Electron localization in disordered systems: critical behavior and macroscopic manifestations

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The review deals with new results in the theory of electron localization in disordered systems and with experiments which have been stimulated by these results. An elementary scaling theory of localization and its consequences are discussed. Attention is concentrated on the phenomenon of localization in two-dimensional systems. The main experiments on the conductivity of thin metal wires and films, demonstrating qualitatively new behavior associated with localization, are discussed. The review concludes with a brief look at the problems of deriving a self-consistent scaling theory of critical behavior at a mobility edge.

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CONTENTS

INTRODUCTION

The concept of electron localization in disordered systems is central to the modern theory of such systems: the main ideas on the energy spectrum, and on the transport and other electron properties of disordered systems are based on this concept.[1,2] The concept was first formulated in the cornerstone paper of Anderson[3] and later, after a period of neglect, it was developed qualitatively by Mott, who used it to formulate the main laws of the electron theory of disordered systems.[1,2] Several good reviews[4-7] have recently been devoted to the phenomenon of localization and its fundamentals are now well known. Moreover, in spite of its importance, the problem of localization is still far from the state of a fully satisfactory solution. This applies particularly to our ideas on the behavior of electron states near what is known as the mobility edge and the associated problem of the physical properties of a system in which the electron Fermi level lies in the vicinity of a mobility edge. Difficulties in the understanding of these properties are related to the extreme mathematical difficulty of the problem under discussion and due to the fact that experimental evidence on electron localization is fairly indirect.[1,2,5]

The present review describes briefly the progress made in the last few years, both in the theoretical respect and in the formulation of new experiments. We shall concentrate on those theoretical results which relate to the "critical" behavior of a system in the vicinity of a mobility edge, using a certain analogy which clearly exists between such behavior and the usual critical behavior in the vicinity of a second-order phase transition; we shall also discuss new experiments on the conductivity of thin metallic wires and films in which the phenomenon of localization (and the corresponding critical behavior) should be manifested most clearly. The special attention currently given to two-dimensional disordered systems is due to the circumstance that practically all the devices used in modern microelectronics are examples of such systems. However, we shall ignore completely the problems and experiments relating to the localization in quasione-dimensional systems (for details see Refs. 8–11).

We shall proceed as follows. We shall begin by recalling the main ideas of the theory of localization and the relevant terminology. We shall then consider elementary scaling theory of localization based mainly on the ideas of Thouless,[4] including newest developments of his results and their applications to the description of the conductivity of thin metallic wires and two-dimensional metallic systems. Next, we shall discuss the main experiments aimed to verify these ideas. We shall conclude with a brief review of the attempts to construct a rigorous scaling theory of localization on the basis of the above-mentioned analogy with the conventional critical phenomena.

1. ANDERSON MODEL AND MINIMUM METALLIC CONDUCTIVITY

We shall recall the main results of the theory of localization. This is usually done on the basis of the Anderson model[3] in which one electron propagating in a regular lattice of sites in a d-dimensional space is considered; at each site there is a random energy level E_j (j is the number of the lattice site). It is also assumed that there is a definite amplitude of the probability V_{ij} of a transition from a jth to an ith site. It is usually postulated[3,12,13] that this transition amplitude differs from zero and is equal to a certain constant V for transitions between the nearest neighbors. We

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shall not consider the role of "nondiagonal" disorder of the matrix elements V_{ij} (for details see Ref. 14).

The Hamiltonian of the model is therefore

$$H=\sum_j E_j a_j^+ a_j+\sum_{ij} V_{ij} a_i^+ a_j. \qquad (1.1)$$

where a_j and a_j^+ are, respectively, the usual operators of annihilation and creation of an electron at a site j. The energy levels E_j are assumed to be distributed independently from one another at different sites. The distribution at a given site is usually assumed to be[3]

$$P(E_j)=\begin{cases}\frac{1}{W} & \text{for } |E_j|<\frac{1}{2}W,\\ 0 & \text{for } |E_j|>\frac{1}{2}W,\end{cases} \qquad (1.2)$$

i.e., it is assumed to be uniform in a certain range of energies of width W. Qualitative results are clearly independent of the nature of the distribution $P(E_j)$ and instead of Eq. (1.2) we can use any other distribution with an effective width W.

If there is no disorder in the system ($W=0$), the solution of the problem of the electron spectrum of the Hamiltonian (1.1) is elementary. Electron states form a band of width $B=2ZV$ (for a simple cubic lattice with Z nearest neighbors) and the wave function of each of the states in a band is represented by a Bloch wave, i.e., it is shared equally by all the lattice sites. If $W\neq 0$, the situation changes and the dimensionality of space d has a considerable influence. For example, if $d=1$ (a one-dimensional disordered chain), an infinitesimally small value of W is sufficient to alter completely the nature of the electron states: they all become localized, i.e., their wave functions decrease exponentially in the coordinate space and at $T=0$ the static conductivity of the system vanishes.[8-11] Two-dimensional systems ($d=2$) represent clearly a special case and are discussed in detail below. If $d>2$, it has been established reliably[3-7,12-13] that for high values of the ratio $W/V>(W/V)$ (i.e., in the case of sufficiently strong disorder) all the electron states in a band are again localized. Typical values of this ratio are $(W/V)_c=8-15$ (for three-dimensional lattices).[3-6]

The physical meaning of localization is fairly simple. Quantum tunneling from site to site can occur only between sites with identical (degenerate) energy levels. In a random system the probability of such degeneracy is generally small for sites sufficiently close in coordinate space and if the disorder W is sufficiently strong, this probability vanishes.[3,15,16]

If $W/V<(W/V)_c$, electron states become localized at the band edges, whereas at the center of a band the states remain delocalized. This gives rise to critical energies E_c and E_c', which separate the regions of localized and delocalized states, usually called mobility edges. This is illustrated in Fig. 1. The term "mobility edge" is due to the fact that localized states make no contribution to the static conductivity at absolute zero ($T=0$). If at $T=0$ the fermi level E_F of a many-electron system lies in the energy range corresponding to localized states, the system is an insulator: conduction is possible only at $T\neq 0$ or when electrons are excited by an alternating electromagnetic field. Conduction then takes place by the hopping mechanism.[1,7,17] However, if the Fermi level lies in the region of delocalized states, conduction is metallic. This metal–insulator transition, which occurs when the Fermi level crosses a mobility edge, is usually called the Anderson transition.

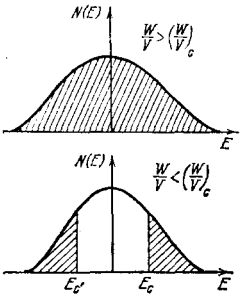

FIG. 1. Density of states in an Anderson lattice. The shaded regions are the localized states. Here, E_c and $E_{c'}$ are the mobility thresholds.

One of the central problems in the theory under discussion is the nature of changes in the metallic (static, at $T=0$) conductivity that take place when the Fermi level E_F crosses a mobility edge. Possible alternative representations are shown in Fig. 2. The conductivity may abruptly vanish after reaching a certain minimum value $\sigma_{m.m}$, which denotes the minimum metallic conductivity (maximum metallic resistivity). This behavior has been suggested by Mott[1,2] on the basis of a qualitative analysis of the conduction process in the Anderson model, and also on the basis of an analysis of some experimental data. We can logically expect also a continuous decrease of the conductivity to zero and in this case $\sigma_{m.m}$ retains in a sense the meaning of a characteristic measure of the conductivity beginning from which the conductivity decreases quite rapidly to zero. This alternative model was put forward by Cohen[18] using basically an analogy between localization and percolation.[7,17] Clearly, both alternatives can have the same experimental consequences (at $T\neq 0$), but this does not solve the corresponding theoretical problem.

An estimate of the value of $\sigma_{m.m}$ can be obtained quite simply.[1,2,5] In fact, this estimate is based on the ideas put forward some time ago by Ioffe and Regel,[19] according to whom the mean free path of an electron l in a metallic system cannot be less than the interatomic distance a. It then follows from the usual Drude formula $\sigma=(ne^2/m)\tau$ (n is the electron density, τ is the mean free time between collisions, and m is the electron mass) and from the expressions relating the elec-

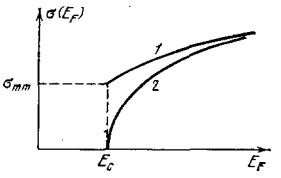

FIG. 2. Changes in the metallic conductivity when the Fermi level passes through a mobility edge: 1) according to Mott[1,2]; 2) according to Cohen.[18]

tron density and the Fermi momentum to the interatomic distance ($n \propto a^{-d}$, $p_F \propto \hbar/a$, $\tau \propto lm/p_F \propto am/p_F \propto ma^2/\hbar$) that

$$\sigma_{m.m} \approx \text{const} \cdot \frac{e^2}{\hbar} a^{2-d}. \quad (1.3)$$

According to Mott,[1,2,5] the constant in Eq. (1.3) is determined by the dimensionless ratio $(W/V)_c$ and lies in the range 0.025–0.06 for $d=3$, which gives $\sigma_{m.m} \sim (1-5) \times 10^{-2} \Omega^{-1}$ cm^{-1} for typical values $a \sim 2-3$ Å.

It is clear already from Eq. (1.3) that $d=2$ is a special case: the minimum metallic conductivity is governed only by the fundamental physical constants and the characteristics of a system exhibiting such conductivity affect only the fairly universal dimensionless ratio $(W/V)_c$. The question arises: is this true? Moreover, a detailed understanding of the changes occurring in the wave functions near a mobility edge is important: how does the localization radius of a wave function change: does it diverge in the limit $E \to E_c$ on the localized state side? Are the wave functions of localized states necessarily exponential or can we expect a fairly rapid (ensuring normalization to unity) power-law fall in the coordinate space? Can these problems be considered by analogy with the problem of phase transitions and is not the behavior near a mobility edge a characteristic "critical" phenomenon? Finally, what is the role of the electron–electron interaction (correlation) in the phenomenon of localization? Our review will, to a greater or lesser extent, try to answer these questions; we shall consider the current level of understanding of these problems and some of the experiments stimulated by their formulation.

2. ELEMENTARY SCALING THEORY OF LOCALIZATION

Behavior of a system in the vicinity of a mobility edge can be understood by a scaling analysis similar to that employed in the theory of critical phenomena[20,21] or, to use the modern language, by constructing some variant of transformations of the renormalization group.[22-24] The main physical idea behind this approach is the gradual transition from small cells (or scales) of a system in the coordinate space, for which the problem can be solved (at least approximately!), to larger cells (scales) which are described tentatively in terms of the same physical variables as the small-scale cells. The chain of formulas performing this transition from small to large scales is known as the transformations of the renormalization group in real space[23,24] and they are

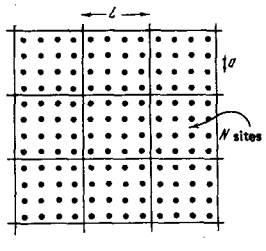

FIG. 3. Analog of the Kadanoff construction in the Anderson problem, plotted in accordance with the Licciardello–Thouless treatment.

modern variants of the Kadanoff scaling transformations.[20] In the theory of critical phenomena the construction of such transformations is usually motivated[20-24] by an increase in the correlation length of fluctuations of the order parameter in the vicinity of a critical point. An analog of this phenomenon in the theory of localization is the increase in the localization radius on approach to a mobility edge from the localized-state side. However, it should be stressed that in the range of delocalized states an analog of such a diverging length is not known[1)] (just as the "order parameter" associated with the Anderson transition is not known), which limits seriously this analogy and makes it difficult to carry out a simple qualitative analysis.

We shall now present a variant of scaling transformations proposed for the Anderson problem by Thouless *et al.*[4,25-29] We shall consider an Anderson lattice with the period a. We shall make a transition from a cell of side a to a new cell of side L, containing N sites of the original lattice. Then, the original lattice can be described as consisting of new cells, each of which has a set of N random levels (Fig. 3). We shall consider a lattice constructed by periodic repetition of one such cell. Then any one of the N levels in a cell spreads into a band of width $2\Delta E$. The quantity ΔE is defined in Refs. 25–29 numerically by a shift of the levels on transition from the periodic to antiperiodic boundary conditions for wave functions at the boundary of the cell L. Although the lattice formed by such continuation is not identical with the scaling-transformed Anderson lattice, in a qualitative analysis we can assume that the effective coupling[2)] between electrons at two levels in neighboring cells of the Anderson lattice (i.e., an analog of the overlap integral V for a system constructed from the new cells) is of the order of

$$V_L \approx \frac{1}{Z} \Delta E, \quad (2.1)$$

since each such coupling or bond produces a band of width $2\Delta E$ when the cell is continued periodically. The average scatter of the levels in the neighboring cells is governed by the reciprocal of the density of states calculated per unit volume in the original lattice and by the cell size L:

$$W_L \approx L^{-d} N^{-1}(E). \quad (2.2)$$

If we consider one of the N energy bands in the new lattice, we can regard W_L as an analog of the parameter W of the original lattice. In this way we make the transition from the original Anderson problem with the ratio W/V to a new (scaling-transformed) problem of the same type with an "effective" Hamiltonian characterized by a new ratio W_L/V_L. The process of such

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1) See, however, a discussion of this question in a recent paper.[135]

2) The quantity $V_L \sim \Delta E$ is a measure of the mutual influence of the wave functions in neighboring cells. If the states are localized, they are practically insensitive to a change in the boundary conditions in a large cell [see Eq. (2.3)]. On the other hand, delocalized states are sensitive to changes in the boundary conditions and this results in a considerable shift ΔE of the levels[25-29] [see Eq. (2.4)].

scaling transformations can be continued along the chain $L \to 2L \to 4L \to \ldots$ or, in general, we can make the transformation $L \to bL$ going over to increasingly larger cells containing more and more of the sites in the original lattice.

If a given state of energy E is localized, then

$$V_L \propto e^{-\alpha(E)L}, \tag{2.3}$$

where $\alpha(E) = R_{\text{loc}}^{-1}(E)$ is the reciprocal of the localization radius of a given state. Consequently, in this case the ratio V_L/W_L also decreases exponentially on increase in L. If we assume the existence of mobility edges, then for any value of E in the original band there is a maximum ratio $(V_L/W_L)_{\max}$, for which such exponential decrease is still valid. Such a localization criterion has been used[25-29] to determine numerically the positions of mobility edges in the original energy band.

In a system of cells of size $L \gg l$, where l is the mean free path, a delocalized electron can equiprobably become displaced (can diffuse) from one cell to a neighboring one[3)] in a time $\tau_L \sim \hbar/V_L$. It follows from elementary transport theory that the coefficient of such diffusion is $D_L \sim L^2/\tau_L$, so that we obtain

$$V_L \approx \hbar D_L(E) L^{-2}, \tag{2.4}$$

where $D_L(E)$ is the diffusion coefficient of an electron of energy E. Then, in the case of delocalized states, we have

$$g(L) \equiv \frac{V_L}{W_L} \approx \hbar D_L(E) N(E) L^{d-2} \approx \frac{\hbar}{2e^2} \sigma_L(E) L^{d-2}, \tag{2.5}$$

where $\sigma(E) = 2e^2 D(E) N(E)$ is the electrical conductivity of the system expressed in terms of the diffusion coefficient and the density of states, which follows from the general Einstein relationship[30] $\sigma = e^2 D dn/d\mu$ (μ is the chemical potential) at $T = 0$. A more rigorous derivation of Eq. (2.5) is given in the original papers.[4,25-28]

It is clear from Eq. (2.5) that if $d = 2$, the universal minimum metallic conductivity does exist if scaling transformations of $(V_L/W_L)_{\max}$ give a certain universal constant. A numerical analysis carried out by Licciardello and Thouless[27,28] showed that such a universal (independent of the nature of the original lattice) constant does exist (in Ref. 27, they considered a hexagonal honeycomb lattice, as well as triangular and square lattices for $N = 64$, 100, 144, and 196). Licciardello and Thouless obtained the following expression for the metallic conductivity in the $d = 2$ case:

$$\sigma_{\text{m.m}} = \frac{2e^2}{\hbar} \left(\frac{V_L}{W_L}\right)_{\max} = (0.12 \pm 0.03) \frac{e^2}{\hbar} \approx 3 \cdot 10^{-5}\ \Omega^{-1}. \tag{2.6}$$

However, later numerical calculations of the same authors[29] gave somewhat unexpected results. Roughly speaking, they found that the value of $\sigma_{\text{m.m}}$ decreases as the dimensions L of the system increase. This raised doubts about the existence of $\sigma_{\text{m.m}}$ for $d = 2$ and led Licciardello and Thouless[29] to the hypothesis that in the case of two-dimensional systems there may be complete localization even in the case of infinitesimally weak disorder (by analogy with the one-dimensional case). On the other hand, the results of Ref. 31 confirmed the earlier results of Licciardello and Thouless.[27] Numerical calculations of three-dimensional systems are far too indefinite in Refs. 25–29; the fullest analysis of the three-dimensional case can be found in Ref. 32 for a diamond-type lattice.

Considerable progress in our understanding of the phenomenon of localization was made[4)] by Abrahams, Anderson, Licciardello, and Ramakrishnan.[33] It was found that a simple qualitative analysis can be made of the behavior of the function $g(L)$ defined by Eq. (2.5) by postulating the simplest renormalization group equation for this function [this equation gives the change in $g(L)$ due to the transition $L \to bL$][5)]

$$\frac{d \ln g(L)}{d \ln L} = \beta_d(g(L)). \tag{2.7}$$

At this stage the most important assumption is that the function $\beta_d(g)$ on the right-hand side depends only on the variable $g(L)$, which is known as one-parameter scaling. Equation (2.7) is an analog of the well-known Gell-Mann–Low equation [the function $\beta_d(g)$ is an analog of the Gell-Mann–Low function] in the renormalization group used in quantum field theory.[34-36] The behavior of the function $\beta_d(g)$ is easily understood from simple physical considerations. It is clear from the definition (2.5) that $g(L) \propto \sigma_L L^{d-2}$, i.e., that this quantity is proportional to the total conductance (and not specific conductivity!) of the system (which is a cube of side L). In the case of large values of g (weak disorder, $V_L/W_L \gg 1$), we should obtain the static metallic conductivity $\sigma_{L\to\infty} = \sigma$, i.e., the following condition should be satisfied:

$$\lim_{g\to\infty} \beta_d(g) \to d-2, \tag{2.8}$$

since it follows from $d \ln g(L)/d \ln L = d - 2$ that $g(L) = \text{const} \cdot L^{d-2}$ and the constant of integration is simply $(\hbar/2e^2)\sigma$. At low values of $g(V_L/W_L \ll 1)$, we should obtain exponential localization, i.e., $g(L) = g_c(d)\exp(-\alpha L)$ [compare with Eq. (2.3)]. Then,

$$\lim_{g\to\infty} \beta_d(g) \to \ln \frac{g}{g_c(d)}, \tag{2.9}$$

since it follows from $\ln g = \ln g_c - \alpha L$ that $d \ln g(L)/d \ln L = -\alpha\, dL/d \ln L = -\alpha L = \ln(g/g_c)$.

We shall now *assume* the existence of two "perturbation theory" expansions:

$$\beta_d(g \to 0) = \ln\left[-\frac{g}{g_c(d)}\right](1 + ag + \ldots), \tag{2.10}$$

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3) The quantity V_L defines, as mentioned above, the effective coupling of electrons occupying random levels in neighboring cells. This coupling or interaction results in broadening of the levels, i.e., it is responsible for the finite (of the order of τ_L) lifetime of electrons in a given cell.

4) These authors have been called collectively as the "gang of four."

5) Equation (2.7) is a differential variant of the postulated general scaling transformation of variables in the problem in question: $g(bL) = f(b, g(L))$. The differential form of Eq. (2.7) is most convenient for a qualitative analysis and it implies a transition from cells of side L to cells of side $bL = L + dL = (1 + \lambda)L$, where $dL/L = \lambda \to 0$.

$$\beta_d(g\to\infty)=d-2-\frac{b}{g}+\dots\ . \qquad (2.11)$$

Arguments are given in Ref. 33 to show that $a>0, b>0$. Using the asymptotic expressions given by Eqs. (2.8) and (2.9), and the expansion represented by Eqs. (2.10) and (2.11), we can easily deduce the behavior of $\beta_d(g)$ throughout the range of g (on the assumption that it is monotonic and continuous!). The behavior of this function is shown in Fig. 4. We can see that $\beta_d(g)$ does not have zeros for $d<2$. If the expansion (2.11) is valid, there is no zero either for $d=2$. If $d>2$, the function $\beta_d(g)$ definitely has a zero, $\beta_d(g_c)=0$, but its position cannot be found in such a simple way as above (obviously, $g_c\sim 1$). The existence of this zero implies the existence of an *unstable* fixed point for Eq. (2.7). We shall mention in this connection that in the theory of critical phenomena the scaling behavior is usually governed by stable fixed points of the renormalization group equations.[22-24] Near g_c, we can use the approximation (shown by circles in Fig. 4)

$$\beta_d(g)\approx\frac{1}{\nu}\ln\frac{g}{g_c}, \qquad (2.12)$$

where $\nu<1$, because $a>0$ in Eq. (2.10). Then, if Eq. (2.7) is integrated beginning from $g_0\gtrsim g_c$ [$g(L=a)=g_0$, where a is a distance of the order of the lattice period], then for $L\to\infty$ the value of σ_L will exhibit the following asymptotic behavior[6)]:

$$\sigma\approx A\frac{e^2}{\hbar}\frac{g_c}{a^{d-2}}\left(\ln\frac{g_0}{g_c}\right)^{(d-2)\nu}\approx A\frac{e^2}{\hbar}\frac{g_c}{a^{d-2}}\left(\frac{g_0-g_c}{g_c}\right)^{(d-2)\nu}\quad\text{for } g_0\gtrsim g_c, \qquad (2.13)$$

where $A=\text{const}\sim 1$. Clearly, the existence of a fixed point implies the existence of a mobility edge and the behavior of $\beta_d(g)$ near absolute zero determines the critical behavior at the mobility edge. Under these assumptions, the conductivity decreases *continuously* to zero in the limit $g_0\to g_c$ and the quantity $e^2/\hbar a^{d-2}$ (the Mott value of $\sigma_{m.m}$) is simply a characteristic conductivity scale. In this sense the results of Ref. 33 confirm, to some extent, the Cohen variant. A Mott conductivity jump can be obtained at a mobility edge if the function $\beta_d(g)$ behaves as shown by the dashed curve in Fig. 4 (in the case when $d=2$). According to the authors of Ref. 33, such behavior is not very likely because it is in conflict with the expansion (2.11). However, it should be stressed that the existence of such an expansion is an assumption made in the theory.[7)]

[6)] Strictly speaking, integration in Eq. (2.7) subject to the indicated initial conditions and with Eq. (2.12) on the right-hand side gives

$$\sigma(L)=\frac{2e^2}{\hbar}\frac{g_c}{a^{d-2}}\left(\ln\frac{g_0}{g_c}\right)^{(d-2)\nu}\left\{\frac{g(L)}{g_c}\left(\ln\frac{g(L)}{g_c}\right)^{(2-d)\nu}\right\}.$$

Since because of the instability of the fixed point in the limit $L\to\infty$, we reach the range $g\gg 1$, where the true function $\beta_d(g)$ reaches the constatn value $d-2$, the factor in the braces reduces to the constant A of Eq. (2.13) in the limit $L\to\infty$. The dependence on the initial conditions in Eq. (2.13) is governed by the function $\beta_d(g)$ linearized near the fixed point.

[7)] The existence of such an expansion can be justified by considering the region of weak disorder on the basis of the conventional perturbation theory,[38-40] which becomes inapplicable for $g\sim g_c$.

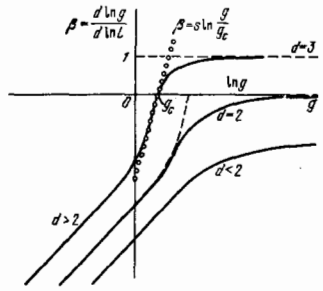

FIG. 4. Qualitative form of the Gell-Mann–Low function for various values of d. The dashed lines are used to represent the behavior needed to ensure a conductivity jump at a mobility edge in the $d=2$ case.

Similarly, we can show that integration of Eq. (2.7) from the initial point $g_0\leq g_c$ gives (for large values of L)

$$g\approx g_c\exp\left(-A'\left|\ln\frac{g}{g_c}\right|^{\nu}\frac{L}{a}\right). \qquad (2.14)$$

Hence, [compare with Eq. (2.3)] it follows that

$$R_{loc}\sim a\left|\frac{g-g_c}{g_c}\right|^{-\nu}, \qquad (2.15)$$

i.e., ν plays the role of the critical index of the localization radius.[8)]

If $d\leq 2$, it follows from Ref. 33 that $\beta_d(g)<0$ throughout the range of g. Then, $\sigma\propto g(L\to\infty)\to 0$ for any initial conditions, there is no mobility edge, and all the states are localized (the static conductivity of an infinite system vanishes!). For $d=1$, this agrees with the known exact results.[8-11] For $d=2$, this is a completely new result (although suspected earlier[29]). Subject to the initial condition $g_0(L=a)$ in the range of sufficiently large values of g where the expansion (2.11) is valid (for $d=2$ the first term of this expansion vanishes), we obtain from Eq. (2.7)

$$g(L)\approx g_0-b\ln\frac{L}{a}\approx\frac{\hbar}{2e^2}\sigma_L, \qquad (2.16)$$

i.e., the conductivity σ_L decreases on increase in L because of a *logarithmic* correction, until the correction term becomes of the same order as the main term; then, the reduction in the conductivity becomes expo-

[8)] An attempt has been made[135] to give a physical meaning to the effective length $\xi\sim a[(g-g_c)/g_c]^{-\nu}$ also in the range $g\gtrsim g_c$. We can easily see that Eq. (2.13) can be written in the form $\sigma\approx Ae^2/\hbar\xi^{d-2}$ so that the length ξ describes the behavior of the conductivity farily close to a mobility edge in a situation when ξ becomes much larger than the interatomic distance or the mean free path in the case of inelastic scattering processes. According to Ref. 135, this length governs the scale for which the conductivity becomes "ohmic" in the sense that the resistance of a cube ($d=3$) of side L decreases proportionally to L^{-1}. Near a mobility edge only samples of increasing dimensions ($\xi\to\infty$) can be regarded as macroscopic and in the limit $L<\xi\sigma_L$ they depend on L. It is possible that these considerations allow us to give a definite physical meaning to the diverging length in the region of delocalized states which occur in the scaling pattern. In Ref. 135, this is used to propose an original explanation of the negative temperature coefficient of the resistance, which is exhibited by many poorly conducting metallic systems.

nential. The existence of a weak logarithmic dependence in Eq. (2.16) has the effect that the behavior of the conductivity can indeed be close to that expected in the case of existence of an abrupt mobility edge. We can observe experimentally a logarithmic tendency of the "metallic" phase to become insulating (at high temperatures), followed by a fairly rapid drop of the conductivity in the vicinity of $\sigma_{m.in}$ in accordance with Eq. (2.6). The processes which determine the effective lengths L, which govern the experimentally observed conductivity, are discussed below (Sec. 3).

The result (2.16) can be justified[33] by direct perturbation theory calculations. We have to consider (for $d=2$) a series of Feynman graphs for a two-particle Green function introduced some time ago by Langer and Neal.[37] This sequence of graphs predominates in the $d=2$ case and gives rise to a logarithmic correction in Eq. (2.16). A similar result is obtained also in Ref. 38, where summation of the Langer–Neal graphs gives the following expression for the coefficient of two-dimensional diffusion in an external field of frequency ω:

$$D \approx D_0\left(1-\frac{\hbar}{2E\tau}\ln\frac{1}{\omega\tau}\right), \qquad (2.17)$$

where $D_0=v^2\tau/2$ is the usual transport formula for the two-dimensional diffusion coefficient (v is the electron velocity, E is the electron energy, and τ is the mean free time; it is assumed that $E\tau\gg 1$). We can see that in the limit $\omega\to 0$, the correction reduces the diffusion coefficient and this may indicate complete localization in a two-dimensional system. It should be stressed that Eq. (2.17) is valid as long as the correction is small compared with the main term. In the case of a finite sample in the frequency range $D_0L^{-2}\gg\omega\gg\hbar/mL^2$, Eq. (2.17) reduces to a result of the type given by Eq. (2.16) (Ref. 38). Generalization of the analysis given in Ref. 38 to the case when allowance is made for the scattering accompanied by spin flipping, and also for the effect of a weak external magnetic field,[39,40] shows that these processes suppress the logarithmic correction in Eq. (2.17), i.e., they destroy the two-dimensional localization. Allowance for the spin–orbit interaction of an electron with impurities alters the sign of the logarithmic correction to the conductivity,[41] i.e., it gives rise to a conductivity that becomes infinite in the limit $\omega\to 0$. It is at present difficult to give a quantitative interpretation of these unexpected results.[9)]

The above simple description of the effects near a mobility edge has not yet been proven. Moreover, after the appearance of the work of Abrahams, Anderson, Licciardello, and Ramakrishnan,[33] a numerical calculation of the conductivity was made by Lee[42] using the Anderson model and this calculation was in a sense similar to an analysis of the Kondo problem carried out by Wilson.[23] Without going into details, we shall simply point out that because of the special scheme of scaling transformations (renormalization group) in real space, Lee was able to calculate the conductivity for "computer" samples of much larger dimensions than in Refs. 25–29. In his calculations the effective number of sites was $N=256^2$! Nevertheless, these calculations reproduced almost exactly the results of earlier studies.[27,28] According to Lee,[42] the minimum metallic conductivity does exist in the $d=2$ case and the constant in Eq. (2.6) is 0.13. Moreover, Lee was able—in a sense—to calculate also the "Gell-Mann–Low function" $\beta_{d=2}(g)$. His results are given in Fig. 5. We can see that there is a clear tendency for a kink and intersection of the abscissa, and this is followed by $\beta_{d=2}(g)=0$, i.e., "nonphysical" behavior predicted and this is represented by the dashed curve in Fig. 4; this behavior ensures the existence of minimal metallic conductivity.[10)]

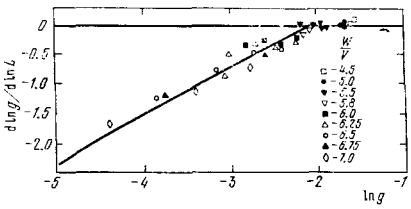

FIG. 5. Gell-Mann–Low function $\beta_{d=2}(g)$ found numerically by Lee.[42]

The question whether the one-parameter scaling scheme of Abrahams, Anderson, Licciardello, and Ramakrishnan[33] is valid is thus still open. It is equally unclear whether the renormalization group of Lee, associated with a numerical calculation, can reveal such fine logarithmic effects which result in complete localization in a two-dimensional space. In this connection, we should mention an earlier investigation,[43] where a different scheme of numerical calculations yielded a continuous metal–insulator transition for $d=2$, contradicting the existence of the minimum metallic conductivity. However, no indications were obtained there for a complete localization in a two-dimensional space. Stein and Krey[32] also found mobility edges in two-dimensional lattices but they pointed out that the precision of their calculations was clearly insufficient to detect weak logarithmic effects. In a later investigation,[136] the same authors made detailed numerical calculations of the conductivity of two-dimensional lattices and of a three-dimensional lattice of the diamond type. Their calculations were made by direct application of the Kubo formula for the conductivity expressed in terms of exact wave functions of an electron in an Anderson lattice, employing a recurrent algorithm proposed in Ref. 137. The results obtained *supported* the existence of the minimum metallic conductivity in the $d=2$ and $d=3$ cases. For $d=2$, Stein and Krey[136] obtained a universal value $\sigma_{m.in}=(0.11\pm 0.02)e^2/\hbar$, whereas for $d=3$ they found that $\sigma_{m.in}=(0.07\pm 0.01)e^2/\hbar a$, where a is the lattice constant. Stein and Krey[136] made an attempt to detect a logarithmic dependence of the conductivity on the dimensions of the investigated system in the quasimetallic region assuming that $d=2$, which was postulated in Ref. 33. Such a dependence was not observed

[9)] The influence of these processes was also considered in a recent study,[142] where similar results were obtained.

[10)] The work of Lee[42] begins with the slogan "Down with the gang of four!"

within the limits of precision of the numerical method as the dimensions of the system were increased from 50×50 to 100×100. However, this did not exclude completely the possibility that a logarithmic dependence may be observed for large systems.[11)]

Haydock[138] used a recurrent method[137] in an analytic study of the behavior of the wave functions of systems with a weak disorder. In the $d=2$ case he found that complete localization was possible for any disorder no matter how weak. Interesting suggestions were made concerning a possible physical meaning of the mobility edges obtained in the majority of numerical and analytic calculations for $d=2$. In the $d=2$ case these edges probably separated regions of exponentially localized states distributed, for example, at the band edges, from regions of localized states with a power-law attenuation of the wave functions with distance.[26] Complete localization is then retained for $d=2$ but the transition to the quasimetallic region acquires a clearer meaning, which should help in removing some of the contradictions mentioned above.

3. CONDUCTIVITY OF THIN WIRES AND FILMS

We shall now consider the experimental manifestations of the localization effect. The recent progress has been to predict a number of pronounced effects in which localization plays the dominant role and these effects differ from fairly indirect manifestations of localization considered earlier.[1,2,5] One of such pronounced effects permitting direct experimental verification is the prediction[27] of a universal minimum of metallic conductivity for two-dimensional systems of Eq. (2.6).

Another remarkable result was obtained by Thouless[44] in an analysis of the conductivity of thin metallic wires[44] (similar ideas were also put forward by Adkins).[45] This result was extremely simple: in any metallic wire with transverse dimensions much smaller than the length and with an extrinsic resistance exceeding about 10–20 $k\Omega$ all the electron states are localized. Thus, any sufficiently long metallic wire is "effectively" *an insulator* (in the meaning of the ground state, i.e., at $T=0$)! In fact, we must now return to Eq. (2.5). We have pointed out earlier that the right-hand side of Eq. (2.5) is proportional to the conductance (reciprocal of the resistance) of a finite system, i.e., the resistance of this system is

$$R_L \approx \frac{\hbar}{2e^2}\frac{W_L}{V_L}. \tag{3.1}$$

In the case of an ordinary metallic wire we have $R_L=(1/\sigma)L/A$ (L is the length of the wire and A is its cross section), which is proportional to its length. Then, $(1/\hbar)V_L \propto D/L^2$ represents the reciprocal of the diffusion time of an electron along the whole length of the wire. Hence, we find that [$N(E)$ is the density of states per unit volume]

$$V_L \sim \hbar\frac{D}{L^2} \sim \frac{\hbar}{2e^2}\frac{\sigma}{L^2}\frac{1}{N(E)} \sim \frac{\hbar}{2e^2}\frac{\sigma}{L}A\frac{1}{LA}\frac{1}{N(E)} \sim \frac{\hbar}{2e^2}\frac{W_L}{R_L},$$

which gives Eq. (3.1). Therefore, it we select a sufficiently long wire, we can easily satisfy the condition $W_L/V_L > (V_L/W_L)^{-1}_{\max}$, i.e., we can achieve complete localization. Beginning from this length, the resistance increases exponentially with an increase in the length of the wire. The length in question is governed by the condition $R_L > \text{const}\cdot\hbar/e^2$, where the constant is[44] ~2–4, i.e., $R_L > 8$–16 $k\Omega$. This surprising result requires an explanation. One must point out the conditions when the effect can become observable. Naturally, if the temperature is sufficiently high, the localization of electron states is unimportant because scattering by phonons (and other forms of inelastic scattering) results in transitions between localized states well before an electron diffuses over a distance of the order of the localization length and "learns" that it is localized. However, cooling freezes out the inelastic scattering processes. The localization length in our case is of the order of the length of a wire whose resistance exceeds $2\hbar/e^2$. If we use the Drude formula $\sigma=(ne^2/m)\tau = (e^2/\hbar)p_F^2 l/3\pi^2$ (p_F is the Fermi momentum of an electron), it follows directly from this condition that

$$R_{\text{loc}} \propto \frac{1}{\pi^2}Ap_F^2 l. \tag{3.2}$$

An electron diffuses over this distance in a time

$$\tau_{\text{diff}} \propto \frac{R_{\text{loc}}^2}{D} \sim \frac{1}{\pi^4}A^2 p_F^2 \tau, \tag{3.3}$$

where the diffusion coefficient is $D \approx (p_F^2/3m^2)\tau$. Localization of electron states becomes manifest if

$$\tau_{\text{inel}} > \tau_{\text{diff}}, \tag{3.4}$$

where $1/\tau_{\text{inel}}$ is the frequency of the inelastic scattering events. This frequency is usually proportional to some power of the temperature T^p (p is an integer). For example, the frequency of electron–phonon collisions in contaminated samples is proportional to T^4, whereas in the electron–electron scattering case we have proportionality to T^2 (Ref. 44). It then follows from Eqs. (3.3) and (3.4) that the temperature at which localization becomes manifest is inversely proportional to $\sqrt{A}$ in the case when scattering by phonons predominates, and inversely proportional to A when the electron–electron scattering processes are more important. Below the relevant temperature an electron may diffuse over a distance of the order of R_{loc}, but not further as long as a phonon (or another electron) does not produce a transition to a different state. Consequently, in this range of temperatures the resistance increases as a result of cooling proportionally to some reciprocal of the power of temperature: $T^{-p/2}$ (Ref. 44). Cooling to temperatures corresponding to the energy separating random electron levels results in a transition to an exponential temperature dependence of the conductivity. However, the estimates of Thouless for a wire of $A \approx 2.5\times10^{-11}$ cm^2 cross section, an impurity-controlled mean free path $l \approx 5\times10^{-6}$ cm, and $p_F/\hbar \approx 1.2\times10^8$ cm^{-1} give $R_{\text{loc}} \approx 0.012$ mm, and the condition (3.4) is already satisfied at temperatures of the order of 1 °K for reasonable estimates of the inelastic scattering frequen-

[11)] It is pointed out in Ref. 143 that in the one-dimensional case one should treat with caution the average Kubo formula for the conductivity; further discussions of this problem are given in Refs. 144 and 145.

cies.[12]

We shall now discuss a two-dimensional case when again complete localization occurs according to Ref. 33, i.e., when the static conductivity vanishes at $T=0$. The conductivity at $T\neq 0$ is discussed in Ref. 46. We shall again carry out a qualitative analysis in the spirit of Thouless.[44] At $T\neq 0$, an electron subject to inelastic scattering processes may diffuse over a distance[13]

$$L_1^2 \propto D\tau_{\text{inel}} = \frac{1}{2} ll_{\text{inel}}, \tag{3.5}$$

where l_{inel} is the mean free path governed by inelastic processes. Over a distance given by Eq. (3.5) the coherence of electron states is lost (an electron loses information on its own state). This length can be regarded as the effective length governing the conductivity given by the two-dimensional formula (2.16) valid at finite temperatures. It is possible that some other distance is the effective length.[46] In fact, the processes of inelastic scattering result in broadening of electron levels by an amount of the order of $\hbar/\tau_{\text{inel}}$. Then, the discrete nature of the levels with $W_L \propto N^{-1}(E)$ [see Eq. (2.2) for $d=2$] is unimportant when $W_L \propto \hbar/\tau_{\text{inel}}$ and this determines the length:

$$L_2^2 \sim \frac{1}{\hbar}\tau_{\text{inel}} N^{-1}(E). \tag{3.6}$$

The lengths L_1 and L_2 are proportional but not equal: $(L_1/L_2)^2 \propto \sigma\hbar/e^2$. Bearing in mind that $\tau_{\text{inel}} \propto T^{-p}$, we find from both estimates that $L_{\text{eff}}^2 \propto T^{-p}$ and it follows from Eq. (2.16) that the temperature dependence of the conductivity of a two-dimensional film is[46]

$$\Delta\sigma(T) \sim \text{const}\cdot\frac{e^2}{\hbar}\ln\frac{T}{T_0}, \tag{3.7}$$

i.e., the conductivity decreases logarithmically as a result of cooling. Similar behavior follows from generalization of the work in Ref. 38 to the case of finite temperatures. Further cooling should result in a transition from the logarithmic dependence (3.7) in the "metallic" region to the exponential temperature dependence of the conductivity in the "insulating" region.

We shall now review the experiments designed deliberately to check these theoretical predictions. We shall begin by considering evidence in favor of 30 000 Ω/□ is the maximum resistance of a metallic film.[27] The first indication of the existence of such a limit was in fact obtained back in 1914 (Ref. 47). A study was then made of the resistance of platinum films as a function of their thickness, governed by the duration of evaporation. The resistance increased on reduction in the thickness (reduction in the deposition time) and in all cases did not exceed a value of about 28 000 Ω/□, which was followed by an abrupt increase to more than 10^9 Ω/□. These results were later rediscovered by Licciardello[48] and interpreted by him as supporting Eq. (2.6). Similar results were obtained more recently.[49,50] In particular, Liang *et al.*[49] studied the conductance of Bi films; the dependence of the conductance of the film thickness is plotted in Fig. 6. We can see that there is a jump in the conductance in the region of 10^{-4} Ω^{-1}. Similar results were also obtained by Anderson[50] for Au films.

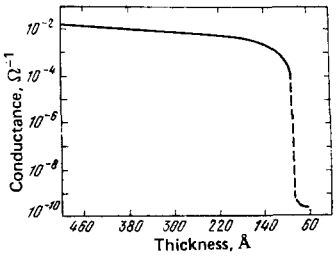

FIG. 6. Dependence of the conductivity of a Bi film on its thickness.[49]

One of the first investigations designed deliberately to check the predictions of Ref. 27 was reported by Vul *et al.*,[51] who studied conduction in thin high-conductivity layers formed at grain boundaries in germanium bicrystals.[14] A two-dimensional network of edge dislocations formed on these boundaries and such a network represented (in germanium) a negatively charged surface of partly filled bonds with adjoining layers with p-type conduction and a thickness of a few tens of angstroms. The electrical conductance of such layers was investigated by Vul *et al.*[51] as a function of the grain misorientation angle. A transition from the metallic to the activated conduction was observed at $\sigma_{m.m} \sim 4\times 10^{-5}$ Ω^{-1}, in agreement with the estimates of Ref. 27.

The most thorough investigation of the conductance of metallic films carried out with the aim of checking the predictions of Ref. 27 was reported by Dynes, Garno, and Rowell.[52] They studied films prepared by the evaporation of Pb, Sn, Au, Al, or Cu on a substrate whose temperature was kept near 4.2 °K. The conductance was measured directly in the apparatus used to evaporate the films, which made it possible to increase gradually the film thickness until it exhibited

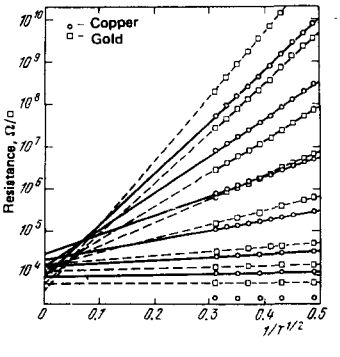

FIG. 7. Temperature dependences of the resistance of thin copper and gold films. The results demonstrate a transition from the activated to the metallic conduction near 30 000 Ω (Ref. 52).

[12] These estimates have been found to be overoptimistic[146]: in the case of real systems the values of τ_{inel} are considerably smaller.

[13] The length $L_1 \propto \tau_{\text{inel}}^{1/2} \propto T^{-p/2}$ determines, according to Thouless,[44] the above-mentioned temperature dependence of the resistance of a one-dimensional wire: $\propto T^{-p/2}$.

[14] See also a later investigation reported in Ref. 147.

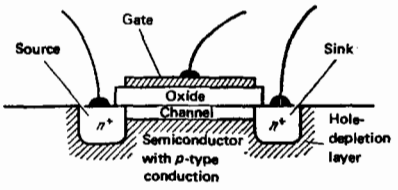

FIG. 8. Field-effect transistor with an n-type channel. An inversion layer is produced by applying an external potential to the gate electrode.

metallic conduction. The transition from the activated to the metallic conduction mechanism occurred near a resistance of 30000 Ω/□. Typical data obtained in Ref. 52 are reproduced in Fig. 7. In the region of activated conduction the resistance of Sn films varied as $\exp(1/T)$, whereas in the case of Au and Cu films the dependence was $\exp(1/\sqrt{T})$. Dynes *et al.*[52] pointed out that films with such a high resistance could hardly be homogeneous, i.e., they probably consisted of metal "islands" linked by narrow channels or separated by tunnel barriers. The predictions of Ref. 27 are generally applicable to a homogeneous system, but according to the ideas put forward in Ref. 48, they are valid also in the case of inhomogeneous systems.

Another system which would seem to be ideally suited for investigating two-dimensional conduction is a metal–oxide–semiconductor field-effect transistor (MOSFET) with an inversion layer of carriers (Fig. 8). This layer is created by the application of an external potential to the gate electrode and the conductance of the layer is measured directly between the source and sink. Variation of the gate voltage makes it possible to alter (within wide limits) the properties of the inversion layer and, in particular, the carrier density in the layer. An important property of such a system is the fact that the carrier density can be found from electrostatic calculations and it does not have to be deduced from, for example, the Hall effect. A detailed review of the experiments and physical phenomena in inversion layers was published by Adkins,[53] and we shall not discuss them in detail. We shall simply mention that the results of such experiments generally *disagree* with the predictions of the theory of localization in general and with the conclusions of Ref. 27 in particular. The transition from the activated to the metallic conduction may occur at conductance values considerably higher

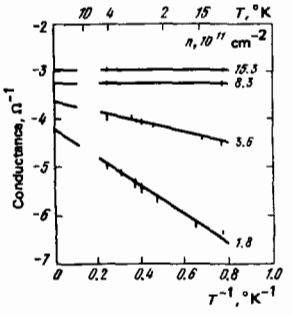

FIG. 9. Typical temperture dependences of the conductance of an inversion layer.[53] The different dependences correspond to different electron densitites in the inversion layer.

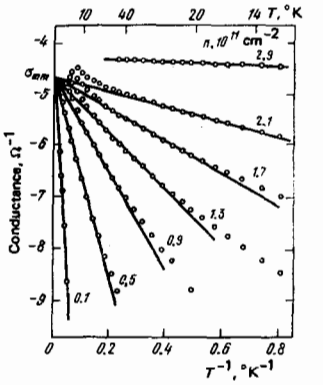

FIG. 10. Behavior of the conductance of inversion layers with low electron densities.[53]

than 3×10^{-6} Ω^{-1} (Fig. 9). However, results in agreement with Ref. 27 are sometimes obtained (Fig. 10). Clearly, these contradictory results arise from the fact that the role of the electron–electron correlations in inversion layers is important[48,53] and this, in particular, gives rise to phenomena of the Wigner crystallization type.[53] Therefore, it is possible that inversion layers are not a very convenient object for verifying the theory of localization, which unfortunately does not allow for the role of the electron–electron interaction.

The first experimental investigations designed deliberately to test the predictions of Thouless[44] on the conductivity of thin metallic wires have appeared recently. The very first studies[54,55] failed to observe the effect. The most convincing evidence in support of the localization in wires was obtained by Giordano, Gilson, and Prober.[56] They prepared samples by an original lithographic method (Fig. 11). A glass substrate, half-covered by a metal film [Fig. 11(a)], was bombarded with Ar^+ ions and this produced a step in the glass [Fig. 11(b)]; then, the metal film was removed by chemical means [Fig. 11(c)]. Next, a new metal film was evaporated [Fig. 11(d)] and then the sample was subjected again to the bombardment with Ar^+ ions incident at an angle such that the metal behind the step remained in the shadow [Fig. 11(d)]. This produced a thin "wire" [Fig. 11(e)]. An investigation of such wires with a scanning electron microscope showed that they were continuous strips of uniform cross section. The wires used in that investigation were made of $Au_{60}Pd_{40}$ films formed by two different evaporation methods, which yielded films with a resistivity 3.7×10^{-4} Ω. cm (and ratio of the resistances at room temperature and at 12 °K amounting to 1.03), which were referred to as "contaminated" films, as well as films with a resistivity of

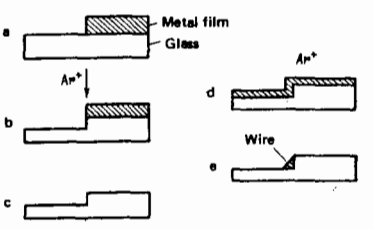

FIG. 11. Method of preparation of thin wires.[56]

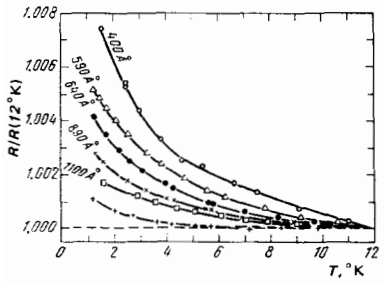

FIG. 12. Temperature dependences of the resistance of wires of different cross sections. The results given for each sample are normalized to the resistance at 10°K. The numbers above the curves give the values of $\sqrt{A}$. The lowest curve shows the behavior of a continuous film.[56]

1.0×10^{-4} Ω.cm (and a resistance ratio 1.06), which were referred to as "pure" films. The cross sections of the wires varied from 1×10^{-11} cm^2 to 3×10^{-10} cm^2. The values of the cross section A were deduced from the known lengths and resistances of the wires and the values of A found in this way were in good agreement with those expected from the known heights of the steps and film thicknesses. The $Au_{60}Pd_{40}$ films had thicknesses from 200 to 1000 Å before the Ar^+ bombardment. The resistance of the wires was from 15 to 500 kΩ and their length was from 90 to 450 μ.

Figure 12 shows the temperature dependences of the resistance of various contaminated wires with different cross sections. Figure 13 demonstrates how the resistance rises (at a fixed temperature $T=1.5$ °K) as a function of the cross section A in the case of contaminated and pure wires. The increase in the resistance of a continuous film is subtracted here (the origin of this increase is not clear and it may even be due to two-dimensional localization).[15)] We can see that the resistance increases in accordance with the law A^{-1} and that the resistance is higher for the contaminated wires. Thus, the results obtained are in qualitative agreement with the predictions of the theory of Thouless.[44]

The temperature dependence of the resistance of a typical wire is shown in Fig. 14. Here again the effect of a continuous film is subtracted. We can see that the temperature dependence is approximately logarithmic

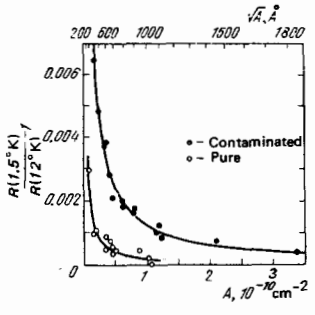

FIG. 13. Resistance plotted as a function of the cross section for contaminated (black dots) and pure (open circles) samples. The curves are proportional to A^{-1} (Ref. 56).

15) This is confirmed by recent investigations.[146]

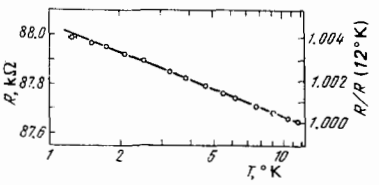

FIG. 14. Resistance plotted as a function of the logarithm of the temperature of a contaminated wire with $A = 3.6\times10^{-11}$ cm^2 ($\sqrt{A}=590$ Å).[56]

although, according to Ref. 56, we cannot exclude the possibility that the dependence is of the $T^{-1/2}$ type. This behavior is not in agreement with the power-law increase in the resistance (of the T^{-2} type due to the scattering by phonons) as a result of cooling, predicted in Ref. 44. This disagreement may reflect insufficient understanding of the inelastic scattering processes and not shortcomings of the theory of localization, especially as the dependence of the effect on the geometric dimensions (cross section A) clearly supports the theory. It should be stressed that the ideas of Thouless on the power-law rise of the resistance are somewhat qualitative.[16)]

We shall now consider the results obtained in another investigation[57] designed deliberately to check the predictions of Thouless.[44] "Wires" studied in Ref. 57 were actually thin films of length L, width B, and thickness C, where $L\gg B\gg C$.[17)] Such samples should either exhibit one-dimensional behavior predicted by Thouless[44] or two-dimensional behavior predicted by Abrahams, Anderson, Licciardello, and Ramakrishnan.[33] Dolan and Osheroff[57] investigated films whose resistance was $R_c < 30\,000$ Ω/□ (known as the Licciardello–Thouless limit[27]). The behavior of these films should be two-dimensional if the resistance is below this limit (metallic region) but $L_1\propto\sqrt{D\tau_{inel}} < B$ and/or $L_2\propto\sqrt{\tau_{inel}/\hbar}/\sqrt{N(E)} < B$ [compare with Eqs. (3.4), (3.5), and Ref. 46], i.e., under the conditions such that an electron "does not know" that it is in a "one-dimensional" wire.

Films of the composition 58 wt.% Au and 42 wt.% were prepared on substrates which were mostly made of polished sapphire (and kept at room temperature). Long and narrow strips were obtained by a modification of the lithographic method suggested in Refs. 58 and 59. The film thicknesses were $(2–4)\times10^{-7}$ cm and, therefore, they almost certainly consisted of separate metal islands, the resistance being governed by the tunneling between these islands.[52] This inhomogeneity was unimportant provided the films were homogeneous over distances of the order of R_{1oc} of Eq. (3.2), as well as L_1, L_2, and B. Measurements were carried out down to temperatures of ~10×10^{-3} °K. Nonmetallic behavior was exhibited by the temperature dependences of the

16) As pointed out earlier, simple estimates[44] clearly exaggerate the value of τ_{inel}, which explains the relative difficulty of observation of the localization effects and their small magnitude.[146]

17) Samples with L = 2–7 mm, B = 0.1–1 μ, and $C\approx3\times10^{-7}$ cm were investigated.

resistance and by the nonlinear dependence of the resistance on an external electric field (voltage). These resistance dependences could be described by the following formulas:

$$R(T,\ E)=R(T_0,\ E)\left(1-S_T\ln\frac{T}{T_0}\right) \qquad (3.8)$$

in the case of low fields E, and

$$R(T,\ E)=R(T,\ E_0)\left(1-S_V\ln\frac{E}{E_0}\right) \qquad (3.9)$$

in the case of low values of T, where T_0 and E_0 are arbitrary normalization points, and S_T and S_V are experimentally determined parameters.

Figures 15 and 16 give the experimental results obtained for one of the samples in Ref. 57.

This behavior was interpreted in Ref. 57 in the spirit of predictions of Abrahams, Anderson, Licciardello, and Ramakrishnan,[33] i. e., as associated with complete localization in a two-dimensional system [compare with Eq. (3.7)]. It was concluded that the investigated samples which exhibited behavior of the type described by Eqs. (3.8) and (3.9) were two-dimensional. The nonlinear dependence on the field of Eq. (3.9) was explained in Ref. 46 by a mechanism involving electron heating. It should be pointed out that no nonlinear dependences on the external field were observed in Ref. 56; it was concluded there that this difference was due to inhomogeneities of the samples in Ref. 57. A surprising result reported in Ref. 57 was the saturation of the logarithmic dependence of $R(T, 0)$ at temperatures below 70×10^{-3} °K. This could not be explained theoretically on the basis of the interpretation of Abrahams *et al.*[33] The effect in question could be due to quite different factors (for example, the role of electron–electron correlations).

The results discussed above were obtained for films with a resistance of 1000–5000 Ω/□ and a width of the order of 1 μ. In the case of two samples with a resistance exceeding 10 kΩ/□ and several samples with a lower resistance but a width of the order of 0.1 μ the resistance $R(T, 0)$ increased exponentially with $1/T$, but saturation was again observed in the region of 70×10^{-3} °K. An exponential dependence of the localization at such very low resistances was another aspect that

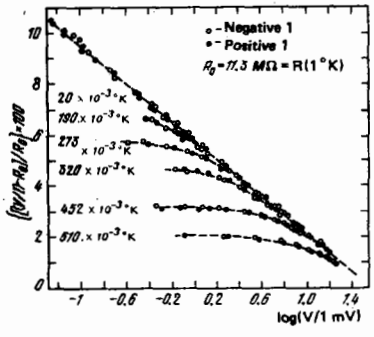

FIG. 15. Dependence of $\Delta R=[(V/I)-R_0]/R_0$ on the logarithm of an external voltage $\ln V$ plotted for different temperatures. Here, I is the current, V is the voltage, and R_0 is the resistance of a sample at $T=1$°K (Ref. 57).

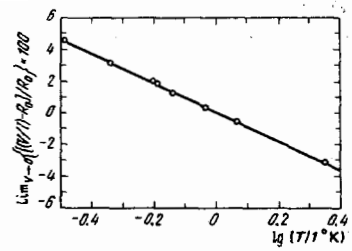

FIG. 16. Dependence of the resistance at low values of V, deduced from Fig. 15, on $\ln T$ (Ref. 57).

could not be explained from the theoretical point of view.

There have been preliminary reports[60] of observations of the logarithmic temperature dependence of the resistance of inversion layers in field-effect transistors.[18)] According to Ref. 60, an additional analysis of the results of Dynes *et al.*[52] obtained for the "metallic" region revealed a similar logarithmic dependence. This slight rise of the resistance on cooling was not explained earlier, because the results were plotted on a logarithmic scale (of the conductance; see Fig. 7).

It is thus clear that the available experimental data indicate a qualitative agreement with the predictions of the theory of localization, but—in our opinion—these results are insufficient for reliable selection of a specific theoretical model. We cannot exclude also the possibility of a different interpretation. We should mention in this connection the work of Al'tshuler and Aronov[61] proposing an original mechanism of the increase in the resistance as a result of cooling, associated with interference between the Coulomb interaction of electrons and the impurity scattering. Generalization of the treatment given there to the $d=2$ case gives rise to corrections to the conductance, which are logarithmic functions of temperature *of the same type* as the effects of two-dimensional localization.[62,63] This demonstrates the need for further theoretical analysis and new experiments. In particular, it would be very useful to carry out measurements in a magnetic field, including those of the Hall effect in the "metallic" range of two-dimensional films. Measurements of the Hall effect in inversion layers have in fact stimulated a proposal for a new interpretation of the phenomena in such systems.[53] The negative magnetoresistance exhibited by these systems[64] has been attributed[41] to the suppression of two-dimensional localization by an external magnetic field and by the effects of the spin–orbit scattering as well as the scattering accompanied by spin flipping.[39-41] We should mention, however, a recent investigation[139] of the resistance of wires made of an amorphous W–Re alloy. These wires were produced from amorphous W–Re films of 50 Å thickness by electron-beam lithography.[140] The wires were divided into two groups in accordance with their width; from 700 to 5000 Å and 2×10^{-3} cm. A detailed investi-

[18)] The results of experiments on inversion layers in Si were reported in detail in Ref. 141. They were very similar to the results given in Ref. 57.

gation confirmed in practice all the results of Ref. 56 and demonstrated a nonmetallic temperature dependence of the conductance of these wires[19] as well as a dependence of the conductance on the wire cross section for samples whose resistance considerably exceeded 10 $k\Omega$. Measurements in an external magnetic field gave particularly interesting results. Application of a 4×10^4 G field (either parallel or perpendicular to a wire) resulted at $T < 20$ °K (i. e., in the range of temperatures where the localization effects appeared) in a fairly strong *rise* of the resistance. The effect of a perpendicular field was greater. These results supported the conclusion that an external magnetic field aids the manifestation of localization effects.[20]

4. DOES SCALING EXIST AT A MOBILITY EDGE?

The presentation of the theory in the preceding sections was fairly elementary. We will try to avoid a false impression that the theory of localization is simple by reviewing briefly attempts to construct a "consistent" scaling theory of localization in the vicinity of a mobility edge and by reviewing the problems encountered in this approach. A scaling theory is understood to be a consistent derivation and verification (or rejection) of assumptions of the elementary theory presented in Sec. 2, calculation of the critical index of the localization radius and of other critical indices describing the behavior of the correlation functions of the electron system (if such indices can be introduced at all), calculation of such functions (Green's functions), etc. In other words, it would be desirable to "raise" the scaling theory of localization from the qualitative level of the Kadanoff constructions[20] to the level of a quantitative theory and to the level of the modern theory of critical phenomena.[21-24] We shall show that this task is far from being complete and this is due to the fact that difficulties of fundamental nature are encountered.

The ideas for a consistent scaling description of a mobility edge were put forward soon after the appearance of the modern theory of critical phenomena and were formulated practically simultaneously by a number of authors.[65-72,24] There is now an extensive literature on various aspects of this problem.[73-86]

There are two alternative ways of constructing a theory of localization rigorously. One of them (used until recently to obtain all the main conclusions) is based on the pioneering work of Anderson[3] (Anderson approach). The main feature of this approach is its familiar departure from tradition involving a discussion of random configurations of the Green's function system of an electron in the model of (1.1) not averaged over an ensemble. In a sense, this approach considers *the most probable* Green's function.[15,16] This is due to the fact that a one-particle *averaged* Green's function does not (as is well known[3-6,12,15,18]) give, in principle, any information on a possible localization of electrons, but we can study localization by considering the divergence of a stochastic perturbation theory series for a nonaveraged one-particle Green's function.[3,12] The condition for convergence of this series determines, in particular, the position of the mobility edges in a band.[3-6,12,13] This condition is so far the only way of determining the mobility edges analytically (a somewhat different method proposed in Refs. 87 and 88 is related genetically to the same original Anderson approach).

Another approach to a theory of localization, put forward in the well-known paper of Edwards[89] (Edwards approach), uses Green's functions averaged over an ensemble of random configurations of a system. The advantage of this approach is its familiar "automatism" (availability of the diagram technique). Averaged Green's functions determine, in principle, all the physical quantities in the theory (this is not true of the most probable Green's function in the Anderson approach), but in order to investigate localization (static conductivity at $T = 0$) we have to find the averaged two-particle Green's function of an electron.[89,90] This immediately makes the problem very complex. It has been solved only for the one-dimensional case.[10,11] Nevertheless, the traditional nature and convenience of this approach makes it a favorite with the majority of theoreticians.[21] Recent investigations[91,92] carried out using a self-consistent variant of the traditional method have yielded a number of interesting results (including complete localization in the $d = 2$ case). However, some of these conclusions (such as the presence of a gap in the absorption spectrum at low frequencies) have met with objections.

The Anderson approach allows us to obtain quite simply scaling at a mobility edge[68] in the sense of the usual theory of critical phenomena.[21-24] This is because the spatial behavior of the Anderson most probable Green's function is governed entirely by the statistics of intersection-free paths in the investigated lattice,[3,12,93,94] i. e., it is a purely geometric problem. This is related to a summation of the perturbation theory series for V in the case of the Green's function of the Hamiltonian (1.1) allowing for multiple scattering processes.[3,12] The main approximation made after this summation is the restriction of the discussion to just one initial Green's function, although one of the results of the summation is an infinite number of similar series for the auxiliary Green's functions.[3,12] This is due to the assumption of identical stochastic properties of all these series. The higher approximations[3,12,13] relating to the statistical properties of a series and various ways of estimating its convergence clearly influence only the positions of the mobility edges in a band and are unimportant in our case. Consequently, a nondiagonal (with respect to the lattice sites) matrix element of

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[19] A subsequent investigation of W–Re wires in the superconducting state[148] made it possible to determine directly the value of τ_{inel}, which agreed well with the localization effect.

[20] A similar result was obtained in Ref. 146 for films, but no influence of a magnetic field on wires was detected.

[21] This duality in the theory of disordered systems is reflected in the dialogue between the "author" and "a theoretician" in the review by Éfros.[6]

a one-electron Green's function can be represented in the form[93,68]

$$G_{ij}(E) \sim \sum_{N=1}^{\infty} Z_N(\mathbf{R}_i - \mathbf{R}_j) K^{-N} F^N\left(E, \frac{W}{V}\right), \tag{4.1}$$

where $F(E, W/V)$ is the localization function determining the position of mobility edges in the band:[12,13]

$$F\left(E_c, \frac{W}{V}\right) = 1, \tag{4.2}$$

and $Z_N(\mathbf{R}_i - \mathbf{R}_j)$ is the number of intersection-free paths consisting of N steps connecting a site i to a site j; K is the connectivity constant of the lattice $\ln K = \lim_{N\to\infty} N^{-1} \ln Z_N$, Z_N is the total number of intersection-free paths consisting of N steps. The values $Z_N(\mathbf{R}_i - \mathbf{R}_j)$ obtained as a result of computer calculations were used in Ref. 93. The main idea of Ref. 68 was to deal with Eq. (4.1) by applying an analytic theory of intersection-free paths based on the theory of critical phenomena proposed by de Gennes[95] and des Cloizeaux.[96] According to Refs. 95 and 96,

$$Z_N(\mathbf{R}_i - \mathbf{R}_j) \sim \int_{c-i\infty}^{c+i\infty} \frac{d\tau}{2\pi i} e^{Na\tau} G(\mathbf{R}_i - \mathbf{R}_j; \tau), \tag{4.3}$$

i.e., this quantity is found by inverse Laplace transformation of $G(\mathbf{R}_i - \mathbf{R}_j; \tau)$ which is a correlation (Green's) function in the theory of critical phenomena, governed by the usual fluctuation-free Landau energy with an n-component order parameter[22-24]:

$$F = \frac{1}{2} \sum_{j=1}^{n} [(\nabla\Phi_j)^2 + \tau\Phi_j^2] + \frac{1}{8} g_0 \left(\sum_{j=1}^{n} \Phi_j^2\right)^2; \tag{4.4}$$

here, $g_0 > 0$, which corresponds to "repulsion" of intersection-free paths and a is the lattice constant. The statistics of intersection-free paths is derived from Eqs. (4.3)–(4.4) by going to the limit $n \to 0$ (Refs. 95 and 96). This eliminates loop graphs, which do not occur in the problem of intersection-free paths. Using Eq. (4.3), we obtain directly from Eq. (4.1)

$$\left.\begin{aligned} G_{ij}(E=0)|_{\frac{W}{V} \geqslant \left(\frac{W}{V}\right)_c} &\sim G\left(\mathbf{R}_i - \mathbf{R}_j; \frac{W - W_c}{W_c}\right), \\ G_{ij}(E)|_{E \sim E_c;\ \frac{W}{V} \leqslant \left(\frac{W}{V}\right)_c} &\sim G\left(\mathbf{R}_i - \mathbf{R}_j; \frac{E - E_c}{E_c}\right), \end{aligned}\right\} \tag{4.5}$$

i.e., the spatial behavior of the Anderson Green's function in the theory of localization is governed by the correlation function of the standard theory of critical phenomena (4.4) (with $n = 0$) and the mobility edge corresponds to the phase transition point. Near the mobility edge [i.e., for $W/V \gtrsim (W/V)_c$ or for $E \lesssim E_c$] the Green's function decreases exponentially with the distance:

$$G_{ij} \propto \exp\left(-\frac{|\mathbf{R}_i - \mathbf{R}_j|}{R_{loc}}\right), \quad |\mathbf{R}_i - \mathbf{R}_j| \gg R_{loc}, \tag{4.6}$$

where the localization radius is

$$R_{loc} \sim a\left[\frac{(W/V) - (W/V)_c}{(W/V)_c}\right]^{-\nu}, \tag{4.7}$$

for localization at the center of a band and

$$R_{loc} \sim a\left[F'\left(E_c, \frac{W}{V}\right) E_c\right]^{-\nu} \left(\frac{E_c - E}{E_c}\right)^{-\nu} \tag{4.8}$$

for localization in the $E \lesssim E_c$ case. The critical index of the localization radius ν is identical with the critical index of the correlation length in the theory of critical phenomena. In terms of the Wilson ε expansion[22] ($d = 4 - \varepsilon$, $n = 0$) we have

$$\nu \approx \frac{1}{2} + \frac{\varepsilon}{16} + \frac{15\varepsilon^2}{512} + \ldots, \approx 0.592 \text{ for } \varepsilon = 1\ (d = 3). \tag{4.9}$$

A numerical calculation of the statistics of intersection-free paths gives $\nu \approx 0.6$ (Ref. 93). For $W/V = (W/V)_c$ or $E = E_c$, we have

$$G_{ij} \sim |\mathbf{R}_i - \mathbf{R}_j|^{-(d-2+\eta)}, \tag{4.10}$$

where the index η obtained within the framework of the ε expansion ($d = 4 - \varepsilon$, $n = 0$) is[22]

$$\eta \approx \frac{\varepsilon^2}{64}\left(1 + \frac{17}{16}\varepsilon + \ldots\right) \approx 0.032 \text{ for } \varepsilon = 1\ (d = 3). \tag{4.11}$$

Such very small values of η mean, in particular, that the power-law localization[68] proposed for the Anderson model in Ref. 26 is impossible. The values of the index ν can also be very important. According to Mott,[97] a conductivity jump at a mobility edge (i.e., the minimum metallic conductivity) can occur if the inequality $\nu > 2/d$ is satisfied. It follows from Eq. (4.10) that $\nu < 2/d$; this may mean a continuous fall of the conductivity to zero, analogous to that discussed in connection with Eq. (2.13).

Unfortunately, since the Anderson Green's function does not determine (as pointed out above) physical quantities such as the density of states or the conductivity, we cannot go far beyond the results obtained. Moreover, although the localization radius does occur in several formulas (for hopping conduction, high-frequency conduction between localized states, etc.),[1] it is not very clear whether there are direct experimental methods for determining it and the index ν; this applies even more to η. It should be mentioned only that an increase in R_{loc} on approach to a mobility edge may affect the diamagnetic susceptibility of a system.[98]

We can find directly the most important physical quantities by turning to the calculation of the averaged Green's functions (Edwards approach). By way of example of the problems encountered here we can describe the familiar model of an electron in a Gaussian random field (see, for example, Refs. 99–102), which proceeds from the usual problem of an electron in a system of randomly distributed point scatterers in the limit $\rho \to \infty$, $V \to 0$, $\rho V^2 \to \text{const}$, where ρ is the density of the scatterers and V is the scattering amplitude.[101] It is easy to formulate this model in terms of a language similar to the theory of critical phenomena,[24,67,68,70-72,75,76] or, more exactly, similar to the unstable field theory of a specific type. The main conclusions reached in these investigations largely overlap and we shall follow the treatment in Refs. 70 and 76. It is found that the averaged Green's function of an electron in a Gaussian random field (with a "white noise" correlation function) can be defined as the Green's function of a Euclidean scalar field theory with the following Lagrangian [compare with Eq. (4.4)]

$$\mathcal{L} = \frac{1}{2} \sum_{j=1}^{n} \left[\frac{1}{2m}(\nabla\Phi_j)^2 - E'\Phi_j^2\right] - \frac{1}{8}\rho V^2 \left(\sum_{j=1}^{n} \Phi_j^2\right)^2, \tag{4.12}$$

$$E' = E \pm i\delta,$$

where m is the mass of an electron and E is its energy. The number of field components approaches the limit

$n \to 0$, which again excludes "loop" graphs which do not occur in the Edwards diagram technique.[89] It should be pointed out that Eq. (4.12) does not contain random parameters. This is an "effective" Lagrangian, which yields directly the diagram technique for the averaged Green's functions. A Green's function is defined by the functional integral

$$G(\mathbf{r}-\mathbf{r}'|g=-\rho V^2)=-\frac{1}{Z}\lim_{n\to 0}\frac{1}{n}\sum_{j=1}^{n}\int\{\delta\Phi(\mathbf{r})\}\,\Phi_j(\mathbf{r})\,\Phi_j(\mathbf{r}')\,e^{-S[\Phi]} \equiv \lim_{n\to 0}\frac{1}{n}\sum_{j=1}^{n}\langle\Phi_j(\mathbf{r})\,\Phi_j(\mathbf{r}')\rangle, \tag{4.13}$$

where

$$S[\Phi]=\int d^d r\,\mathcal{L}(\mathbf{r}) \tag{4.14}$$

is the action in the field theory in Eq. (4.13) and

$$Z=\int\{\delta\Phi(\mathbf{r})\}\,e^{-S[\Phi]}. \tag{4.15}$$

The functional integral (4.13), corresponding to the Lagrangian (4.12), diverges in the case of formal calculation because of the "incorrect" sign of the coupling constant $g=-\rho V^2<0$ (attraction!). This reflects the familiar instability of the ground state in field theory [Eq. (4.12)].[103] Therefore, the functional integral should be regarded as *an analytic continuation* with respect to the coupling constant from arbitrary values $g>0$ to $g=-\rho V^2<0$. Analytic properties of the Green's functions in the field theory $g\Phi^4$ in the complex plane of the coupling constant are given by the following dispersion relationship applicable to this constant:[104-106]

$$G(g)=\frac{1}{\pi}\int_{-\infty}^{0}dz\,\frac{\Delta(z)}{z-g}, \tag{4.16}$$

where

$$\Delta(g)=\frac{1}{2i}[G(g+i\varepsilon)-G(g-i\varepsilon)]=\operatorname{Im}G(g), \tag{4.17}$$

i.e., the Green's functions in this theory are analytic with respect to the coupling constant in the complex plane with a cut along the negative part of the real axis and $\Delta(g)$ is a discontinuity at the cut (which differs from zero for $g<0$) which in this approach governs all the principal properties of the Green's functions.

The functional integral (4.14) can be analyzed by the steepest-descent method.[108-110] This can be done by considering first the classical field equations corresponding to the Lagrangian (4.13):

$$\frac{1}{2m}\Delta\Phi_j=-E\Phi_j-\frac{1}{2}\rho V^2\Phi_j\Big(\sum_{j=1}^{n}\Phi_j^2\Big). \tag{4.18}$$

In the method of steepest descent we shall be interested in the solutions of Eq. (4.18) with *finite action* (4.14) (instantons).[106-110] In this problem the important solutions of Eq. (4.18) are spherically symmetric of the type

$$\Phi_j(r)=\Phi_0(r)\,\mathbf{u}, \tag{4.19}$$

where $\mathbf{u}^2=1$ is a unit vector in "isotopic" spin space of the $O(n)$-symmetric field theory under discussion. In this model such solutions exist for $d<4$ if $E<0$ (Refs. 75 and 76). These solutions have the form

$$\Phi_0(r)=\sqrt{\frac{2|E|}{\rho}}\,\chi(t),\qquad r=\frac{t}{\sqrt{2m|E|}}, \tag{4.20}$$

where χ and t are dimensionless variables. The explicit form of $\chi(t)$ is known only for $d=1$ but it is easy to show that

$$\Phi_0(r)\underset{r\to\infty}{\propto}\frac{(2m|E|)^{-1/4}}{r^{(d-1)/2}}\exp[-(2m|E|)^{1/2}r],\qquad \Phi_0(r)\underset{r\to 0}{\to}\text{const},\quad \frac{d\Phi_0(r)}{dr}\Big|_{r=0}=0. \tag{4.21}$$

If $E>0$ there is only a trivial solution of Eq. (4.18) with a finite (equal to zero!) action $\Phi_0(r)\equiv 0$. Therefore, the steepest-descent method of calculation of the functional integral (4.13) reduces in this case to the conventional perturbation theory. In the space with $d=4-\varepsilon$ dimensions there is a dominant sequence of diagrams known as the parquet pattern.[111,112] If we analyze the problem in this parquet approximation,[70] we find that because of the incorrect sign of the coupling constant the effective vertex (a "four-leg" vertex) has an apparent pole in the theory under discussion and this pole corresponds to the inapplicability of perturbation theory in the energy range ($\hbar=1$)

$$E\ll E_{sc}=\frac{1}{2ma^2}\left(\frac{u}{4-d}\right)^{2/(4-d)}, \tag{4.22}$$

where $u=(m^2/2\pi^2)\rho V^2 a^{4-d}$ is a dimensionless interaction constant and a is a constant with dimensions of length (the shortest length in the problem, corresponding to the correlation radius of a random field[70,76]). The quantity E_{sc} is an exact analog of the "Ginzburg" critical range in the theory of phase transitions.[24,113] The existence of such a range in the problem of an electron in a random field was first pointed out in Ref. 65. However, in contrast to the theory of critical phenomena, the transition to this energy range does not result in weakening[22-24] but in enhancement of the effective interaction[70] (transition to the tight binding range). The perturbation theory parameter is the dimensionless ratio $(E/E_{sc})^{(d-4)/2}$ and in the limit $E\to 0$ we find that perturbation theory is no longer valid and its formal application gives nonphysical results.[70,71] This has been pointed out already in Ref. 68 and was expressed in Ref. 72 in terms of the recurrent formulas of Wilson,[22,23] because of the absence of fixed points of these equations in the problem under discussion.

In the range $E<0$ an important role is played by nonlinear solutions (instantons) given by Eq. (4.20). The occurrence of these solutions gives rise, in the calculation of the functional integral by the steepest-descent method,[75,76] to the appearance of contributions which are nonanalytic with respect to the coupling constant and are of the type ($A_d=\text{const}$)

$$\exp(-S[\Phi_0])=\exp\left(-A_d\frac{m^{-d/2}}{\rho^{1/2}}|E|^{2-(d/2)}\right), \tag{4.23}$$

first obtained by Zittartz and Langer.[114] In particular, in the momentum representation, we have

$$\operatorname{Im}G(E\mathbf{p}|-\rho V^2)=C(E\mathbf{p})\frac{1}{(\rho V^2)^{(d+1)/2}}\exp\left(-A_d\frac{m^{-d/2}}{\rho^{1/2}}|E|^{2-(d/2)}\right), \tag{4.24}$$

where $C(E\mathbf{p})$ is some function of E and $\mathbf{p}$ independent of the coupling constant and the power of the coupling constant in the preexponential factor is governed by the

number of "zeroth modes" of the instanton solution.[75,76] Knowing $\operatorname{Im} G(Ep|-\rho V^2)$, which represents the jump across a cut, we can reconstruct the real part from Eq. (4.16). The method of calculation of $C(Ep)$ based on the use of the dispersion relationship for the coupling constant (4.16) and on the correspondence with the theory of critical phenomena is proposed in Ref. 76. The form of this function is determined in Ref. 75 from dimensional considerations. If $d=1$, both methods give the result for the density of states identical with the exact solution of Halperin.[115] If $d>1$, the results for the preexponential factor given in Refs. 75 and 76 are different.[22)]

The criterion of validity of the results of the (4.24) type is the large value of the argument of the exponential function, which again reduces to the condition $|E| \gg E_{sc}$ (Refs. 70, 71, 75, and 76), which should be compared with Eq. (4.22). Thus, a "Ginzburg" critical range appears also on the negative energy side. It is the range $|E| \lesssim E_{sc}$ that is of interest from the point of view of the scaling theory of localization (the mobility edge of the problem in question is located there), but an analysis of the phenomena in this range of strong (or intermediate) coupling is inaccessible to the current theory. A hypothesis of the possibility of the conventional scaling behavior of the theory in this range of energies (in spite of the "incorrect" sign of the coupling constant) is put forward in Ref. 76 on the basis of the universality of a jump across a cut in the dispersion relationship (4.16) [the same function $\Delta(z, Ep)$ governs the behavior of the correlation function in the problem of an electron in a random field corresponding to $g<0$ in Eq. (4.16) for $|E| \lesssim E_{sc}$ and the usual scaling behavior of the correlation function in the theory of critical phenomena corresponding to $g>0$ in Eq. (4.16)]. This would lead, in particular, to the usual values for the critical indices (4.9) and (4.11). Then, the derivative of the density of states with respect to the energy acquires a singularity of the type encountered in the specific heat[76] in the theory of phase transitions. However, Wegner[84] suggested that such values of the indices are in conflict with the inequalities that follow from analytic properties of the Green's functions.[23)]

The problem of calculating the conductivity (two-particle averaged Green's function) requires an analysis not of Eq. (4.12) but of a different effective Lagrangian of two interacting zero-component ($n \to 0$, $m \to 0$) fields[72,75] ($E' = E + i\delta$, $E'' = E - i\delta$):

$$\mathcal{L} = \frac{1}{2}\sum_{j=1}^{n}\left[\frac{1}{2m}(\nabla\Phi_j)^2 - E'\Phi_j^2\right] + \frac{1}{2}\sum_{i=1}^{m}\left[\frac{1}{2m}(\nabla\varphi_i)^2 - E''\varphi_i^2\right] - \frac{1}{4}\omega\left(\sum_{j=1}^{n}\Phi_j^2 - \sum_{i=1}^{m}\varphi_i^2\right) - \frac{1}{8}\rho V^2\left(\sum_{j=1}^{n}\Phi_j^2 + \sum_{i=1}^{m}\varphi_i^2\right)^2, \quad (4.25)$$

where ω is the frequency of an external field [we are speaking here of the conductivity $\sigma(\omega)$]. This Lagrangian yields directly (for $n \to 0$, $m \to 0$) a sequence of diagrams that determine conductivity in the Edwards problem.[89,90] The expression (4.25) has the $O(n)\times O(m)$ symmetry in the isotopic spin space. In the limit $\omega \to 0$ this symmetry reduces to the $O(m+n)$ symmetry.[116] The importance of this circumstance for an electron in a random field was recently stressed by Wegner[81,82] (see below).

Unfortunately, the use of perturbation theory in the problem (4.25) gives rise, as indicated by a direct generalization of the results of Ref. 116, in the parquet approximation for $n \to 0$ and $m \to 0$ to the same nonphysical singularities in the vertex parts as in the case of the simpler problem (4.12) with the one-electron Green's function, which again indicates that perturbation theory is inapplicable. This result was obtained in Ref. 72 in terms of the recurrent Wilson formulas.

The range $E<0$, $|E| \gg E_{sc}$ for Eq. (4.25) was considered in Ref. 75, where electron localization was demonstrated for this range but the conclusions were basically preliminary. It is interesting to note that the localization is associated with the appearance of zeroth modes of the instanton solution.

We can envisage also a different approach to the problem of calculation of a two-particle averaged Green's function (conductivity) based on the use of a different effective Lagrangian and of an analogy between localization and a transition to a spin glass state.[117-119] A "fluctuating field" ("order parameter") is then of tensor (matrix) nature. Unfortunately, when variants of this approach are used, the direct link to the usual Edwards diagram technique[89] is lost, contrary to the case when the effective Lagrangians (4.12) and (4.25) are used and this link is retained. The renormalization group analysis of the tensor Lagrangian proposed in Ref. 73 demonstrated the existence of a stable fixed point in the recurrent formulas corresponding to the usual critical Wilson indices ($n=0$) of Eqs. (4.9) and (4.11), but this point is inaccessible by physical variation of the parameters of the system.

It is pointed out in Ref. 74 that the effective Lagrangian[73] does not include all the possible invariants of the corresponding tensor field. An attempt is made in Ref. 74 to relate the problem of conduction in the two-dimensional Anderson model to the critical behavior of a planar XY model.[120] In particular, a conductivity jump at a mobility edge is obtained there and for the $d=2$ case it is given by

$$\sigma_{mm} = \frac{e^2}{\hbar}\frac{K(T_c)}{2\pi} = \frac{1}{\pi^2}\frac{e^2}{\hbar} \approx 0.101\frac{e^2}{\hbar}, \quad (4.26)$$

which is in good agreement with the Licciardello–Thouless result of Eq. (2.6) (Ref. 27). Such a jump is associated in Ref. 74 with the universal jump of the coefficient of spin-wave rigidity $K(T_c)=2/\pi$ in the two-dimensional XY model, discovered by Nelson and Kosterlitz.[121] However, it should be pointed out that the arguments used in Ref. 74 are fairly approximate and the result (4.26) is obtained from calculated estimates. The attainment of the relevant fixed point in the XY

[22)] See also a discussion of the problem of the preexponential factor in Ref. 149.

[23)] The author of the present review does not agree fully (see Ref. 150) with the arguments in Ref. 84, but the assumptions in Ref. 76 remain unproven and should be regarded as a hypothesis.

model is not demonstrated in Ref. 74. Moreover, there are some doubts about the validity of the results of Nelson and Kosterlitz (see Ref. 122). Qualitative arguments in favor of the scaling law (2.13) for the conductivity in the $d>2$ case are also given in Ref. 74.

An original approach to the problem in question not based on perturbation theory is proposed in Refs. 77 and 78. A detailed presentation of the proposed method is given in Ref. 123, where it is applied to statistics of polymer chains. In these investigations it is also pointed out that there is a region of strong coupling of the E_{sc} type given by Eq. (4.22). Moreover, inequalities are obtained there for the critical index of the conductivity in the law (2.13). These inequalities admit[77] the existence of a minimal metallic conductivity in the case of $d=2$ but not $d=3$.

Wegner[69] was the first to derive the scaling law for the conductivity (2.13). He developed an analytic scheme for transformation of the renormalization group in real space applicable to the Anderson model and analogous to the Thouless scheme.[4,25-29] Two alternative types of critical behavior are proposed in Ref. 69, but the selection between them cannot be made by the methods discussed in that paper. The scaling law (2.13) is obtained in both variants but in what is known as the inhomogeneous case the derivative of the density of states has a singularity at a mobility edge and this singularity is of the same type as that encountered in the specific heat in the theory of critical phenomena (compare with Ref. 76). In the homogeneous case the density of states has no singularity at a mobility edge; for this reason in more recent papers[81-83] Wegner prefers this variant of critical behavior. However, it should be pointed out that there are no rigorous theorems (see Ref. 4) which would forbid a weak singularity in the density of states at a mobility edge, although most authors regard the occurrence of this singularity as not very likely. A method for calculating the critical indices is not given in Ref. 69 and there is no proof that the corresponding fixed points of the equations of the renormalization group are attainable.

In recent investigations[79,80] Wegner discussed an interesting model of an electron in a random lattice with n levels at each site. For each site there are n eigenfunctions $|j\alpha\rangle$ $(\alpha=1,2,\ldots,n)$, where n is regarded as large and $1/n$ is the small parameter of the theory. The intention is to construct an analog of the $1/n$ expansion in the theory of critical phenomena[24] and apply it to the problem of localization.

The Hamiltonian of the model is

$$H=\frac{1}{\sqrt{n}}\sum_{\substack{ij\\ \alpha\beta}} f_{i\alpha,\,j\beta}\,|i\alpha\rangle\langle j\beta|, \tag{4.27}$$

where the matrix elements $f_{i\alpha,\,j\beta}$ are regarded as independent Gaussian random variables satisfying the condition of symmetry and reality: $f_{i\alpha,\,j\beta}=f^{*}_{j\alpha,\,j\beta}=f_{j\beta,\,i\alpha}$. Their average values are regarded as zero and the variance is given by

$$\langle f_{i\alpha,\,j\beta} f_{k\gamma,\,l\delta}\rangle=(\delta_{\alpha\delta}\delta_{\beta\gamma}\delta_{il}\delta_{jk}+\delta_{\alpha\gamma}\delta_{\beta\delta}\delta_{ik}\delta_{jl})\,M_{i-j}. \tag{4.28}$$

The Hamiltonian (4.27) represents a generalization of the Anderson model and it includes, not only n orbitals per site, but also disorder in the transport integrals.

Since the matrix elements $f_{i\alpha,\,j\beta}$ are independent and have a symmetric distribution, the probability of finding a system with the specific Hamiltonian given by Eq. (4.27) is equal to the probability of finding a system with

$$H=\frac{1}{\sqrt{n}}\sum_{\substack{ij\\ \alpha\beta}} S_{i\alpha} f_{i\alpha,\,j\beta} S_{j\beta}\,|i\alpha\rangle\langle j\beta|, \tag{4.29}$$

where $S_{i\alpha}=\pm 1$. If $\Psi_\alpha(j)$ is an eigenfunction of Eq. (4.27), then $S_{i\alpha}\Psi_\alpha(j)$ is an eigenfunction of Eq. (4.29). The system is invariant under local changes in the sign of the wave functions at a site. This variant of the model discussed in Refs. 80 and 81 is known as a local gauge-invariant model. The local gauge invariance imparts the following property to the Green's functions:

$$\langle G_{i\alpha,\,j\beta}(E)\rangle = S_{i\alpha}S_{j\beta}\langle G_{i\alpha,\,j\beta}(E)\rangle, \tag{4.30}$$

which shows that

$$\langle G_{i\alpha,\,j\beta}(E)\rangle = G(E)\,\delta_{ij}\delta_{\alpha\beta}; \tag{4.31}$$

here, the angular brackets represent averaging over random variables of the system. Thus, in this model a one-particle Green's function is of the "point" type because of random fluctuations of the phases of the wave functions. Similarly, a two-particle Green's function vanishes if the four sites defining it are not pairwise coincident.

In the limit $n=\infty$ the model has an exact solution[80] and the results of this solution are identical with the results of application of the coherent potential approximation[124] to this model. In this case *there are no mobility edges* and the density of states is

$$\frac{1}{n}N_\infty(E)=\begin{cases}\dfrac{2}{\pi E_0^2 v_0}\sqrt{E_0^2-E^2}, & |E|\leqslant E_0,\\ 0, & |E|>E_0,\end{cases} \tag{4.32}$$

where $E_0^2=4\sum_j M_{i-j}$, i.e., the density of states is proportional to the variance of the matrix elements of the Hamiltonian (4.28), and v_0 is the volume of a unit cell. The static conductivity (at $T=0$) is

$$\sigma_\infty(E)=\frac{ne^2}{\pi\,dv_0}\left(1-\frac{E^2}{E_0^2}\right)R^2\sim\frac{ne^2}{a^{d-2}}\left(1-\frac{E^2}{E_0^2}\right), \tag{4.33}$$

where $R^2=\sum_j R_{ij}^2 M_{i-j}/\sum_j M_{i-j}$ represents the radius of action of random correlations and a is the lattice constant. The last equality in Eq. (4.33) applies to the case of the "short-range" variance of the matrix elements $R\sim a$. Thus, the conductivity vanishes at a band edge.

If we begin from the exact solution for $n=\infty$, we can develop a systematic (although cumbersome) procedure for calculating the corrections in powers of $1/n$ (Ref. 80). This gives rise to an interesting but not fully understood analogy of the behavior of *two-particle* Green's functions in this model with the behavior of the longitudinal and transverse susceptibilities of an isotropic ferromagnet below the phase transition point.[80,82]

The conductivity may be represented by the following series in powers of $1/n$:

$$\frac{1}{n}\sigma=\sigma_\infty+\frac{1}{n}\sigma_1+\ldots+\frac{1}{n_t}\sigma_t+\ldots, \tag{4.34}$$

where the coefficients σ_t are finite for $d>2$ and diverge no faster than $(d-2)^{-t}$ in the limit $d\to 2$. If we retain in Eq. (4.34) only the most diverging terms of the order of $n^{-t}(d-2)^{-t}$ and ignore terms of the order of $n^{-t}(d-2)^{-t'}$ with $t'<t$, we find that

$$\sigma = n\sigma_\infty(1-2\gamma+O(\gamma^3)), \tag{4.35}$$

where

$$\gamma=\frac{1}{2\pi n[1-(E^2/E_0^2)](d-2)R^2/a^2}. \tag{4.36}$$

This expansion is valid in the range of delocalized states [this is the zeroth approximation for Eq. (4.33)]. A mobility edge should appear at some value $\gamma=\gamma_c$, when the static conductivity vanishes; the metallic region corresponds to $\gamma<\gamma_c$. Clearly, in view of the convergence of Eq. (4.36) in the limit $d\to 2$, the latter inequality is never satisfied in the two-dimensional case. This is in agreement with the results of Ref. 33. If we assume that

$$\sigma=n\sigma_\infty\left(1-\frac{\gamma}{\gamma_c}\right)^s, \tag{4.37}$$

we find that a comparison with Eq. (4.35) gives $\gamma_c=1/2$ and the critical index of the conductivity is

$$s=1. \tag{4.38}$$

If, following Refs. 33 and 69, we assume that $s=(d-2)\nu$, we find that the critical index is

$$\nu=\frac{1}{d-2}, \tag{4.39}$$

which is identical with the first term of the expansion $(d=2+\varepsilon)$ for an isotropic ferromagnet.[125,126] However, as pointed out earlier, the reason for this analogy is not understood.

In Refs. 81–83, Wegner draws attention to the circumstance, mentioned above in connection with the effective Lagrangian (4.25), that the frequency of an external field ω in the conductivity problem acts as a source ("field") which disturbs the $O(n+m)$ symmetry of the problem and

$$\frac{1}{4}\lim_{\substack{n\to 0\\ m\to 0}}\left(\frac{1}{n}\sum_{j=1}^{n}\langle\Phi_j^2\rangle-\frac{1}{m}\sum_{i=1}^{m}\langle\varphi_i^2\rangle\right)=\frac{1}{4}[G(rr, E+i\delta)-G(rr, E-i\delta)]=\frac{i\pi}{2}\nu_0 N(E), \tag{4.40}$$

i.e., the density of states acts as the "order parameter" which disturbs this symmetry. This analogy is unusual, since it is normally assumed that the density of states $N(E)$ remains finite at the mobility threshold[81,82] in contrast to the order parameter at the phase transition point. Then, assuming that the critical index β of the order parameter vanishes and employing the usual scaling relationship $\beta=(\nu/2)(d-2+\eta)=0$ in the case when $\nu\neq 0$, Wegner obtains

$$\eta=2-d. \tag{4.41}$$

He established this analogy using a somewhat different formalism based on the matrix model[81,82] of the type employed in Refs. 73 and 74. Wegner suggested a matrix effective Lagrangian for the description of the mobility edge allowing for the above symmetry breaking and representing a generalization of the so-called nonlinear σ model. In this model there is no mobility edge for $d=2$, in agreement with Ref. 33. Renormalization group calculations based on these models were reported in Ref. 127. These calculations give ε expansions for the critical indices $(d=2+\varepsilon)$ whose zeroth terms coincide with Eqs. (4.38) and (4.39) in the case of a problem symmetric relative to the orthogonal transformations of a fluctuating matrix field. In a second variant of the same model (invariant under unitary transformations) the indices are different. At present it is not clear which physical situations correspond to these two variants of critical behavior (however, see Refs. 40 and 41).[24] This is clearly associated with the fact that in the Wegner model,[81-83] as well as in earlier matrix models,[73,74] there is no direct correspondence with the standard apparatus for calculating the averaged Green's functions of an electron in a random system,[89,90] which makes it difficult to obtain specific physical results and to make comparisons with the known simple approximations to the theory. In this sense, the formalism based on the effective Lagrangians of the (4.12) and (4.25) type seems to be preferable. At present it is not clear either how the above difficulties associated with the inapplicability of perturbation theory can be avoided in the matrix formalism. It is likely that these difficulties are indeed avoided when the problem is considered near the "lower" critical dimensionality $d=2$, whereas we have considered above—in connection with Refs. 68–73, 75–78—the vicinity of the "upper" critical dimensionality $d=4$. It should be mentioned that in recent papers there have been some arguments to suggest that the "upper" critical dimensionality in the localization problem is $d=8$ (Refs. 85 and 86). However, it seems to us that these conclusions are premature.

This review of various investigations of scaling at a mobility edge gives a somewhat checkered pattern in which the final and reliable results are difficult to identify. Only one thing is certain: the problem of justifying scaling at a mobility edge is much more difficult than the corresponding problem in the conventional theory of critical phenomena. In the mobility edge case we are encountering such obstacles as the need to use an unstable theory of fields, inapplicability of perturbation theory, etc. The absence of universal methods for solving problems of this kind is familiar. Therefore, the problem in question may require development of basically new approaches and methods.

We shall conclude this section by mentioning one other extremely important theoretical problem for a discussion of which we simply have no space. This is allowance for the Coulomb correlations in connection with the problem of localization or the so-called theory of Fermi glasses.[15,128-130] This problem is of fundamental importance, but so far only the first and frequently contradictory results have been obtained.[61,128-134] Clearly, the next important stage in the development of

[24] Further developments of this subject can be found in recent interesting papers.[142,151-153]

the theory will involve this direction. However, the review of the results published so far and especially formulation of some overview of the problem would be, in our opinion, even more premature than the corresponding procedure in the case of the theory of critical behavior at a mobility edge discussed above.

CONCLUSIONS

We have discussed new results obtained recently in the theory of electron localization in disordered systems and a number of experiments stimulated by these results. In the main, the reviewed ideas on the physics of disordered systems have been developed by the application of ideas and methods taken from the theory of critical phenomena. This approach has been found to be quite successful and has given a number of new and unexpected results, such as transition of long metallic wires to the insulating state and complete localization in two-dimensional systems. From this point of view, any sufficiently long metallic wire is strictly speaking a nonmetal. Similarly, there are serious grounds for expecting nonmetallic properties in the case of two-dimensional films prepared by the evaporation of metals, although in this case the theoretical and experimental situations are still fairly contradictory. In this sense one can speak of a macroscopic manifestation of the quantum phenomenon of electron localization in disordered systems. Experimental observations of these effects are, however, only possible at very low temperatures and on very small samples. The very possibility of investigating such systems has arisen only because of the developments in microelectronics and technology of large integrated systems, which has required suitable lithographic methods. Therefore, there is no need to stress the importance of the new results from the point of view of microelectronics itself.

We have also tried to demonstrate the incompleteness of the results obtained and to stress the unresolved problems. The theory is still far from complete clarity and this applies to such fundamental problems as the very existence or otherwise of the minimum metallic conductivity. This situation is primarily due to the fact that the problem of localization and the description of behavior of electron states near a mobility edge is a much more complex problem than the corresponding tasks in the theory of phase transitions. A complete solution of these problems may require the development of new theoretical methods and formulation of subtle experiments whose importance may go beyond the problem under consideration. Although at present it is difficult to make definite forecasts, there is no doubt that a deeper understanding of the problem of electron localization in disordered systems will foster further theoretical and experimental studies of disordered systems, and practical applications of these systems.

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