this point we wish to point out that although extrapolation was proposed by Chew and Low back in 1959 the extrapolation procedure has not yet been perfected, and each particular reaction must be treated separately.

Another method for determining the $\pi\pi$ phase shifts is to pick out the enriched signal corresponding to OPE and then use an extrapolation. The background amplitudes are separated out by amplitude analysis. In this method the experimental quantities are expressed in terms of the partial helicity amplitudes $H^{m}_{\lambda\lambda';\mu}$ (λ and λ' are the helicities of the primary and secondary nucleons, respectively; μ is the spin of the dipion; and m is the projection of the dipion spin). Then the amplitudes with $m=0$, caused by OPE, are found. This problem is not fully determined, so that additional assumptions must be made; one possibility is to assume that the nucleon has no spin-flip amplitudes.[81] The values found for $H^{0}_{\lambda\lambda';\mu}$ are extrapolated to the pole.

Essentially all the work on extrapolation has been based on the old Chew–Low equation,[5] which relates the cross section $\sigma_{\pi\pi}$ to the differential cross section for the reaction $\pi N \to \pi\pi N$:

$$\frac{d^2\sigma}{dt\,dm^2_{\pi\pi}} = K\frac{f^2}{2\pi}\frac{1}{p^2_\pi}\frac{t}{(t-\mu^2)^2}\,m\sqrt{\frac{m^2_{\pi\pi}}{4}-\mu^2}\;\sigma_{\pi\pi}(m_{\pi\pi}). \qquad (5)$$

Here K is a numerical coefficient, taking on a value of 1 or 2 for reactions of the type $\pi N \to \pi_\pm \pi^0 N$ and $\pi N \to \pi^+\pi^- N$, respectively; $f^2=0.08$ is the πN interaction constant; and p_π is the momentum of the incident pion.

Strictly speaking, Eq. (5) holds only for $t=\mu^2$, but the pole nature of the diagram suggests that it can also be used in the physical region at small values of $|t|$. The usual approach for finding the pion–pion cross section is to construct the auxiliary function

$$F(t) = \frac{d^2\sigma_{\pi\pi N}}{dt\,dm^2_{\pi\pi}}\frac{2\pi p^2_\pi}{Kf^2} \times \frac{(t-\mu^2)^2}{m_{\pi\pi}\sqrt{(m^2_{\pi\pi}/4)-\mu^2}} \qquad (6)$$

or $F'(t)=F(t)/t$ (this is the pseudoperipheral approximation), which is then extrapolated to the pole, where $\sigma_{\pi\pi} = F(t)/t\,|_{t=\mu^2}$.

Questions involved in the extrapolation of these functions were discussed by Leksin[18]; here we wish to recall that the pseudoperipheral approximation requires $F(0)=0$, i.e., it is a particular case of the more general Chew–Low method [which permits $F(0)\neq 0$]. If the experimental $F(t)$ curve satisfies the condition $F(0)=0$, then both methods are applicable; the linear approximation of $F'(t)$ corresponds to a quadratic $F(t)$, etc. In this case, the $F'(t)$ would naturally be preferable, since it leads to smaller errors. In using this method, however, we must make sure that the condition $F(0)=0$ holds. The question of the crossing of zero by $F(t)$ has been studied by Hagopian *et al.*[82] They mention that experiments at low momenta, $p_\pi \sim 2$ GeV/c, yield values of $F(0)$ which are definitely negative, especially at $m_{\pi\pi} \gtrsim 700$ MeV; at large values, $p_\pi \gtrsim 15$ GeV/c, on the other hand, we have $F(0)>0$. These results can be explained

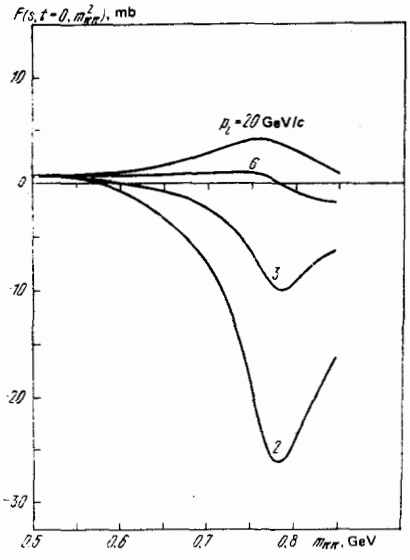

FIG. 2. Calculation of values of the extrapolation function $F(s, t=0, p_\pi)$ from the OPEA model from Ref. 82.

by the OPEA model, whose results have $F(0)$ depending on p_τ. Figure 2 shows curves from Ref. 82 showing the behavior of $F(s, t=0, m_{\pi\pi})$ for various values of p_τ. We see that we have $F(0) \approx 0$ for any p_τ if $m_{\pi\pi} < 0.6$ GeV. At large masses the condition $F(0) \approx 0$ holds only in a small momentum interval, $4 < p_\tau < 6$ GeV/c. For other momenta of the incident pion we should apparently use the ordinary Chew–Low extrapolation procedure.

Another approach for choosing the extrapolation function is the conformal-mapping method. This method was apparently first applied to the $\pi\pi$ scattering problem by Batusov *et al.*[83] Some theoretical aspects of the method were subsequently discussed in Ref. 84. The method may be outlined as follows: The problem of reconstructing an analytic function over the entire plane from its values in a certain continuous region is a complicated one, since the resulting function is sensitive to small changes in the data in the given region. The convergence of the process can be improved by making use of knowledge of the analytic structure of the function. For this purpose one uses a convenient function W, which uses a conformal mapping to transform the complex energy plane into the interior of the circle $|W| \leq 1$. The amplitude is written as a polynomial series in W. It has been shown that this series converges better than a series in t does. Figure 3 shows some conformal mappings. Baton *et al.*[28a] have shown that for the reaction $\pi^- p \to \pi\pi N$ it is sufficient to use the variable $x(t) = (at+b)/(t+c)$ (a, b, and c are the transformation constants), since the elliptical-transformation parameters k^2 (Fig. 3) are small in this case. This method can be used to make the following improvements:

1) to achieve extrapolation results which are less sensitive to the way in which partitioning with respect to t and $m_{\pi\pi}$ is carried out.

2) to use a polynomial of lower degree to describe the experimental data.

The net result may of course be a decrease in the extrapolation error. Figure 4 illustrates the use of the

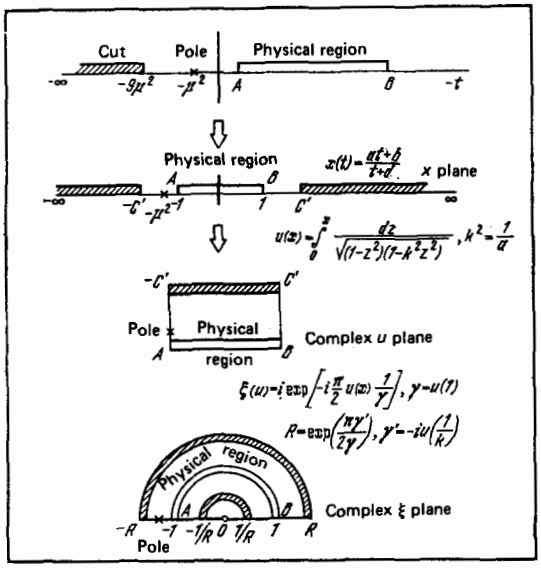

FIG. 3. Examples of conformal mappings.

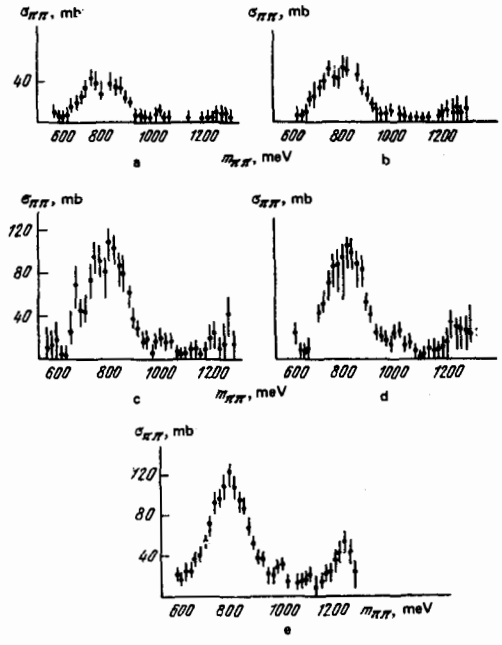

FIG. 4. Extrapolated values of $\sigma_{\pi\pi}$ from Ref. 111. a—Extrapolation through the use of a function quadratic in t; b—extrapolation through the use of a function linear in x; c—extrapolation through the use of a function quadratic in x; d—extrlapolation through the use of a function quadratic in x by the maximum-likelihood method; e—extrapolation using a function linear in t and the introduction of the Dürr–Pilkuhn and Benecke–Durr form factors.

various extrapolation methods for the reaction $\pi^- p \to \pi^- \pi^+ n$ from Ref. 111.

There is also the so-called Pade approximation, in which $F(t) = a + [bt/(1+ct)]$. This parametrization, however, apparently differs only slightly from the conformal-mapping variable x.

In order to use these extrapolation methods it is necessary to have a clear understanding of the dynamics of the processes involved, primarily the nature of the possible background of mechanisms other than OPE. To demonstrate the importance of this understanding, we will compare some possible extrapolation methods for certain reactions. For the reaction $\pi N \to (\pi\pi)N$ there is a paradoxical situation, in which a very large statistical base may degrade the situation: The use of an extrapolation to very small $|t| \lesssim \mu^2$ may lead to pronounced distortions. This occurs because the OPE contribution to $d\sigma/dt$ vanishes in the limit $t \to 0$ for this reaction, but the background is not zero at $t=0$. For the reaction $\pi N \to (\pi\pi)N$ the extrapolation should not come very close to $t=0$ (it usually does), or we should work entirely in the physical region. This difficulty does not arise for the reaction $\pi^+ p \to (\pi^+ \pi^-)\Delta^{++}$, and as smaller values of $|t|$ are used in the extrapolation the result becomes more accurate. In the reaction $\pi^- p \to (\pi^- \pi^-)(\pi^+ p)$ the background of other diagrams is important over a broad range of $|t|$, so that it is particularly important to take this background into account.[85]

We thus see that the problem of eliminating or taking into account the effects of diagrams other than one-pion

diagrams is exceedingly important. To solve this problem it is necessary to find a suitable approach for each reaction, and the solution itself depends on the momentum of the primary particle and of the t region under consideration.

Let us consider in more detail the situation regarding the background for reactions of the type $\pi N \to \pi\pi N$. For such reactions the background of other diagrams can be reduced slightly by choosing a suitable momentum. Specifically, the momentum must not be too small, because there is a significant production of nucleon isobars at $p_\pi \lesssim 1.5$ GeV/c, and it must not be too large because the high-lying Regge trajectories become increasingly important. Even at the optimum momentum, however, the background does not disappear completely.

In experiments with an unpolarized target it is not possible to distinguish exactly the contributions of various types of exchange with the same parity. Because of the predominance of the OPE diagram at small $|t|$ it is usually assumed that exchange with an anomalous spin–parity $P=(-1)^{J+1}$ is caused completely by π exchange, and it is assumed that the possible exchange of an A_1 (1^+) meson can be ignored. This question was discussed by Kimel and Owens,[86] who used experimental data on $\pi p\uparrow \to \pi^-\pi^+ n$ with a polarized target.[87] The calculation results show that the A_1-exchange effect can be ignored within 10% for $|t| < 0.5$ (GeV/c)^2. In a recent experimental study with a polarized target,[88] however, a partial-wave analysis of the reaction $\pi^- p\uparrow \to \pi^-\pi^+ n$ at $p_\pi = 17.2$ GeV/c showed that the A_1 effect in the vicinity of the ρ meson was ~30%. This result seems to mean that A_1 exchange must be taken into account in order to refine the $\pi\pi$ scattering characteristics.

The effect of exchange with a natural parity $P=(-1)^J$, e.g., the exchange of an A_2 (2^+) meson, can be estimated from experimental data with unpolarized targets. In Ref. 39, with a large statistical base, it was shown that this exchange amounts to <10% at $|t| \sim t_{\min}$ and increases to ~90% at $|t| \sim 1$ (GeV/c)^2. This conclusion agrees with the calculations by Kimel and Reya,[89] who showed that the OPE model is sufficient for a description of experimental data at $|t| < 0.2$ (GeV/c)^2 and that A_2 exchange must be taken into account at higher values of $|t|$.

The situation is worse with regard to the incorporation of ω exchange. There have been only a few efforts at a phase-shift analysis of the $\pi^-\pi^0$ and $\pi^+\pi^0$ states from the reactions $\pi N \to \pi\pi N$, and the attitude to ω exchange of the authors of these papers has been different. This exchange was simply ignored in Refs. 43 and 44, while in Ref. 27 it was emphasized that ω exchange is important, especially at the small values of $|t| \sim 0.1$ (GeV/c)^2. It may be that it is ω exchange which is distorting the picture and which leads to discrepancies in the values of phase shifts with $I=2$ found from the $\pi^+\pi^0$ states and by other methods.

Opinion is also divided regarding the importance of diffractive-dissociation diagrams (Figs. 1c and 1d). In a report to a Tallahassee conference,[90] Walker showed that diffractive dissociation had an effect of about 10% in the reaction $\pi^- p \to \pi^-\pi^+$ at small values of $|t|$ (for a high initial momentum, $p_\pi = 25$ GeV/c). Working with the same reaction at $p_\pi = 4$ GeV/c, Oh *et al.*[27] showed that diffractive dissociation could be ignored in the vicinity of the ρ^0 and f^0 resonances. This conclusion is also in agreement with the results of a study[91] of the reaction $\pi^+ p \to \pi^+\pi^-\Delta^{++}$ at $p_\pi = 16$ GeV/c with a large statistical base (~30 000). It was concluded in Ref. 91 that the diffractive-dissociation admixture was small, and negligible in the ρ region. At the same time, several studies (for example, Ref. 92) have shown that diffractive dissociation is important for the reactions $\pi^\pm p \to \pi^\pm\pi^0 p$, because of both the large contribution of diffractive dissociation (~20–30%) and strong interference effects. Particular note should be taken of Ref. 93, where an effect ~95% was assigned to diffractive dissociation in the reaction $\pi^+ p \to \pi^+\pi^+ n$ at $p_\pi = 3.9$ GeV/c. It is hardly possible to accept this estimate, but the very fact that it has been reported indicates the need for a careful consideration of diffractive-dissociation diagrams.

We emphasize that precisely what distortions are caused in the extrapolation by the admixture of diffractive-dissociation diagrams is not known at present, and this question requires further study.

To summarize this section we may say that although the predominant role of one-pion exchange is obvious at small values of $|t|$ a further improvement in the accuracy of the results on $\pi\pi$ scattering will require a further development of a procedure for correctly taking into account all possible background diagrams.

4. DESCRIPTION OF PION RESONANCES

Elementary-particle physics has dealt for a long time with unstable particles or resonances, which participate in strong interactions not only in the course of their production but also in the course of their decay. A study of resonances can contribute to our understanding of strong interactions and can help us test the various models. If the characteristics of the resonances have been determined, it is possible to study the interaction in definite quantum states and to study partial-wave amplitudes. Experimental research in the $\pi\pi$ interaction itself began with the discovery of certain distinctive features in the two-pion system, i.e., $(\pi\pi)$ resonances. Another important motivation for a study of these resonances is that many methods which have been developed for studying the pion–pion interaction (the Dürr-Pilkuhn form factors, several versions of the OPEA model, etc.) are based on the assumption that the reaction $\pi N \to \pi\pi N$: $\pi + N \to (\pi\pi)_{\text{res}} + N$ is a quasi-two-particle reaction. The possible existence of resonances which are multiquark molecules (of the type $q\bar{q}q\bar{q}$) and gluon formations has recently been the subject of a lively discussion. The first possible candidates are meson resonances with the quantum numbers 0^+, that is, S^* and δ mesons (Refs. 94–96).

What are these resonances from the standpoint of the experimentalist? How can they be studied?

We know that resonances have the same sets of quantum numbers as ordinary "long-lived" elementary particles. The instability of resonances with respect to a strong-interaction decay, however, leaves an important imprint on this class of particles. The short lifetime ($\tau \approx 10^{-23}$ s) and the correspondingly large width $\Gamma \approx \hbar/\tau \approx 100$ MeV determine the method which must be used to study a resonance experimentally.

From the standpoint of the experimentalist, a resonance is a statistical concept. In contrast with ordinary particles, which may be identified on the basis of a single event (the Ω^- hyperon, for example), a resonance can be identified only from a large statistical base of events over a broad mass region ($\Delta M \gg \Gamma$).

It might be thought (and in fact it was thought at one time) that a broad peak (with $\Delta M \sim \Gamma$) protruding from the background on the mass diagram, with a height equal to several standard deviations, would be a manifestation of a resonance. This conclusion turns out to be correct, however, only for so-called leading resonances, e.g., $\rho(1^-), f(2^+)$, or $g(3^-)$. If, in contrast, we are dealing with less obvious resonances, which are masked by the strong background in the same wave, then we evidently need some different tests.

Let us examine this question in more detail.

Resonances are described most thoroughly in a partial-wave analysis. Omitting the kinematic factors, we can write the partial amplitude for the case of a purely elastic interaction as

$$f_l(E) = \frac{1}{2i}(e^{2i\delta_l(E)} - 1) = \frac{1}{\cot\delta_l(E) - i}. \tag{7}$$

A resonance corresponds to a maximum of this function at a certain value of E_R, which then determines the mass of the resonance. The spin and parity of the resonance are determined by the value of l for the given partial wave; the various charges of the resonance (the electric, baryon, strangeness, etc., charges) are determined by the corresponding charges of the particles making up the resonance; etc.

If we expand $\cot\delta_l$ in a Taylor series around the resonant energy E_R (at which $\delta_l = \pi/2$ and $\cot\delta_l = 0$), we easily find the Breit–Wigner formulas for the amplitude,

$$f_l(E) = \frac{\Gamma/2}{(E_R - E) - (i\Gamma/2)}, \tag{8}$$

and for the cross section,

$$\sigma_{el}(E) = 4\pi\lambda^2(2l+1)\frac{(\Gamma/2)^2}{(E_R - E)^2 + (\Gamma/2)^2}. \tag{9}$$

If other channels are also open, then expression (7) should be replaced by

$$f_l(E) = \frac{\eta_l(E)\, e^{2i\delta_l(E)} - 1}{2i}, \tag{10}$$

where the coefficient $\eta_l(E)$ determines the probability that the process will be elastic ($\eta_l = 1$ corresponds to the case without absorption, while $\eta_l = 0$ corresponds to total absorption).

In this case the Breit–Wigner formula becomes

$$f_l(E) = \frac{\Gamma_{el}/2}{(E_R - E) - (i\Gamma/2)} = \frac{x}{\alpha - i}; \tag{11}$$

where Γ_{el} is the partial width of the elastic decay of the resonance

$$x = \Gamma_{el}/\Gamma, \quad \alpha = 2(E_R - E)/\Gamma.$$

It is convenient to inspect the behavior of the partial-wave amplitudes on so-called Argand diagrams. Here the imaginary part of the amplitude, $\mathrm{Im} f_l = (1 - \eta_l \times \cos 2\delta_l)/2\delta_l)/2$, is plotted along the ordinate, while the real part, $\mathrm{Re} f_l = (\eta_l/2)\sin 2\delta_l$, is plotted along the abscissa (Fig. 5). The circle of radius 1/2 centered at $i/2$ is the unitary boundary.

The relationship between real parameters (phases and absorption coefficients) and $\mathrm{Re}\, f_l$ and $\mathrm{Im}\, f_l$ can be conveniently demonstrated on such a diagram. Each point on the diagram corresponds to a definite value of the partial amplitude. As the energy is changed, this point moves, tracing out some curve. In the case of elastic scattering, for example, we have $\eta_l = 1$, and the amplitudes traces out the unitary circle; if the potential is attractive, the point moves counterclockwise. In general, the curve can be described by

$$\left(\mathrm{Im}\, f_l - \frac{1}{2}\right)^2 + (\mathrm{Re}\, f_l)^2 = \left(\frac{\eta_l}{2}\right)^2. \tag{12}$$

When $\eta_l < 1$, the amplitude lies within the unitary circle; for $1/2 \leq x_l < 1$ the phase passes through 90° at the resonance; and for $0 < x_l < 1/2$ it passes through zero.

The Argand diagram can also be used to study the more general case of scattering through the channel $a \to b$ (for example, the scattering $\pi\pi \to K\bar{K}$). Then the Breit–Wigner formula becomes

$$f_l(E) = \sqrt{x_a x_b}\,\frac{1}{\alpha - i}, \tag{13}$$

where $x_i = \Gamma_i/\Gamma$ is the elasticity corresponding to channel i. This is not the only possible way to describe the resonances, but it is convenient if there is not a strong background in the given wave. It follows from this expression that in the case of a resonance we will see the following behavior of the amplitude on the diagram:

1. As the energy increases, the amplitude moves along the circle, reaching a maximum value $f_{max} = i\sqrt{x_a x_b}$ at $E = E_R$.
2. Near $E = E_R$, the change in the amplitude is at a maximum:

$$\frac{df_l}{dE} = \frac{\sqrt{x_a x_b}}{\Gamma}\cdot\frac{2 + \alpha(d\Gamma/dE)}{\alpha^2 + 1}.$$

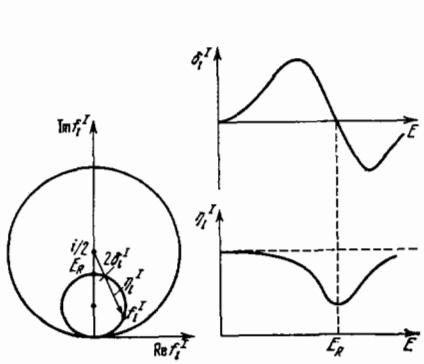

FIG. 5. A resonance with $x < 1/2$ on the Argand diagram in the case in which there is no background.

At resonance, the "velocity" $df_l/dE\big|_{E=E_R}$ reaches its maximum value. If there is no background, the resonance takes the form shown in Fig. 5 ($\sqrt{x_a x_b} < 1/2$).

We can summarize this discussion by stating the conditions under which we may conclude that a resonance has been identified on the Argand diagram:

1. The phase passes through a singularity, 0° or 90°.
2. The parameter η_l is at a minimum (the inelastic cross section is at a maximum).
3. The magnitude $|f_l|$ has a maximum; i.e., the elastic cross section is at its maximum.
4. $\mathrm{Im} f_l$ is at its maximum; i.e., the total cross section is at its maximum.
5. The "velocity" df_l/dE is at its maximum.

Unfortunately, it would be rare to find a case as "clean-cut" as this. In the first place, the picture may be changed slightly by an E dependence of the resonance width Γ. *This is not a very important point, since equations are available for handling such a dependence.* For the elastic resonance, for example, we have

$$\Gamma = \Gamma_0 \left(\frac{k}{k_R}\right)^{2l+1} \frac{E}{E_R}, \tag{14}$$

where $k(k_R)$ is the momentum of the scattered particle (of the resonance) in the c.m. system (see, for example, Refs. 13 and 17). If the resonance is inelastic, then Γ_a and Γ_b may depend on the energy in different ways. There is also a change in the Breit–Wigner formula. The "relativistic" formula is usually written in the form

$$f_{ab} = \frac{E\sqrt{\Gamma_a \Gamma_b}}{(E^2 - E_R^2) - i\Gamma E}. \tag{15}$$

This question is discussed in more detail in Refs. 97–99.

The picture is made significantly worse by the following factors:

1. a substantial background in the same partial wave
2. the presence of several resonances which overlap on the energy scale
3. the approximate equality of the energy of the resonance to the energy at which an inelastic channel comes into play.

When the background is taken into account, the amplitude may be written as

$$f_l(E) = \frac{\eta_l' e^{2i\delta_l'} \cdot \eta_l e^{2i\delta_l} - 1}{2i}, \tag{16}$$

where the factor $\eta_l' e^{2i\delta_l'}$ describes the background. Using $\eta_l e^{2i\delta_l} = 1 + [2ix/(\alpha - i)]$ for the resonant part, we find

$$f_l(E) = \frac{\eta_l' e^{2i\delta_l'} - 1}{2i} + \frac{x}{\alpha - i} \eta_l' e^{2i\delta_l'}. \tag{17}$$

If the background is elastic and does not depend on the energy, then it displaces the resonance circle along the unitarity boundary (the circles are tangent). If the background is inelastic, and $\eta_l' < 1$, the resonance circle lies within a circle of unit diameter. If, on the other hand, the background also depends on the energy, i.e., if η_l' and δ_l' vary with increasing E, then the resonance circle becomes distorted into a loop. Figure 6 illustrates the behavior on the Argand diagram of an

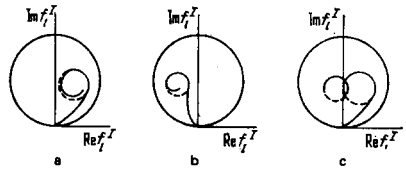

FIG. 6. Examples of the behavior of an inelastic resonance on the Argand diagram in the case in which there is nonresonant background scattering. a—Attractive; b—repulsive; c—case in which two resonances are superimposed.

inelastic resonance in the presence of (*a*) an attractive and (b) a repulsive inelastic background which depends on the energy. As before, the parameter η_l has a minimum, but the position of this minimum is no longer at the resonance, and the phase may not go through the singularities. Figure 6c shows the Argand diagram for the case in which two resonances are superimposed. The presence of a loop on an Argand diagram is a strong argument for the existence of a resonance, while a peak in the cross section for the process may correspond to either a resonance or kinematic enhancement.[2)]

It can be seen from these examples that the background must be taken into consideration. The resonance wave is usually described as a resonance (e.g., with the Breit–Wigner shape) which varies rapidly with the energy plus a slowly varying background, and then this parametrization is used in an energy-dependent analysis. If the energy-independent method reveals the position at which the phase passes through the singularities and the minimum of the parameter η_l, and if the energy-dependent method successfully describes the experimental distributions in terms of a resonance plus a background, then we can determine the parameters of the resonance at hand. This is not always sufficient, however, since a nearby threshold for an inelastic channel or for another resonance may complicate the picture. In such cases it is useful to apply the K-matrix formalism. This formalism was developed in connection with the need to consider all possible states and transitions of coupled channels. It is difficult to work with the amplitude for multichannel transitions, however, because of the various thresholds and cuts for each channel. It is convenient to remove all the unitary singularities by introducing a so-called reaction amplitude or K-matrix.[101–103] This new function is determined in terms of the integral equations, which take a particularly simple form for two-particle channels. This approach has the advantages that it can be used to calculate any quantity pertaining to the given interaction, not only those quantities which pertain to some particular reaction. It has proved quite successful in describing the particular features of the S_0 wave for $\pi\pi$ scattering near the $K\bar{K}$ threshold. This method has been used in various modi-

[2)] In general there may be loops on the Argand diagrams which do not correspond to a resonance. Inelastic channels containing a resonance and a particle may lead to features in the partial amplitude which generate closed loops on the Argand diagrams. In such a case it is useful to compare the diagrams for forward and backward scattering (see, for example, Ref. 100).

fications.[33,39,40,55,104] It has also been used in other cases, e.g., to describe the $\rho'(1600)$ resonance in the P wave.

In addition to using Argand diagrams, one could describe the resonances in terms of the poles of the amplitude. Let us assume that the partial-wave amplitude is an analytic function of the energy E and that it can be continued into the complex plane. Then the Breit-Wigner formula tells us that there is a pole at $E=E_R-i\Gamma/2$. If the system has a bound state, then the amplitude has a pole on the real axis, at $E=E_B<0$.

This approach is very convenient, since it describes both resonances and stable particles. However, the results which emerge from this method are ambiguous when applied to the problem in which we are interested, because not every pole may be assumed to be a resonance. If a pole lies far from the physical axis, we would hardly be justified in interpreting it as a resonance. In the S_0 wave, for example, there is a singularity which can be described as a pole at[55] $E=[(600\pm100)-i(250\pm70)]$ MeV. A detailed study of the singularity shows, however, that the other conditions for a resonance (listed earlier in this section) are not satisfied at this energy, so that the singularity cannot be identified as a resonance. It should also be noted that the definition of the width of a resonance is ambiguous. In addition to the usual definition $\Gamma_R=-2(d/dE)(\cot\delta_l)^{-1}$ we could define the width by working from the concept of a pole: $M_R=E_R-i\Gamma_0/2$. We could also take the difference between the points at which the phase passes through 135° and 45°, that is, $\delta(E+\Gamma_{\text{high}}/2)=135°$, $\delta(E+\Gamma_{\text{low}}/2)=45°$, $\Gamma=(1/2)(\Gamma_{\text{high}}-\Gamma_{\text{low}})$. The numerical difference can be seen in the example of the value of Γ_ρ from Ref. 28: $\Gamma_R=133$ MeV, $\Gamma_0=131$ MeV, $\Gamma(45°-135°)=127$ MeV.

We see that it is a complicated matter to determine resonances. All the available methods and tests must be considered, and those most applicable for the given case must be selected.

5. ANGULAR DISTRIBUTIONS OF SECONDARY PIONS

The preceding section has demonstrated the need to find the phase shifts and elasticities for pion–pion scattering, so that the angular characteristics of the pions must be known. There are various ways to characterize angular distributions: by means of Legendre polynomials or simply a power series in[3)] $\cos\theta$, in terms of the average spherical harmonics or the elements of the spin density matrix. Any distribution which gives a good description of the experimental angular distributions will incorporate the information necessary on the $\pi\pi$ scattering phase shifts.

For any type of description, the coefficients of the angular distribution must of course be known on the mass shell, at the pion pole. There is thus the question of how to extract these coefficients from the experimental data in the physical region.

Some experimentalists[33] believe that the angular distributions of the secondary pions at $|t|<0.15$ (GeV/c)^2 are approximately the same as the distribution at the pole. Others,[55] working from a comparison of the extrapolated and nonextrapolated average spherical harmonics for $|t|<0.10$ (GeV/c)^2, note a ~15% difference. This difference is important for phase-shift analysis, so that the extrapolation is apparently necessary.

As in the case of $\pi\pi$ scattering, the extrapolation law is unknown. Linear and quadratic extrapolations are ordinarily tried, the χ^2 test being used to determine which is better. As a rule, these functions yield approximately the same results. It is customary to extrapolate normalized coefficients, i.e., coefficients normalized to the number of events in the given interval. In this case the kinematic factors are less important, and the use of form factors such as the Dürr–Pilkuhn form factors has only a slight effect on the results. As Protopopescu has pointed out,[55] the use of this form factor changes the value by ~1%, while the extrapolation errors are ~10%. A conformal-mapping method has been used successfully in certain studies,[28] but in other cases it has yielded unstable results, frequently going beyond the unitary limit.[55]

In extrapolating the angular distributions we must evidently take into account the results calculated from the OPEA model of Williams.[105] Williams has shown that the functions $\langle Y_L^0\rangle$ change the curvature in the nonphysical region, making it impossible to extract the correct result in an extrapolation from the physical region. The linear extrapolation, the one most frequently

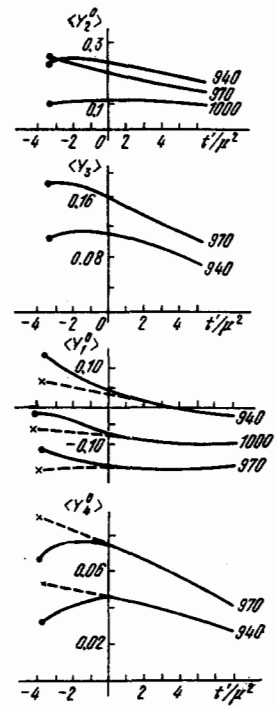

FIG. 7. Calculations of the behavior of $\langle Y_L^0\rangle$ as a function of t from the OPEA model from Ref. 77. Dashed curves—Extrapolation from the physical region.

[3)] Here θ is the angle between the primary and secondary pions in the rest frame of the dipion (the so-called Jackson angle).

used, leads to systematic deviations ~10% from the predictions of the Williams calculations for the first two harmonics ($\langle Y_1^0\rangle$ and $\langle Y_2^0\rangle$) and ~30% for $\langle Y_3^0\rangle$ and $\langle Y_4^0\rangle$. The value of $\langle Y_1^0\rangle$ decreases, and the values of all the other harmonics increase (Fig. 7). We will conclude this section with two examples of the use of the method of average spherical harmonics.

Figure 8 shows values of $\langle Y_L^0\rangle$, $L=1,\ldots,4$, for the $\pi^-\pi^+$ and $\pi^-\pi^0$ states from Refs. 44 and 45. From the general behavior of the harmonics we can extract some qualitative information about the $\pi\pi$ scattering phase

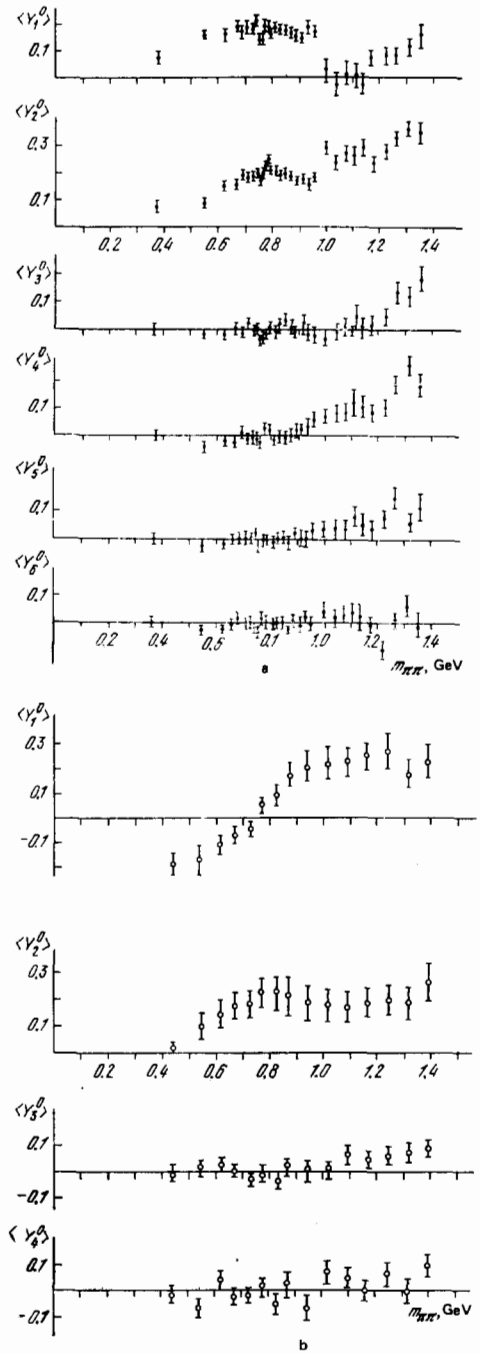

FIG. 8. Behavior of $\langle Y_L^0\rangle$ as a function of the dipion mass. a—For the $\pi^-\pi^+$ state from Ref. 55; b—for the $\pi^-\pi^0$ state from Ref. 44.

shifts. For example, comparing $\langle Y_1^0\rangle$ for the $\pi^-\pi^+$ and $\pi^-\pi^0$ states, we can draw conclusions about the signs and relative magnitude of the phase shifts δ_0^0 and δ_0^2; from the small values of $\langle Y_3^0\rangle$ and $\langle Y_4^0\rangle$ we conclude that the higher-wave contributions are slight in the region $m_{\pi\pi} < 1$ GeV. The maxima in $\langle Y_2^0\rangle$ and $\langle Y_4^0\rangle$ indicate the positions of the ρ and f^0 resonances. This is clear from the relations $\langle Y_2^0\rangle \sim \sin^2\delta_1^1$ and $\langle Y_4^0\rangle \sim \sin^2\delta_2^0$.

The first step of a phase-shift analysis is a qualitative analysis of the characteristics of the angular distributions, and some interesting results can be obtained from this first step; the reader is referred to Refs. 39, 44, and 106 for illustrations.

6. PHASE-SHIFT ANALYSIS

All the methods for determining partial amplitudes from experimental data may be classified as energy-dependent, energy-independent, or model-dependent methods. Each has its own advantages and disadvantages.

In an energy-dependent analysis the experimental data for each mass interval are fitted simultaneously through an energy-dependent parametrization of the partial-wave amplitudes. The parametrization reflects certain ideas regarding the behavior of the waves, and the success of the fit tells us how closely these ideas correspond to experiment. The best formula is selected on the basis of some prespecified criterion, usually the χ^2 test. The experimental data may be any characteristics of the angular distributions.

This approach has the advantages that the analysis is carried out over a broad energy range and that a smooth solution is found which includes known resonances. In an energy-dependent analysis, the statistical errors at the individual points are less important than in an energy-independent analysis. As a rule, this method yields only one solution. The cost, however, is quite high: The systematic errors may be large if the parametrization is chosen poorly. The number of adjustable parameters is limited, and the analysis is correspondingly simplified, but some assumptions must be made beforehand about the nature of the solution.

The parametrization is usually selected to incorporate known resonances and the background in the given wave, in an attempt to achieve a "model-independent" fit. In some cases the data are grouped in energy intervals, and each interval is assigned its own parametrization. For $\pi\pi$ scattering the simplest parametrization is developed by assuming that the P_1 and D_0 waves contain only resonances of Breit–Wigner shape and that the phase shifts with $I=2$ are smooth power-law functions of the energy. This parametrization was used, for example, to describe $\pi^-\pi^0$ scattering in Ref. 44. In a first approximation, this parametrization is clearly valid. It seems better, however, to consider the background even for the waves with the leading resonances, particularly in a study of the region $m_{\pi\pi} \gtrsim 1$ GeV, where elasticities must also be introduced. The number of adjustable parameters required of course increases. To illustrate the method for incorporating the background

we may cite the paper by Protopopescu *et al.*,[55] who used the following expression to describe P_1 and D_0 waves:

$$T_l^I = \frac{\eta_{\text{back}} e^{2i\delta_{\text{back}}} - 1}{2i} + e^{2i\delta_{\text{back}}} \frac{\Gamma_{\pi\pi}/2}{(m_R - m_{\pi\pi}) - (i\Gamma/2)}, \quad (18)$$

where η_{back} and δ_{back} describe the background (cf. Section 3).

The S_0 wave always causes the greatest complications. It has no clearly defined resonances, although it does have much structure. Furthermore, there is a definite coupling with other channels, for example, $\pi\pi \to K\bar{K}$ (Section 8). It is thus a complicated matter to parametrize the S_0 wave. The K-matrix formalism is used for this purpose.[107] Here

$$\tilde{T}_0^0 = \begin{vmatrix} T_{\pi\pi} & T_{\pi K} \\ T_{K\pi} & T_{K\bar{K}} \end{vmatrix}, \quad (19)$$

where $T_{\pi\pi}$ describes the S-wave $\pi\pi \to \pi\pi$ scattering, $T_{\pi K}$ describes the $\pi\pi \to K\bar{K}$ scattering, and $T_{K\bar{K}}$ describes the $K\bar{K} \to K\bar{K}$ scattering.

It is convenient to describe threshold effects by expressing the amplitude matrix $\tilde{T}$ in terms of the K-matrix:

$$\tilde{T} = \frac{\tilde{K}}{\delta_{ik} - i\tilde{q}\tilde{K}},$$

where $\tilde{q} = \delta_{ik}\sqrt{(m_{\pi\pi}^2/4) - m_i^2}$. The energy dependence of the K-matrix is given, for example, in the following form[107]:

$$\tilde{K} = (\delta_{ik}\sqrt{q_i})\,\tilde{K}'\,(\delta_{ik}\sqrt{q_i}), \quad (20)$$

where $\tilde{K}'$ is a real, symmetric matrix whose elements are treated as adjustable parameters. As Hyams *et al.* have pointed out,[33] this representation also reflects other possible channels (4π, K^*K, etc.), so that they need not be introduced explicitly. A similar description is sometimes used for other waves, e.g., for the P_1 wave in the region of a possible ρ' resonance or for the D_0 wave to allow for coupling of the f^0 resonance with the 4π channel.

In an energy-independent analysis the data for different energies are analyzed separately, and the solutions for the phase shifts and elasticities are found for each mass interval. Waves with spin above a certain $l_{\max}$ are assumed negligibly small and are ignored. The parameters of the other waves, with $l \leq l_{\max}$, are treated as adjustable within the confines of unitarity. Several solutions are usually found, and a choice among them is made on the basis of energy continuity, the χ^2 test, and agreement with the known behavior of the phase shifts at adjacent energies—or in accordance with some theoretical ideas. This method has the advantage that there are no restrictive assumptions regarding the behavior of the waves, so that new structures may be discovered. In fact it was an approach of this type which revealed certain features in the behavior of the phase shift δ_0^0 near the ρ^0 resonance and at the $K\bar{K}$ threshold. The difficulties of this method are in choosing the actual solution and in joining the values obtained in different mass intervals. Furthermore, in this method one usually has to deal with several simplifying assumptions, because for a given charge channel in each energy interval there are $(2l_{\max}+1)$ independent experimental points, so that in the case of an $S-P$ analysis there are only three equations. There are, however, more unknowns (the phase shifts of the S_0, S_2, and P_1 waves and the elasticities). Some of the unknowns must be fixed in order to solve the system of equations. The question of incorporating higher-order waves is thus particulary important here. The value of $l_{\max}$ is usually determined by approximating the angular distributions by Legendre polynomials. This approximation shows that at $m_{\pi\pi} < 1$ GeV waves with $l \geq 2$ may be ignored in a first approximation. If it is also assumed that the interaction is completely elastic in this region, the number of unknowns is reduced to three: the phase shifts of the S_0, S_2, and P_1 waves. It is not possible to distinguish S_0 and S_2 in those states in which both waves are possible, so that the phase shift δ_0^2 is usually fixed in a study of $\pi^-\pi^+$ interactions, and this shift is assumed to be known from other studies. Under these approximations the angular distributions may be approximated by a power law of the type

$$\frac{d\sigma_{\pi\pi}}{d(\cos\theta)} \sim \frac{dN}{d(\cos\theta)} = a_0 + a_1\cos\theta + a_2\cos^2\theta, \quad (21)$$

followed by an extrapolation of a_i to the pole.[38a] If the angular distributions are characterized by average spherical harmonics, the latter are also extrapolated to the pole.[44a,45] From the values found at the pole one can calculate the phase shifts δ_1^1 and δ_0^0 for the $\pi^-\pi^+$ state or δ_1^1 and δ_0^2 for the $\pi^\mp\pi^0$ state. Because of the interference term, however, for which the values δ_0^0 and $\delta_0^{0'} = \delta_1^1 + (\pi/2 - \delta_0^0)$ are equally possible, the solution is double-valued (the "up–down" uncertainty). In an energy-independent analysis of the $\pi^-\pi^+$ state, with a large δ_1^1, we cannot avoid this uncertainty unless we make some auxiliary assumptions. It should be noted that the phase-shift analysis method described above is not completely accurate at $m_{\pi\pi} \lesssim 1$ GeV, since a higher-order wave may be very important—even if its amplitude is small—because of the strong interference with lower-order waves. This is the situation with D waves in this energy range.[37b,44] For work at larger values of $m_{\pi\pi}$, higher-order waves with $l \geq 2$ must be taken into account completely, i.e., must be extrapolated to the pole $\langle Y_L^0 \rangle$ with $L = 1, 2, \ldots, 2l_{\max}$.

In some recent papers, with a good statistical base, a phase-shift analysis has been approached through an amplitude analysis.[33,37,40]

The spin correlations for a particular process are customarily described in terms of the helicity amplitudes $H_{\lambda'\lambda}$ in the t channel, where λ and λ' are the helicities of the nucleon in the initial and final states. For a reaction of the type $\pi N \to \pi\pi N$ the intensity may be written (for an unpolarized target)

$$\frac{d^4\sigma}{dm_{\pi\pi}\,dt\,d\Omega_{\pi\pi}} = \frac{1}{2}\sum_{\lambda'\lambda} |H_{\lambda'\lambda}|^2. \quad (22)$$

The complete helicity amplitude is written in terms of the sum of the amplitudes for the production of an intermediate dipion with spin l and helicity m:

$$H_{\lambda'\lambda}(\theta, \varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \sqrt{2l+1}\, H^l_{\lambda'\lambda;\,m} d^l_{m0}(\theta)\, e^{im\varphi}. \quad (23)$$

In general, for the reaction $a + b \to c + d$, we have

$$H_{\lambda_c\lambda_d;\,\lambda_a\lambda_b} = \eta(-1)^{\sum_i (l_i+\lambda_i)} H_{-\lambda_c-\lambda_d;\,-\lambda_a-\lambda_b}, \tag{24}$$

where η is the product of the internal parities, and l_i is the spin. For the reaction $\pi p \to (\pi\pi)n$ this expression is equivalent to

$$H_{\lambda'\lambda;\,m} = (-1)^{\lambda'+\lambda+m} H_{-\lambda'-\lambda;\,-m}. \tag{25}$$

It follows from (25) that it is sufficient to know half of the amplitudes, e.g., those with $\lambda' = +1/2$.

It is convenient to describe exchange reactions by the following combinations of the amplitudes $H_{\lambda'\lambda;m}$, rather than by these amplitudes themselves:

$$\begin{aligned} H^{(\mp)}_{\lambda'\lambda;\,m} &\equiv \frac{1}{\sqrt{2}}\left[H_{\lambda'\lambda;\,m} \mp (-1)^m H_{\lambda'\lambda;\,-m}\right] \quad \text{for} \quad m \neq 0, \\ H^{(-)}_{\lambda'\lambda;\,0} &\equiv H_{\lambda'\lambda;\,0}, \quad H^{(+)}_{\lambda'\lambda;\,0} \equiv 0 \qquad\qquad \text{for} \quad m = 0. \end{aligned} \tag{26}$$

Cohen-Tannoudji *et al.*[108] have shown that exchange with the natural parity is predominant for a fixed t and large values of s for the $H^{(+)}$ amplitudes, while exchange with anomalous parity is predominant for $H^{(-)}$. Accordingly, if we single out the $H^{(-)}$ amplitudes with $m = 0$ we can hope to distinguish the OPE signal, to separate out the background, and then to extrapolate the "pure" OPE components. This method was used in Refs. 33, 37, and 40. The experimental quantities are usually unnormalized average spherical harmonics $N\langle Y^m_L\rangle$:

$$\frac{d^4\sigma}{dt\,dm\,d\Omega} = N \sum_{L=0}^{L_{\max}} \left[\langle Y^0_L\rangle Y^0_L(\Omega) + 2\sum_{m=1}^{L} \langle \mathrm{Re}\, Y^m_L\rangle \,\mathrm{Re}\, Y^m_L(\Omega)\right], \tag{27}$$

where N is the number of events in the given interval. The spherical harmonics are related to the partial-wave amplitudes through the Clebsch–Gordan coefficients. For high-exchange processes the pion–pion state has a zero helicity, so that only $\langle Y^0_L\rangle$ harmonics should be present.

Figure 9 shows values of $\langle Y^m_L\rangle$ from Ref. 39. We see that $\langle Y^1_L\rangle \neq 0$, so that the mechanisms involving the exchange of more than one pion are also important. It is for this reason that Estabrooks and Martin[37] have given preference to an extrapolation of A_0 amplitudes, rather than spherical harmonics; even $\langle Y^0_L\rangle$ is affected by amplitudes with a nonzero helicity, i.e., by mechanisms other than OPE.

Unfortunately, a model-independent determination of the amplitudes for reactions $\pi N \to \pi\pi N$ is not possible without experiments with a polarized target. In a "poor man's model" which Williams[81] worked out for the case of an unpolarized target it is assumed that the helicity system in such reactions is dominated by spin-flip processes and that other processes may be ignored. This assumption has apparently been supported by recent work with a polarized target.[88] Working from this hypothesis, Estabrooks *et al.*[37,109] found six unknowns from six observables ($\langle Y^m_L\rangle$ for $l \leq 2$). These six unknowns are the moduli of the S-wave amplitude, $|A_S|$; the P-wave components $|A_0|$, $|A_+|$, and $|A_-|$ (the subscript gives the helicity of the dipion); the relative phase δ_{SP} between A_S and A_0; and the relative phase φ_{10} between the P-wave components, as functions of t and $m_{\pi\pi}$. Then the t dependence is parametrized, and the parameters are found in each $m_{\pi\pi}$ interval by comparison with the experimental values of $\langle Y^m_L\rangle$. Different functions have been used for this purpose in different papers. In Refs. 37 and 109, for example,

$$A^l_0 = \frac{\sqrt{-t}}{\mu^2 - t} e^{b^l_0 t} T_l(m_{\pi\pi}), \tag{28}$$

while for amplitudes with $m = 1$

$$\frac{1}{\sqrt{2}}(A^l_1 - A^l_{-1}) = A^l_- = \frac{C^l(m_{\pi\pi})}{\sqrt{2}\, m_{\pi\pi}} e^{b^l_- t} \sqrt{l(l+1)}\, T_l(m_{\pi\pi}), \tag{29}$$

where b^l_m and C^l are the slope and absorption parameter for spin l and angular momentum projection m.

The most complete parametrization is that carried out by Hyams *et al.*[40] Three types of parameters were sought:

1. absorption parameters which determine the t dependence but which vary only slowly with $m_{\pi\pi}$
2. the parameters of the resonances and the coefficients of the K-matrix for higher-order waves (D and F waves)
3. the phase shifts and inelasticities of the lower-order waves.

The experimental values of 13 average spherical harmonics were used for the analysis. The experimental results were partitioned into 40-MeV mass intervals, each of which was further partitioned into 19 intervals

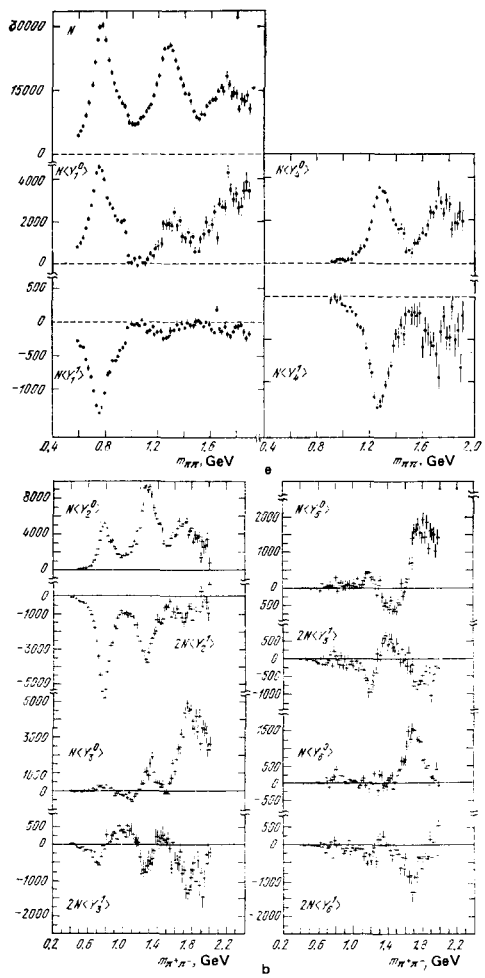

FIG. 9. Behavior of $\langle Y^m_L\rangle$ as a function of $m_{\pi\pi}$ for the $\pi^-\pi^+$ state from Ref. 32.

in t. Figure 10 shows four versions of the solutions for δ_0^0 from Ref. 40. The large number of solutions results from the fact that only the cross section, not the amplitude, can be determined experimentally. If we write the amplitude as

$$T(m_{\pi\pi}, \theta) = \sum_{l=0}^{l_{\max}} a_l(m_{\pi\pi}) P_l(\cos\theta) = f(m_{\pi\pi}) \prod_{i=1}^{l_{\max}} (\cos\theta - z_i), \quad (30)$$

where θ is the angle through which the pions are scattered, and z are the roots of the polynomial, then

$$\sigma_{\pi\pi} \sim \prod_{i=1}^{l_{\max}} (\cos\theta - z_i)(\cos\theta - z_i^*). \quad (31)$$

It follows from (31) that with $l_{\max} = 3$ there are $2^{l_{\max}} = 8$ solutions, since we cannot determine from the cross section whether an amplitude has a zero at z_i or at z_i^*. If we assume that the amplitudes for the D_0 and F_1 waves are positive and are described well by the Breit–Wigner formulas, then we are left with two possible solutions in the region of the f resonance and four in the region of the g resonance. Although there are no such uncertainties in $\langle Y_L^m \rangle$ for $m \neq 0$, the slow variation of these harmonics with z_i prevents a choice.

Energy-independent and energy-dependent methods are ordinarily used together; while the energy-independent method can be used to find structure in the partial-wave amplitudes, the energy-dependent method can explain this structure by introducing new resonances and by choosing an appropriate parametrization.

The third type of analysis, the model-dependent analysis, is based on an exact energy dependence of the amplitudes as predicted by some theoretical model.

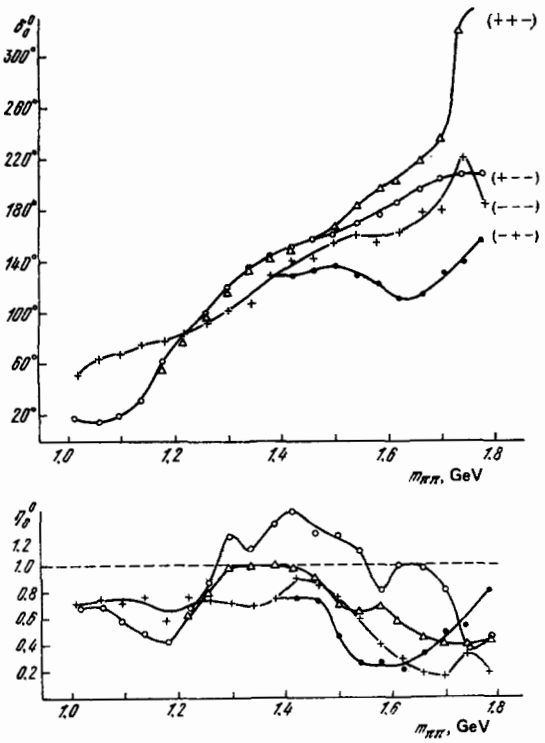

FIG. 10. Four versions of the behavior of the phase shift δ_0^0 and of the parameter η_0^0 (from Ref. 30).

This approach is ordinarily used for reactions for which only fragmentary experimental data are available. This method is good in that it allows us to test the given model and to examine the mechanisms for the various processes. Methodologically, this approach is similar to the energy-dependent method. Specifically, in both cases one specifies formulas and then chooses parameters—the phase shifts and inelasticities—from a comparison with experiment. The experimental spectra are chosen in different ways; most frequently, they are distributions in $\cos\theta$ and $m_{\pi\pi}$. Distributions in t and φ are not very sensitive to the choice of phase shifts, although the use of two-dimensional angular distributions (in $\cos\theta$ and φ) does yield some useful information.[110] Since there are many adjustable parameters, so that a comparison becomes difficult, some simplifying assumptions are made: The analysis is restricted to a certain number of waves (usually $l \leq 2$); the interaction is assumed to be completely elastic; etc. In a model-dependent analysis, the data in the physical region are used, without extrapolation.

Combined analysis methods are sometimes used. Engler *et al.*,[34] for example, have used a model-dependent, energy-independent analysis. Experimental data on the reaction $\pi^- p \to \pi^- \pi^+ n$ were analyzed by the OPEA model separately in three regions of the dipion mass; a distinct parametrization was used for each region. This method reduces the number of adjustable parameters but it presents difficulties in joining the solutions.

Finally, we note that at present it is difficult to draw the line clearly between model-dependent and energy-dependent analyses. Papers using amplitude analysis (for example, Refs. 37 and 40) have used the assumptions of the Williams model[81] that spin-flip processes are predominant, and several papers have used the assumption that the absorption coefficient is independent of $m_{\pi\pi}$ (this assumption is evidently at odds with experiment[37b]), so that these papers are not completely model-independent.

7. RESEARCH ON THE $\pi\pi$ INTERACTION AT LOW ENERGIES

There are some distinctive features in research on the $\pi\pi$ interaction near the threshold. Here there are more sources of information—not only reactions of the type $\pi N \to \pi\pi N$ at a low momentum of the incident pion but also K decays, $\eta \to 3\pi$ decays, etc. Furthermore, there is the possibility of extrapolating the results of phase-shift analyses at intermediate energy into this region. Work at low energies has its own particular difficulties, however, primarily the small interaction cross section. Because of this small cross section it is difficult to accumulate a large statistical base, and it is frequently necessary to appeal to some model or another. At the same time, it is very important to study the characteristics of $\pi\pi$ interactions near the threshold. For example, the threshold behavior of the scattering amplitude is characterized by a constant called the scattering length:

$$a_l^I = \lim_{k \to 0} \frac{f_l^I(k)}{k^{2l+1}} \quad (32)$$

(k is the pion momentum in the rest frame of the dipion). This scattering length is required for problems of general physical interest and also choosing a model in π-condensate calculations.[112]

As Migdal has shown,[20,113] in strong external fields or in a dense nucleonic medium there should be a restructuring of the pion vacuum of the nature of a phase transition: a pion condensation. The energy released in the phase transition of the pion field may exceed that required to compress the nucleonic matter, so that there is the possibility in principle of finding a second minimum on the energy-density curve; i.e., there is the possibility of ultradense nuclei. When the polarization of the medium under the influence of pions is taken into account it is concluded that a pion condensate may exist in ordinary nuclear matter also,[114] at a density above the critical density (which may be larger or smaller than the nuclear density). Some interesting new theoretical and experimental problems arise here; they are discussed, for example, in Ref. 21. The choice of the parameters for the pion-condensate theory is determined by the experimental information on the elementary processes, in particular, $\pi\pi$ scattering.

Let us examine in more detail the basic possibilities for studying the $\pi\pi$ interaction near the threshold. The most natural approach would seem to be to analyze reactions of the type $\pi N \to \pi\pi N$ at a low momentum of the incident pion. Until very recently, however, there had been no study of such reactions in which the statistical base was good enough to extract angular distributions of the secondary pions.[4)] Essentially all the information on $\pi\pi$ interactions has been extracted from cross sections, and usually on the basis of some model (the isobar model,[117-120] models incorporating triangle diagrams,[121-123] etc.). Among these studies we would like to single out one carried out at the Joint Institute for Nuclear Research, which encompassed all channels of the reaction $\pi^* p \to \pi\pi N$ reaction and which was carried out extremely close to the threshold,[124] at $E_\pi < 300$ MeV. Isobars cannot be produced at such energies, and the experimental results may be compared directly with the predictions of the models for the low-energy $\pi\pi$ interaction. Figure 11 shows the cross sections for certain reactions and calculations based on various models which take the effective-Lagrangian approach and the current-commutator approach. The basic conclusion reached in this work is that it is not possible to describe simultaneously the cross sections for all reaction channels in any single version of the models.

Makarov[29] has attempted to find the $\pi\pi$ scattering phase shifts near the threshold. He worked from the data from several studies of the $\pi\pi$ cross sections obtained by the Chew-Low method for this region. The cross sections were approximated, and then the phase shifts were determined from the resulting averages. The resulting phase shifts cannot be judged very reliable, since they were determined from a combination of data from studies at various energies, carried out with a low statistical base.

4) The sole exception is Ref. 116, and this paper will be discussed at the end of this section.

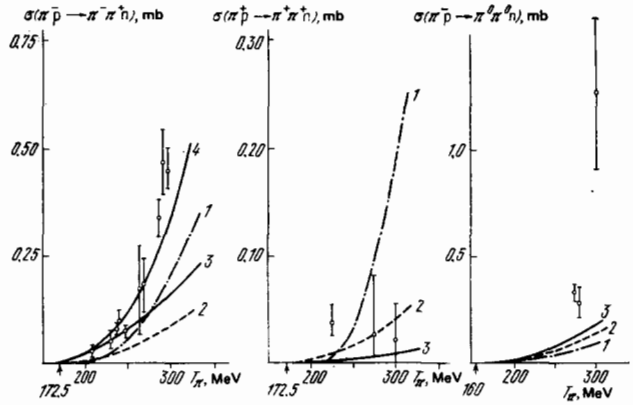

FIG. 11. Experimental values of $\delta(\pi p \to \pi\pi\pi)$ near the threshold from Ref. 124. 1—Calculation from the statistical model of Ref. 172; 2—effective-Lagrangian method with $\xi = 0$ (Ref. 174); 3—effective-Lagrangian method with $\xi = 1.4$ (Ref. 174); 4—current-commutator method.[172]

Another important source of information on the $\pi\pi$ interaction near the threshold is study of the decays $K^* \to \pi^+\pi^- e^* \nu$. As early as 1968, Pais and Treiman[125] showed that the K_{e4} decay is extremely convenient for studying the low-energy $\pi\pi$ interaction. There are no hadrons in the final state of this decay aside from pions; a minimum number of additional assumptions is required; and in principle a completely model-independent analysis is possible. Unfortunately, the cross section for this process is small, and it is very difficult to accumulate an adequate statistical base. It is usually assumed that in decays of the type $K \to \pi\pi e\nu$ the rule $|\Delta I| = 1/2$ holds; then only S_0 and P_1 waves are possible (the D_0 and F_1 waves are of minor importance because of the condition $m_{\pi\pi} \leq m_K = 494$ MeV). This conclusion is completely reasonable, since the rule $|\Delta I| = 1/2$ holds quite well in most nonlepton decays of hyperons, in $K \to 2\pi$ decays, etc. (an exceptional case is the decay $K \to 3\pi$, where this rule does not hold and where it is necessary to assume $|\Delta I| = 3/2$). This analysis is based on a study of the interference between the S_0 and P_1 waves, from which one can determine the value of $(\delta_0^0 - \delta_1^1)$.

It was mentioned above that much information on pion–pion interactions is available in the region of large dipion masses ($m_{\pi\pi} > 500$ MeV). It seems extremely interesting to use these results to study the low-energy region through an extrapolation to the threshold. Here we need to know the behavior of the $\pi\pi$ scattering phase shifts in the pertinent region. Near the threshold the scattering length can be approximated by

$$k^{2l+1} \operatorname{ctg} \delta_l^I = \frac{1}{a_l^I}, \tag{33}$$

and in a slightly broader region we can use the effective-radius approximation

$$k^{2l+1} \operatorname{ctg} \delta_l^I = \frac{1}{a_l^I} + \frac{1}{2} k^2 r_l^I. \tag{34}$$

The applicability limit of Eq. (34) can be assumed to be $m_{\pi\pi} \sim 700$ MeV. Then by approximating the phase shifts in the region 500–700 MeV by this expression we can easily find the scattering lengths. Both the ordinary

expression for the effective radius[28b, 55, 126-128] and its relativistic modifications[28b, 30] have been adopted widely by experimentalists, but there are some objections to the use of this method. The objections stem from the hypothesis that the real part of the S-wave amplitude, $\mathrm{Re}\, f_0$, goes through zero not far from the physical region, i.e., has a subthreshold zero. In this case a direct extrapolation of the quantity $k\cot\delta_0$ from the physical region ignores the sharp increase in this quantity near the zero and leads to an overestimate of the scattering length. This question has been discussed, for example, by Gareevanishvili and Shirkov.[129]

Franklin[130] took up the theoretical problem of continuing the amplitudes found in the Weinberg model into the low-energy physical region through the use of dispersion relations for scattering into the forward hemisphere. Franklin concluded that the calculated curve agrees well with experimental data at $m_{\pi\pi} < 550$ MeV and that the scattering length found by extrapolating the data from the region above 400 MeV is overestimated.

Two recent experimental papers[131,116] seem to support this point of view. Bel'kov and Bunyatov[131] reported a joint description of data from K_{e4} decay at $m_{\pi\pi} < 350$ MeV (Ref. 132) and from the reaction $\pi^- p \to \pi^-\pi^+ n$ at $500 \leq m_{\pi\pi} \leq 700$ MeV (Ref. 137b), by means of an expression which incorporates a subthreshold zero. The value found for χ^2 by this approach is significantly better than that found by using the expression for the ordinary effective radius. In Ref. 116 Bel'kov *et al.* found δ_0^0 for $\pi\pi$ scattering from the reaction $\pi^- p \to \pi^-\pi^+ n$ at $m_{\pi\pi} < 330$ MeV through an extrapolation in the pseudoperipheral approximation. Comparison of these results with data obtained for the region $m_{\pi\pi} > 450$ MeV by the same group by a similar procedure[178b] reveals a change in the slope of the function $k\cot\delta_0^0$ (Fig. 12). This slope change may be direct evidence for the existence of a subthreshold zero in the S-wave amplitude near the physical region and also direct evidence that the effective-radius formula cannot be used at $m_{\pi\pi} > 400$ MeV.

A powerful method for studying low-energy $\pi\pi$ scattering is to use the properties of analyticity, unitarity, and crossing symmetry. Many attempts have been made to use these properties both for theoretical models and for analyzing experimental data. This work is reviewed, for example, in Refs. 15 and 133.

Particular interest was attracted to this approach after the Indian physicist Roy[134] derived equations for

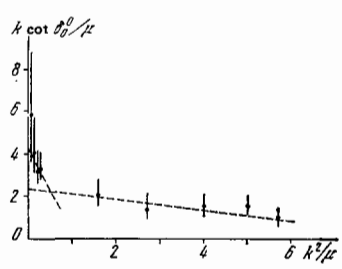

FIG. 12. Experimental values of $k\cot\delta_0^0$ from Ref. 116 (near the threshold) and Ref. 178b. Dashed lines—Approximations by the effective-radius formula in the corresponding regions.

the pion–pion amplitudes which bear his name. A fundamental distinction between these equations and the others available at the time was that they gave partial wave amplitudes which satisfied the conditions of analyticity and crossing symmetry in the region[5)] $-4 \leq s \leq 60$, which includes the nonphysical part $-4 \leq s \leq 4$, in terms of quantities in the physical region. Then it is possible in principle first to construct amplitudes in the low-energy region from a knowledge of their behavior at high energies and then choose among the available experimental data. In particular, it becomes possible to resolve the "up–down" problem and to find the scattering lengths.

In deriving the equations, Roy used dispersion relations with two subtractions at fixed t and then projected them onto partial waves. For the case of charged pions, the Roy equations are

$$\mathrm{Re}\, f_l^I(s) = \lambda_l^I(s) + \sum_{I'=0}^{2}\sum_{l'=0}^{1}\int_4^\infty K_{ll}^{I'I'}(s,s')\,\mathrm{Im}\, f_{l'}^{I'}(s')\,ds' + \varphi_l^I(s); \tag{35}$$

here $\lambda_l^I(s)$ are subtraction terms, given for the S and P waves by

$$\lambda_0^0(s) = a_0^0 + \frac{1}{12}(2a_0^0 - 5a_0^2)(s-4),$$

$$\lambda_0^2(s) = a_0^2 - \frac{1}{24}(2a_0^0 - 5a_0^2)(s-4),$$

$$\lambda_1^1(s) = \frac{1}{72}(2a_0^0 - 5a_0^2)(s-4),$$

where a_0^0 and a_0^2 are the S-wave scattering lengths; $\varphi_l^I(s)$ is the sum of the contributions of higher-order partials with $l \geq 2$,

$$\varphi_l^I(s) = \sum_{I'=0}^{2}\sum_{l'=2}^{\infty}\int_4^\infty K_{ll}^{I'I'}(s,s')\,\mathrm{Im}\, f_{l'}^{I'}(s')\,ds'; \tag{36}$$

and $K_{ll}^{I'I'}$ are the kernels of the integral equations, which are quite complicated. Exact expressions for $K_{ll}^{I'I'}$ are given, for example, in Ref. 133. As mentioned above, the region $-4 \leq s \leq 60$ is considered.

For the practical use of these equations, the integral in (35) is usually broken up into two parts:

$$\begin{aligned}\mathrm{Re}\, f_l^I(s) = \lambda_l^I(s) &+ \sum_{I'=0}^{2}\sum_{l'=0}^{1}\int_4^N K_{ll}^{I'I'}(s,s')\,\mathrm{Im}\, f_{l'}^{I'}(s')\,ds' \\ &+ \sum_{I'=0}^{2}\sum_{l'=0}^{1}\int_N^\infty K_{ll}^{I'I'}(s,s')\,\mathrm{Im}\, f_{l'}^{I'}(s')\,ds' + \varphi_l^I(s).\end{aligned} \tag{37}$$

The value of N is chosen such that in the region $N < s < \infty$ the Regge representation can be used; i.e., it can be assumed that the $\pi\pi$ amplitudes do not contain resonances in this region. Comparison of the quantities in Eq. (35) reveals that the integrals for $4 < s < N$ and $l < 2$ play a leading role. The terms which incorporate large masses and higher-order waves are small, but it is necessary to determine just how important these terms are. This problem was taken up in Refs. 135 and 136; it was found that for practical purposes a rough estimate of these terms is sufficient at present, but a more detailed study must be made for a future refinement of these results.

There are two basic approaches for using the Roy equations to study $\pi\pi$ interactions, and these approaches

[5)] Here and below, $s \equiv m_{\pi\pi}^2$ in units of μ^2.

may be called the "general theoretical" approach and the "calculation–experimental" approach. In the general theoretical approach, the Roy equations are supplemented with the requirement of unitarity (the equations themselves satisfy only the properties of analyticity and crossing symmetry), and a solution of the resulting system of equations is sought. It turns out that a broad class of solutions can be singled out. From this class one selects a subclass containing amplitudes which satisfy well-established experimental results of a general nature. Usually included among these general results are[136]

1. the existence in the P wave of a ρ resonance with $m_\rho \approx 770$ MeV and $\Gamma_\rho \approx 150$ MeV
2. the absence of resonances in waves with $I = 2$ at $m_{\pi\pi} \leq 1$ GeV
3. the existence of an inelasticity beginning at the $K\bar{K}$ threshold (and not at $s = 16$, where this inelasticity is assumed to be negligibly small).

These completely natural requirements are not enough, however; to reduce the size of the subclass it is necessary to adopt some further requirements, regarding the behavior of the phase shift δ_0^0: δ_0^0 passes through 180° near the $K\bar{K}$ threshold, and there is no major discrepancy with the data from phase-shift analyses. Basdevant, Froggatt, and Petersen[136] compared the solutions of the Roy equations with the phase curves from the papers[6)] by Baton,[28a] Protopopescu,[55] and Estabrooks–Martin.[109] The one possible set of solutions on the (a_0^0, a_0^2) plane (the "universal curve") was found for each of these papers; these curves are shown in Fig. 13. It is not possible to identify a unique solution i.e., a definite point on each of the universal curves. Figure 14, taken from Ref. 17, shows phase curves calculated by this method from the experimental data of Ref. 28a for three values of a_0^0: -0.05, 0.2, and 0.6 μ^{-1}. Comparison with the experimental values shows that all three curves have approximately the same—and good—value of χ^2, which does not exceed 2 per degree of freedom. An analogous result was found for other experimental data (see Ref. 136 for a comparison with Refs. 55 and 109). Basdevant, Froggatt, and Petersen[136] thus conclude that over the interval $500 \leq m_{\pi\pi} \leq 1000$ MeV, at the present level of experimental accuracy, only a broad range of values can be found for a_0^0: $-0.5 < a_0^0 < 0.6\ \mu^{-1}$. Correspondingly, for a_0^2 we find the range $-0.10 < a_0^2 < 0.03\ \mu^{-1}$. The value found for a_1^1 is quite stable and approximately equal to that which follows from the Weinberg model: $a_1^1 \sim 0.03\ \mu^{-3}$.

In addition to the general theoretical approach, it is also possible to take a "calculation–experimental" approach. In this case some specific experimental data on the phase shifts are adopted. These values are assumed to be the actual values, and a search is made for an analytic and unitary continuation to the vicinity of the threshold. This continuation must satisfy the Roy equations. This approach was used, for example, in Ref. 137. The experimental data used there were the S_0 and P_1 amplitudes from Ref. 55 and the S_2 amplitudes from Ref. 43 over the range $500 \leq m_{\pi\pi} \leq 1050$ MeV. The integral in (35) was broken up into three parts:

1. A near-threshold part for $4 < s < 10$. The amplitudes in this part were assumed unknown and were varied in the course of the calculation.
2. An experimental part, $10 \leq s \leq 60$. In this region the amplitudes were specified and fixed.
3. The region of large masses, $s > 60$. The integral over this region was calculated by means of Regge representations and was assumed constant.

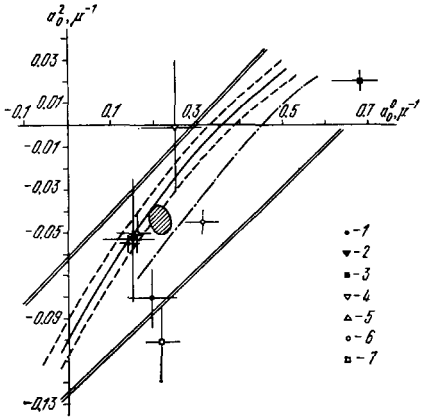

FIG. 13. "Universal curve" on the (a_0^0, a_0^2) plane. Double lines: Boundaries of the region of possible values from Ref. 136. Dot-dashed line and region between the dashed lines: Results calculated from the same study for the data of Refs. 38 and 28a, respectively. Hatched region: Predictions of the Weinberg model. Experimental points: 1—Ref. 152; 2—Ref. 137; 3—Ref. 128; 4—Ref. 128; 5—Ref. 138; 6—Ref. 124; 7—Ref. 152.

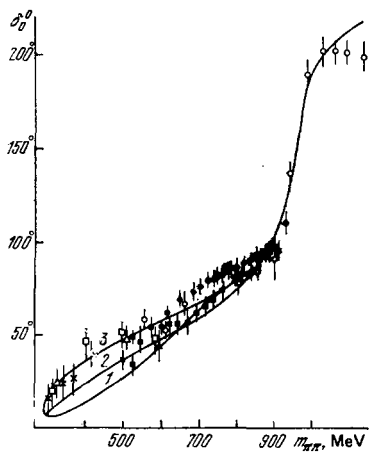

FIG. 14. Illustrative behavior of the phase shift δ_0^0 from calculations by the Roy-equation method[136] for the data of Ref. 28a. Curves 1-3—Results calculated for $a_0^0 = -0.05\,\mu^{-1}$, and $0.6\,\mu^{-1}$, respectively. The results of other phase-shift analyses are shown for comparison.

[6)] As a rule, the results of other phase-shift analyses agree with one or another of those mentioned here. For example, the results calculated by the CERN–Munich group[33] agree well with the data of Ref. 55. The values of δ_0^0 from the study by Baton *et al.* are 5°—10° lower than those reported by Protopopescu *et al.*, which are in turn about the same amount smaller than the values give by Estabrooks and Martin in region $500 \leq m_{\pi\pi} \leq 900$ MeV.

For each value of a_0^0 an iteration method was used to find a solution which satisfied Eqs. (35) and which agreed best with the experimental points. Then the optimum solution was found by comparing the solutions for various values of a_0^0 in the experimental region on the basis of the χ^2 test. The scattering lengths found by this method are given in the following section.

In this section we have briefly reviewed the methods ordinarily used to study the low-energy $\pi\pi$ interaction experimentally. The theoretical aspects of the problem are reviewed in Ref. 115.

8. EXPERIMENTAL RESULTS

a) Scattering lengths

Let us examine in more detail the $\pi\pi$ scattering lengths known at present. We will begin with the most important and relatively large S-wave scattering lengths a_0^0 and a_0^2.

An effort has been made to determine a_0^0 in most studies of the $\pi\pi$ interaction at low energies, but the situation remained unclear for a long time. Most theoretical papers predict values in the range $0.10 < a_0^0 < 0.25\ \mu^{-1}$ (Refs. 138–142), i.e., values approximately the same as those predicted by Weinberg's soft-pion model,[143] $a_0^0 \sim 0.2\ \mu^{-1}$. There are some exceptional cases, however. Daly,[144] for example, found a dependence of a_0^0 on Γ_ρ, working in the Veneziano model. With a ρ-resonance width $\Gamma_\rho \sim 150$ MeV, the value is $a_0^0 \sim 0.4\ \mu^{-1}$. In a study[145] based on data on $K \to 2\pi$ decays and the Cabibbo structure for the weak-interaction Hamiltonian, the result $a_0^0 \sim 0.6\ \mu^{-1}$ was found. Even greater discrepancies were found among the experimental results (Table II and Fig. 15). Even studies carried out by the same procedure frequently yielded contradictory results. It was not too long ago that the range of possible experimental values was so broad ($0.05 < a_0^0 < 0.8\ \mu^{-1}$) that even the sign of a_0^0 was uncertain. Substantial progress has apparently been made recently, because both of improvements in the experimental accuracy and the appearance of studies confirming the existence of a subthreshold zero in the S amplitude near the physical region (Refs. 131 and 116; see Section 7). It has now become clear that the relatively large value found for a_0^0 through the use of the effective-radius formula for phase shifts at $m_{\pi\pi} > 500$ MeV can be attributed to the overestimated scattering length in this method. Most of the studies carried out recently by other methods, on the other hand, yield consistent results. A recent paper on K_{e4} decay,[132] a paper in which the dispersion relations are applied to the data from a phase-shift analysis for πN scattering,[146] a study in which the effective-radius formula was applied to the phase shifts in the threshold region,[116] $m_{\pi\pi} \leqslant 320$ MeV, and calculations based on the equation for a subthreshold zero[131] all yield values in the range $a_0^0 \sim 0.23$–$0.28\ \mu^{-1}$. Figure 16 compares the experimental values of the difference $\delta_0^0 - \delta_1^1$ with calculations from the Roy equations. We see that the solution with $a_0^0 \sim 0.25\ \mu^{-1}$ seems to be the best. A smaller value is predicted by certain studies of the cross sections for the reaction $\pi N \to \pi\pi N$ near the threshold,[120,123] but these studies have generally been model-dependent. The only results which stand out at all are those of a

TABLE II.

Reaction	Method for finding lengths	a_0^0, μ^{-1}	a_0^2, μ^{-1}	Reference
	1. From phase shifts in the region $m_{\pi\pi} > 400$ MeV			
$\pi^-p \to \pi^-\pi^+n$	Effective radius	0.44 ± 0.10	—	126
$\pi^-p \to \pi^-\pi^+n$	» »	$0.35^{+0.06}_{-0.04}$	—	35
$\pi^+p \to \pi^+\pi^-\Delta^{++}$	» »	0.34 ± 0.18	—	55
$\pi^-p \to \pi^-\pi^+n$	» »	0.39 ± 0.07	—	127
$\pi^-p \to \pi^-\pi^+n$	Relativistic effective radius	0.67 ± 0.06	—	128
$\pi^-p \to \pi^-\pi^0p$	Relativistic effective radius	—	-0.052 ± 0.005	286
$\pi^\pm p \to \pi^\pm\pi^0p$	Effective radius	—	-0.131 ± 0.014	148
$\pi^+p \to \pi^+\pi^+n$	» »	—	-0.16 ± 0.03	48
$\pi^-n \to \pi^-\pi^-p$	» »	—	-0.092 ± 0.015	59
	2. From K decays			
$K \to 3\pi$	Semiphenomenological model	0.59 ± 0.07	-0.20 ± 0.03	124a
$K \to \pi\pi ev$	Effective radius with the P wave neglected	0.17 ± 0.13		149
$K \to \pi\pi ev$	Effective radius with the P wave neglected	0.55 ± 0.27		150
$K \to \pi\pi ev$	Approximation by the model of Ref. 133	0.28 ± 0.05		122
	3. From the $\pi N \to \pi\pi N$ reaction near the threshold			
$\pi^-p \to \pi^-\pi^+n$	Isobar model	$-0.06 < a_0^0 < 0.03$		120
$\pi^-p \to \pi^-\pi^+n$	Isobar model	$0.20^{+0.08}_{-0.10}$		119
$\pi N \to \pi\pi N$	Effective-Lagrangian calculation	0.17 ± 0.01	-0.045 ± 0.005	124b
$\pi^-p \to \pi^-\pi^+n$	Incorporation of triangle diagram	0.33 ± 0.08		122
$\pi^-p \to \pi^-\pi^+n$		0.05 ± 0.01		123
$\pi^-p \to \pi^-\pi^+n$	Effective radius	0.24 ± 0.10		116
	4. Other methods			
$\pi N \to \pi N$	Dispersion relations	$-0.15 < a_0^0 < 0.10$		151
$\pi N \to \pi N$	Dispersion relations	0.25 ± 0.09		146
$\pi^+p \to \pi^+\pi^-\Delta^{++}$	Roy equations	0.15 ± 0.07	-0.053 ± 0.028	137
K_{e4} и $\pi^-p \to \pi^-\pi^+n$	Formula with a subthreshold zero	0.23 ± 0.05		131
$\pi^+p \to \pi^+\pi^+n$	Formula with a subthreshold zero		-0.06 ± 0.07	121
$\pi^+p \to \pi^+\pi^+n$			-0.07 ± 0.03	176

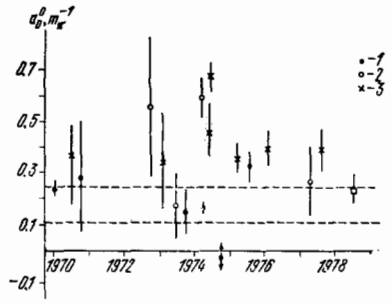

FIG. 15. Summary of the experimental data on a_0^0 from Refs. 30, 35, 55, 120, 122, 124a, 126–128, 131, 132, 149, 150, 174, 175, and 180. 1—Data on the reaction $\pi N \to \pi\pi N$ near the threshold; 2—data from K_{e4} decays; 3—results of the use of the effective-radius formula.

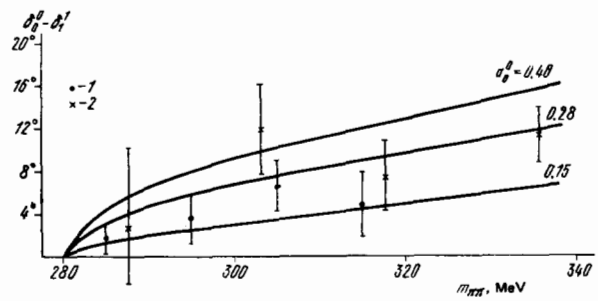

FIG. 16. Values of the difference $\delta_0^0 - \delta_1^1$ from Refs. 116 (1) and 132 (2). Solid curves—results calculated in Ref. 136.

series of studies[52-54] on the reaction $\pi^-\pi^+ \to \pi^0\pi^0$ from the reaction $\pi^-p \to \pi^0\pi^0 n$, where large values of δ_0^0 were found near the threshold, which led to large scattering lengths $a_0^0 \sim (0.6-0.8)\mu^{-1}$. This discrepancy can probably be attributed to the difficulties of the corresponding experiments. At any rate, a large δ_0^0 was not observed near the threshold in a similar study.[147] An accurate experimental study of the $\pi^0\pi^0$ state at low energies is seen to be one of the most urgent problems in the physics of the $\pi\pi$ interaction.

The S-wave scattering lengths are strongly correlated with each other. At present most physicists accept the "universal curve" for a_0^0 and a_0^2 (see Section 7 and Fig. 13). It follows from this universal curve that $0.2 < a_0^0 < 0.3\ \mu^{-1}$ then a_0^2 must be small and negative. Theoretical predictions based on the soft-pion model yield $a_0^2 \approx (-0.05)-(-0.04)\ \mu^{-1}$. Analyses based on the dispersion relations expand the possible range of a_0^2 to[138] $-0.2 < a_0^2 < 0.03\ \mu^{-1}$ or to[136] $-0.12 < a_0^2 < 0.04\ \mu^{-1}$. As a rule, experimental studies yield negative values of a_0^2, but there is a rather large scatter (Table II). If the existence of a nearby subthreshold zero in the S_2 wave is confirmed, then the results of studies based on an effective radius will clearly be unreliable, and it will be necessary to work solely from the results of Refs. 131 and 176, the equation for the subthreshold zero, and calculations by the Roy-equation method.[137]

At first glance the situation would seem to be best for the P-wave scattering length a_1^1. Theoretical predictions based on the soft-pion model yield values in the range $0.028 < a_1^1 < 0.035\ \mu^{-3}$. The use of the dispersion relations expands this range to[136, 138, 152] $0.02 < a_1^1 < 0.05\ \mu^{-3}$, but there is essentially no change in the average value. The experimental values of a_1^1 found in the effective-radius approximation are also concentrated around the average value $a_1^1 \sim 0.03\ \mu^{-3}$ (see, for example, Ref. 30, where the value $a_1^1 = 0.05 \pm 0.03\ \mu^{-3}$ was found, and Ref. 128, where $a_1^1 = 0.027 \pm 0.002\ \mu^{-3}$ was found). The use of an "experimental" method for carrying out calculations from the Roy equations[137] has yielded the value $a_1^1 = 0.036 \pm 0.002\ \mu^{-3}$.

At a 1974 conference in Boston, however, Männer[126] reported that a CERN-Munich group had found a scattering length $a_1^1 \sim 0.10\ \mu^{-3}$. Here the usual approximation of the effective radius was applied to the phase shifts found from a large statistical base on the reaction $\pi^-p \to \pi^-\pi^+ n$ at $p_\pi = 7$ GeV/c. If we accept this result as correct, then we will have to reexamine the accepted conclusion that the P wave has a smooth behavior near the threshold. Figure 17 shows experimental values of the function $2k^3 \cot\delta_1^1/m_{\pi\pi}$ from the paper by the CERN-Munich group[126] and theoretical curves calculated from the Roy equations.[153] We see that the experimental points do not agree with the calculations, which predict a sharp decay at small $m_{\pi\pi}$. In order to explain this behavior of the P wave near the threshold it is necessary to assume some feature, possibly of the nature of a resonance, near the nonphysical region. New experiments are required to confirm or reject the Männer results. In particular, this question might be clarified by a study of the $e^+e^- \to \pi^+\pi^-$ reaction in the pertinent region near the threshold. For the time being, on the other hand, it is believed that the scattering length is $a_1^1 \sim 0.035\ \mu^{-3}$.

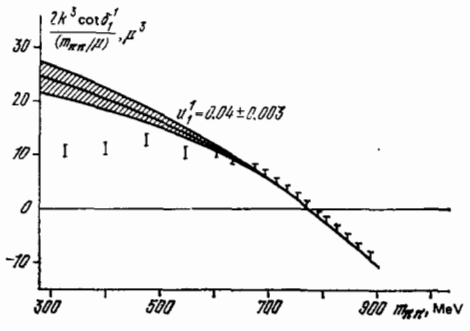

FIG. 17. Behavior of the P_1 wave near the threshold. Experimental values of the function $2k^3\cot\delta_1^1/m_{\pi\pi}$ from Ref. 126. The hatched region corresponds to $a_1^1 = 0.040 \pm 0.003\ \mu^{-3}$ from calculations by the Roy-equation method.[153]

The scattering lengths for waves with $l \geq 2$ are small. The theoretical values[136-138, 141, 152] are approximately as follows: $a_2^0 \sim 16 \cdot 10^{-4}\mu^{-5}$, $a_2^2 \sim 2 \times 10^{-4}\mu^{-5}$, $a_3^1 \sim 10^{-4}\mu^{-7}$. Experimental studies of this question have been few in number and low in accuracy, but on the whole the results are consistent with the theory. As an example, we can give the results calculated by the Roy-equation method for data on K_{e4} decay[154]: $a_2^0 = (17 \pm 3) \cdot 10^{-4}\mu^{-5}$; $a_2^2 = (1.3 \pm 3) \cdot 10^{-4}\mu^{-5}$, $a_3^1 = (0.6 \pm 0.2) \cdot 10^{-4}\mu^{-7}$. In addition, we note that the value $a_2^2 = (3.8 \pm 1.4) \cdot 10^{-4}\mu^{-5}$ was recently found[176] in the effective-radius approximation from data on the reaction $\pi^+\pi^+ \to \pi^+\pi^+$. The question of the scattering lengths with $l = 2, 3$ can be considered answered, although quantitative refinements are of course possible.

b) The elastic-interaction region and scalar resonances

Strictly speaking, the elastic-interaction region is bounded by the 4π threshold: $m_{\pi\pi} = 560$ MeV. In practice, however, it turns out that the $\pi\pi$ interaction can be assumed elastic up to $m_{\pi\pi} < 1$ GeV. Protopopescu

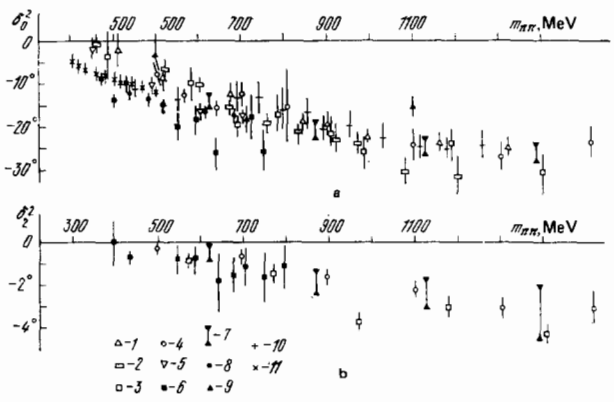

FIG. 18. Experimental values of δ_0^2 (a) and δ_2^2 (b). Open symbols: Data on $\pi^-\pi^-$ scattering. 1—Ref. 85; 2—Ref. 59; 3—Ref. 62; 4—Ref. 61; 5—Ref. 58. Filled symbols: Data on $\pi^+\pi^+$ scattering. 6—Ref. 49; 7—Ref. 47; 8—Ref. 48; 9—Ref. 46; 10—values from an analysis of $\pi^-\pi^0$ scattering (Ref. 3); 11—Data of Ref. 29.

et al.,[55] for example, have estimated the inelasticity for the reaction $\pi^+p \to \pi^+\pi^-\Delta^{++}$ at $m_{\pi\pi} < 1$ GeV to be $|1-\eta_1^I| \lesssim 2\%$. At the present level of experimental accuracy, effects of this order of magnitude can be ignored, and the interaction in this region can be considered completely elastic.

We will begin our discussion with the waves with $I=2$. The phase shifts for these waves vary smoothly with the dipion mass and are small in absolute value. Information on these phase shifts is extracted from reactions of the type $\pi^+p \to \pi^+\pi^+$ and $\pi^-p \to \pi^-\pi^-\Delta^{++}$. In most of the analyses it has been assumed that the $\pi\pi$ scattering with $I=2$ is described by elastic S_2 and D_2 waves. Inelastic effects should evidently come into play above 1 GeV, but information on this region is scanty at present. Figure 18 shows the experimental values of δ_0^2 and δ_2^2. We see that the agreement between the data from different studies is satisfactory on the whole. However, since the information on these waves is introduced and fixed in a study of the $\pi^-\pi^+$ interaction (which is the primary and most accurate source of information at present), we would like to know the phase shifts for the S_2 wave within $\pm 2°$ and those for the D_2 within $\pm 0.5°$. Here is it is pertinent to recall that interference effects make it necessary to consider waves with $l=2$ even where the amplitudes of these waves are small, at $m_{\pi\pi} < 1$ GeV (Refs. 37b and 44). It is interesting to note that the shift δ_0^2 found from an analysis of the $\pi^-\pi^-$ state (the open symbols in Fig. 18a) has smaller values near the threshold and a larger slope than that for the $\pi^+\pi^+$ state (the filled symbols). The difference may result from a systematic error which stems from the different backgrounds for the non-OPE diagrams for the initial reactions.

Figure 19 shows the results on the shift δ_1^1 in the region $500 < m_{\pi\pi} < 1000$ MeV. The agreement between the experimental data is seen to be good. The curve here is the result of an energy-dependent analysis with a Breit–Wigner resonance and the background from Ref. 55. The ρ resonance with $m_\rho = 773 \pm 3$ MeV and $\Gamma_\rho = 152 \pm 3$ MeV (Ref. 155) determines δ_1^1 essentially completely in this region, and it seems that there are no unresolved questions here.

Of primary interest in the phase-shift analysis of the $\pi\pi$ interactions below 1 GeV are the behavior of δ_0^0 and the existence of scalar resonances. For a long time, δ_0^0 could not be found reliably at $2\mu < m_{\pi\pi} \lesssim 500$ MeV. After the appearance of studies on the K_{e4} decay[132] and

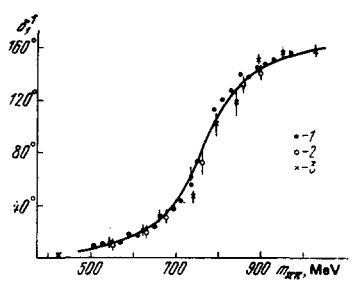

FIG. 19. P_1-wave phase shift from Refs. 37b (1), 55 (2), and 44 (3). Curve—Results of an energy-dependent analysis.[55]

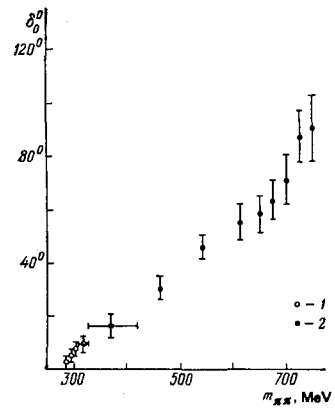

FIG. 20. Values of δ_0^0 from threshold to $m_{\pi\pi} = 750$ MeV obtained from the reaction $\pi^-p \to \pi^-\pi^+n$. 1—Values from Ref. 116; 2—from Ref. 178.

a comparison[178b] of data on the reaction $\pi^-p \to \pi^-\pi^+$ near the threshold and at $p_\pi = 4.5$ GeV/c, analyzed by a common method, it could be asserted that the near-threshold behavior of δ_0^0 was known (Fig. 20). At larger values of $m_{\pi\pi}$, as was mentioned in Section 6, there is an uncertainty of the type $\delta_0^{0'} = \delta_1^1 + (\pi/2 - \delta_0^0)$—the familiar "up–down" problem—because δ_0^0 must be extracted from the coefficient of the isotropic term. Up to $m_{\pi\pi} \sim 700$ MeV, this uncertainty is resolved on the basis of data on the cross section $\sigma_{\pi\pi}$, but in the vicinity of the ρ resonance, where σ_ρ is much larger than σ_s, this method cannot be used. We are left with two possibilities (Fig. 21): the "down" solution, in which δ_0^0 increases slowly from ~80° at $m_{\pi\pi} \approx 770$ MeV to ~90° at $m_{\pi\pi} \approx 950$ MeV and then increases sharply to ~150° at the $K\bar{K}$ threshold; 2) the "up" solution, in which δ_0^0 increases rapidly from ~60° at $m_{\pi\pi} = 700$ MeV to 130° at 800 MeV and then changes slope and increases slowly to ~150°. The second solution leaves us with the possible existence of an $\varepsilon(750)$ resonance in the S_0 wave.

It cannot be said that the actual solution has been finally resolved. There are papers which give preference to the "down" solution,[33,55,37] and there are also papers which speak in favor of the "up" solution.[25,156,157] Recently, however, there is an increasing tendency to

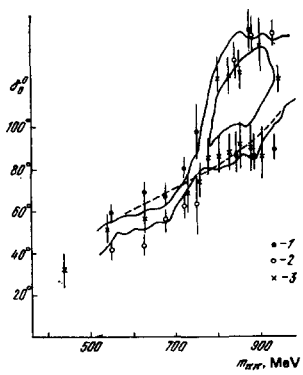

FIG. 21. The phase shift δ_0^0 in the elastic region. 1, 2—two solutions from Ref. 55; 3—values from Ref. 38a; dashed curve—energy-dependent solution from Ref. 33; region between solid curves—region of possible values Ref. 37b.

adopt the "down" solution as the physical solution. Let us review the arguments for this conclusion.

As was mentioned earlier, the choice of a solution in the region $m_{\pi\pi} < 1$ GeV is related to the question of whether the $\varepsilon(700)$ meson exists. The problem of whether this meson exists has been under discussion for a long time; in fact, its parameters have been found in certain papers on phase-shift analysis.

In the well-known paper by Baton *et al.*,[28] for example, the values $m_\varepsilon = 736$ MeV and $\Gamma_\varepsilon = 181$ MeV were found. The particular features of the S_0 wave in this region have been discussed in detail, for example, at a conference in Philadelphia in 1972.[158] In studies of the mass spectra of secondary mesons, however, the ε meson was not found (see, for example, Ref. 159). A particularly important study was carried out on the reaction $d + d \to {}^4\mathrm{He} + X$ by a French–Italian group.[160] In an accurate study of the pure S_0 state—there were no admixtures of other waves besides the D_0 wave—this group found no ε-meson signal. Phase-shift analyses in this region have found a pole at the S_0 amplitude, but far from the physical axis.[33,55] In Ref. 55, for example, the value $m_\varepsilon = 660 - i\cdot 320$ MeV was found. Although the concept of a resonance has not been rigorously defined (Section 4), if we use the generally accepted meaning of this term we must acknowledge that there is no ε meson in the S_0 wave near the ρ resonance.

The "up–down" problem might be resolved by data on the $\pi^0\pi^0$ cross section. Since odd values of l are forbidden for this reaction, we have $\sigma(\pi^-\pi^+ \to \pi^0\pi^0) \sim |A_s|^2$ for small values of $m_{\pi\pi}$, at which the D wave can be ignored; in other words, δ_0^0 can be determined directly from the cross section. Unfortunately, the present level of experimental accuracy on $\pi^0\pi^0$ experiments is such that we cannot yet draw any final conclusion. Figure 22 shows experimental values of $C_0 = |A_S|^2$ from Ref. 52 and curves corresponding to the "up" and "down" solutions. David *et al.*[52] found the cross sections

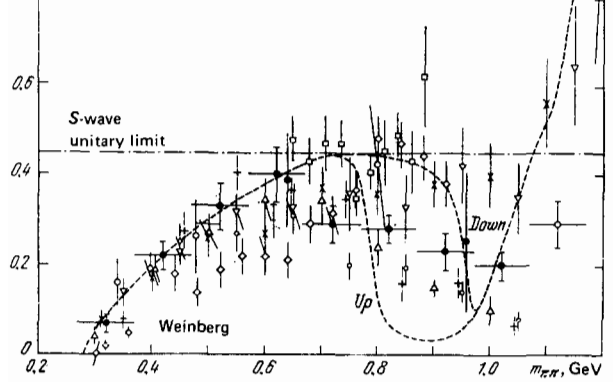

FIG. 22. Behavior of $C_0(m_{\pi\pi})$ from experiments on $\pi^-\pi^+ \to \pi^0\pi^0$ scattering. Here C_0 is determined from $\delta_{\pi\pi} = 2\pi\hbar^2 C_0$; i.e., $C_0 = |A_S|^2$. The figure is taken from Ref. 52, and the filled symbols show values from the paper.

$\sigma(\pi^-\pi^+ \to \pi^0\pi^0)$ for the region $550 < m_{\pi\pi} < 950$ MeV through a Chew–Low extrapolation. The data agree with the results of phase-shift analyses in the region $m_{\pi\pi} < 750$ MeV, but at $m_{\pi\pi} > 750$ MeV the results cannot be described by either the "up" or the "down" solution. Studies by Carroll *et al.*[60] and (especially) Apel *et al.*[50] do not reveal the sharp change in σ_s at $m_{\pi\pi} \approx 700$ MeV which follows from the "up" solution.

Most of the energy-dependent[55] and model-dependent[27,34,60] analyses give preference to the "down" solution. Pennington and Protopopescu[135] have shown that attempts to describe the "up" solution in calculations from the Roy equations lead to very large values of χ^2 and must be abandoned.

The discovery and study of features in the behavior of the S_0 wave near the $K\bar{K}$ threshold played a special role in the choice of a solution. Figure 8a shows that $\langle Y_1^0 \rangle$ for the $\pi^-\pi^+$ system undergoes a sharp decay, nearly to zero, at the $K\bar{K}$ threshold. In the unnormalized harmonics (Fig. 9) we can clearly see a knee in this region. The harmonic $\langle Y_1^0 \rangle$ describes an interference of S and P waves. Since there is no decay of $\langle Y_1^0 \rangle$ for the $\pi^-\pi^0$ state (Fig. 8b), and there are no distinctive features in $\langle Y_2^0 \rangle$ describing the P_1 wave, it is natural to suggest that the structure in $\langle Y_1^0 \rangle$ for the $\pi^-\pi^+$ system is a consequence of the S_0 wave. Assuming $\delta_1^1 \approx 160°$ (Fig. 19), we easily see that the vanishing of $\langle Y_1^0 \rangle$ at the $K\bar{K}$ threshold requires $\delta_0^0 \sim 180°$ at $m_{\pi\pi} = 2m_K$, i.e., the "down" solution.

Let us briefly review the basic arguments in favor of the "down" solution:

1. The "up" solution predicts a decay of the cross section for the $\pi^0\pi^0$ state, and this decay is evidently not confirmed by the experimental data available.[50,52,60]
2. The "down" solution explains the behavior of the S_0 wave near the $K\bar{K}$ threshold.[31,161]
3. The search for the $\varepsilon(700)$ meson, corresponding to the "up" solution, in the mass spectra has been unsuccessful.[159,160]
4. The "up" solution cannot be described in the Roy-equation approach[137]; the model-dependent and energy-dependent analyses yield the "down" solution.[27,34,55,60]

Let us examine in more detail the structure at the $K\bar{K}$ (980) threshold. This structure has been found by essentially all experimentalists working in this region. The anomalous behavior of the S_0 wave at the $K\bar{K}$ threshold was discussed in most detail in Refs. 106, 161, and 162. The behavior in this region can be summarized as follows:

1. There is a sharp increase in the cross section for $K\bar{K}$ production, and this increase can also be seen in the sharp decrease in the elasticity η_0^0 from 1 to ~0.5,

$$\sigma_0^0(\pi\pi \to K\bar{K}) = 2\pi\lambda^2(1-\eta_0^0)^2. \tag{38}$$

2. The cross section $\sigma_{\pi^+\pi^-}$ decreases between 900 and 990 MeV, implying that the S_0 amplitude should be at the vertex of the Argand diagram, i.e., $\delta_0^0 \approx 90°$, at $m_{\pi\pi} \approx 900$ MeV.
3. The quantity $\langle Y_1^0 \rangle$ vanishes at the $K\bar{K}$ threshold,

TABLE III.

Authors	Year	S^* mass, MeV	S^* width, $\Gamma = 2\mathrm{Im}E_R$, MeV	Reference
B. Hyams et al.	1973	1007±20	30±10	33
S. D. Protopopescu et al.	1973	997±5	54±16	35
P. Estabrooks, A. D. Martin	1973	997	10	109
G. Grayer et al.	1973	1012±6	34±10	107
D. M. Binnie et al.	1973	987±7	48±14	161
M. Buttram et al.	1976	970±5	44±10	162
R. J. Leeper et al.	1977	969±5	30±8	41

implying that δ_0^0 is approximately equal to 180° at $m_{\pi\pi} = 2m_K$.

The fact that the phase shift δ_0^0 passes through a singularity (a multiple of 90°), on the one hand, and the resonant behavior of η_0^0, on the other, forces us to assume that there is some resonance state, an S^* meson, below the $K\bar{K}$ threshold or a virtual bound state above $m_{\pi\pi} = 2m_K$. This region was studied by Leeper *et al.*[41] with a large statistical base (~60 000 $\pi^+\pi^-$ events from the reaction $\pi^-p \to \pi^-\pi^+n$ and ~500 events from the reaction $\pi^-p \to K^+K^-n$). A sharp increase in the cross section for the reaction $\pi^-\pi^+ \to K^+K^-$ was clearly seen near the threshold, as was a sharp decay in the $\pi^+\pi^-$ mass spectrum near 950 MeV. This effect was analyzed in terms of the parameters of the S^* resonance with the help of both the ordinary Breit–Wigner formulas and the K-matrix formalism. The results found by the two methods were in agreement. The parameters of the S^* resonance from this and other studies are listed in Table III. Whether the scalar resonances (ε, S^*) actually exist has not yet been resolved.

As we will see below, δ_0^0 again passes through a singularity in the vicinity of the f resonance (the ε' singularity). The loop on the Argand diagram is not sufficient evidence for the definite conclusion that the ε' resonance exists, but the parameters of this feature have nevertheless been found in several studies. Carroll *et al.*,[60] for example, found $m_{\varepsilon'} = 1250 \pm 40$ MeV and $\Gamma_{\varepsilon'} = 300 \pm 100$ MeV. Morgan[164] has suggested that the passage of δ_0^0 through 90° near 900 MeV and through 270° near 1250 MeV may be evidence of a broad (Γ ~600 MeV), elastic ($\eta_0^0 \sim 1$)ε resonance with a mass ~1200 MeV, accompanying an $S^*(1000)$ meson. This question is discussed in detail in Ref. 17. Achasov *et al.*,[96] from the Institute of Mathematics, Novosibirsk, have recently suggested that S^* might be caused by a threshold effect (a "cusp"). In this case δ_0^0 for $\pi\pi$ scattering above the $K\bar{K}$ threshold does not increase sharp-

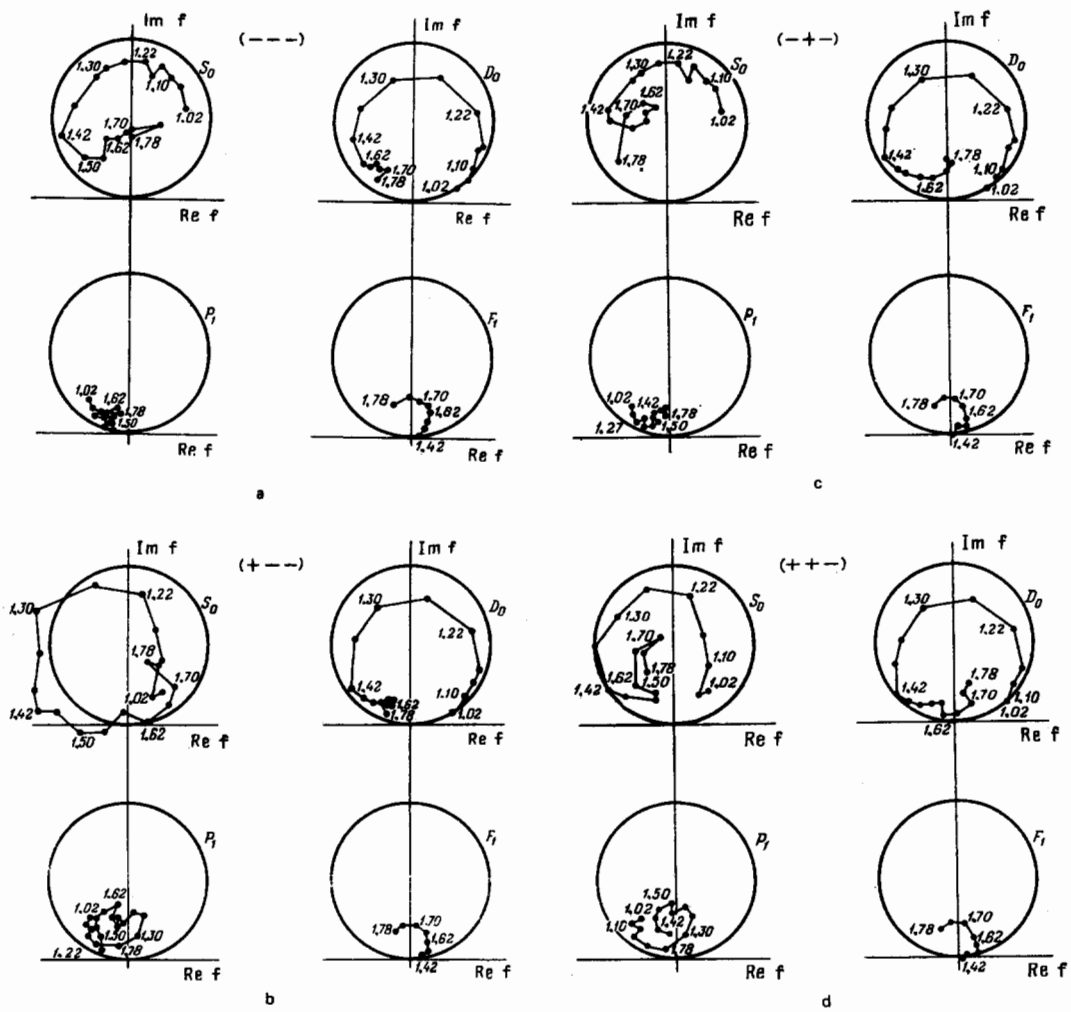

FIG. 23. Four possible versions of the behavior of the partial-wave amplitudes in the region $1 < m_{\pi\pi} < 1.8$ GeV from Ref. 36. a—The solution (---); b—the solution (+--); c—the solution (-+-); d—the solution (++-).

ly; instead it continues to remain near ~90°, having a spike at $m_{\pi\pi} = 2m_K$. A conclusive interpretation of the behavior of the S_0 wave in this region must await improvements in experimental accuracy.

c) The inelastic-interaction region. Choosing the actual solution.

Inelastic channels cannot be ignored above $m_{\pi\pi} = 1$ MeV. Experiments have shown that the reactions $\pi\pi \to K\bar{K}$ and $\pi\pi \to 4\pi$ both have rather large cross sections in this region and must be considered. As a result, the phase-shift analysis becomes much more complicated. It was mentioned in Section 6 that if the analysis is restricted to waves with $l \leq 3$ there will be $2^{I^{\max}} = 8$ possible versions of the phase-shift behavior: eight solutions. Half of them can be discarded if we assume that the D_0 and F_1 phase shifts are positive and are described by Breit–Wigner resonances (for proof, see, for example, Ref. 165). Then we are left with two possible solutions in the region of the f resonance and four in the region of the g resonance. These solutions were found by Hyams *et al.*[40] and are shown in Fig. 23. The solutions were classified on the basis of the spin of the imaginary part, $\mathrm{Im}Z_i(m_{\pi\pi})$ [see Eq. (31)], at $m_{\pi\pi} = 1500$ MeV. Since $\mathrm{Im}Z_3 < 0$ for $m_{\pi\pi} < 1.8$ GeV, there are four possibilities: (– – –), (+ – –), (– + –), and (+ + –). Data on the $\pi^- p \to \pi^-\pi^+ n$ reaction obtained by the CERN–Munich group were used in this study. At roughly the same time, Estabrooks and Martin[37c] reported results on the same reaction. They also found four possible solutions (A, B, C, and D), which differ from the solutions reported by Hyams *et al.* only in details. Estabrooks and Martin concluded that the correct solution could not be chosen on the sole basis of data on the $\pi^-\pi^+ \to \pi^-\pi^+$ reaction, but they did mention that solutions of the type (– – –) appeared slightly more promising. A reliable choice of solution requires data on other charge channels, primarily $\pi^+\pi^- \to \pi^0\pi^0$. As Shimada[165] has pointed out, comparison of the data obtained by the CERN–IHEP–Karlsruhe–Pisa–Vienna collaboration[50b] on the $\pi^- p \to \pi^0\pi^0 n$ reaction with the solutions of Hyams *et al.* and Estabrooks and Martin allows us to discard the solutions (– + –) and (+ + –)

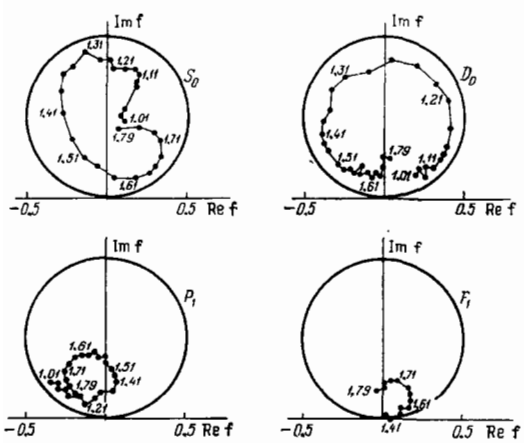

FIG. 24. Results of the energy-dependent pahse-shift analysis in the region $1 < m_{\pi\pi} < 1.8$ GeV from Ref. 36.

(*C* and *D*, respectively). The critical question for the two remaining solutions is the magnitude of the signal from $\rho'(1650) \to 2\pi$, which is visible in $B(+--)$ but not observed in $A(---)$.

In a study of the behavior of the $\pi\pi$-scattering phase shifts at $m_{\pi\pi} > 1$ GeV, Froggatt and Petersen[36] attempted to reduce the number of possible solutions by using a "formalism for amplitude analysis with fixed-t analyticity." This technique uses the requirements of analyticity and unitarity, and the experimental data are the coefficients of the Legendre polynomials for the reaction $\pi^-\pi^+ \to \pi^-\pi^+$ from the results of Ref. 39 in the s channel and the data of Hoogland *et al.*[47] for $\pi^+\pi^+ \to \pi^+\pi^+$ in the u channel, all extrapolated to the pion pole. It was concluded that the actual solution should contain the $\rho'(1600)$ resonance, since this is required by the pronounced forward-backward asymmetry of the imaginary part of the amplitude in the region 1500–1700 MeV. One solution was found which satisfied all the requirements and described the experimental data. It is a smoother version of solution B of Estabrooks and Martin and solution (+ – –) of Hyams *et al.*, and it contains a ρ' resonance with the parameters $m_{\rho'} = 1600 \pm 50$ MeV and $\Gamma_{\rho'} = 220 \pm 70$ MeV (Fig. 24). Among earlier studies, the values from the study by Protopopescu *et al.*[55] agree well with this solution.

As mentioned earlier, the ρ' meson can be seen in photoproduction[163] and also in e^+e^- experiments.[166] Alles-Borelli *et al.*[167] found the probability for decay through the $\rho' \to 2\pi$ channel from the reaction $\gamma + \mathrm{Be} \to \mathrm{Be} + \pi^+\pi^-\pi^+\pi^-$ to be $F \approx 10\%$. The solution shown in Fig. 24 requires $F = 35 \pm 10\%$. The experimental accuracy, however, is not good enough to definitely decide whether the photoproduction data agree with the results of the phase-shift analyses. Comparing the behavior of the phase shift for the solutions of Hyams *et al.* with data on the reaction[27] $\pi^-\pi^0 \to \pi^-\pi^0$, Shimada[165] concluded that the (+ – –) solution does not correspond to the experimental data. The discrepancies, however, do not seem all that convincing, in view of the low accuracy of the data of Ref. 27; we believe that Shimada's conclusion is premature. The magnitude of the S wave with respect to the known D wave was recently determined[168] from a study of the reaction $\pi^- p \to \pi^0\pi^0 n$ at $p_\pi = 25$ GeV/c. Two solutions were found. In one of them, there is a sharp violation of unitarity above 1200 MeV. The

TABLE IV. Resonances in the $\pi\pi$ channel.

State	$J^{PC}I^G$	Mass, MeV	Width, MeV	Probability for decay into 2π, %
ρ (750)	$1^{--}1^+$	776±3	158±5	≈ 100
f (1250)	$2^{++}0^+$	1273±5	178±20	83±2
g (1680)	$3^{--}1^+$	1690±20	180±30	24
h (2040)	$4^{++}0^+$	2040±20	193±50	Seen
S* (980)	$0^{++}0^+$	993±3	40±8	Seen
ε (900)	$0^{++}0^+$	The phase shift δ_0^0 goes through 90° at about 900 MeV and through 270° at about 1250 MeV with $\eta_0^0 \approx 1$. Interpretation not clear at this point.		
ε′ (1250)	$0^{++}0^+$			
f' (1514)	$2^{++}0^+$	1516±12	67±10	≈ 0*)
ρ′ (1600)	$1^{--}1^+$	1600	200–600	Seen to 4π

*Seen in the reaction $\pi^- p \to K^+K^- n$. Not observed in phase-shift analyses of $\pi\pi \to \pi\pi$.

second solution contains an S_0 wave which is nearly elastic up to $m_{\pi\pi}=1450$ MeV, with a resonance at about 1300 MeV. This solution is similar to the (– – –) solution, although there is hardly any firm basis for rejecting the (+ – –) version. At the moment we can evidently say that we can describe the behavior of the phase-shift curves only up to $m_{\pi\pi}\sim 1400$ MeV. Beyond this point there are several possible solutions, and at the moment we cannot determine which is correct.

Recent years have seen studies of the region of large $m_{\pi\pi}$ on the basis of the annihilation reaction[65,66] $\bar{p}p \to \pi\pi$. Dulude *et al.*,[66] for example, carried out an energy-dependent partial-wave analysis of the angular distributions of secondary pions in the reaction $\bar{p}p \to \pi^0\pi^0$ and found a resonance with $J^P=2^+$, $m=2.15$ GeV, and $\Gamma=250$ MeV. In a review of data on the reaction $\bar{p}p \to \pi^-\pi^+$, Martin and Morgan[169] noted that an amplitude analysis supported by the requirements of analyticity and crossing symmetry leads to the conclusion that resonances with spins $J=0$, 1, 2, and 3 and a width $\Gamma\sim 200$ MeV exist in the region $m_{\pi\pi}\approx 2.1$–2.3 GeV and that there are possible broad ($\Gamma\sim 400$ MeV) features with even spin $J=0$, 2, 4 at $m_{\pi\pi}\approx 2.43$ GeV. All this is of course based on the first results obtained at large dipion masses—results which have not been solidly confirmed as yet. Table IV summarizes the $\pi\pi$ resonances.[177]

9. CONCLUSION

Striking progress has been achieved recently in the physics of pion–pion interactions. The progress can be attributed to improvements in experimental apparatus, the improved accuracy of the experimental data (especially for the $\pi^-\pi^+ \to \pi^-\pi^+$ channel), the development of new phase-shift analysis methods, and the more general use of the principles of analyticity, unitarity, and crossing symmetry. We can now say that we know a lot about $\pi\pi$ scattering.

The question of scattering lengths has basically been resolved. It can evidently be asserted that we know the behavior of the phase shifts in the elastic region from the threshold up to $m_{\pi\pi}\approx 1$ GeV within ~15–20%. The basic characteristics are known, and we have the basis for choosing the correct solution in the very complicated region $1 < m_{\pi\pi} < 1.8$ GeV. Several resonant formations have been found; their characteristics are listed in Table IV. Study at high energies $m_{\pi\pi} > 2$ GeV has begun, and the first results have been obtained.

There are, of course, some unresolved questions. Among them are an accurate determination of the P-wave scattering length a_1^1, a correct interpretation of the anomalous feature in the S_0 wave near the $K\bar{K}$ threshold, and a final choice of the solution for $0.7 < m_{\pi\pi} < 1.8$ GeV. We would like to have more accurate values for the phase shifts and elasticities.

How can these problems be approached?

1. It is necessary to improve the reliability with which OPE can be distinguished from other mechanisms. If the phase shifts are to be determined within a few degrees, we must know the contributions of the background diagrams at various energies and how these contributions distort the results of the phase-shift analysis. This is a particularly important point for studying resonance-free $\pi\pi$ states, i.e., $\pi^-\pi^-$ and $\pi^+\pi^+$.

2. Experiments with polarized targets are required for reducing the uncertainties and for carrying out a model-independent analysis. Useful information can be obtained from study of reactions of the type $\pi N \to \pi\pi\Delta^{++}$, accompanied by a study of the angular distributions of the decay products of the Δ^{++} isobar. In principle, such a study can reveal the amplitudes with a definite nucleon helicity; i.e., experiments similar to experiments with polarized targets can be carried out. Such studies are already being carried out,[157,170] and they should evidently be expanded.

3. Sources of information other than the $\pi^-\pi^+ \to \pi^-\pi^+$ channel should be used more extensively. For example, the reaction $\pi^-\pi^+ \to \pi^0\pi^0$ can be used to determine the behavior of the δ_0^0 phase shift; the reaction $e^+e^- \to \pi^+\pi^-$ can be used to study the P_1 wave; accurate information on the channels $\pi^+\pi^0 \to \pi^+\pi^0$ and $\pi^+\pi^+ \to \pi^+\pi^+$ will make it possible to resolve the uncertainties in the solutions for the region

$$1 < m_{\pi\pi} < 1.8 \text{ GeV};$$

and the annihilation reactions $\bar{p}p \to \pi\pi$ are necessary for a study of the region of large dipion masses, $m_{\pi\pi} > 2$ GeV.

4. It would be extremely useful to study inelastic channels such as $\pi\pi \to 4\pi$, $\pi\pi \to 6\pi$, $\pi\pi \to K\bar{K}$, $\pi\pi \to \pi\omega$, etc. We have already said a lot about the importance of the channel $\pi\pi \to K\bar{K}$. Data on the process $\pi\pi \to \pi\omega$ may help explain the change in the parameter η_1^1 for the P_1 wave at $m_{\pi\pi}\sim 1100$ MeV. Study of the "multiple" channels $\pi\pi \to 4\pi$, $\pi\pi \to 6\pi$ is important because the results of the phase-shift analyses in the region $m_{\pi\pi}\sim 1$ GeV are not very stable with respect to the incorporation of these channels.[171] There is a particular need for information on inelastic $\pi\pi$-scattering channels in order to determine in which solutions unitarity is violated and how.

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