The uncertainty relation between energy and time of measurement

Yu. I. Vorontsov

Lomonosov State University, Moscow
Usp. Fiz. Nauk 133, 351-365 (February 1981)

Contrary to a wide-spread impression, the possibility of measuring an energy in a finite time without changing its initial value ($E' = E_0$) is not in contradiction with the principles of quantum mechanics. The relation $\Delta(E'-E_0)\Delta t \geq \hbar$ holds only in the case when the energy of interaction between the quantum system in question and the apparatus is a function of a coordinate of the system. The condition for a nonperturbing energy measurement is that the interaction energy H_I of the system and the apparatus depend on the energy operator $\hat{E}$ and that the operators $\hat{H}_I$ and $\hat{E}$ commute. It is also possible to have a nonperturbing measurement in which the error in measuring the energy is so small that $\Delta E \ll \hbar/\Delta t$. Measurement of the energy of a given system is accompanied by an increase in the uncertainty $\Delta\varepsilon$ of the energy of the apparatus. The error ΔE in the measurement of the system's energy and the perturbation $\Delta\varepsilon$ of the energy of the apparatus are connected by the relations $(\Delta E + \Delta\varepsilon)\cdot\Delta t \geq \hbar$ and $\Delta E \cdot \Delta\varepsilon \geq (\hbar/2\Delta t)^2$.

PACS numbers: 03.65.Bz

CONTENTS

1. DIFFERENCES OF OPINION ABOUT THE FUNDAMENTAL CHARACTER OF THE HEISENBERG-BOHR RELATION

There is a conviction among most physicists that the uncertainty $\Delta(E' - E_0)$ of the change in the energy of a system during a measurement is connected with the duration Δt of the measurement by the relation

$$\Delta (E' - E_0)\cdot \Delta t \geqslant \hbar, \tag{1}$$

which is called the Bohr relation or the Heisenberg-Bohr relation.[1-4] (E_0 and E' are the energies of the system before and after the measurement.) The relation (1) has the following fundamental consequences:

a) The energy can be measured without changing its value only to the accuracy

$$\Delta E > \frac{\hbar}{\Delta t}. \tag{1a}$$

Consequently, the law of conservation of energy can be verified only to this sort of accuracy.

b) In a measurement of the momentum of a free particle there is an unavoidable change of its velocity such that

$$(v' - v_0)\cdot \Delta p\cdot \Delta t \geqslant \hbar \tag{1b}$$

[Δp is the error in the measurement of the momentum, and $v' - v_0$ is the known change of the velocity of the particle during the momentum measurement].

The possibility of exact measurement in a finite time of E' alone or of E_0 alone is affirmed in Refs. 1–3 and denied in Ref. 4.

There is, however, no united opinion about the fundamental character of the relation (1) among specialists concerned with the quantum theory of measurements. The reason for the existence of different opinions about the Heisenberg-Bohr relation is that it has not been possible to establish it as unambiguously as the uncertainty relation between coordinate and momentum. Since Eq. (1) relates to an act of measurement, it cannot be a consequence of Schrödinger's equation, and requires for its justification the use of some further arguments associated with the quantum theory of measurements, which is as yet far from being perfected. In Ref. 4 the relation (1) is regarded as a physical principle which can be confirmed only through the analysis of examples of conceptual experiments. [A comparison of various views about the meaning of Eq. (1) is given in Ref. 5.] The relation (1) must be distinguished from relations of a different type between energy and time, which relate the half-life for decay of an almost stationary state and the width of the level,[4] or the time of displacement of a wave packet and the energy uncertainty in the free evolution of a system.[5] These are not associated with measurement and have rigorous proofs. On the other hand, the derivation of Eq. (1) in Refs. 1–4 rests on a single example of a collision of two particles. In 1961 Aharonov and Bohm[6] proposed an example which, in their opinion, disproved the Heisenberg-Bohr relation. After a categorical denial of this by Fock[7] the authors of Ref. 6 gave further, but still not completely convincing, arguments in defense of their position.[8] The result was that Fock came to the conclusion that the propositions of Bohm and Araronov are in contradiction with the principles of quantum mechanics, and that nonacceptance of the relation (1) is equivalent to denial of the whole of quantum mechanics.[9] In a review article that appeared in 1968,[10] however, it was again asserted that the question of the energy-time uncertainty relation is still not resolved finally and that the correct interpretation remains to be found. Another article[11] gave the example of a conceptual experiment showing that the energy of a conservative system can be measured without changing its value with an error not limited by the relation (1). It was also shown there that for a nonperturbing measurement, i.e., one that does not change the value, the error ΔE is connected with the increase of the energy uncertainty $\Delta\varepsilon$ of the quantum read-out system (QRS) by the relations

$$(\Delta E + \Delta\varepsilon)\cdot\Delta t \geqslant \hbar, \quad \text{(a)} \qquad \Delta E\cdot\Delta\varepsilon \geqslant \left(\frac{\hbar}{2\Delta t}\right)^2. \quad \text{(b)} \tag{2}$$

[The QRS is designated as the quantum link in the indirect measurement scheme. The numerical factors in Eq. (2) are determined approximately.]

The present article takes the positions of the present theory of quantum measurements[12] and gives a critical analysis of the treatment of the relation (1) in Refs. 1–4, 7, points out weaknesses and mistakes in the demonstrations in Refs. 6, 8, and gives a comprehensive justification of the propositions stated in Ref. 11, with examples of various schemes of nonperturbing energy measurements.

2. CRITICAL ANALYSIS OF THE FOUNDATIONS OF THE HEISENBERG-BOHR RELATION

The process of a quantum measurement requires that one of the links of the measuring system be classical (more precisely, quasiclassical); that is, it must be such as to accomplish a "dequantization" of the signal, so that thereafter the quantum uncertainties in other links do not affect the accuracy of the measurement. If the quantum system being investigated interacts directly with the classical link, the procedure is called a direct measurement. When there is a quantum link (a quantum read-out systems or QRS) between the system and the classical link, the measurement is called indirect.[13] If the QRS does not interact with the classical link during the time it interacts with the system under investigation, the latter interaction obeys the Schrödinger equation and is reversible. A direct measurement is irreversible, and its description requires the use of a special mathematical apparatus.[14] Not one of the works on the foundations of the Heisenberg-Bohr relation, to which we have referred, treats consistently the entire scheme of a measurement. All that is analyzed is the process of interaction with the QRS. The only sort of QRS considered is a free particle; the change of its momentum (or energy) in a collision with the particle under study is considered. The change of the probe particle's momentum (or energy) is calculated, and from this change it is proposed to determine the momentum (or energy) of the original particle. The result of the investigation of this single example, even though conducted in various ways, cannot be a proof of the fundamental relation (1). Besides this, the approach to the problem is itself not entirely consistent, since conversion of momentum of one particle into momentum of another particle is still not a real measurement of momentum.

In the quantum theory of measurement it is well known that the error of a measurement depends on principle on the nature of the interaction of the apparatus with the system and on the structure of the apparatus.[12] The Hamiltonian must be a function only of the operator $\hat{A}$ whose characteristic values we wish to measure. If it indeed depends on an operator which does not commute with $\hat{A}$, then naturally an exact measurement of $\hat{A}$ will be impossible. Moreover, as will be shown below, even with a suitable choice of the interaction Hamiltonian the precision of the measurement may be limited owing to incorrect choice of the QRS and a nonoptimal direct measurement.

In the case when the interaction Hamiltonian is a function of a coordinate operator of the system (the only sort of interaction regarded as possible in Ref. 1) the effect on the apparatus will be determined by the coordinate system. Consequently, the directly measured quantity will be a coordinate. The energy (or momentum) of the system can be calculated only in terms of coordinate values at different times. The momentum of the system must be determined via the velocity, which is defined as $v=[x(t+\Delta t)-x(t)]/\Delta t$. Because the Heisenberg operators $x(t+\Delta t)$ and $x(t)$ do not commute, the quantity v cannote be determined exactly. In the case of a free particle $[x(t+\Delta t), x(t)] = i\hbar\cdot\Delta t/m$, and consequently[15]

$$\Delta v \geqslant \sqrt{\frac{\hbar}{m\cdot\Delta t}}. \tag{3}$$

The energy uncertainty corresponding to such an uncertainty of the velocity of a free particle is

$$\Delta E \geqslant \frac{m\,(\Delta v)^2}{2} \geqslant \frac{\hbar}{2\Delta t}. \tag{3a}$$

In this case the relation (1) is satisfied for any value of the average velocity; i.e., instead of Eq. (1b) we have $\Delta v\cdot\Delta p\cdot\Delta t > \hbar$.

A violation of the relation (1) can occur only in experiments in which the interaction Hamiltonian depends on the energy of the system (or on the momentum in the case of a free particle).[1,14] The result of the analysis of the interaction of particles in Refs. 1–3 was predetermined in advance, since the interaction Hamiltonian was regarded in those papers as a function of a coordinate of the particle.

The collision of an object particle with a test particle was considered in Ref. 4 from a somewhat different standpoint. Starting from the fact that in a measurement of momentum the increase of the coordinate's uncertainty must satisfy the uncertainty relation $\Delta x\cdot\Delta p$

$> \hbar/2$, Krylov and Fock arrived, as in Ref. 1, at the relation (1b).

The consequences of Eq. (1b) do not seem entirely logical. Starting from this relation, the authors consider that the momentum (in the nonrelativistic region) "can be measured arbitrarily quickly, the measurement being accompanied by a large increase of the momentum, which indeed is subject to control" (Ref. 4). The analogous conclusion that exact energy measurement is possible is regarded as incorrect, however, since $\Delta E' = v' \cdot \Delta p$, and v' increases as Δp is made smaller.

On the other hand, only the process of conversion of momentum of the object particle into momentum of the test particle is considered, and not the process of obtaining information about the momentum. The question again arises: Can one now measure the momentum, that of the test particle, rapdily and exactly? For example, in the case when the probe particles are light quanta the error in measuring the momentum with spectral devices is inversely proportional to the time of measurement. On the other hand, if in some way one can measure the momentum of the object particle accurately and quickly, then the conclusion that accurate and rapid measurement of the energy is impossible is incorrect, since the measuring system can be supplemented with a device which, after the momentum measurement, will compensate the known change of the velocity of the object in a time smaller than Δt.

3. THE DISCUSSION OF V. A. FOCK WITH Y. AHARONOV AND D. BOHM

The starting point for Aharonov and Bohm[6] was as follows. We suppose that the Hamiltonian of a system consisting of two interacting particles can be given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + \hat{y}\hat{p}g(t) + \frac{\hat{p}_y^2}{2M}, \tag{4}$$

where $\hat{p}, \hat{x}$ are the momentum and coordinate operators of the object particle, and $\hat{p}_y, \hat{y}$ are the momentum and coordinate operators of the test particle. (The motion is regarded as one-dimensional.) $g(t)$ is different from zero only in the interval from t_0 to $t_0 + \Delta t$. Then $\hat{p}(t)$ is constant over the entire range of time, despite the turning on $[g(t) \neq 0]$ and off $[g(t) = 0]$ of the interaction between the particles. However, the velocity operator of the particle, which in this case is given by

$$\dot{\hat{x}} = \frac{\hat{p}}{m} + \hat{y}g(t), \tag{5}$$

changes when the coupling is turned on and off. Meanwhile, both before and after the interaction we have $\dot{\hat{x}} = \hat{p}/m$. Consequently, if $\hat{p}$ can be measured exactly at the instant of interaction, then we can determine exactly the energy that the object particle has before and after the interaction with the test particle.

The momentum of the test particle changes proportional to $\hat{p}$ during the interaction:

$$\dot{\hat{p}}_y = -\hat{p} \cdot g(t) \tag{6}$$

and its change during the time Δt is

$$\hat{p}_y - \hat{p}_y^0 = -\hat{p}g \cdot \Delta t. \tag{7}$$

It is further assumed in Ref. 6 that p_y can be measured exactly, and the uncertainty $\Delta(p_y - p_y^0)$ is taken to be equal to just the original uncertainty Δp_y^0. Then from Eq. (7) one gets $\Delta p \approx p_y^0/g \cdot \Delta t$. Consequently, for $g \to \infty$ one has $\Delta p \to 0$.

V. A. Fock did not agree with this proof.[7] He thought that there was a logical mistake in Ref. 6 because of the assumption that g has a definite value during the short time Δt; that is, a proposition that still requires proof has been taken as fact and used in the demonstration. In their answer[8] to Fock, Aharonov and Bohm asserted that if the coupling is determined by the motion of an auxiliary body of large mass, then this body's coordinate (z) and its velocity can be fixed arbitrarily accurately. Then the coefficient g will be an explicit function of the coordinate z, and different from zero only in some range from z_0 to $z_0 + \Delta z$. The weakness of this argument is as follows: It is logical to suppose that the coupling will depend not on z, but on differences of the coordinates of the object particle, x, the test particle, y, and the auxiliary body, z, i.e., $g = g(x, y, z)$. But then $\hat{p}$ is not a constant, since $\hat{H}$ depends on $\hat{x}$.

Nevertheless, V. A. Fock's assertion that there is a logical mistake in Ref. 6 is not a justification for rejecting the conclusions of Aharonov and Bohm. Even if the relation (1) applies to $g(t)$, it affects only the absolute error. The fractional error can be arbitrarily small. Therefore for $g \to \infty$ the uncertainty of the quantity g will be unimportant.

In analyzing the error of measurement of the quantity $E = \frac{1}{2}m\dot{x}^2$ in the example of Ref. 6, Fock does not rely on the indefiniteness of $g(t)$, and considers E not before or after the interaction but during the interaction; he shows that during this time the uncertainty of E satisfies the relation (1). This result comes from the fact that for $g(t) \neq 0$ the quantity $\dot{x}$, even with fixed p, is undetermined by the amount $\Delta y \cdot g \geq \frac{1}{2}\hbar g/\Delta p_y$, where Δy is the uncertainty of $y(t)$ during the interaction. Since the error in the measurement of p is $\Delta p_y/g \cdot \Delta t$, the uncertainty of $\dot{x}$ is actually

$$\Delta\dot{x} \geq \sqrt{\left(\frac{\Delta p_y}{m \cdot g \cdot \Delta t}\right)^2 + \left(\frac{\hbar g}{2\Delta p_y}\right)^2} \geq \sqrt{\frac{\hbar}{m\Delta t}}. \tag{8}$$

But this is the uncertainty of $\dot{x}$ at the moment of interaction. What we are interested in is the value of $\dot{x}$ before and after the interaction. In this case the uncertainty of $\dot{x}$ during the interaction is immaterial and can be arbitrarily large. Nevertheless, after the interaction the uncertainty of $\dot{x}$ will be equal to only the error of the measurement of the quantity p/m. The proposition requiring proof in Ref. 6, that an exact measurement of p_y can be made in a finite time is not necessary. Let us assume that the change of the test particle's momentum is determined by measuring its coordinate and is therefore known with error $\Delta p_y = (\hbar M/\Delta t)^{1/2}$. Then the momentum p will be determined to accuracy $\Delta p = (1/g\Delta t)(\hbar M/\Delta t)^{1/2}$. In the case $g \to \infty$ we have $\Delta p \to 0$. Aharonov and Bohm did not examine the process of direct measurement, and this was one of the points of Fock's criticism of their position.[9] However, to determine the error of the measurement of p it is sufficient to note that as the result of a direct measurement of the coordinate of the QRS with accuracy to Δy there is an increase of the uncertainty of the momentum of the QRS by an amount $\Delta p_y \geq \hbar/2\Delta y$. Inasmuch as in the case of the Hamiltonian (4) the changes of the operators $\hat{y}(t)$ and $\hat{p}_y(t)$ do not affect the evolution of the operator $\hat{p}(t)$, the interaction of the QRS with the classical link will not change $p(t)$. In the measurement process only a reduction of the values of the system's momentum occurs. If the system was in a state with a definite momentum before the measurement, then it will remain in that state after the measurement. If, on the other hand, the initial state of the system was arbitrary, then after the measurement it goes into a state with a certain momentum, which is one of the possible momentum values in the original state.

The most important point of principle in these papers is the assumption that systems with Hamiltonians of the form (4) can be realized physically. For some reason, during the discussion of Fock with Aharonov and Bohm attention was never called to the fact that accepting this Hamiltonian, with any nondeterminacy of the quantity $g(t)$, leads to the conclusion that momentum can be measured without changing the velocity, which contradicts the relation (1b). It must be emphasized that the result of the formal analysis in the papers of Aharonov and Bohm was not unexpected. In the very first paper devoted to the justification of the Bohr relation,[1] Landau and Peierls pointed it out as obvious that if there were in nature possible interactions for which the Hamiltonian would commute with the momentum operator, then it would be possible to measure the momentum exactly and quickly without changing the velocity. For reasons that are not clear, however, Landau and Peierls considered such an interaction impossible. Accordingly, the analysis carried out by Aharonov and Bohm will have meaning only in case the realizability of the Hamiltonian (4) is established.

Aharonov and Bohm supposed that the Hamiltonian (4) corresponds to a system of two interacting free particles. This is not so, however; the expression (4) is not positive definite, i.e., it does not describe a closed system. As a conceptual experiment, Ref. 6 considers the measurement of one of the momentum components of a charged particle by means of a freely moving massive charged plane-plate condenser. The momentum of the particle is to be measured by observing the change in the mechanical energy of the condenser. There is no need to describe this experiment in detail, since its inadequacy is quite evident; the energy of the interaction of a charged particle with an electrostatic field is a function of the particle's coordinates. Consequently, the Hamiltonian of this system fails in principle to correspond to the expression (4). (It is not written out in Ref. 6.) The disagreement between the result of the calculation made in Ref. 6 and the relation (1) is due to mistakes in the calculation; the effects of the field at the edge of the condenser on the motion of the particle and the displacement of the condenser during the interaction are not taken into account. Accordingly, the realizability of the Hamiltonian (4), and therefore the practical significance of the formal analysis in Refs. 6 and 8, have not been properly discussed. The articles by Aharonov and Bohm have, however, raised the question of a critical reevaluation of the generally accepted treatment of the Heisenberg-Bohr relation, and some of the ideas they expressed have helped, at a certain stage of the development of the theory of nonperturbing quantum measurements[12] in making a new approach to the solution of this problem.

4. SCHEME OF A NONPERTURBING MEASUREMENT OF ENERGY

An example of a scheme whose Hamiltonian satisfies the condition for a nonperturbing measurement of energy (momentum) is shown in Fig. 1.[11] This is a scheme for measuring the energy of the current in a short-circuited superconducting coil L. Beside the coil is placed a loop which can be rotated freely around its axis. The loop serves as the QRS; its motion depends on the field of the coil. The current $i(t)$ is supplied by a source with a high enough internal resistance that the influence of the field of the coil on $i(t)$ can be neglected.

Taking as generalized coordinate the charge q that has flowed through some cross section in the coil and the angle φ of rotation of the loop, we find as the Lagrangian of the scheme

$$\mathcal{L} = \frac{1}{2} L\dot{q}^2 - M(\varphi)\cdot i(t)\cdot \dot{q} + \frac{1}{2} I\dot{\varphi}^2 \tag{9}$$

[I is the moment of the inertia of the loop, and $M(\varphi)$ is the coefficient of mutual inductance]. Accordingly, the generalized momenta are

$$p_q \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}} = L\dot{q} - M(\varphi)\cdot i(t), \quad p_\varphi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = I\dot{\varphi}, \tag{10}$$

and the Hamiltonian operator is

$$\hat{H} = \frac{\hat{p}_q^2}{2L} + \hat{p}_q\hat{\varphi}g(t) + \frac{\hat{p}_\varphi^2}{2I} + \frac{Lg^2(t)\hat{\varphi}^2}{2}. \tag{11}$$

Here we have used the approximation $M(\varphi) = M_0\varphi$, $\varphi \ll \pi$ and the abbreviation $g(t) \equiv M_0 i(t)/L$; $\hat{p}_q, \hat{p}_\varphi, \hat{\varphi}$ are momentum and coordinate operators. The Hamiltonian (11) differs from (4) by the term $Lg^2(t)\hat{\varphi}^2/2$, which corresponds to the introduction of a stiffness $Lg^2(t)/2$ in the motion of the loop, but nevertheless satisfies the conditions for a nonperturbing measurement of the momentum $\hat{p}_q$, since $\hat{H}$ does not depend on the coordinate operator $\hat{q}$, and the interaction Hamiltonian is proportional to $\hat{p}_q$. To eliminate the last term in Eq. (11) the loop should be connected to a negative stiffness $-Lg^2$. There is no need for this, however.

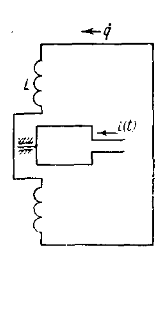

FIG. 1.

Using Eq. (11), we obtain the Heisenberg equations

$$\dot{\hat{p}}_q(t)=0 \quad (a), \qquad \dot{\hat{q}}(t)=\frac{\hat{p}_q(t)}{L}+\hat{\varphi}(t)\cdot g(t) \quad (b),$$
$$\dot{\hat{p}}_\varphi(t)=-Lg^2(t)\,\hat{\varphi}(t)-\hat{p}_q(t)\,g(t) \quad (c), \qquad \dot{\hat{\varphi}}(t)=\frac{\hat{p}_\varphi(t)}{I} \quad (d). \tag{12}$$

Consequently, the momentum $\hat{p}_q$ does not change when the current in the loop is turned on or off. During the time of interaction with the QRS the current $\dot{q}$ in the coil changes and is not definite because of the uncertainty of the quantity φ. After the coupling is turned off [$g(t)=0$], however, the value of $\dot{q}$ that existed before the interaction with the QRS is reestablished. The constancy of p_q and the restoration of the value of $\dot{q}$ after the interaction are reflections of the well known phenomenon of the conservation of magnetic flux through a superconducting ring. A scheme similar to this one can be used to measure the energy of a rotating charged dielectric ring. In this case also the generalized momentum

$$p_q=\left(\frac{m}{\rho^2}+L\right)\dot{q}-M(\varphi)\cdot i(t)$$

($\dot{q}=l\rho\dot{\alpha}$, m is the mass, l is the radius of the ring, ρ is the linear density of the charge, and α is the angular velocity of the rotation) will remain constant, but the magnetic flux through the ring will no longer be an integral of the motion in this case.

Accordingly, by observing the change of φ during the time of the interaction one can determine p_q, $\dot{q}$, and the energy E of the coil before and after the interaction with the QRS. After such a measurement the velocity and the energy retain their original values, i.e., $\dot{q}'-\dot{q}_0=0$ and $E'-E_0=0$. The relations (1) and (1b) are violated in this case.

Let us find the accuracy of the measurement of p_q through measurement of some operator of the QRS. Equations (12c) and (12d) are analogous to the equations of motion of an oscillator with the stiffness $Lg^2(t)$ acted on by a force $F=-p_q\,g(t)$. If during the interval Δt the quantity $g(t)$ is constant and equal to g, one can obtain information about the value of p_q by measuring the mean value over a period of the angle φ, which is $\varphi_0=-p_q/Lg$. This quantity can be measured in two ways: 1) By continuous measurement of the coordinate $\varphi(t)$; 2) by measurement of the instantaneous coordinate at two times differing by an odd number of half-periods of the vibrations (the stroboscopic method).[16,17] In the first case the limiting accuracy of the measurement of φ_0 corresponds to[18]

$$\Delta\varphi_0=\sqrt{\frac{\hbar}{I\omega^3\Delta t}}=\sqrt{\frac{\hbar}{Lg^2\Delta t}} \tag{13}$$

[$\omega=(Lg^2/I)^{1/2}$ is the natural frequency of the operator, $\omega\cdot\Delta t\gg 1$]. Consequently, $\Delta p_q\geqslant(\hbar L/\Delta t)^{1/2}$, $\Delta E\geqslant\hbar/2\Delta t$. In this way we again arrive at the relation (1).

The stroboscopic method of measuring the mean value over a period is as follows: The Heisenberg operator for the coordinate of a harmonic oscillator depends on the time by the law

$$\hat{\varphi}(t)=\varphi_0+\hat{\varphi}(0)\cos\omega t+\frac{1}{I\omega}\cdot\hat{p}_\varphi(0)\sin\omega t, \tag{14}$$

and

$$[\hat{\varphi}(t),\hat{\varphi}(t+\tau)]=\frac{i\hbar}{I\omega}\sin\omega\tau. \tag{15}$$

Consequently, the perturbation of the momentum by the measurement of the coordinate at the time t does not affect the value of the coordinate at times $t_k=(k\pi/\omega)+t$ (k is an integer). Therefore the sum of the values of $\varphi(t)$ measured at the times t and $t+(\pi/\omega)$ will be equal to $2\varphi_0$. If the value of the parameter ω is known exactly, the accuracy of the measurement of φ_0 will be limited only by relativistic effects, which affect the error of measurement of an instantaneous coordinate and also the vibration period. In our case the definiteness of ω depends on that of g. Let us suppose that the quantity g can be determined during a time Δt only to within an error Δg. Then the uncertainty of the vibration period of the loop will be $\Delta T=(\pi\Delta g/g^2)\times(I/L)^{1/2}$. The error in the determination of φ_0 owing to ΔT will be

$$\Delta\varphi_0=\sqrt{(\Delta\varphi_1)^2+(p_\varphi\cdot\Delta T/I)^2}\geqslant\sqrt{(\Delta\varphi_1)^2+\left(\frac{\hbar}{2\Delta\varphi_1}\,\frac{\Delta T}{I}\right)^2}\geqslant\sqrt{\frac{\pi\hbar\Delta g}{\sqrt{LI}\,g^2}}, \tag{16}$$

where $\Delta\varphi_1$ is the uncertainty of the instantaneous coordinate after its first measurement. Since $\Delta p_q=Lg\cdot\Delta\varphi_0$, we have

$$\Delta p_q\geqslant\sqrt{\frac{L\hbar}{\Delta t}\cdot\frac{\pi^2\Delta g}{g}}, \tag{17}$$

where $\Delta t=T/2=\pi(I/Lg^2)^{1/2}$ is the time of measurement of the momentum p_q.

Since the quantity $\Delta g/g$ can be arbitrarily small, the error of measurement of the momentum p_q can be much smaller than $(L\hbar/\Delta t)^{1/2}$. Correspondingly, the error in the measurement of the energy of the coil before and after the interaction, $\Delta E=q\cdot\Delta p_q+(\Delta p_q)^2/2L$, can be smaller than $\hbar/\Delta t$. Besides this, if before the measurement the system was in a state with a given energy, then also after the measurement it is in the same state [$\Delta(E'-E_0)=0$]. In this sense this measurement of the energy is nonperturbing.

5. CONNECTION BETWEEN THE ERROR OF THE MEASUREMENT, THE DURATION OF THE MEASUREMENT, AND THE UNCERTAINTY OF THE ENERGY OF THE APPARATUS

A comparison of the results of the two methods for measuring φ_0 shows that the condition that the interaction Hamiltonian commute with the energy operator of the system is the necessary and sufficient condition for a nonperturbing measurement, but is not sufficient to allow the accomplishment of a nonperturbing measurement with an error $\Delta E<\hbar/\Delta t$. A direct measurement in the QRS must be such that the action by the system on the QRS could be measured with a prescribed accuracy in a minimal time. Moreover, the operator measured in the QRS must be chosen so that through it one can unambiguously determine the momentum (energy) of the system. If, for example, in the scheme we have discussed we measure not φ_0 but the energy of the QRS, then it will be impossible to determine the momentum p_q, since the change of the energy of the QRS depends not only on p_q, but also on the random initial phase of the oscillations of the loop.

The process of measuring φ_0 by continuous observation of the coordinate φ differs from that of stroboscopic measurement in the value $\Delta\varepsilon$ of the random change of the energy of the QRS. In the case of continuous measurement, during a time $\Delta t \gg \omega^{-1}$ there will be a change $\Delta\varepsilon \geq \frac{1}{2} I\omega^2(\Delta\varphi_0)^2 \geq \hbar/2\Delta t$. In the stroboscopic measurement the error of the measurement of φ_0 is $\Delta\varphi_0 = \frac{1}{2}[(\Delta\varphi_1)^2 + (\Delta\varphi_2)^2]^{1/2}$ [$\Delta\varphi_1$, $\Delta\varphi_2$ are the errors in measuring the instantaneous coordinates at the times t and $t + (T/2)$]. The uncertainty of the measurement of the momentum of the QRS will be

$$\Delta p_\varphi \geqslant \frac{\hbar}{2\Delta\varphi_1}, \tag{18}$$

after the first measurement of φ. If $\Delta\varphi_1 = \Delta\varphi_2$, then

$$\Delta\varepsilon \geqslant (\Delta p_\varphi)^2/2I \geqslant \frac{\hbar^2}{16I(\Delta\varphi_0)^2}. \tag{19}$$

Noting that $\Delta\varphi_0 = \Delta p_q/Lg$, $\Delta E \geq (\Delta p_q)^2/2L$, $\Delta t = \pi(I/Lg^2)^{1/2}$, we have from Eq. (19)

$$\Delta\varepsilon \geqslant \frac{\pi^2}{8} \cdot \frac{\hbar^2}{(2\Delta t)^2} \cdot \frac{1}{E}. \tag{20}$$

The numerical factor in Eq. (20) depends on the relation between $\Delta\varphi_1$ and $\Delta\varphi_2$. When $\Delta\varphi_2 \ll \Delta\varphi_1$, it will be $\pi^2/16$ instead of $\pi^2/8$. Accordingly, in the stroboscopic measurement of φ_0 the perturbation of the energy of the QRS is inversely proportional to the error ΔE of the energy measurement. This calculation is one of the foundations of the relation (2); in both ways of measuring φ_0 the relations $(\Delta E + \Delta\varepsilon) \geq (\hbar/2\Delta t)^2$ hold. [The numerical factors in Eq. (2) are approximate.]

We shall make an analogous analysis of a somewhat different measurement scheme, in which the loop is not free, but connected to a negative stiffness $-Lg^2$. In this case the Hamiltonian of the scheme is analogous to Eq. (4):

$$\hat{H} = \frac{\hat{p}_q^2}{2L} + \hat{\varphi}\hat{p}_q g(t) + \hat{p}_\varphi^2/2I. \tag{21}$$

During the interaction with the field of the coil the loop will behave as if it were free under the action of the torque $\hat{p}_q \cdot g$. By measuring the instantaneous values of φ twice, at an interval Δt with precision corresponding to $(\Delta\varphi)^2 = \hbar\Delta t/2I$, we determine the momentum p_q with the smallest possible error[15] $\Delta p_q \geq (1/g)[4I\hbar/(\Delta t)^3]^{1/2}$, which corresponds to $\Delta E \geq 2I\hbar/Lg^2(\Delta t)^3$. By increasing g, we can find p_q to arbitrarily high accuracy with a given value of Δt.

Measurement of φ with accuracy to $\Delta\varphi = [\hbar\Delta t/2I]^{1/2}$ increases the momentum uncertainty of the QRS by the amount $\Delta p_\varphi \geq (\hbar I/2\Delta t)^{1/2}$, which corresponds to a change of the kinetic energy of the loop by $\Delta\varepsilon_k \geq \hbar/4\Delta t$. The uncertainty of the potential energy of the loop in the field, caused by the negative stiffness $-Lg^2$, is $\Delta\varepsilon_p \geq Lg^2\hbar\Delta t/4I$. Consequently, the total change of the energy of the QRS, as in the examples considered before, is connected with ΔE by the relations (2).

A general demonstration of the relations (2) for the case of measurement of the momentum of a free particle in one-dimensional motion can be given as follows: Measurement of the momentum to precision Δp must be accompanied by an increase of the uncertainty of the coordinate by the amount $\Delta x \geq \hbar/2\Delta p$ [otherwise it would be possible to prepare a state with $\Delta x \cdot \Delta p < \hbar/2$]. The increase of Δx is a consequence of motion with velocity uncertain by $\Delta\dot{x}(t)$ during the time Δt of the momentum measurement. Putting the dependence of Δx on $\Delta\dot{x}(t)$ in the form $\Delta x = \gamma \cdot \Delta\dot{x}_m \cdot \Delta t$, where $\Delta\dot{x}_m$ is the maximum value of the uncertainty $\Delta\dot{x}(t)$ in the interval Δt, and γ is a numerical factor smaller than unity, we obtain

$$\gamma \cdot \Delta\dot{x}_m \cdot \Delta t \geqslant \frac{\hbar}{2\Delta p} \tag{22}$$

or

$$\frac{m(\Delta\dot{x}_m)^2}{2} \cdot \Delta E \geqslant \frac{\hbar^2}{4\gamma^2(2\Delta t)^2}. \tag{22a}$$

But $m(\Delta\dot{x}_m)^2/2$ is the uncertainty of the change of the energy of the system under investigation during the measurement process. Since the energy in the system comes from the QRS, the uncertainty $\Delta\varepsilon$ of the energy of the QRS will not be smaller than $\frac{1}{2}m(\Delta\dot{x}_m)^2$. Consequently we obtain from Eq. (22a)

$$\Delta\varepsilon \cdot \Delta E \geqslant \frac{\hbar^2}{4\gamma^2(2\Delta t)^2}. \tag{23}$$

If $\Delta\dot{x}(t)$ depends on time in the interval Δt almost according to a harmonic law, then we have $\gamma \approx 2/\pi$. Then the numerical factor in Eq. (23) will be half as large as that in Eq. (20). This value corresponds to a type of stroboscopic measurement of φ_0 in which $\Delta\varphi_2 \ll \Delta\varphi_1$.

In this we have taken $\Delta\varepsilon$ to mean the uncertainty of the energy of the QRS during the process of measurement, in particular, before the second measurement of the value of φ in the stroboscopic method. What will the uncertainty of the energy of the QRS be like after the measurement? If φ_0 is determined by direct measurement of φ, then each measurement of φ will increase the uncertainty of the energy of the QRS. Then after the determination of p_q it will be larger than $\Delta\varepsilon$. By changing the scheme of the measurement we can get information about φ_0 without finding instantaneous values of φ. To do so we must introduce another quantum link, which interacts with the loop in two pulses at an interval $T/2$. For example, if a short electromagnetic pulse is first reflected from the loop, then from a stationary mirror, and then from the loop again, it will then carry information not about an instantaneous value of the coordinate, but about a sum $\varphi(t) + \varphi(t + \Delta t)$. When $\Delta t = T/2$, this is $\varphi(t) + \varphi(t + \frac{1}{2}T) = 2\varphi_0$. Since such a measurement does not give information about the state of the oscillator, it will not perturb it. In the present example the perturbation of the mechanical momentum of the loop in the first reflection will be cancelled in the second, since after the time $T/2$ the momentum of the QRS will have the opposite sign. (The degree of cancellation will be the greater, the smaller the ratio of the linear velocity v_φ of the loop to the speed of light c_0.) To get $\Delta t = T/2$, one must place the mirror at a distance $l = c_0 T/4$ from the equilibrium position of the loop. In our case, however, the equilibrium coordinate value x_0 is the unknown quantity. However, if $x_0 \ll 1$, then by laying off the distance l from the point corresponding to $\varphi = 0$ we obtain information about the value of φ_0 with accuracy up to terms of order v_φ/c_0.

After such a measurement the uncertainty of the energy of the loop can be smaller than $\Delta\varepsilon$, but the uncertainty of the energy of the electromagnetic pulse will be larger than $\Delta\varepsilon$. Therefore in the general case we can take $\Delta\varepsilon$ in the relations (2) to mean the change of the entire measuring chain up to the classical link.

6. NONPERTURBING MEASUREMENT OF THE ENERGY OF A HARMONIC OSCILLATOR

Measurement of the energy of the oscillator means measuring the operator $\hat{E}=\hat{p}^2/2m+k\hat{x}^2/2$. Obviously we must not do this by measuring $\hat{x}$ and $\hat{p}$ separately, since measuring one of these quantities causes an increase of the uncertainty of the other. For an exact measurement it is necessary that the Hamiltonian of the interaction between the oscillator and the QRS commute with $\hat{E}$, i.e., in the simplest case we must have

$$\hat{H}_I=\left(\frac{\hat{p}^2}{2m}+\frac{k\hat{x}^2}{2}\right)f(\hat{y}) . \tag{24}$$

(Here $\hat{y}$ is the coordinate operator of the QRS.)

Consequently the Hamiltonian of the oscillator and the coupled QRS will be of the following form

$$\hat{H}=\frac{\hat{p}^2}{2m}(1+f(\hat{y}))+\frac{k\hat{x}^2}{2}(1+f(\hat{y}))+\hat{H}_k(\hat{y},\hat{p}_y) \tag{25}$$

($\hat{H}_k$ is the Hamiltonian of the QRS.) Accordingly, the effective mass and the stiffness of the oscillator will, during the time of the measurement, be operators that depend on the coordinate operator of the QRS.

Let us consider, for example, a measurement of the energy of an electromagnetic LC circuit.[16,18] The capacitance of the circuit can depend on the distance between the plates of a condenser or on the position of a dielectric insert, and the inductance can depend on the distance between windings or on the position of a magnetic core. By rigidly connecting the movable parts of the coil and the condenser we can get an interaction of the required type. Let $L=L(y)$, $C=C(y)$, and let the QRS consist of a mass M bound with a stiffness k_0. Then the Lagrangian of the setup is

$$\mathcal{L}=\frac{1}{2}L(y)\dot{q}^2-\frac{1}{2}q^2/C(y)+\frac{1}{2}M\dot{y}^2-\frac{1}{2}k_0y^2. \tag{26}$$

Consequently,

$$\hat{H}=\frac{\frac{1}{2}\hat{p}^2}{L(\hat{y})}+\frac{\frac{1}{2}\hat{q}^2}{C(\hat{y})}+\frac{\frac{1}{2}\hat{p}_y^2}{M}+\frac{1}{2}k_0\hat{y}^2. \tag{27}$$

We can choose the parameters of the interaction in such a way that

$$\frac{1}{L(y)}=\frac{1}{L_0}(1+f(y)),\quad \frac{1}{C(y)}=\frac{1}{C_0}(1+f(y)),$$

where L_0, C_0 are the values of the circuit's parameters before the interaction with the QRS.

In this case the Hamiltonian (27) will satisfy the condition (25):

$$\hat{H}=\left(\frac{\hat{p}^2}{2L_0}+\frac{\hat{q}^2}{2C_0}\right)[1+f(\hat{y})]+\frac{\hat{p}_y^2}{2M}+\frac{k_0\hat{y}^2}{2}. \tag{28}$$

The operator $\hat{E}_0=(\hat{p}^2/2L_0)+(\hat{q}^2/2C_0)$ is the energy operator of the circuit with fixed values of L and C equal to their values before the interaction with the QRS. We now express E_0 in terms of the operator $\hat{N}$ of the number of quanta: $\hat{E}_0=(\hat{N}+\frac{1}{2})\hbar\omega_0$ $[\omega_0=(L_0C_0)^{1/2}]$. Then from Eq. (28) we obtain

$$\hat{H}=\left(\hat{N}+\frac{1}{2}\right)\hbar\omega_0(1+f(\hat{y}))+\frac{\hat{p}_y^2}{2M}+\frac{k_0\hat{y}^2}{2}. \tag{29}$$

Since $[\hat{N},\hat{H}]=0$, $d\hat{N}/dt=0$, and the quantum number for the energy of the resonator does not change in the measurement process. However, the frequency depends on the coordinate operator of the QRS and therefore is itself an operator:

$$\hat{\omega}_e=\omega_0(1+f(\hat{y})). \tag{30}$$

Accordingly, what does not change during the measurement process is not the current energy of the resonator, but the quantum number (and E_0). The energy cannot in principle stay constant during the measurement process, since in the process of measuring the energy the uncertainty of the phase must increase. In the case of the Hamiltonian (25) a change of the uncertainty of the phase can only be a consequence of an uncertainty of the frequency. Consequently, in the process of measurement the number of quanta and the frequency cannot be constant simultaneously. But after the measurement of N the original value of the frequency can be restored. For this it suffices to make the permeabilities of the cores equal to unity, or to hold them fixed in their original positions. Having measured N, we can determine the energy that the resonator had before the measurement began, and having restored the original values of the parameters L and C, we determine the value of the energy E after the measurement. Since the values of the energy before and after the measurement are the same, i.e., $E'=E_0$, in this sense we can speak of a nonperturbing measurement of the energy.

Let us find the connection between the error in determining N and that in measuring the coordinate of the QRS. We consider the simplest case, when the shape of the cores is such that $f(y)=-\beta y$. Then from Eq. (29) we have

$$\begin{aligned}\dot{\hat{p}}_y&=-k_0\hat{y}+\beta\hat{E}_0,\\ \dot{\hat{y}}&=\frac{\hat{p}_y}{M}\end{aligned} \tag{31}$$

(this sort of approximation can be valid only for $1-\beta y>0$, since otherwise L and C would have to change sign). The motion of the QRS is like that of a harmonic oscillator under the action of a constant force βE_0. Therefore $\Delta E_0=k_0\cdot\Delta y_0/\beta$, where Δy_0 is the error in measuring the equilibrium position. With a stroboscopic measurement the value of the instantaneous coordinate taken a half vibration period of the QRS after the first measurement is equal and opposite to the value of the coordinate in the first measurement. Consequently, a necessary condition for satisfying the condition $1-\beta y>0$ is $\beta\Delta y_1<1$, where Δy_1 is the error of the first measurement of the instantaneous coordinate. Although during the motion of the QRS its wave packet reaches a width $\Delta y_m=[(\Delta y_1)^2+(\hbar/2\Delta y_1M\omega)^2]^{1/2}$ and can be much larger than Δy_1, the condition $\beta\Delta y_1<1$ is a sufficient one. The point is that during the first measurement a negative momentum can be imparted to the QRS which on the average is large enough so that the quantum-mechanical average value $\langle y\rangle$ between the measurements will have an amplitude larger than y_m. Noting that $\Delta y_0\geqslant\Delta y_1/2$ and that the increase of the energy uncertainty of the QRS is $\Delta\varepsilon>(1/2M)(\hbar/2\Delta y_1)^2$, we obtain

$$\Delta E_0 \cdot \Delta\varepsilon \geqslant \frac{k_0 \Delta y_1}{2\beta} \cdot \frac{1}{2M}\left(\frac{\hbar}{2\Delta y_1}\right)^2 = \frac{\hbar^2\omega^2}{16\beta\Delta y_1} > \frac{\hbar^2\omega^2}{16} = \frac{\pi^2}{4}\left(\frac{\hbar}{2\Delta t}\right)^2. \quad (32)$$

Accordingly, in this case also the relations (2) hold and it is possible that $\Delta E_0 \ll \hbar/\Delta t$.

Let us find the relation between the error of the measurement of N and the uncertainty of the change of phase in the measuring process. The uncertainty of the frequency of the oscillations in the resonator during the time of the measurement is determined by the uncertainty of the coordinate y. The uncertainty of the phase change is

$$\Delta\psi = \int_0^{\Delta t} \Delta\omega_e(t)\, dt = \gamma (\Delta\omega_e)_m \cdot \Delta t. \quad (33)$$

Since

$$(\Delta\omega_e)_m = \omega_0 \beta \Delta y_m > \omega_0 \beta \frac{\hbar}{2\Delta y_1 M\omega},$$

we then have

$$\frac{\Delta E_0}{\hbar\omega_0} \cdot \Delta\psi \geqslant \frac{k_0 \gamma \cdot \Delta t}{4M\omega} = \frac{\gamma\pi}{4}. \quad (34)$$

For the stroboscopic measurement in the case $\Delta y_1 \ll \Delta y_m$, we have $\gamma \approx 2/\pi$. Consequently, the error of the nonperturbing measurement of the number of quanta in the resonator, $\Delta N \equiv \Delta E_0/\hbar\omega_0$, is connected with the uncertainty of the phase change of the oscillations during the time of the measurement by the relation

$$\Delta N \cdot \Delta\psi > \frac{1}{2}. \quad (35)$$

The relation (35) is the same as the traditional number-frequency uncertainty relation (which was not too accurately interpreted) in the form only.[10] In the traditional relation $\Delta\psi$ is taken to be the uncertainty of the phase of an oscillator, and its values are confined to the interval 0 to 2π. The operator for the phase, which is not Hermitean[10] is also defined very differently. In the relation (35) $\Delta\psi$ is the uncertainty of the change of phase in the process of energy measurement and can have any value from 0 to ∞. The phase operator itself is expressed in terms of the Hermitian frequency operator. In the usual description of the state of an oscillator the frequency is a number.

Nevertheless the relation (35) is a necessary consequence of the traditional number-phase uncertainty relation. If (35) were not obeyed, then one could prepare a state in which the product of the phase and the quantum number would be smaller than $\frac{1}{2}$. Let us suppose that by some method a state of an oscillator has been prepared with a small phase uncertainty $\Delta\psi_0$ and with $\Delta N_0 > 1/2\Delta\psi_0$. We then make a measurement of the number of quanta with an accuracy of $\Delta N \ll N_0$. If during the measuring process there is not an increase in the uncertainty of the phase by an amount $\Delta\psi \gtrsim 1/2\Delta N$, the state appearing after the measurement of N will have $\Delta N \cdot \Delta\psi < \frac{1}{2}$.

As was already stated, all calculations in this paper are valid in the nonrelativistic approximation, i.e., when the speeds of mechanical motions in all links of the measurement chain are much smaller than the speed of light c_0. Consequently, the minimum value of Δy_1, for example, is limited by the condition $\hbar/2\Delta y_1 M \ll c_0$. Then we have

$$\Delta E_0 \geqslant \frac{k_0 \Delta y_1}{2\beta} \gg \frac{\pi^2}{4\beta c_0 \Delta t} \cdot \frac{\hbar}{\Delta t}. \quad (36)$$

7. CONCLUSION

The possibility of measuring an energy in a finite time without changing its initial value (i.e., with the condition $E' = E_0$) is not, contrary to a widespread opinion, in contradiction with the principles of quantum mechanics. The relation $\Delta(E' - E_0) \cdot \Delta t > \hbar$ is valid only in the case when the interaction between the quantum system under investigation and the apparatus is a function of the coordinates of the system.

The condition for a nonperturbing energy measurement is that the Hamiltonian $\hat{H}_1$ of the interaction between the system and the apparatus depends on the energy operator $\hat{E}$ of the system and that the operators $\hat{H}_1$ and $\hat{E}$ commute. A nonperturbing measurement is possible in which the error of the energy measurement is $\Delta E \ll \hbar/\Delta t$.

Measurement of the energy of the system being studied is accompanied by an increase of the uncertainty $\Delta\varepsilon$ of the energy of the apparatus. The error ΔE of the measurement of the system's energy and the disturbance of the energy of the apparatus are connected by the relations

$$\Delta E + \Delta\varepsilon \geqslant \frac{\hbar}{\Delta t}, \quad \Delta E \cdot \Delta\varepsilon \geqslant \left(\frac{\hbar}{2\Delta t}\right)^2.$$

It has not been absolutely proved that these relations, as found in particular examples, are generally valid, but the writer has found no example that contradicts them.

The relations (2) obtained do not deny the fundamental character of relations of the type of $\Delta H \cdot \Delta t > \hbar$, which is a special case of a more general relation found by Mandel'shtam and Tamm,[5] if ΔH is the uncertainty of the energy of a free particle and Δt is the time in which its wave packet is displaced through a distance equal to its width.

The writer expresses his sincere gratitude to V. E. Braginskiĭ and F. Ya. Khalili for conversations which were very helpful in writing this paper.

[1]L. Landau and R. Peierls, Z. Phys. **69**, 56 (1931).

[2]L. D. Landau and E. M. Lifshits, Kvantovaya mekhanika (Quantum mechanics), Moscow, Fizmatgiz, 1963; English translation, Addison-Wesley, Reading, Mass., 1965.

[3]D. I. Blokhintsev, Osnovy kvantovoĭ mekhaniki (Principles of quantum mechanics), Moscow, Vysshaya Skhola, 1963; English translation, D. Reidel Pub. Co., Dordrecht, Holland, 1964.

[4]N. S. Krylov and V. A. Fock, Zh. Eksp. Teor. Fiz. 17, 93 (1947).

[5]a) L. I. Mandal'shtan Polnoe sobranie trudov (Complete collected works), Vol. 3, Moscow-Leningrad, Izd. AN SSSR, (1950) (concerning energy in wave mechanics).
b) L. I. Mandel'shtam and I. E. Tamm, Izv. Akad. Nauk SSSR, ser. fiz. **9**, Nos. 1, 2 (1945).

[6]Y. Aharonov and D. Bohm, Phys. Rev. **122**, 1649 (1961).

[7]V. A. Fock, Zh. Eksp. Teor. Fiz. 42, 1135 (1962) [Sov. Phys. JETP **15**, 784 (1962)].

[8]Y. Aharonov and D. Bohm, Phys. Rev. **134**, B1417 (1964).

[9]V. A. Fock, Usp. Fiz. Nauk 86, 363 (1965) [Sov. Phys. Usp. 8, 628 (1966)].
[10]P. Carruthers and M. Nieto, Rev. Mod. Phys. 40, 411 (1968).
[11]Yu. I. Vorontsov, Dokl. Akad. Nauk SSSR 251, 1110 (1980) [Sov. Phys. Dokl. 25, 278 (1980)].
[12]V. Braginskiĭ, J. Vorontsov, and K. Thorn, Science 209, 547 (1980).
[13]L. I. Mandel'shtam, Lektsii po optike, teorii otnositel'nosti i kvantovoĭ mekhaniki (Lectures on optics, relativity theory, and quantum mechanics) Moscow, Nauka, 1972.
[14]J. von Neumann, Mathematical foundations of quantum theory, Princeton University Press, 1955. [Russ. Transl. Nauka, M., 1972].
[15]V. B. Braginskiĭ and Yu. I. Vorontsov, Usp. Fiz. Nauk 114, 41 (1974) [Sov. Phys. Usp. 17, 644 (1975)].
[16]V. B. Braginskiĭ, Yu. I. Vorontsov, and F. Ya. Khalili, Zh. Eksp. Teor. Fiz. 73, 1340 (1977) [Sov. Phys. JETP 46, 705 (1977)].
[17]V. B. Braginskiĭ, Yu. I. Vorontsov, and F. Ya. Khalili, Pis'ma Zh. Eksp. Teor. Fiz. 27, 296 (1978) [JETP Lett. 27, 276 (1978)].
[18]F. Ya. Khalili, Author's abstract of candidate dissertation, Moscow State University, Department of Physics, 1979.

Translated by W. H. Furry