# Maxwell's "Treatise on Electricity and Magnetism"

M. L. Levin and M. A. Miller

Usp. Fiz. Nauk 134, 425–440 (November 1981)

Some of the features of the "Treatise on Electricity and Magnetism" (on which Maxwell continued to work until the end of his life after the first editon of 1873) are discussed. They relate to the language, style, construction, and mode of reasoning. The text of the "Treatise" provides an indication of the arguments that led Maxwell to his equations of electrodynamics, and his subsequent editing throws light on his further intentions. Close examination of the "Treatise" reveals that some deeply rooted existing views on the original content and form of Maxwell's equations, and also their history, are in fact fallacious.

PACS numbers: 01.60. + q, 03.50.De

In this short article (more a sketch than a review) we should like to share with the reader certain impressions that, in our view, should arise when the "Treatise" is examined by those who, a century later, are still preoccupied with Maxwell's electrodynamics. Soviet readers will soon be helped in this endeavor by a new complete translation of the "Treatise" promise by Nauka Press for the near future[1)].

## 1. BIBLIOGRAPHIC INFORMATION

James Clerk Maxwell (May 13, 1831–November 5, 1879)[2)] completed and published his "Treatise" in 1873. This was the only edition published in his lifetime.[7] Just before his death, when he was already seriously ill, he undertook the preparation of the second edition but got only as far as the fundamentals of electrostatics, although he also rewrote the introductory chapter (Preliminary). The latter provides us with some clues as to his further intentions, but it is not entirely clear what they were, and it is probable that Maxwell intended to return to some of these ideas later.

The second, posthumous edition of the "Treatise" (1881) was prepared by Professor W. D. Niven[3)] who appears to have transferred all Maxwell's own corrections to the first nine chapters from the original manuscript and provided the remaining chapters with his own explanations. The latter may well have been given in the spirit of Maxwell's lectures, given after the publication of the first edition. Unfortunately, Niven did not always explicitly mark his own insertions, thus preventing us from separating "factual" from "editorial" matter. The third edition[8] appeared relatively rapidly (in 1891), even by our standards, but Professor Niven was not able to contribute to it because, as one might say today, of the pressure of administrative and teaching duties[4)], and the publishers turned to J. J. Thomson whose encounter with the "Treatise" turned out to be historic. He verified all Maxwell's results and, without altering the original text (or, more precisely, the text of the second edition), provided generous commentaries, corrections, additions, and even a "Supplement Volume," although the working life of Maxwell's electrodynamics began much earlier. All the subsequent editions were sterotype reissues of the first.

This is hardly the place (nor would it be possible) to examine the many subsequent years of methodological and mathematical "polishing" and elucidation of the fundamental ideas laid down in the "Treatise." The result of this process was the familiar (indeed, routinely familiar) form of Maxwell's electrodynamics used today, although the value of this modernization has generally tended to be somewhat overstated. There is little doubt, however, that even without the efforts of people such as Heaviside and Hertz (canonical form of the equations, 1889–1890), Poynting (conservation of the energy of the electromagnetic field, 1884), H. A. Lorentz (1875), Fitzgerald (boundary value problem and derivation of the Fresnel formulas, 1878), and Hertz again (field due to an elementary source, 1889), all these and subsequent generalizations would have been introduced by others[5)].

## 2. LANGUAGE

Maxwell first became interested in electrical and magnetic phenomena in 1885 when he was already 24 and could confidently regard himself as a physicist. But the realization of a closed reciprocal connection between these phenomena came to him, probably, in 1861. As far as the final "Treatise" is concerned, it was written

[1)]Maxwell's work has been available in Russian only in the form of collections of excerpts, including fragments of the "Treatise"[1] with J. J. Thomson's notes and Boltzmann's very expressive commentaries. The Preface to the "Treatise" was newly translated in Ref. 2. All these editions have long been out of print.

[2)]Many of the topics that we shall cover below were in one way or another intimately connected with the circumstances of Maxwell's life. Biographical details can be found in Niven's preface to Ref. 9, the biographies of Campbell and Garnett[10], Smith-Rose[3], and MacDonald[4], and the chapter by Claus[5], in Ref. 5 and the fascinating paper by V. L. Kartsev in Ref. 6.

[3)]The French (1889) and German (1883) translations were based on the Niven edition of the "Treatise".

[4)]This may have been only a polite excuse: Niven had only just published the two-volume collected works of Maxwell[9] (of which a Russian translation is sadly lacking) and it could not have been easy for him to follow this immediately with yet another, very laborious, editorial task.

[5)]It is interesting that Maxwell had time to become familiar with the work of Lorentz and of Fitzgerald, but this had no influence on the initial chapters of the "Treatise", although it may be conjectured that their results could have been incorporated in the chapter on waves.

 © 1982 American Institute of Physics 

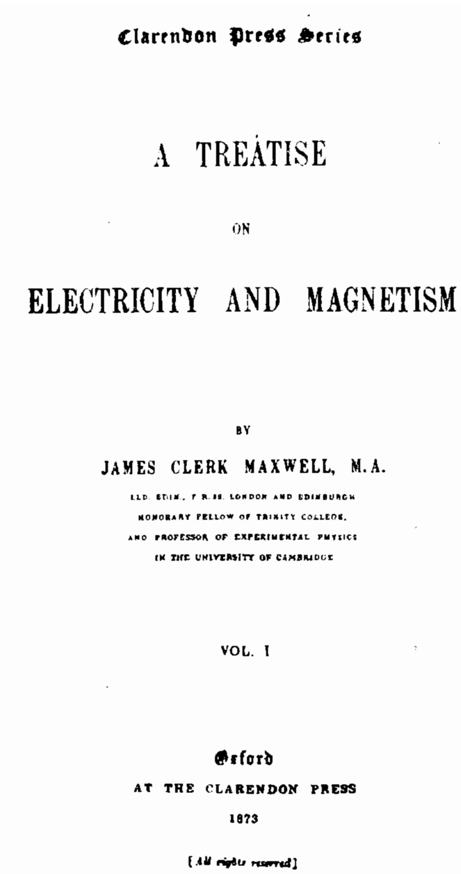
at the peak of his powers. Indeed, it even is possible that this was an (alas, not an infrequent) case when a maximum coincides with the boundary of an interval. The frankness of a man's utterances without doubt increases with his achievements, for one who has tasted success rises above fear of making mistakes. From the modern point of view, the "Treatise" is indeed a marvelous (and hence pedagogically instructive) example of scientific honesty. It can rightly be compared to a building which, though complete and free from scaffolding, still bears the traces of work in progress. This may well have been the reason why the greatness of this magnificent edifice was not immediatley acknowledged by all.

Now that we know that the answers are correct, this quality, i.e., the hint of incompleteness and, at times, sketchiness, makes the "Treatise" into a unique document, enabling us to examine the various "risk factors," to trace the development of doubts, and to appreciate the perseverance of the author in avoiding tempting blind alloys. It is instructive to consider in such cases not only the tactics and techniques of approach, but even the semantics—the run of phrase and discussion, their cause-and-effect structure, the way reservations are expressed, and so on. To some extent, much of this is dictated by the standards and traditions of the language (which is Maxwell's case go back to Shakespeare and Sterne, and influence his way of thinking), but it is also true to say that these characteristics transcend language and involve the personality of the author, and can therefore be used to diagnose his individual qualities.

Maxwell's style is unusual. It is unhurried and prone to repetition as he gradually develops a particular thesis and examines it verbally from different angles. He prefers to use conditional phrases such as "if we suppose," "if we adopt," "if we take," or even "if we denote"—as if trying to leave the way open for other possibilities. This insinuating caution is then suddenly interrupted by bursts of inspired enthusiasm, frequently in the form of unusual and precise expressions of ideas.

Together with the frequent and often unexpected subdivision of the text into verse-like paragraphs, all this gives the impression of a solemn text, full of digressions in the form of "sermons" of independent interest: on measurement, on galvanometers, on the state of polarization, on solid angles, on analogies, and so on.

Maxwell's language is both concise and rich. It is particularly diverse terminologically. Some of his concepts live, develop, and then vanish altogether. To others he remains faithful to the very end but retains in reserve a few equivalent phrases. For example, permittivity initially appears as specific inductive capacity, then as dielectric constant, then as dielectric capacity, and so on. It is interesting that these vacillations have persisted to this day despite desperate efforts of standardizers. His terminology relating to fields is exceedingly diverse. Electric field strength acquires a new name whenever it appears in a new guise: it is the electric force when it originates in one charge and



Clarendon Press Series

A TREATISE

ON

ELECTRICITY AND MAGNETISM

BY

JAMES CLERK MAXWELL, M.A.

LL.D. EDIN., F.R.SS. LONDON AND EDINBURGH
HONORARY FELLOW OF TRINITY COLLEGE,
AND PROFESSOR OF EXPERIMENTAL PHYSICS
IN THE UNIVERSITY OF CAMBRIDGE

VOL. I

Oxford
AT THE CLARENDON PRESS
1873

[All rights reserved]

acts on another, it is the electric intensity when it is the field in a medium (but its direction is still along the line of force), it is the intensity of electromotive force or simply the electromotive intensity or even electromotive force at a point, i.e. density, when the field arises from a change in magnetic introduction. [Translator's note: I have not been able to verify all these occurrences in the Third Edition of 1904. "Electromotive intensity" appears, for example, in Articles 44 and 598, "Electromotive force" in Article 45, "Electric intensity" in Article 68, "Intensity of force" in Article 122. In general, the word "intensity" is associated with the electric force per unit charge.] This diversity is not encountered in the case of the magnetic field strength: it always appears as the "magnetic force" (although we note that it does not have the dimensions of mechanical force[6)]). All this is, of course, a reflection of the process of consolidation and amalgamation of concepts (fields **E** of different origin have combined into a single concept, but the magnetic field **H** has remained itself), which is inseparable from the process

[6)]The Russian equivalent of "force" is "sila". It has been greatly overworked as the result of unforseen inbreeding and thoughtlessness of translators). It has also been used as a translation of the English word "strength" in phrases such as "strength of current", "pole strength" and so on. "Zhivaya sila" (*vis viva*) is still (somewhat archaically) used for "kinetic energy", but occasionally it appears with dimensions of power as in "loshadinaya sila" (horse power).

of modification of interrelations between physical phenomena.

It is very likely that Maxwell was not worried by these temporary, intermediate infelicities or by the free use of relatively informal language. Here he followed Faraday and used his ideas and pictorial representations with the intention to develop rather than to trim them. For example, he states that "In his [Faraday's] published researches we find these ideas expressed in language which is all the better fitted for a nascent science, because it is somewhat alien from the style of physicists who have been accustomed to establish mathematical forms of thought" (Article 528 of the "Treatise"). And again, with still greater conviction in the protective function of noncanonical language: "... he did not feel called upon either to force his results into a shape acceptable to the mathematical taste of the time, or to express them in a form which mathematicians might attack. He was thus left at leisure to do his proper work, to coordinate his ideas with his facts, and to express them in natural, untechnical language" (Article 528).

## 3. MODEL OF THINKING

Maxwell was after all a mathematician by education, so that his mode of thinking and research was closest to that of "model" physicists (in the at that time uncorrupted meaning of the word "model"). He was apparently able to visualize all that he could understand. And at the slightest provocation he resorted to models borrowed from the fundamental science of Dynamics (always treated with deference and spelled with upper case initial letter), against which he verified every electrodynamic result. This has been partly responsible for the persistent legend (which has outlived its usefulness) of the Maxwell aether which still surfaces in modern textbooks in the guise of the "cogwheel picture"[7)] of his early exploratory papers (see Refs. 1 and 9). However, these auxiliary constructs are already absent from the first edition of the "Treatise," and the electromagnetic field appears as an independently measurable physical entity in its own right. Since, at the time, dynamics was perhaps the only branch of physics with a logically closed theoretical description (postulates → measurements → rules → measurements → conclusions → measurements → postulates), comparison of other phenomena with dynamics was desirable (and probably necessary) even if these phenomena could not be reduced to dynamics. It is interesting to note that the role of the "standard of comparison" for the examination and interpretation of different phenomena (including mechanical phenomena!) is now assumed by Maxwell's macroelectrodynamics which is now a closed subject, whose parameters can readily be measured, and which, most importantly, is a ready source of intuitive images and possible experiments. It is thus a happy hunting ground for analogies. In a certain sense Maxwell put forward some suggestions for analog machines even before he established his equations, i.e., without using the generality of the mathematical description. On the contrary, he established this description on the basis of physical similarity between different phenomena. We are so used to using analogies that we tend to miss the subtlety of Maxwellian model comparisons (made before the equations were available!)[8)]

Let us now consider another, more specific, example of a Maxwellian model-based analysis which throws unexpectedly revealing light on a problem that would appear to be completely solved in a purely formal fashion. We shall take the liberty of translating it into modern language and notation.

Consider the currents induced by external sources in a thin (finite thickness $d$) plane conducting (conductivity $\sigma$) sheet ($z=0$). If we suppose that the external sources vary slowly with time (so that the skinlayer depth is very large in comparison with $d$), we may suppose that the current is distributed uniformly within the thickness of the sheet, i.e., its effects are equivalent to those of a surface current $\mathbf{i}$ flowing in the $z=0$ plane with surface density $\sigma'=\sigma d$. The vector potential $\mathbf{A}$ which gives the magnetic field due to the current ($\mathbf{H}=\mathrm{curl}\mathbf{A}$) is then an even function of $z$, so that only its transverse ($\perp\mathbf{z}_0$) components are nonzero [$A_z=0, A_\perp^+(z>0)=A_\perp^-(z<0)\neq 0$], which means that the potential can be written in terms of a scalar function $P$ (the prototype of the scalar magnetic potential, i.e., satisfying outside the sources the Laplace equation):

$$\mathbf{A}=\mathbf{z}_0\times\nabla P. \qquad (\alpha)$$

---

7) We recall the celebrated and frequently cited phrase of Poincare's: "The complicated structure which he attributed to the ether rendered his system strange and unattractive; one seemed to be reading the description of a workshop with gearing, with rods transmitting motion and bending under the effort, with wheels, belts, and governors". This was reproduced in the Russian collection entitled "Maxwell's theory and Hertzian oscillations"[11] and may have been an indication of the fact that Poincare was not fully familiar with the "Treatise" although the latter was published while Poincare was a student. It is also possible that Kelvin himself, who contributed to the involvement of Maxwell in electromagnetism, had not got around to reading the "Treatise", at least not before his Baltimore lectures (1884).[12]

8) Mechanical analogs of the electrodynamic system, based on the "ready-made" Maxwell equations, are not in principle unique. They have frequently been considered (and continue to be so even today), sometimes with "applied" intentions and sometimes for their own sake (see, for example, the paper by Kelly in Ref. 5). Here is an obvious way of doing it. All space is first divided into cells whose size depends on the precision of reproduction but at any rate must be less than the wavelength. The electromagnetic field within each such cell may be concealed in quasistationary discrete elements, namely, self-inductors, capacitors, and resistors, which have their mechanical analogs in mass and the coefficients of elasticity and damping. This correspondence was already known to Maxwell. The reverse analogy is sometimes more naturally realized because mechanical models of gyrotropic, resonance, and dispersive (especially spatially dispersive) electromagnetic media require a certain amount of ingenuity.

The current in the sheet can be found from the discontinuity in the tangential component of the magnetic field, $H_T$:

$$\mathbf{i}=\frac{c}{2\pi}\mathbf{z}_0\times\mathbf{H}^+=\frac{c}{2\pi}\mathbf{z}_0\times\operatorname{curl}\mathbf{A}\,|_{z=+0}. \qquad (\beta)$$

It can also be expressed in terms of the electric field:

$$\mathbf{i}=\sigma'\mathbf{E}_T=-\frac{\sigma'}{c}\frac{\partial}{\partial t}(\mathbf{A}_\perp+\mathbf{A}_\perp^{\mathrm{ext}})|_{z=+0}, \qquad (\gamma)$$

where $\mathbf{A}^{\mathrm{ext}}$ is the vector potential of the field due to the external sources (in the absence of the conducting sheet).

Since the right-hand sides of the last two equations must be equal, we have the following expression for the positive side of the sheet:

$$\mathbf{z}_0\times\operatorname{curl}\mathbf{A}=-\frac{2\pi\sigma'}{c^2}\frac{\partial}{\partial t}(\mathbf{A}_\perp+\mathbf{A}_\perp^{\mathrm{ext}})|_{z=+0}. \qquad (\delta)$$

Finally, if we substitute ($\alpha$) into ($\delta$), we obtain the required expression derived by Maxwell:

$$\frac{c^2}{2\pi\sigma'}\frac{\partial P}{\partial z}=\frac{\partial}{\partial t}(P+P^{\mathrm{ext}})\,. \qquad (\varepsilon)$$

This is actually the boundary condition for the problem, whose solution can be completed by any standard method (for example, by separating the variables). However, at this point Maxwell departs from the expected line of approach. He considers that, since the coefficient $V=c^2/2\pi\sigma'=c^2/2\pi\sigma d$ has the dimensions of velocity, one should be able to assign to the system some motion with this particular constant velocity. In fact, it turns out that the relaxation of the currents in the conducting sheet (say, after a magnetic field is suddenly turned off at time $t=0$) occurs in such a way that the field in the region $z>0$ is identical to the field that would be produced if one were to freeze the initial currents in a sheet moving away with velocity $z=-Vt$.

Next, it is well known that, in the limiting case of a perfect screen ($\sigma\to\infty$, $V\to 0$), the normal component of the magnetic field is zero on the screen ($H_n|_{z=0}=0$), and the field due to the sources can be obtained by the usual method of images. Proceeding from this, Maxwell leads us with a certain natural inevitability to the method of receding images for screens of finite conductivity ($\sigma\neq\infty$, $V\neq 0$) whereby in each interval of time $\delta t$ an element $(\partial P^{\mathrm{ext}}/\partial t)\delta t$ moves away from the image of the source and recedes from the sheet with velocity $V$. In Maxwell's own words: "If we suppose that in every succesive element of time an image of this kind is formed, and as soom as it is formed it begins to move away from the sheet with velocity $V$, we shall obtain the conception of a trail of images, the last of which is in the process of formation, while all the rest are moving like a rigid body away from the sheet with velocity $V$." [Translator's note: quotation from "Treatise" added in translation; Maxwell uses $R$ instead of $V$; Article 662.] The field due to this trail of images gives us the solution of the boundary-value problem with the boundary condition ($\varepsilon$)[9]. In this example, the model (again a dynamic model) does not substitute for the entire process, but contrasts one electrodynamic system against another (no less electrodynamic), thus enabling us to recognize qualitiatively the possibilities of the method of images, and to predict the result for cases that cannot be solved exactly, for example, when the size of the trial becomes comparable with the width of the screen. All this may well make us realize that we often respond to dimensional clues and use "toy models" which at first sight contribute little to the rigorous solution but, in fact, enrich our intuition, and therefore turn out to be more significant than the original special solution from which they derive.

Maxwell himself is sometimes able to go directly to the heart of the phenomena in hand and to see the similarity between them without resorting to models. For example, the analogy between electrostatics and magnetostatics (and also between electrostatics and stationary currents in conducting media), which had been known in general terms before, assumes a "field significance" in Maxwell's work, and actually looks like a principle of duality (permutation duality) for the electromagnetic field ($\mathbf{E}\to\mathbf{H}$, $\mathbf{H}\to\mathbf{E}$, $\varepsilon\rightleftharpoons\mu$). The only probable reason that he did not formulate this in its modern form is that he simply physically had no time (he died) to bring together all his diverse analogies and run them through his equations.

## 4. STRUCTURE

"I have therefore thought that a treatise would be useful which should have for its principal object to take up the whole subject in a methodical manner, and which should also indicate how each part of the subject is brought within the reach of methods of verification by actual measurement" ("Treatise," Preface). In implementing this program, Maxwell collected together all the experimental and theoretical advances in the field of electricity and magnetism that were known at the time. The structure of the "Treatise" is historical and problem oriented. It begins with electrostatics, then goes on to conduction (including electrolysis), and devotes particular attention to the principles of sources and instruments. The same scheme is used for magnetic phenomena. He next considers what is now called the theory of quasistationary circuits. This is a key point in the "Treatise" because it marks the beginning of the remarkable generalizations that led to the development of the complete electrodynamics. After chapters devoted to the equations for fields and potentials, Maxwell again returns to a discussion of basic quantities, their measurability, dimensions, and correspondence with those introduced earlier in the historical sequence. At the end of the "Treatise" there are "test" solutions of the field equations: plane electromagnetic waves in homgeneous media and in crystals,

[9]This is, of course, not only an example of the analog approach, but also an "instrumental" problem in its own right. We note that the method of receding images has continued to appear and disappear in classical texts on electrodynamics (see Appendix). It can be found in Smythe[13] with a reference to Jeans.[14] But, in spite of this, it apparently had to be rediscovered in connection with certain modern problems involving plasma rings in systems used in controlled thermonuclear fusion.

and the Faraday rotation of the plane of polarization. The concluding chapter is devoted to the wave theory of light.

In his methodological plane, Maxwell is always faithful to Faraday: "...I resolved to read no mathematics on the subject till I had first read through Faraday's 'Experimental Researches in Electricity'" ("Treatise," Preface). He also adopted this approach in his review of all that was known about electromagnetism at that time.

## 5. MEASURABLE QUANTITY

"The most important aspect of any phenomenon from a mathematical point of view is that of a measurable quantity" ("Treatise," Preface). Thus begins the "Treatise." It is also the basic position of its creator. Here Maxwell appears as philosopher (although this had more "applied" connotations than it has today), formulating the principles for the construction of a physical theory. And thereafter he does not allow himself any departure from this standpoint: a physical quantity must be measurable, i.e., it must admit of a direct or indirect comparison with a standard. From this there follows the concept of physical meaning as a relation between measurable quantities. There is no doubt that this firm attitude facilitated the acceptance of Faraday's idea of a field as a continuously distributed and everywhere measurable object. It is interesting that Maxwell follows each important symbolic expression with a verbal statement, as if once again correlating a logical operation with the prescription of measurement.

These brief remarks do not do justice to Maxwell's attitude to measurement. Maxwell was not one of those "high-power" theoretical physicists to whom measurement is in principle desirable but is preferably done by others. Apparatus and the precision of measurement attracted his concern and participation. At the time, the notorious division of physicists into experimenters and theoreticians had not reached the present discordant level. Research was done by means best suited to it. Maxwell in fact becomes absorbed in dozens of different designs of galvanometers, magnetometers, and other instruments, taking on the role of both an inventive physicist and consummate engineer. He perceives each measurable quantity as very real and tangible, and knows how to repeat or verify each operation upon it. He was equally able to immerse himself in either theoretical calculations or experimental methods of increasing the precision of absolute measurement: the Cavendish Laboratory which he established in 1874 began with a series of measurements of the electrodynamic constant. On the other hand, he was greatly impressed by the almost complete elucidation of the physical nature of phenomena on the basis of purely relative (independent of standards) measurements, for example, in the development of Ampere of his electrodynamics. Both before and after the completion of the "Treatise," Maxwell demonstrated his mastery of, and passion for, not only metrology but also—as one would say today—the maintenance of the cult of measurement in physics. It is no accident that his last published work—the Introduction to the collected works of the great 18th century experimenter Henry Cavendish (sent to the printer on June 14, 1879!)— contains a very explicit statement in praise of measurement.[18] It begins as follows:

"Let us suppose that we have been admitted by Cavendish into his laboratory in Great Marlborough Street, as it was arranged for his electrical experiments in 1773, and let us make the best of an opportunity rarely, if ever, afforded to any scientific man of his own time, and examine the apparatus by which the electric fluid, instead of startling us with the brilliant phenomena, new instances of which were then every day being discovered, was made to submit itself, like everything else which entered that house, to be measured."

## 6. FIELD

Maxwell himself has a very modest view of his own contribution to the subject. Scalar and vector quantities had been understood and used prior to his arrival on the scene. On the other hand, in contrast to hydrodynamics, where spatially separated scalar and vector fields had a clear "micromodel" interpretation (through the relationship between Eulerian and Lagrangian descriptions), the potentials and their variations employed in the theory of the gravitational, electrostatic, and magnetostatic interactions were regarded mainly as convenient tools for obtaining results rather than aids to the understanding of the mechanism responsible for the transmission of disturbances through space. With hindsight, it is, of course, surprising that the idea of an electromagnetic "continuum" was proclaimed as the fruit of imagination, intuition, and flair of the Great Experimenter whilst theoreticians, who already had at their disposal well-tried constructs and physical analogies,[10)] maintained a deafening silence. Maxwell performed the necessary unification of Faraday's idea of a field and the virtually ready-made formalism that had reached a degree of invariant generality as the result of the application of Hamilton's quaternions (which is not, in general, essential in this case). Incidentally, this gave rise to another myth that is not confirmed by an examination of the "Treatise" but which reproaches Maxwell for using only the coordinate form of his equations. In fact, he frequently does use this approach to solve specific problems (just as, a hundred years later, we still do today—and are glad to take advantage of the separation of variables). At the same time, Maxwell was the first to introduce the invariant vector form of the field equations and used the Hamilton operator $\nabla$. This may well have been responsible for its acceptance in physics generally. The final equations of the "Treatise" are actually given in terms of the coordinate components and also in terms of the same divs, grads, and

[10)] "The whole theory, for instance, of the potential, considered as a quantity which satisfied a certain partial equation, belongs essentially to the method which I have called that of Faraday" ("Treatise", Preface).

rots (curls) that are so familiar and useful today.[11)]

Following Faraday, Maxwell saw the field as a "tangible" entity endowed with its own properties, independent of the sources of the field. As we have already mentioned, the analogy with hydrodynamics was almost inevitable: it led to a finite velocity of propagation of the independent field, and there was no longer any need to introduce the principle of contact forces as an *ad hoc* postulate. After reflecting on the topological properties of vector fields, Maxwell showed that they could be comprehensively divided into conservative and rotational.[12)] And since it turned out that the curl of a vector field exhibits special properties under the reversal of the axis of rotation (or under replacement of a left-handed frame of reference by a right-handed one), Maxwell was probably the first to appreciate the difference between $\mathbf{E}$ and $\mathbf{H}$ with respect to rotational properties, i.e., the difference between polar (or true) vectors and axial (or pseudo) vectors. The next step taken by Maxwell is more difficult for us to appreciate. He distinguishes between vectors characterized by intensity ("defined with reference to lines;" Article 12)—these are $\mathbf{E}$, $\mathbf{H}$, the vector potential $\mathbf{A}$, and so on, and vectors associated with flux ("defined with reference to areas;" Article 12), i.e., $\mathbf{D}$, $\mathbf{B}$, $\mathbf{j}$, and so on. This classification is difficult to understand in view of the relations $\mathbf{j}=\sigma\mathbf{E}$, $\mathbf{D}=\varepsilon\mathbf{E}$, $\mathbf{B}=\mu\mathbf{H}$. But it is precisely this classification that enables Maxwell to perform in a manner beyond criticism the separation of the purely electromagnetic relationships that do indeed contain only quantities of the form $\oint E_l\, dl$, $\int \mathbf{B}\cdot d\mathbf{s}$, $\oint \mathbf{H}_l\cdot d\mathbf{l}$, $\int \mathbf{D}\cdot d\mathbf{s}$, $\int \mathbf{j}\cdot d\mathbf{s}$, from the constitutive (material) relations that are introduced into electromagnetism from outside. It must be admitted, however, that he did not rigidly adhere to this classification, and now and then modified his solutions by interchanging "intensity" and "flux," even in proof. But even if we regard this as a puzzle, we must acknowledge it as an example of an unwillingness to conceal the use of an intermediate model, which he did not have to publicize (since it made no difference to the final result)[13)] but which he nevertheless frankly shares with us despite its inconclusive nature.[14)]

## 7. RECONSTRUCTION OF MODE OF THINKING

"Maxwell's equations are Maxwell's theory." Hertz's frequently cited dictum[2] suffers from the exaggeration of a catch-phrase by underlining the independence of the final result from the grouping meanderings on the way to it. However, if we consider that Maxwell's "Treatise" is in fact an account of his theory, this phrase does not refect its content at all. The "Treatise" contains many trains of thought, approaches, and methods which have remained in physics (by now, some of them anomously) and have indeed become part of the general scientific folklore.[15)]

Of course, the principal aim of the author was to lead the reader to the equations that provide a unified (and self-consistent) description of electric and magnetic fields. It is therefore interesting to follow the lines of reasoning—with the aid of a little reconstruction and extrapolation—that led to this closed description. The important point is that there is a number of them. Here again one must take care to avoid being trapped by "legends" such as the heuristic guesswork that was allegedly involved in the idea of the displacement current, the disheartening liberties that were supposedly taken in overcoming logical difficulties, and so on.[16)] This willy-nilly distracts the attention of later generations who are almost totally unaware of the instructive—though occasionaly contradictory and understated—arguments whose traces remain in the "Treatise."

Let us first consider the direct introduction of the displacement current.[17)] Conductivity relates to trans-

---

[11)]There are two "notation-worshipping religions" which are not at war with each other but are nonetheless emphatically caste-conscious. One sect regard themselves as the "true believers" and use verbal operators curl and div, whilst others, the dissidents, use the symbolic notation $\nabla\times$ and $\nabla$ for the vector and scalar products of the operator. The Teacher himself takes an eclectic stand: he is more inclined toward the use of symbols but in a somewhat different form, namely, (V. $\nabla$) and (S. $\nabla$), where V and S are the vector and scalar parts of a single product of the operator with a vector.

[12)]The "Treatise" does not give a rigorous proof of the result that an arbitrary (in general differentiable) vector field can be divided into conservative and rotational parts, but Maxwell uses it as if it were obvious.

[13)]"The method of Ampere, however, though cast into an inductive form, does not allow us to trace the formation of the ideas which guided it. We can scarcely believe that Ampere really discovered the law of action by means of experiments which he describes. We are led to suspect.......that he discovered the law by some process which he has not shown us.........."(Article 528).

[14)]Following the example of Faraday who ".......on the other hand, shews us his unsuccessful as well as his successful experiments, and his crude ideas as well as his developed ones, and the reader, however inferior to him in inductive power, feels sympathy even more than admiration, and it tempted to believe that, if he had the opportunity, he would be a discoverer" (Article 528).

[15)]Thus, many of the sections on statics are so well presented that they have not become out of date in the sense that, with some refurbishment of terminology, they would not look out of place in a modern (i.e., not so modern after all) text.

[16)]As an example, here is an extract from a book recently translated into Russian[15]: "Maxwell thinks nothing of excluding some unwanted term, reversing an inconvenient sign, changing the meaning of some letter. ..........Physicists have not succeeded in reducing his theory to an orderly structure, i.e., free it from logical errors and inconsistencies. But, on the other hand, they could not dismiss a theory which......provided an organic link between optics and electricity".

[17)]It is interesting that the almost obvious discussion that follows attracted so much "digging", apparently timed to coincide with the centenary of Maxwell's equations. The associated reconstruction of Maxwell's thoughts was performed by Peirels[5] and then by Shapiro[16] on the basis of the three succesive papers by Maxwell: "On Faraday's lines of force" (1855–1856)[1,9] in which the displacement current is absent, "On the physical lines of force" (1861–1862)[1,9] where it appears for the first time, and "A dyport phenomena, i.e., the transport of electric charge carriers, and although the rate of this transport is not clear, the analogy with hydrodynamics suggests that the carriers must obey the continuity equation

$$\frac{\partial \rho}{\partial t}+\operatorname{div} \mathbf{j}=0.$$

Since the electric charge is the source of the divergence of the electric displacement vector in the sense that $\rho=(1/4\pi)\operatorname{div}\mathbf{D}$, the equation of continuity assumes the form

$$\operatorname{div}\left(\frac{1}{4\pi}\frac{\partial \mathbf{D}}{\partial t}+\mathbf{j}\right)=0.$$

There is no way of satisfying the equation of continuity other than by introducing an additional term into what might be described as Ampere's equation:

$$\operatorname{curl}\mathbf{H}=\frac{4\pi}{c}\mathbf{j}\rightarrow\frac{4\pi}{c}\left(\mathbf{j}+\frac{1}{4\pi}\frac{\partial \mathbf{D}}{\partial t}\right).$$

A quite separate point is that Maxwell tried to penetrate the significance of all the intermediate consequences of this stunning additional term to the usual quasistationary expressions, but the resulting scheme (unfortunately, we do not know at what stage this came to him) is complete and is not so much a striking idea as an expression of a conservation law.[5]

This is generally a very important point: the appeal to physical analogy and, in particular, dynamical analogy, in the course of development of a new theory ensures that general physical principles, including conservation laws, will be satisfied even though they may not be fully appreciated at that particular stage of understanding of the natural world. "In forming ideas and words relating to any science, which, like electricity, deals with forces and their effects, we must keep constantly in mind the ideas appropriate to the fundamental science of Dynamics, so that we may, during the first development of the science, avoid inconsistency with what is already established, and also that when our views become clearer, the language we have adopted may be a help to use and not a hindrance" (Article 567).

Another argument that led Maxwell to the necessary generalization of the equation for the curl of the electric field, involved an analysis of fields in isolation from charges and currents. He had at his disposal the equation for the magnetic induction (which could be referred to as Faraday's equation)

$$\oint \mathbf{E}\,d\mathbf{l}=-\frac{1}{c}\frac{\partial}{\partial t}\int \mathbf{B}\,d\mathbf{s},$$

which relates the intensity-type electric vector $\mathbf{E}$ to the flux-type magnetic vector $\mathbf{B}$. The only other analogous relation was between the intensity vector $\mathbf{H}$ and flux vector $\mathbf{D}$. In contrast to the former scheme, the latter involves intuition because, as we have noted before (and as still seems correct to us now), these relations

namical theory of the electromagnetic field"[1,9], which is the clearest and, in effect, a resume of the other two. In this context the "Treatise" appears as a later work (in fact, *the last, after corrections were inserted*), and can be regarded as evidence.

constitute the basis of the intensity-flux classification of vectors.

The intuitive argument (apart from the sign which is determined by demanding that the electromagnetic field must be stable) might rely on the analogy between the electric and magnetic fields, which was investigated by Maxwell very meticulously and which initiates the dual invariance (at least in source-free regions) under

$$\operatorname{curl}\mathbf{E}=-\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t},$$
$$\operatorname{div}\mathbf{D}=0,$$

which had been known (in different notation) before Maxwell then "automatically" generates the "magnetic pair"

$$\operatorname{curl}\mathbf{H}=\frac{1}{c}\frac{\partial \mathbf{D}}{\partial t},$$
$$\operatorname{div}\mathbf{B}=0,$$

the first of which was the required relation.

Finally, the third way which has essentially been ignored by Maxwellists (this species probably does not exist in pure form—it consists simply of physicists who have had the opportunity temporarily to absent themselves from the rat race) consists of Maxwell setting up the Lagrangian description of the electromagnetic field. He gave this a fair amount of space in the "Treatise." Although we shall not pause to consider the interesting discussion that led Maxwell to the relations for the energy densities associated with the electric and magnetic fields, and the stress tensors along the lines of force (without striction corrections which were introduced later), we note that it was this that enables him to construct the Lagrangian function for the electromagnetic field:

$$\mathscr{L}=\int_V\left(\frac{\mathbf{E}\cdot\mathbf{D}}{8\pi}-\frac{\mathbf{H}\cdot\mathbf{B}}{8\pi}\right)dV,$$

from which, as we now know, it is possible to derive, by the usual variational techniques, the equations of electrodynamics in the complete Maxwellian form. The essential point here is that the quasistationary Lagrangian function for $LC$ circuits

$$\mathscr{L}=\sum\left(\frac{Q^2}{2C}-\frac{LI^2}{2}\right),$$

($I$ represents currents and $Q$ charges) which he obtained from the electromechanical analogy,[18)] is the local, i.e., expressed in terms of fields in a small region of space, and accurate expression for the action of the electromagnetic field, provided the continuity of

[18)]"It is difficult, however, for the mind which has once recognized the analogy between the phenomena of self-induction and those of the motion of material bodies, to abandon altogether the help of this analogy, or to admit that it is entirely superficial and misleading. The fundamental dynamical idea of matter, as capable by its motion of becoming the recipient of momentum and of energy, is so interwove with our forms of thought that, whenever we catch a glimpse of it in any part of nature, we feel that a path is before us leading, sooner or later, *to the complete understanding of the subject*" (Article 550.

charge (in this case, $I = dQ/dt$) is satisfied. The equations of electrodynamics are then automatically augmented with the displacement current, as we have shown above.

We may therefore conclude that Maxwell was able to write down the equations of electrodynamics also in the Lagrangian formalism, but this cannot be found in the "Treatise" in the final stylized form in which, ideally, one would like to have all historical documents.

## 8. COMPLETENESS BUT NOT COMPLETION

"These may be regarded as the principal relations among the quantities we have been considering. They may be combined so as to eliminate some of these quantities, but our object at present is not to obtain compactness in the mathematical formulae, but to express every relation of which we have any knowledge. To eliminate a quantity which expresses a useful idea would be rather a loss than a gain in this stage of our enquiry" (Article 615). Since the set of equations written down by Maxwell (and it included the equations for the fields and potentials as well as the constitutive relations) was internally consistent, the question of redundancy is essentially a secondary one: it was settled later, after the derivation of existence and uniqueness theorems. The more crucial problem was that of consistency and completeness of the system, and Maxwell did not consider himself entitled (nor was he then) to make any claim about this. He confined himself to certain deductive demonstrations. Firstly, all the then existing descriptions of the electrostatic, magnetostatic, and quasistationary fields obeyed these equations. Secondly, he constructed their solution for arbitrarily rapid variations in time, and obtained plane electromagnetic waves in a homogeneous medium, which transported energy and momentum and propagated in empty space with the velocity of light (a triumph for Faraday's foresight).[19] Finally, he outlined a scheme for the explanation of the "magnetic action on light" (i.e., of the Faraday rotation of the plane of polarization in a magnetized medium). Maxwell thus exhaustively implemented his own program, namely, that of translating "Faraday's ideas into a mathematical form," and this form eventually exhibited self-consistent completeness which, in a way, is a triumph for Faraday as well.

If we leave methodological improvements and instrumentation on one side, all subsequent interference (a word no less appropriate than "generalization") with Maxwell's equations can only involve the constitutive relations, i.e., relations imposed from outside (for example, by the microproperties of the medium), some of which may not even be of purely electromagnetic origin. And any rewriting of the field equations can be reduced to a redefinition of the induction vectors **D** and **B**.[20]

Energy relationships occupy a special place. Changes in the constitutive relations (inclusion of noninstantaneity and nonlocality of interaction, i.e., of temporal and spatial dispersion, nonlinearity, and so on) have had an important influence on the form of the expressions for the electromagnetic energy and momentum densities, frozen or transported by the field. But here also the basic form of the field equations, i.e., the internal electromagnetic relations between the vectors remained unshaken i.e., as Maxwell said (in a different connection it is true). [Translator's note: the remainder of this sentence has been retranslated from the Russian text since it was not possible to identify the quotation given from Article 528 of the "Treatise"]. "They left too much room for the introduction of new ideas as they were suggested by new facts." And since the electrodynamics was created with one eye on Dynamics (if not actually in the image of, or by similarity with, Dynamics), all the conservation laws of dynamics were naturally satisfied. The analogy with mechanics (which we have twice mentioned already) did not therefore force the description of the new class of phenomena into a particular mold, but it did demand that they obeyed certain general norms. This has since been regarded as a necessary condition for any physical theory. The transport of momentum by electromagnetic waves (pressure exerted by light), which was predicted by Maxwell, is thus seen to be of greater scientific significance than the confirmation of "yet another" new effect in physics.

The most difficult question is whether or not the "Treatise" is complete (to some extent it is not even a legitimate question, but, on the other hand, what other than the absence of internal necessity (can serve as a limitation).

Having obtained his equations for the electromagnetic field, Maxwell had as the next step to proceed to the implementation of the methodological program of investigating them that he had just demonstrated in the course of his review of static and quasistationary approximations, and to cover such questions as internal consistency, uniqueness, field due to sources, reciprocity, boundary-value problems, and so on. This sug-

[19] When we determined the velocity of electromagnetic waves in a medium with permittivity $\varepsilon$, Maxwell encountered a very appreciable discrepancy between the refractive index for light ($n = \sqrt{\varepsilon}$) and the corresponding values deduced from static permittivity. However, in contrast to Newton (in the course of his estimates of the force of attraction of the Moon), Maxwell had sufficient faith (and, of course, courage) to maintain that this was no argument against the electromagnetic nature of light and merely indicated that $\varepsilon$ was not universal, i.e., that there was appreciable dipsersion.

[20] It is precisely in this way that one can symmetrize the equations with respect ot the currents. This was done by Heaviside who was once "officially" called a great "propagandist for Maxwellian science".[5] If in the region occupied by currents we substitute $\mathbf{B} \to \mathbf{B} + 4\pi \mathbf{M}_{\text{ext}}$, and replace $\mathbf{M}_{\text{ext}}$ with $\rho^m = +\operatorname{div} \mathbf{M}_{\text{ext}}$, $\mathbf{j}_m = -\partial \mathbf{M}_{\text{ext}}/\partial t$, the field equations will contain phenomenologically equivalent electric and magnetic currents. However, although this operation is convenient for some macroscopic purposes, it is basically anti-Maxwellian in the sense that it is not supported by a micromodel of any kind.

gestion would have looked very strange indeed had it not been based on a certain sterotype that had already been touched upon in certain chapters of the "Treatise" preceding chapters 8 and 9 of volume II, which contain the final equations.

The main "puzzle" for the third and subsequent generations arises from the fact that Maxwell himself hardly ever used a transformation to a moving frame of reference, and did not therefore broach the question of how currents and fields would behave under such transformations. The establishment of group-invariant properties of the equations in the "aether era" would undoubtedly have involved internal obstacles, and could not have been reduced to purely mathematical procedures.[17] Generally speaking, "puzzles" of this kind, "retrospective predictions," and so on, properly belong to virtual history (for some reason, sometimes referred to as alternative history), the "course" of which is arbitrary within certain limits, and which is very dependent on the mysterious mechanism of sociological and psychological maturation of ideas.

## 9. DO WE NEED CLASSICAL PHYSICS (AND WHY) IN ITS ORIGINAL FORM?

"It is of great advantage to the student of any subject to read the original memoirs on that subject, for science is always most completely assimilated when it is in the nascent state ..." ("Treatise," Preface). True, this says nothing about adapted primary sources—something that would not likely occur to Maxwell.

Nowadays, when the interval between the appearance of an idea and its application and exhaustion has become comparable with (and sometimes much less than) the lifetime of a generation (apparently this will for a long time yet remain the characteristic time scale), we have little opportunity to learn even from recent, let alone more remote, history. However, this should not apply to achievements that not only set the world on fire but also mark the beginning of an era. They are so pervading as to affect many generations to come and—by virtue of the principle of complementarity—continue to stand out if they are not too unceremoniously extended beyond their proper range. The crossroads of discord and muddle from which one can sometimes escape by taking one crucial step occurs in science at intervals of roughly two or three generations. The behavior of people and systems in the neighborhood of such states can therefore usefully be studied because this may well help with predicting ways of escaping from such situations in the future. This is almost self evident. To coin a phrase, it constitutes the sociological component of the problem.

There is, however, another component that is more specific and more "applied." By following the thoughts of a classical scientist we become imbued with a particular, practical respect for the Great Man, and since with hindsight we know the eventual course of events, we marvel at his intuitive foresight. And this teaches us to look out even for small clues, i.e., to remember that any generalization in a problem of any kind is usually cost-effective if it is made by carefully considering all eventualities. Modern readers can make such instructive discoveries in the "Treatise" quite readily, including discoveries of immediate personal utility, e.g., those relating to the method of solution (or guessing the behavior of a solution) of many of our own problems ("you name it, Maxwell has it"). At some point one even gains the impression that, at least as far as classical electrodynamics is concerned, the last century was denoted to the development (and, sometimes for reasons of ignorance, rediscovery) of Maxwellian principles which did not reach their ultimate conclusion because of his untimely death.

## APPENDIX

## TEXTBOOKS ON MAXWELLIAN ELECTRODYNAMICS

The "Treatise" may be regarded as the beginning of an extensive and ramified family of monographs and textbooks on Maxwellian electrodynamics. Below, we reproduce a list of English, German, and Russian "language branches" of this "tree." It contains only those texts that have been educationally important in their time.

English Language Branch

[1] J. J. Thomson, Notes on Recent Researches in Electricity and Magnetism, The Clarendon Press, Oxford (1893).
[2] J. H. Jeans, The Mathematical Theory of Electricity and Magnetism, Cambridge University Press, Cambridge (1908); 5th edition (1925).
[3] W. R. Smythe, Static and Dynamic Electricity, McGraw–Hill, New York (1950) (Russian transl., I.L., 1954).
[4] J. A. Stratton, Electromagnetic Theory, McGraw–Hill, New York (1941) [Russian transl., Gostekhizdat, Moscow, 1948].
[5] W. K. H. Panofsky and M. Phillips, Classical Electricity and Magnetism, Addison-Wesley, Reading, Mass. (1962) (Russian transl., I.L., Moscow, 1963).
[6] J. D. Jackson, Classical Electrodynamics, John Wiley, New York (1962) (Russian transl., I. L., Moscow, 1965).

German Language Branch

[1] E. Cohn, Das Elektromagnetische Feld, Leipzig (1900).
[2] A. Föppl and M. Abraham, Einführung in die Maxwellsche Theorie, Leipzig (1904).
[3] M. Abraham and R. Becker, Theorie der Elektrizität (Russian transl., O.N.T.I., 1936).
[4] A. J. W. Sommerfeld, Elektrodynamik (Russian transl., I.L., 1958).

Russian Language Branch

[1] O. D. Khvol'son, Kurs fiziki (A course of Physics), Vols. IV and V, 2nd edition, Gosizdat, Berlin (1923).
[2] A. A. Éikhenval'd, Teoreticheskaya fizika (Theoretical Physics), Vol. I: Teoriya polya (Field Theory), Moscow (1932).
[3] Ya. I. Frenkel', Electrodinamika (Electrodynamics), Vols. I and II, Moscow-Leningrad (1934).
[4] I. E. Tamm, Osnovy teorii élektrichestva (Fundamentals of the Theory of Electricity, 2nd edition, Moscow-Leningrad (1929).

[5]L. D. Landau and E. M. Lifshitz, Teoriya polya (Field Theory), 6th edition, Gostekhizdat (1973); Elektrodinamika sploshnykh sred (Electrodynamics of Continuous Media), Gostekhizdat, Moscow (1957).

The authors are indebted to S. D. Zhernosek and E. B. Suvorov for their friendly help.

[1]P. S. Kudryavtsev (editor), Maksvell Dzh. Klerk - Izbrannye sochineniya po teorii elektromagnitnogo polya (J. Clerk Maxwell— Collected Papers on the Theory of the Electromagnetic Field), Gostekhizdat, Moscow (1950).
[2]L. I. Mandel'shtam (editor), Iz predystorii radio-sbornik original'nykh stateĭ i materialov (from the Prehistory of Radio—a collection of Original Papers and Materals), compiled by S. M. Rytov, Izd-vo AN SSR, Moscow-Leningrad (1948).
[3]R. L. Smith-Rose, James Clerk Maxwell, London (1948).
[4]D. K. C. Macdonald, Faraday, Maxwell, and Kelvin (Russ. Transl., Atomizdat, Moscow, 1967).
[5]Maksvell Dzh. Klerk—Stat'i i rechi (James Clerk Maxwell—papers and speeches) [Russ. Transl., Nauka, 1968].
[6]Vi. Kartsev, Maksvell (Maxwell), Molodaya Gvardiya (1974) (Ser. ZhZL).
[7]J. Clerk Maxwell, A Treatise on Electricity and Magnetism, The Clarendon Press, Oxford (1873).
[8]J. Clerk Maxwell, A Treatise on Electricity and Magnetism, Third Edition (1891).
[9]W. D. Niven (editor), The Scientific Papers of James Clerk Maxwell, Dover, London (1890).
[10]L. Campbell and W. Garnett, The Life of J. C. Maxwell, London (1882).
[11]H. Poincaré, Maxwell's Theory and Hertzian Oscillations, Constable, London (1904) (Russ. Transl. Sankt Peterburg, 1900).
[12]Lord Kelvin, Baltimore Lectures on Molecular Dynamics, Johns Hopkins (1884).
[13]W. R. Smythe, Static and Dynamic Electricity, McGraw-Hill, New York (1950) [Russian transl., I. L. (1954)].
[14]J. H. Jeans, The Mathematical Theory of Electricity and Magnetism, Cambridge University Press, Cambridge (1908).
[15]M. Liozzi, A History of Physics (Russ. Transl., Mir, Moscow, 1970).
[16]I. S. Shapiro, "On the History of the Discovery of Maxwell's Equations," Usp. Fiz. Nauk, 108, 319 (1972) [Sov. Phys. Uspekhi, 15, 651 (1973)].
[17]F. J. Dyson, "Missed opportunities," Bull. Amer. Math. Soc., 78, 635 (1972) [Russ. Transl. Usp. Math. Nauk 35, 171 (1980)].
[18]J. C. Maxwell (editor), The Electrical Researches of the Honourable Henry Cavendish, Cambridge University Press, Cambridge (1879).

Translated by S. Chomet