Low-temperature electrical conductivity of pure metals

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This review analyzes current ideas on the electrical conductivity and galvanomagnetic properties of normal pure metals, in which relaxation processes are governed entirely by collisions of electrons with phonons. Attention is concentrated on a group of closely related topics: the diffusive nature of electron motion on the Fermi surface at low temperatures, mutual influence of the normal collisions and of the umklapp processes, and dependence of the relaxation mechanisms on the topological properties of the Fermi surface. The analysis is based on the diffusion equation method which makes it possible to treat in a unified manner all the topics under discussion and to formulate the results in terms of specific physical mechanisms.

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1. INTRODUCTION

Enormous material is now available on the low-temperature electrical conductivity of metals. This is true both of the experimental data and of the relevant theoretical representations. The present review does not in any way attempt to analyze and classify all this enormous amount of information. We shall consider only the conductivity of typical pure normal metals, in which relaxation processes are entirely due to collisions of electrons with phonons. We shall base the presentation on a certain group of closely related topics: the small-angle nature of electron scattering at low temperatures, mutual influence of the normal and umklapp processes, and dependence of the relaxation mechanisms on the topological properties of the Fermi surface. Our main aim will be to achieve basic clarity in topics of this kind and to discuss various physical consequences. Preference will be given to simple models which make it possible to consider analytically the qualitative aspect of the problem as compared with numerical calculations based on more specific data. No attempt will be made to give a description of the experimental situation in any degree of detail and experimental results will be quoted only to illustrate theoretical conclusions. The treatment will be based on the authors' diffusion equation method which makes it possible, in the authors' opinion, to elucidate in a unified manner the topics under discussion and to formulate the results in terms of specific physical mechanisms.

Some of the results obtained in connection with this review are published for the first time. This applies particularly to the applications and to the galvanomagnetic properties of metals with a two-dimensional network of open orbits.

a) Qualitative considerations

The modern theory of the electrical conductivity of metals was founded in the early thirties mainly by Bloch and Peierls.[1,2] Calculations of the electrical conductivity carried out during this period were based on very rough representations of the spectra of electrons and phonons, and of the interaction between them. However, the qualitative ideas of Peierls, which are extremely clear and complete, have been found very fruitful in the further development of the theory.

An analysis of the transport equation enabled Bloch to show that the electrical resistivity of a metal associated with the electron–phonon interaction is proportional to the fifth power of the absolute temperature:

$$\rho \sim T^5, \quad T \ll \theta, \tag{1.1}$$

where θ is the Debye temperature. Bloch used the simplest model of free electrons interacting with longitudinal phonons, whose dispersion law is linear and isotropic, and he assumed that phonons are described by the equilibrium Bose distribution function.

The result (1.1) is easily understood on the basis of

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the following considerations. Although at low temperatures the density of phonons is proportional to T^3, the number of phonons capable of scattering a given electron with a momentum p is limited by the laws of conservation of energy and momentum $\varepsilon_{p\pm q} - \varepsilon_p = \pm\omega_q$ and, therefore, it is proportional to T^2 (ε_p and ω_q are the electron and phonon energies). Moreover, the square of the matrix element of the electron–phonon interaction is proportional to the phonon momentum $q \approx T/s$ (s is the velocity of sound). Therefore, the mean free time of electrons colliding with phonons is $\tau' \propto T^{-3}$. Moreover, we must bear in mind that a single collision alters the electron momentum only slightly and is therefore not very effective. Collisions with phonons cause an electron to wander at random over the Fermi surface in steps of q at intervals of τ'. An electric field shifts the electron distribution in the p space in the direction of the vector $e\mathbf{E}$. Equilibrium is established by the diffusion of an electron over a distance of the order of the characteristic dimension of the Fermi surface p_F, so that the transport relaxation time is $\tau_{tr} \approx \tau'(p_F/q)^2$. Consequently, the electrical resistivity obeys $\rho \propto (\tau_{tr})^{-1} \propto T^5$.

It is clear from the above discussion that the result (1.1) is not conditional on Bloch's very simple assumptions on the spectra of electrons and phonons: it is sufficient that the thermal momentum of phonons is small compared with all the characteristic dimensions of the Fermi surface. The second assumption of Bloch that the phonon distribution is in equilibrium is of fundamental importance. If we do not make this assumption and confine our attention to normal collisions which conserve the total quasimomentum, we find that the electrical resistivity vanishes. This becomes obvious if we note that in the presence of just normal collisions the electron–phonon system is in a state of internal equilibrium and can move as a whole at an arbitrary velocity. This makes possible a constant (nondecaying) electric current in the absence of an external field.

Peierls demonstrated that the equilibrium of the phonon system can be ensured by phonon–phonon collisions accompanied by spin flipping. However, cooling reduces exponentially the probability of these umklapp (U) processes: the collision frequency obeys $(\tau_{pp}^U)^{-1} \equiv \nu_{pp}^U \propto \exp(-\gamma\theta/T)$, where γ is a numerical coefficient of the order of unity ($2\gamma\theta$ is equal to the smallest value of the sum of energies of all three colliding phonons calculated subject to the laws of conservation of energy and quasimomentum). Moreover, short-wavelength phonons participating in the umklapp processes are most likely to transfer momentum not to thermal phonons but directly to electrons.

It is thus difficult to justify the assumption of phonon equilibrium. In any case, it is obvious that for each metal there is a certain characteristic temperature below which the U processes can no longer ensure this phonon equilibrium. A comparison with the experimental data on the thermal conductivity of insulators (see, for example, Ref. 3) shows that this should occur over a fairly wide range of low temperatures, for example, $T/\theta \leq 1/10$.

The degree of coupling of phonons to electrons is governed by the frequency of phonon–electron collisions $\nu_{pe} \propto T$. [The order of magnitude of the corresponding mean free path is $l_{pe}(T) \approx l_{ep}(\theta)\theta/T$, where $l_{ep}(\theta) \approx 10^{-5}$–$10^{-6}$ cm is the mean free path of an electron at the Debye temperature.] On the other hand, the frequency of the normal phonon–phonon collisions is $\nu_{pp}^N \propto T^5$ and the frequency of the scattering of phonons by microscopic lattice defects is $\nu_{pi} \propto T^4$. Therefore, at sufficiently low temperatures the inequalities $\nu_{pe} \gg \nu_{pp}^N$, ν_{pi}, ν_{pp}^U should be satisfied. This means that the phonon system is tightly coupled to the electron system, i.e., phonons are dragged completely by electrons.

At sufficiently low temperatures when the phonon–phonon U processes can be ignored, the electrical resistivity is governed by the umklapp processes involving collisions of electrons with phonons. Hence, it follows directly that there is a considerable difference between the behavior of the lattice resistivity at $T \to 0$ for metals with closed and open Fermi surfaces. If the Fermi surface is closed and the number of electrons is not equal to the number of holes, the electrical conductivity should increase exponentially as a result of cooling. In fact, the umklapp process requires a phonon with a momentum exceeding the minimum separation Δp between closed electron or hole groups. If the thermal momentum of phonons is less than Δp, the number of such phonons is proportional to $\exp(-\Delta p s/T)$.

According to Peierls, the situation is different in the case of metals with open Fermi surfaces. In this case the Bloch law $\rho \propto T^5$ should be obyed at arbitrarily low temperatures because the umklapp processes in the collisions of electrons with phonons are possible no matter how small is the phonon momentum. The resistivity of a compensated metal depends in a similar way on temperature: the total quasi-momentum supplied by an electric field is then zero and the role of phonons reduces to the exchange of momentum between the electron and hole subsystems via the normal collisions.

Following Peierls, we shall consider in greater detail the mechanism of quasimomentum relaxation in an electron–phonon system in the presence of the umklapp processes. By way of example, we shall consider a Fermi surface of the corrugated cylinder type (Fig. 1).

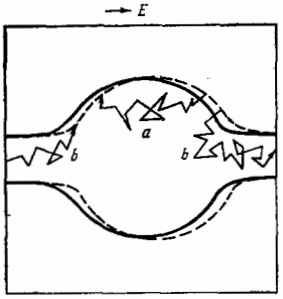

FIG. 1. Two diffusion paths on an open Fermi surface. The dashed curve represents the Fermi surface shifted by an electric field.

If we consider only the normal collisions, i.e., the type a processes, the total momentum of phonons in the direction of the electric field will increase indefinitely with time. Therefore, under steady-state conditions about half the nonequilibrium electrons should follow type b trajectories which involve the umklapp processes. (The absorbed phonon momentum is clearly equal to the vector directed from the point of creation of a nonequilibrium electron to the point of its disappearance in the extended p space.) If the process of one of these types is for some reason hindered, the electrical conductivity is then governed by this process. Clearly, these considerations are applicable also to closed Fermi surfaces except that the type b trajectories now include not only diffusion sections but also "jumps" between the electron or hole groups.

Naturally, these considerations apply only to sufficiently "perfect" metal samples in which the phonon scattering mechanism predominates right down to very low temperatures. The relevant restrictions on the concentration of lattice defects depend strongly on the experimental situation and can easily be formulated in each specific case (see Footnote 7).

b) Transport equation

We shall use the transport equations describing a system of interacting electrons and phonons in an external electric field

$$I^e\{\chi_p,\ \Phi_q\} = e\mathbf{E}\frac{\partial n_p}{\partial \mathbf{p}}, \tag{1.2}$$

$$I^p\{\Phi_q,\ \chi_p\} = 0, \tag{1.3}$$

$$\left.\begin{aligned} &I^e = \sum_{k,q}(\Gamma^+_{pkq} - \Gamma^-_{pkq}), \quad I^p = \sum_{p,k}\Gamma^-_{pkq},\\ &\Gamma^\pm_{pkq} = \frac{2\pi}{\hbar}\,|B_{pkq}|^2\frac{dN_q}{d\omega_q}(n_p - n_k) \quad (\chi_p - \chi_k \pm \Phi_{\pm q})\,\delta(\varepsilon_{p,k}\pm\omega_q),\\ &\varepsilon_{p,k} = \varepsilon_p - \varepsilon_k, \quad B_{pkq} = b_{pkq}\sum_g \delta_{p,k+q+g}; \end{aligned}\right\} \tag{1.4}$$

here, B_{pkq} is the matrix element of the electron–phonon interaction; the δ symbol expresses the law of conservation of quasimomentum to within the reciprocal lattice vector g (since $|qg| < g^2$, then for each pair of states p and k in the sum over g only one term remains and this corresponds to g equal to zero or to one of the minimum reciprocal lattice vectors). The nonequilibrium corrections to the electron and phonon distribution functions are represented by $-\chi_p dn_p/d\varepsilon_p$ and $-\Phi_q dN_q/d\omega_q$, where n_p and N_q are the nonequilibrium Fermi and Bose distribution functions.

We shall now point out some of the properties of Eqs. (1.2) and (1.3), which are consequences of the law of conservation of quasimomentum.

If for any reason there are no umklapp processes, a homogeneous system of transport equations has the solution

$$\chi_p = \mathbf{up}, \quad \Phi_q = \mathbf{uq}, \tag{1.5}$$

which describes the simultaneous drift of electrons and phonons at an arbitrary velocity u.

In the general case, it follows from Eqs. (1.2) and (1.3) that

$$\sum_{pkq}\mathbf{g}\Gamma^+_{pkq} = \sum_p e\left(\mathbf{E}\frac{\partial n_p}{\partial \mathbf{p}}\right)\mathbf{p}. \tag{1.6}$$

The left-hand side of this equation represents the rate of change of the total quasimomentum of the system as a result of the umklapp processes, whereas the right-hand side represents the effects of an electric field.

In the case of a metal with a closed Fermi surface, we have

$$-\sum_p e\left(\mathbf{E}\frac{\partial n_p}{\partial \mathbf{p}}\right)\mathbf{p} = e\mathbf{E}(n_e - n_h), \tag{1.7}$$

where n_e and n_h are the electron and hole densities.

In the case of an open Fermi surface, we can easily show that

$$-\sum_p e\left(\mathbf{E}\frac{\partial n_p}{\partial \mathbf{p}}\right)\mathbf{p} = e^*\mathbf{E}\frac{2}{h^3}V - e^*\frac{2}{h^3}\sum_g \mathbf{g}(\mathbf{S}_g). \tag{1.8}$$

Here, V is understood to represent (depending on the convenience in a given case) either the volume of the cell occupied by electrons or the electron-free "hole" volume. The charge e^* has different signs depending on the case. The absolute value of the vector $\mathbf{S}_g$ is equal to the area of contact between the volume V and the cell boundary and is directed along the external normal to the boundary; g is the reciprocal lattice vector corresponding to this boundary. If the Fermi surface consists of several sheets and some of them are closed, the volumes of the electron and hole groups should be included with the appropriate signs in V.

The fundamental role of the umklapp processes in the electrical conductivity of metals with open and closed (but with $n_e \neq n_h$) Fermi surfaces follows from Eqs. (1.6)–(1.8). When we consider the umklapp processes, we must bear in mind that the difference between them and the normal processes is largely arbitrary. As is known, the selection of the unit cell in the momentum space is arbitrary all that is necessary is that it should include all the nonequivalent states (but not more than once). Clearly, when the unit cell is selected suitably, a transition between any fixed states p and k can be described by a normal or an umklapp process.

This allows us to formulate the general conclusion that the electrical resistivity differs from zero if the umklapp processes cannot be removed by any selection of the unit cell. We can easily show that this is equivalent to the following requirement: under the action of collisions with phonons at least some of the electrons should escape to infinity in the extended p space and, if this escape to infinity is impossible along certain directions, then the electric field E should be perpendicular to these directions. With the exception of compensated metals and the special case of an exactly cylindrical Fermi surface (see Appendix II), this condition is both necessary and sufficient. If the infinite trajectories mentioned above include "jumps" over finite distances, the asymptotic behavior of the resistivity in the limit $T \to 0$ is exponential; otherwise, the Bloch law $\rho \propto T^5$ applies.

We shall conclude this section by considering briefly the role of the umklapp processes in heat conduction in metals with closed Fermi surfaces. The quasimo-

mentum balance condition then becomes

$$\sum_{pkq} g\Gamma^{+}_{pkq} = -\sum_{p} \mathbf{p}\frac{\varepsilon_p-\mu}{T}\left(\nabla T\frac{\partial n_p}{\partial \mathbf{p}}\right)-\sum_{q}\mathbf{q}\frac{\omega_q}{T}\left(\nabla T\frac{\partial n_q}{\partial \omega_q}\right)-e\mathbf{E}'(n_e-n_h), \tag{1.9}$$

where $\mathbf{E}'=\mathbf{E}-e^{-1}(\partial\mu/\partial T)\nabla T$ and μ is the chemical potential.

We can easily see that the role of the umklapp processes in heat conduction in compensated and uncompensated metals is in a sense opposite to that found in electrical conduction.

In fact, if $n_e=n_h$, the balance condition (1.9) clearly cannot be satisfied without allowance for the umklapp processes. This is due to the fact that simultaneous drift of electrons and phonons is accompanied by heat transfer in zero electric field because $\mathbf{j}=e(n_e-n_h)\mathbf{u}$. Consequently, the thermal conductivity increases exponentially as a result of cooling.[1)] However, the thermoelectric field $\mathbf{E}$ deduced from the condition $\mathbf{j}=0$ varies with temperature in accordance with a power law.

If $n_e\neq n_h$, then the balance condition (1.9) does not require allowance for the umklapp processes. In this case we find from Eq. (1.9) that, to within terms which are exponentially small in the limit $T\to 0$, the thermoelectric field is given by

$$e\mathbf{E}'=\left[\alpha\frac{T}{\mu}+\beta\left(\frac{T}{\theta}\right)^3\right]\nabla T, \tag{1.10}$$

where α and β are numerical coefficients of the order of unity. It is important to note that the thermo-emf is then not related to the requirement that the electric field should vanish. The latter condition makes it possible to determine the arbitrary vector $\mathbf{u}$ in the solution (1.5) of the homogeneous system of transport equations considered above. [If $n_e=n_h$, when the umklapp processes have to be allowed for, the solution (1.5) does not apply.] Naturally, the thermal conductivity depends on temperature as a power law.

2. DIFFUSION EQUATION

The majority of calculations of the electrical conductivity of metals have been based on the variational principle applied to the transport equation (see, for example, Ref. 4). The main problem which is encountered in the application of this method is the selection of the trial functions describing the electron and phonon distributions. It is usual to employ the simplest drift functions of the (1.5) type with generally different values of the velocity $\mathbf{u}$ for electrons and phonons. It is difficult to control the precision of the drift approximation. In some cases it overestimates greatly the contribution of the umklapp processes to the electrical resistivity. This is evident most clearly in the case of an open Fermi surface since the function $\chi_p=\mathbf{u}\cdot\mathbf{p}$ does not satisfy the condition of periodicity in p space.

Recently, Kagan, Zhernov, and Flerov carried out a series of investigations[5-7] in which they went beyond the framework of the standard "one-momentum" approximation: the trial function for electrons was selected in the form of an angular polynomial subject to restrictions imposed by the symmetry of a crystal. This made it possible to satisfy the condition of periodicity in the case of open Fermi surfaces and to allow for the anisotropy of the distribution function in the case of closed Fermi surfaces.

Our analysis is based on a fundamentally different (diffusion) approach and it makes consistent use of the principal physical feature of the electron–phonon scattering at low temperatures, which is its small-angle nature. Bethe and Sommerfeld pointed out many years ago[8] that it is possible to expand the transport equation in terms of the small parameter q/p_F. Klements and Jackson[9] used physical considerations to derive an equation describing the diffusion of electrons on the Fermi surface in the case of the simplest model (the dispersion laws of electrons and phonons were assumed to be isotropic, and phonons were assumed to be in equilibrium).

a) Low-temperature expansion

In going over to the calculations, we note that the transport equation (1.3) can be solved for the phonon distribution function Φ_q. Substituting this expression in Eq. (1.2), we obtain a very complex integral equation for the electron function χ_p which cannot be investigated in its general form. The problem simplifies greatly at sufficiently low temperatures when the phonon momentum $q\sim T/s$ is small compared with all the characteristic dimensions of the Fermi surface. We can show that under these conditions the required function χ_p is largely independent of the energy (see Appendix I) and on the constant-energy surface it varies significantly only over distances comparable with the characteristic dimension of this surface.[2)] Therefore, χ_{p-q} and other continuous functions of the quasi-momentum occurring in the collision integral can be expanded as series in powers of q, and the nonzero result is obtained only in the approximation quadratic with respect to q. However, such calculations are very cumbersome and their results cannot be presented in a clear manner.

It is much more convenient to derive the diffusion equation by making an assumption which immediately yields a compact expression for the collision integral. This can be done by summing Eq. (1.2) over a certain region V_0 in the p space. Let us assume that this region includes a part of the Fermi surface whose di-

[1)] We can show that this dependence should appear at lower temperatures than in the electrical conductivity of an uncompensated metal.

[2)] The energy equilibrium is established rapidly—as a result of a single collision with a thermal phonon—whereas complete relaxation requires a large number of collisions needed to displace an electron over the whole Fermi surface (we shall return to this topic in Sec. 3). All this applies to relaxation under the influence of an electric field which—in the principal approximation with respect to the parameter T/μ—does not alter the electron energy distribution. In the presence of a temperature gradient the function X_p is energy-dependent, which has a considerable influence on the nature of the diffusion equation (see Appendix I).

mensions are much greater than the thermal phonon momentum.

Bearing in mind that $B_{\mathbf{pkq}}=B^{*}_{\mathbf{k}\mathbf{p},-\mathbf{q}}$, we find that simple transformations give

$$\sum_{p\subset V_0} I^{e}=\Big(\sum_{p\subset V_0,\ k\notin V_0,\ q}-\sum_{p\notin V_0,\ k\notin V_0,\ q}\Big)\Gamma^{+}_{\mathbf{pkq}}. \tag{2.1}$$

It is important to note that in expanding this approximation in terms of q a nonzero result is obtained already in the linear approximation, whereas in the expansion of the quantity I^e itself it is necessary to include second-order terms. [There is no contradiction here because Eq. (2.1) contains an extra power of q due to the smallness of the summation region: transitions can take place only from a narrow layer near the boundary of the region V_0.] This is the advantage of the proposed method.

We shall divide the collision integral into two parts

$$I^{e}=I^{ep}+I^{epe},$$

where I^{ep} is the integral describing collisions with equilibrium phonons [i.e., without $\Phi_{\mp q}$ in Eq. (1.4) for $\Gamma_{\mp\mathbf{pkq}}$], and I^{epe} describes the interaction between electrons via phonons.

We shall now consider the sum $\sum I^{ep}$. We note that in the first order in q the difference $\chi_{\mathbf{p}}-\chi_{\mathbf{k}}$, where $\mathbf{k}=\mathbf{p}-\mathbf{q}-\mathbf{g}$, can be written in the form $\chi_{\mathbf{p}}-\chi_{\mathbf{k}}=\mathbf{q}\partial\chi_{\mathbf{p}}/\partial\mathbf{p}$ for any value of g (we recall that $\chi_{\mathbf{p}}$ is a periodic and continuous function of the quasimomentum and its period is $\mathbf{g}$). All the other calculations can be carried out in the lowest nonvanishing approximation with respect to q. In particular, the matrix element of the interaction can be written in the form

$$|b_{\mathbf{pkq}}|^2=qM_{\mathbf{p}}(\mathbf{e}),\quad \mathbf{e}=\frac{\mathbf{q}}{q},$$

and since the Hamiltonian is Hermitian, we have $M_{\mathbf{p}}(-\mathbf{e})=M_{\mathbf{p}}(\mathbf{e})$.

The results of calculations can be represented in the form

$$\sum_{p\subset V_0} I^{ep}=\frac{2}{\hbar^3}\oint_{\gamma}(\hat{D}_{\mathbf{p}}\nabla\chi_{\mathbf{p}})\,d\mathbf{l}.$$

Integration is carried out here only along the line of intersection of the Fermi surface with the boundary of the region V_0 and the vector $d\mathbf{l}$ is directed along the outer normal to this line, which lies in a plane tangential to the Fermi surface. The operator ∇ represents a two-dimensional operation applied to the tangential plane: $\nabla=\hat{P}\partial/\partial\mathbf{p}$. When quadratic differential operations are carried out, it must be remembered that the projection operator $\hat{P}$ is a function of the momentum.

The diffusion tensor is then of the form

$$D^{ik}_{\mathbf{p}}=T^5\frac{30\zeta(5)}{\pi^2\hbar^4v^2_{\mathbf{p}}}\int_0^{2\pi}\frac{M_{\mathbf{p}}(\mathbf{e})}{s^5(\mathbf{e})}e^ie^k\,d\varphi, \tag{2.2}$$

where the angle φ is measured in the plane tangential to the Fermi surface [with accuracy to within quantities of the order of q/p_F and s/v, the unit vector $\mathbf{e}(\varphi)$ lies in this plane].

Application of Gauss's theorem gives

$$\sum_{p\subset V_0} I^{ep}=\frac{2}{\hbar^3}\int_{V_0}\operatorname{div}(\hat{D}_{\mathbf{p}}\nabla\chi_{\mathbf{p}})\,dS_{\mathbf{p}},$$

where integration is carried out over the part of the Fermi surface included in the region V_0 and $\operatorname{div}A=\nabla_t A_t$.

Similarly, we obtain

$$\sum_{p\subset V_0} I^{epe}=-\frac{2}{\hbar^3}\int_{V_0}\operatorname{div}(\hat{D}_{\mathbf{p}}\mathbf{a}_{\mathbf{p}})\,dS_{\mathbf{p}},$$

where

$$\mathbf{a}_{\mathbf{p}}=\int\hat{A}_{\mathbf{pp'}}\nabla\chi_{\mathbf{p'}}\,dS_{\mathbf{p'}}, \tag{2.3}$$

$$\hat{A}_{\mathbf{pp'}}=\frac{2\hat{D}^{-1}_{\mathbf{p}}\hat{D}_{\mathbf{p}}(\mu)\,D_{\mathbf{p'}}(\mu)}{\alpha(\mu)\,|[\mathbf{n}_{\mathbf{p}}\mathbf{n}_{\mathbf{p'}}]|}, \tag{2.4}$$

$$\alpha(\mu)=\int D_{\mathbf{p'}}(\mu)\,\delta(\mu\mathbf{n}_{\mathbf{p'}})\,dS_{\mathbf{p'}}.$$

Integration in Eq. (2.3) is over the whole Fermi surface; $\mathbf{n}_{\mathbf{p}}=\mathbf{v}_{\mathbf{p}}/v_{\mathbf{p}}$; μ is the unit vector defined by $[\mathbf{n}_{\mathbf{p}}\times\mathbf{n}_{\mathbf{p'}}]$; $D_{\mathbf{p}}(\mathbf{e})$ is the "partial" diffusion coefficient defined by the relationships

$$\hat{D}_{\mathbf{p}}=\int_0^{2\pi}\hat{D}_{\mathbf{p}}(\mathbf{e})\,d\varphi,\quad D^{ik}_{\mathbf{p}}(\mathbf{e})=D_{\mathbf{p}}(\mathbf{e})\,e^ie^k.$$

Thus, all the terms of the transport equation (1.2) integrated over the region V_0 can be represented in the form of integrals over the part of the Fermi surface included in this region (this transformation is self-evident for the field term $e\mathbf{E}\partial n_{\mathbf{p}}/\partial\mathbf{p}$). Bearing in mind that the region V_0 is chosen in an arbitrary manner, we obtain the required diffusion equation

$$\operatorname{div}\hat{D}_{\mathbf{p}}(\nabla\chi_{\mathbf{p}}-\mathbf{a}_{\mathbf{p}})=-e\mathbf{E}\mathbf{n}_{\mathbf{p}}, \tag{2.5}$$

where the quantities $\hat{D}_{\mathbf{p}}$ and $\mathbf{a}_{\mathbf{p}}$ are given by Eqs. (2.2) and (2.3).

It should be noted that $\hat{D}_{\mathbf{p}}$ in Eq. (2.5) does not have the usual dimensions p^2/t of a diffusion coefficient: $[\hat{D}_{\mathbf{p}}]=p^2/vt$. This is due to the fact that the density of nonequilibrium electrons on the Fermi surface is proportional to $v^{-1}_{\mathbf{p}}\chi_{\mathbf{p}}$, whereas the diffusion flux is proportional to $\nabla\chi_{\mathbf{p}}$.

b) Analysis of the diffusion equation. Allowance for the umklapp processes

Equation (2.5) describes the diffusion of electrons on the Fermi surface in the presence of an electric field, which determines the density of sources and sinks. This equation is integrodifferential and the integral term $\mathbf{a}_{\mathbf{p}}$ is associated with the phonon drag. This is easy to understand. The emission of a phonon by some electron has two consequences: firstly, this electron is displaced on the Fermi surface by an amount equal to the phonon momentum and, secondly, the momentum of the second electron which has absorbed the emitted phonon also changes.

2) *Open Fermi surfaces.* In the diffusion approximation the phonon momentum should be regarded as infinitesimally small. Therefore, in the case of an open Fermi surface the umklapp processes are allowed for fully by the imposition of periodic boundary conditions on the function $\chi_{\mathbf{p}}$:

$$\chi_{\mathbf{p}+\mathbf{q}}=\chi_{\mathbf{p}},\quad \nabla\chi_{\mathbf{p}+\mathbf{q}}=\nabla\chi_{\mathbf{p}}, \tag{2.6}$$

i.e., the function χ_p and its first derivatives should be matched at the equivalent zone faces.

In the subsequent analysis it is convenient to write Eq. (2.5) in the form of a purely integral equation describing a certain vector function $\nabla\chi$ (the index p will be omitted in all those cases when confusion is unlikely):

$$\nabla\chi - \mathbf{a} = \hat{D}^{-1}\psi, \tag{2.7}$$

where the flux ψ satisfies the differential equation

$$\operatorname{div}\psi = -e\mathbf{E}n. \tag{2.8}$$

The homogeneous equation corresponding to Eq. (2.7) has the nontrivial solution $\nabla\chi = \nabla \mathbf{u}p = \mathbf{u}_\parallel$, where $\mathbf{u}_\parallel$ is the component of the drift velocity u parallel to the Fermi surface. It is shown in Appendix II that this solution is the only one if we ignore the case of an exactly cylindrical Fermi surface.[3)]

The integral kernel $\hat{A}$ in Eq. (2.7) is not symmetric but the product $\hat{D}\hat{A}$ is symmetric [see Eq. (2.4)]. It follows that the transposed homogeneous equation has just one solution $\nabla\chi = \hat{D}\mathbf{u}_\parallel = \hat{D}\mathbf{u}$ and, therefore, the condition of solubility of Eq. (2.7) is

$$\int \psi\, dS = 0. \tag{2.9}$$

The displacement of an electron in p space is equivalent to a change in the quasimomentum of the phonon system. The condition (2.9) implies that the total quasimomentum transferred from electrons to phonons is zero. For a Fermi surface with three nonparallel open directions Eq. (2.9) can definitely be satisfied since the solution of Eq. (2.8) can always be supplemented by some flux $\delta\psi$, such that $\operatorname{div}\delta\psi = 0$. In the case of a closed Fermi surface the solution of a homogeneous equation with a nonzero total flux $\int \delta\psi dS$ must have singularities.

The relationship (2.9) allows us to obtain easily the quasimomentum balance condition in the diffusion approximation:

$$\frac{1}{2}\int_L \mathbf{g}(\hat{D}(\nabla\chi - \mathbf{a}),\ d\mathbf{l}) = e\int \mathbf{p}(\mathbf{E}n)\, dS. \tag{2.10}$$

Integration is carried out along the line L of the intersection between the Fermi surface and the boundaries of the unit cell; the vector g is different for different parts of this line; the element dl is directed along the normal to the line L in the tangential plane so that $\mathbf{g}d\mathbf{l} > 0$.

Since Eq. (2.5) does not include a term for the umklapp processes, it is not surprising that the corresponding homogeneous equation has the drift solution $\chi = \mathbf{u}\cdot\mathbf{p}$. In other words, Eq. (2.5) is invariant under the Galilean transformation $\chi = \chi' - \mathbf{u}p$, but the boundary conditions of this equation are not invariant:

$$\chi'_{p+g} - \chi'_p = \mathbf{u}g, \quad \nabla\chi'_{p+g} = \nabla\chi'_p, \tag{2.11}$$

where u is an arbitrary velocity of the reference system.

We shall now consider the following question: is there a reference system in which the nonequilibrium nature of phonons is of little importance, so that it is reasonable to ignore the integral term $a\{\nabla\chi'\}$? It is shown in Appendix II that a particular solution of Eq. (2.7) can be obtained by the method of iterations over the integral term, i.e., the general solution has the form

$$\Delta\chi = C\{\psi\} - \mathbf{u}_\parallel, \tag{2.12}$$

$$C\{\psi\} = \hat{D}^{-1}\psi + a\{\hat{D}^{-1}\psi\} + \ldots, \tag{2.13}$$

where the sum C converges not slower than a certain decreasing geometric progression. The velocity u in Eq. (2.12) should be found from the boundary conditions (2.6) (these also remove the arbitrary nature of the selection of ψ). The reference system in which the problem can be solved by iterations over the phonon nonequilibrium will be called the comoving reference frame. Clearly, if the distribution of phonons is of the drift type, then phonons should be in equilibrium in the comoving reference system. Therefore, higher iterations in Eq. (2.13) (i.e., the difference $C\{\psi\} - \hat{D}^{-1}\psi$) allow for the deviation of the phonon distribution from the drift form. We shall use a specific example [see Eq. (3.8)] to show that the corresponding "nondrift" corrections to the conductivity are numerically small. The point is this: phonons with a given momentum q interact with all the electron states located along the line $\mathbf{q}\cdot\mathbf{v} = 0$. Consequently, the anisotropy of the Fermi surface usually has little effect on the phonon distribution. However, in the case of the Fermi surfaces elongated strongly along a certain direction, the higher terms in an iteration series may be important.

We thus find that in the lowest—with respect to $a\{\nabla\chi'\}$—approximation the distribution function and the velocity of the comoving reference system can be determined from Eqs. (2.8), (2.9), (2.6), and (2.12), where we have to assume that $C\{\psi\} = \hat{D}^{-1}\psi$. An equivalent and more convenient (for subsequent use) is the following system of equations[4)]:

[3)] In the case of a cylindrical Fermi surface the diffusion equation is insoluble. Since this result may be of interest in investigations of the electrical conductivity of two-dimensional systems, we shall consider it in greater detail. In the case of a cylindrical Fermi surface all the phonons can be divided into groups, each of which interacts only with electrons located along one or several pairs of symmetric generators. (The exception to this rule is represented by phonons with momenta parallel to the generators of the cylindrical surface, because these phonons interact with all the electrons on the Fermi surface.) Therefore, in the process of relaxation in an external electric field the momenta of the individual groups rise without limit with time, although the total momentum of the phonon system may remain constant. A steady state is clearly always achieved when an allowance is made for collisions between phonons. Then, depending on the relationship between the frequencies of phonon–phonon collisions ν^N_{pp} and phonon–electron collisions ν_{pe}, we have two possibilities: a) if $\nu^N_{pp} \gg \nu_{pe}$, the diffusion approximation remains valid (see Footnote 4); b) if $\nu^N_{pp} \ll \nu_{pe}$, the rare phonon–phonon collisions govern the electron diffusion velocity and this alters significantly the temperature dependence of the electrical conductivity.

[4)] An analysis within the framework of Eqs. (2.14) and (2.15) is asymptotically accurate under conditions such that the frequent normal phonon–phonon collisions impose the drift distribution on phonons: $\nu^U_{pp} \ll \nu_{pe} \ll \nu^N_{pp}$. At moderately low temperatures these inequalities may be satisfied by some metals, and particularly by semimetals.

$$\operatorname{div} D\Delta\chi' = -e\,\mathrm{En}, \tag{2.14}$$

$$\frac{1}{2}\int_L \mathbf{g}(\hat{D}\nabla\chi',\ \mathrm{dl}) = -e\int \mathbf{p}(\mathrm{En})\,dS \tag{2.15}$$

with the boundary conditions (2.11). The integral on the right-hand side of Eq. (2.15) is given by the formula (1.8).

It should be noted that the system of equations (2.14), (2.15) is in a sense indeterminate. If we use the fact that the quasimomentum balance condition (2.15) is valid for any permissible selection of the unit cell in p space, then Eq. (2.14) follows from this condition (and not the other way round!). Thus, we can use either the system (2.15), assuming that it is valid for any selection of the unit cell, or the system (2.14), (2.15) with a fixed basic region.

In the "one-dimensional" model, in which the Fermi surface is a surface of revolution and all the quantities depend on just one coordinate measured along the axis of revolution, the complete solution can easily be obtained. We then find that

$$|\nabla\chi'| = eE\frac{V-\pi r^2 g}{2\pi r g D}, \tag{2.16}$$

where r is the radius of a certain section of the Fermi surface and D is the value of the diffusion coefficient in this section.

2) *Closed Fermi surfaces.* In this case the umklapp processes (or, more exactly, jumps between connected parts of the Fermi surface) should be allowed for explicitly in the diffusion equation (2.5). The boundary condition (2.6) then loses its meaning because the unit cell in p space can be selected so that it does not intersect the Fermi surface.

First of all, we note that at sufficiently low temperatures, when the characteristic umklapp time $\tau^U \propto \exp(s\Delta p/T)$ as long compared with all the other relaxation times,[5)] an allowance for the umklapp processes presents no difficulty. They can be ignored in the electrical conductivity of a compensated metal. If the metal is uncompensated, the normal collisions result in a simultaneous drift of electrons and phonons whose velocity u can be found from the quasimomentum balance condition. The electric current is then $\mathbf{j} = e(n_e - n_h)\mathbf{u} \propto \exp(s\Delta p/T)$. However, this asymptotic dependence is not observed experimentally because of the influence of the electron–impurity scattering.

Over a wide range of low temperatures a typical situation is one in which the time τ^U is comparable (or even smaller) compared with the time for the diffusion of an electron across a Fermi surface. Nevertheless, even in this case the umklapp processes can in a sense be regarded as rare. This is permitted because the umklapp processes are possible only in the regions of the nearest approach of closed sheets of the Fermi surface and such regions represent a small part of the total surface. (These regions will be called "hot spots" or "sinks.") It is easy to show that under these conditions it is permissible to ignore the influence of the umklapp processes on the phonon distribution and also the nonequilibrium nature of phonons on the umklapp processes. Similarly, in the case of an open Fermi surface which has narrow necks we can ignore the nonequilibrium nature of phonons within the regions of these necks.

Subject to the above comments the procedure described in Sec. 2a gives the diffusion equation (2.5) with the additional term

$$\Pi_{\mathbf{p}} = \frac{1}{\nu_{\mathbf{p}}}\int I_{\mathbf{p}}^U\,d\varepsilon, \tag{2.17}$$

where $I_{\mathbf{p}}^U$ is the part of the collision integral (1.2) corresponding to the umklapp processes in the case when $\Phi_{\mathbf{q}} = 0$.

We shall now obtain a low-temperature expansion of the quantity $\Pi_{\mathbf{p}}$ and we shall do this by considering the model of relatively wide hot spots, which appear when smooth parts of the Fermi surface of radius of curvature of the order of p_F approach one another (Fig. 2a). The characteristic dimension r_0 of such a hot spot can easily be estimated bearing in mind that the inequality $|\mathbf{p}-\mathbf{p}^*| - \Delta p \lesssim q \approx T/s$ ($\mathbf{p}^*$ is the nearest point to p on the neighboring Fermi surface) is satisfied within the hot spot and hence we find that $r_0 \approx \sqrt{qp_F}$. From the point p transitions occur to a region in the vicinity of the point $\mathbf{p}^*$ and the dimensions of this region are $r_1 \approx \sqrt{2q\Delta p + q^2}$. Since $r_0 > r_1$, the "local" approximation can be applied to the umklapp processes: in the expression for I^U we can assume that $\chi_{\mathbf{k}} = \chi_{\mathbf{p}^*}$. With the same degree of precision the points p and $\mathbf{p}^*$ can be regarded as lying on the same horizontal, as shown in Fig. 2a.

Finally, after integration in Eq. (2.17) with respect to energies $\varepsilon_{\mathbf{p}}$ and $\varepsilon_{\mathbf{k}}$, we obtain

$$\left.\begin{aligned} \Pi_{\mathbf{p}} &= -\frac{1}{\nu_{\mathbf{p}}}\nu_{\mathbf{p}}^U(\chi_{\mathbf{p}} - \chi_{\mathbf{p}^*}), \\ \nu_{\mathbf{p}}^U &= \frac{16\pi^3}{h^4}\int |B_{\mathbf{pkq}}|^2\,\omega_{\mathbf{q}}\left|\frac{dN_{\mathbf{q}}}{d\omega_{\mathbf{q}}}\right|\frac{dS_{\mathbf{k}}}{v_{\mathbf{k}}}, \end{aligned}\right\} \tag{2.18}$$

where $\mathbf{q} = \mathbf{k} - \mathbf{p} - \mathbf{g}$; $dS_{\mathbf{k}}$ is an element of the area of the neighboring surface; g is the reciprocal lattice vector corresponding to a given hot spot.

It should be noted that the diffusion approximation is

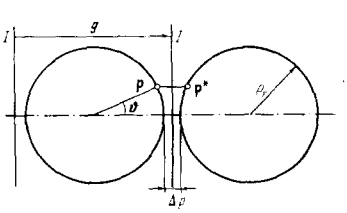

FIG. 2a. Spherical Fermi surface in the periodic zone scheme (I is the boundary of the Brillouin zone).

[5)] Under these conditions the transport phenomena in metals are of hydrodynamic nature and, consequently, several specific effects are observed: the temperature dependence of the electrical conductivity of samples of finite size has a minimum, there are weakly damped vibrations of the second-sound type, etc.[11]

justified for a "wide" hot spot because $r_0 \gg q$.[6] However, when the approaching parts of the Fermi surface have strong curvatures, the hot spots are of small size. Then, the local approximation for the umklapp processes given by Eq. (2.18) and the diffusion approximation for the normal collisions may both be disobeyed within a hot spot. However, the minimum size of a hot spot is $r_0 \gtrsim q$ and, therefore, even in the worst case these approximations are at the limit of their validity. Moreover, the situation is usually such that a change in the distribution function χ_p within a small hot spot can be ignored compared with its change over distances of the order of p_F.

The equations corresponding to Eqs. (2.14) and (2.15) and defining the distribution function in the lowest order of iterations with respect to $a\{\nabla\chi'\}$ are

$$\operatorname{div} \hat{D}\nabla\chi'_p - \frac{1}{v} \nu_p^U (\chi_p - \chi_{p^*}) = -e\mathbf{E}\mathbf{n}, \tag{2.19}$$

$$\frac{1}{h^3} \int_F g\nu_p^U (\chi_p - \chi_{p^*}) \frac{dS}{v} = eE\,(n_e - n_h). \tag{2.20}$$

3. ELECTRICAL CONDUCTIVITY OF METALS

In this section we shall use the diffusion equation method to consider the low-temperature electrical conductivity of metals in the absence of a magnetic field. Since, in general, the diffusion equation cannot be integrated, calculations will be carried out using certain approximate models. Our main task will be to determine the relationship between the electrical conduction mechanisms and the structure of the Fermi surface.

a) Open Fermi surfaces. Fermi surfaces with narrow necks

Let us assume that our Fermi surface consists of large electron or hole pockets (sheets) connected by narrow necks. In general, the Fermi surface occupies several zones and some of these pockets can be isolated. Therefore, the problem has a small parameter $d/p_F \ll 1$ (d is the neck diameter) and a complete solution can be obtained. Clearly, in this model the drag effects are strongest since the electrical conductivity becomes infinite (if we ignore the umklapp processes) when the neck diameter approaches zero. Fermi surfaces with narrow necks are exhibited by, for example, noble and many polyvalent metals. Moreover, the analogy with electrical circuits which applies here will be used frequently in the subsequent discussions.

In calculations of this kind we must note that a diffusion current of high density $D|\nabla\chi| \sim eEp_F^2/d$ flows along a narrow neck [see Eq. (2.16)]. Therefore, in the region of a neck the integral term a can be ignored compared with $\nabla\chi$ [higher iterations in Eq. (2.13) are small since the singularity $1/r$ disappears on integration].

[6] If $q \ll \Delta p$ an electron acquires an energy $\Delta ps \gg T$ as a result of an umklapp collision, i.e., this electron is transferred to a nonthermal energy layer. Normal collisions, which return an electron to the Fermi surface, displace it by a distance Δp and not by q. However, at temperatures sufficiently low to satisfy $\Delta p \gtrsim r_0$, the distribution function changes over distances of the order of p_F and, therefore, if $\Delta p \ll p_F$, the diffusion approximation is not affected.

Moreover, we can ignore the electric field effect because the change in the flux caused by this field is proportional to the cross sectional area of the neck. Therefore, the diffusion equation in a neck region is

$$\operatorname{div} \hat{D}\nabla\chi = 0. \tag{3.1}$$

In the subsequent discussion it is convenient to use the analogy between the present problem and the flow of steady-state currents in a branched electrical circuit. The necks correspond to conductors in the circuit and the large pockets correspond to the nodes of the network. The current flowing through a neck is

$$J = \oint \hat{D}\nabla\chi \, dl,$$

where integration is carried out along a closed line surrounding the neck.

The analog of the potential is the nonequilibrium correction χ'. The drop of the "potential" occurs mainly in narrow necks, whereas in large pockets the potential can be regarded as constant.

The linearity of the equations makes the current J obey "Ohm's law":

$$\chi'_2 - \chi'_1 = JR,$$

where χ'_1 and χ'_2 are potentials in large pockets and the "resistance" R is governed by the properties of the neck itself, and can be found from Eq. (3.1).

Next, it is easy to show (see Ref. 10) that the currents obey the Kirchhoff rules:

$$\sum J = 0, \quad \sum JR = 0.$$

In the first equation the summation sign applies to the currents entering a given large pocket along the necks; the second summation applies to any "internal" circuit which does not intersect the zone boundaries. At a zone boundary the potential χ' has a discontinuity equal to $\mathbf{u}\cdot\mathbf{g}$ [see Eq. (2.11)]. Therefore, the "external" circuits include the sources of emf's:

$$\sum JR = \mathbf{ug}, \tag{3.2}$$

where $\mathbf{g}$ is the reciprocal lattice vector closing the circuit; the sign of this vector is determined by the direction in which the circuit is traversed.

In addition, the quasimomentum balance condition (2.10) yields another condition (the unit cells are selected in such a way that they intersect the Fermi surface only along the necks):

$$\sum_{\mathbf{g}} J\mathbf{g} = e\mathbf{E}V, \tag{3.3}$$

where V is the difference between the electron and hole volumes bounded by the Fermi surface and the zone boundaries.

The above equations are sufficient to determine all the currents and the velocity u of the comoving reference system. In the adopted model the electric current is $\mathbf{j} = 2e\mathbf{u}Vh^{-3}$, because in the reference system linked to the lattice the electron distribution drifts within a large pocket. In the simplest case when all the necks are intersected by the zone boundaries, we find from

Eqs. (3.2) and (3.3) that

$$\hat{\sigma} = \frac{2e^2}{h^3} V^2 \hat{S}, \quad (S^{-1})_{ik} = \sum \frac{g_i g_k}{R},$$

where the summation is taken over all the necks.

The neck resistance can be estimated by using a "one-dimensional" model discussed in Sec. 2. If the typical neck diameter d is of the order of the radius of curvature of the neck $\mathscr{R}$, it follows from Eq. (3.1) that

$$R \approx D^{-1} \ln \frac{p_F}{d}. \tag{3.4}$$

In the limiting case of $d \ll \mathscr{R}$, we have

$$R \approx D^{-1} \sqrt{\mathscr{R} d^{-1}}, \tag{3.5}$$

where the typical neck length is $\sqrt{d\mathscr{R}}$.

It has been assumed so far that the thermal phonon momentum is much less than all the characteristic dimensions of the Fermi surface. In the presence of long necks ($d \ll \mathscr{R}$) it is interesting to consider the range of intermediate temperatures when the thermal momentum of a phonon satisfies the inequalities

$$d \ll q \ll \sqrt{d\mathscr{R}} \tag{3.6}$$

In this case the diffusion approximation remains valid but the tensor $\hat{D}$ changes considerably:

$$D = Ar, \quad A = \frac{4\pi^3}{15\hbar^4} \left(\frac{T}{s(e)}\right)^4 \frac{M_p(e)}{v_p^2}; \tag{3.7}$$

here, D is the diagonal component of the diffusion tensor along the neck axis; $\mathbf{e}$ is a unit vector along this axis; r is the distance of a given point on the Fermi surface from the axis of revolution.

It follows from Eq. (3.7) that

$$R = \frac{1}{4A} \sqrt{\frac{\mathscr{R}}{d^3}}.$$

Thus, under the conditions of Eq. (3.6), we have $R \propto d^{-3/2} T^{-4}$, whereas for $q \ll d$, it follows from Eq. (3.5) that $R \propto d^{-1/2} T^{-5}$. The change in the nature of the temperature dependence is due to the one-dimensional nature of diffusion: only those phonons whose momenta are almost parallel to the neck axis can interact with electrons. The number of such phonons is proportional to T and not to T^2, in contrast to the two-dimensional diffusion case. Thus, in the case of metals with long Fermi-surface necks we find that in the temperature range corresponding to Eq. (3.6) the electrical resistivity is proportional to T^4.

Caution must be exercised when the above results are used in an analysis of the electrical conductivity of specific metals. The small parameter, which justifies the above approximations, is in fact the ratio of the resistance of a large pocket to the resistance of a neck. In the case of a short neck this parameter is of the order of $(\ln p_F/d)^{-1}$ and need not be sufficiently small in real cases. Moreover, as demonstrated by direct measurements of Gantmakher and Gasparov,[12,13] in the case of some metals the probability of electron–photon scattering in a neck is considerably greater than within a large pocket. (This is possibly due to the low velocity of electrons v_p in the region of a neck or due to a considerably greater contribution of the scattering by transverse vibrations in this region.)

Nearly-free electron approximation. We shall consider the simplest model which can be used to obtain the exact solution of the problem. We shall assume that a spherical Fermi surface is intersected by just one pair of the Bragg planes and, consequently, it is located in two energy bands (Fig. 2b; the distortion of the Fermi sphere near the zone boundary is unimportant in the model under discussion).[10] We shall assume that all the characteristics of the investigated metal are isotropic and, in particular, that the diffusion tensor reduces to a constant scalar D. The electric field is naturally perpendicur to the Bragg planes.

In this model the kernel of the diffusion equation (2.4) becomes

$$A_{pp'}^{ik} = \frac{\mu_i \mu_k}{\pi^2 p_F^2 \sin(\hat{nn'})}.$$

We shall omit the calculations of Ref. 10 and give the final result. The density of the electric current is $j = neu$ and the drift velocity u can be expressed in the form

$$u = u_0 - u_0 \gamma^{-1} + u_0 (\gamma^{1/2} - \gamma^{-1/2})^2 + u_1, \tag{3.8}$$

where

$$u_0 = \frac{eEp_F}{2D}, \quad \gamma = \frac{1}{3\cos^2\vartheta_0} \ln \frac{1+\cos\vartheta_0}{1-\cos\vartheta_0},$$

$$u_1 = \frac{2u_0}{3\cos^2\vartheta_0} \sum_{i=3}^{\infty} \frac{2i+1}{i(i+1)} \frac{\lambda_i}{1-\lambda_i} [P_i(\vartheta_0)]^2, \tag{3.9}$$

$$\lambda_i = \frac{1-(-1)^i}{i(i+1)} \left[\frac{i!!}{(i-1)!!}\right]^2;$$

here, $\cos\vartheta_0 = g/2p_F$; $P_i(\vartheta)$ are the Legendre polynominals.

The first two terms in Eq. (3.8) are in agreement with the results of Klemens and Jackson,[9] who ignored the phonon drag. We can easily show that u_0 is the electron drift velocity in the case when phonons are in equilibrium and there are no umklapp processes. The second term ($-u_0\gamma^{-1}$) is associated with the umklapp processes. The third term is due to the phonon drift.

The fourth term u_1 is due to deviation of the phonon distribution from the drift form. An analysis of Eq. (3.9) shows that the contribution of this last term is relatively small. For example, in the case of trivalent fcc metals we have $g/2p_F = 0.89$ and calculations show that then $u_1/u \approx 0.22$. Thus, inclusion of higher terms in the iteration series with respect to $\mathbf{a}_p$ [see Eq. (2.13)] alters the numerical coefficient only slightly.

These results are easily generalized to the case of a more realistic model with several open directions.[10]

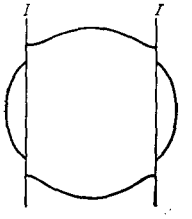

FIG. 2b. Open Fermi surface in the nearly-free electron approximation.

b) Closed Fermi surfaces

As pointed out several times before, the electrical conductivity of a metal with a closed Fermi surface and with $n_e \neq n_h$ should obey—at sufficiently low temperatures—the law $\sigma \sim \exp(T_0/T)$ ($T_0 \approx \Delta p s$, where Δp is the minimum separation between the isolated parts of the Fermi surface). However, there are as yet no reports of experimental observation of the strongly exponential Peierls dependence, although in the case of some metals measurements have been made at temperatures much lower than T_0. The most favorable objects for the observation of this dependence are probably alkali metals for which the separations between the closed electron groups are very large: $\Delta p \approx p_F/3$ (for Na we have $T_0 \approx 20\,^\circ$K whereas for K, we find that $T_0 \approx 10\,^\circ$K). Woods[14] carried out experiments which gave the following temperature dependence of the electrical resistivity of Na: $\rho \sim T^5$ in the range $15\ K > T > 9$ K and a faster variation amounting to approximately $\rho \sim T^6$ in the interval $9\ K > T > 5\ K$; impurity scattering predominates in the range $T \leq 5\ K$. A law close to T^6 was observed also for K at temperatures $4.2\ K > T > 1.3$ K by Tsoĭ and Gantmakher[15]; similar results were reported by Ekin and Maxfield,[16] as well as by other authors.

Two questions arise in this connection. 1) At which temperature does the exponential dependence of the electrical conductivity begin to apply? 2) What is the temperature dependence which precedes the exponential region?

We shall try to answer these questions by considering the electrical conductivity of metals with closed spherical Fermi surfaces.[17] The actual calculations carried out within the framework of this model apply directly to alkali metals, but—as shown below—some general conclusions apply to any closed Fermi surface.

1) *Idealized model.* We shall first consider the simplest model in which Fermi spheres located along the same line approach one another closely in a periodic zone scheme (Fig. 2a). Naturally, the electric field is directed along this line. The distribution function depends only on the angle ϑ, measured along the arc of the great circle from the point of the closest approach of the spheres.

The quantity Δp is assumed to be extremely small so that

$$\ln \frac{p_F}{\Delta p} \gg 1. \tag{3.10}$$

It follows from the above inequality that the transition from the diffusion equation (2.5) to the simplified scheme of Eqs. (2.19) and (2.20) is an asymptotically exact procedure. In the model adopted here, Eqs. (2.19) and (2.20) become

$$\Delta\chi' - A(\vartheta)\chi = -\frac{eE}{D}\cos\vartheta, \tag{3.11}$$

$$\int A(\vartheta)\chi\, dS = \frac{eE}{gD}\cdot\frac{4}{3}\pi p_F^3, \tag{3.12}$$

where

$$A(\vartheta) = \frac{2}{vD}\nu^U(\vartheta) = \frac{f(|\mathbf{p}-\mathbf{p}^*|q^{-1})}{15\zeta(5)q^3} \qquad q = \frac{T}{|s|},$$

$$f(x) = \int_x^\infty \frac{e^y}{(e^y-1)^2} y^3\, dy, \quad D = T^5\frac{30\zeta(5)M}{\pi\hbar^4 v^2 s^5};$$

here, Δ is the angular part of the Laplacian; $|\mathbf{p}-\mathbf{p}^*| \approx \Delta p + {}^2p_F$ is the distance from the point p to the next sphere; an allowance is made for the fact that since χ is an odd function of p and the problem is one-dimensional we have $\chi_{p^*} = -\chi_p$, $\chi = \chi' \pm (ug/2)$, where the plus sign corresponds to the right-hand hot spot and the minus sign to the left-hand one; integration in Eq. (3.12) is carried out within the limits of the right-hand hot spot.

It follows from Eqs. (3.11) and (3.12) that $\int \cos\vartheta\, \Delta\chi' ds = 0$ (this relationship represents the law of conservation of momentum in normal collisions and it is valid in any order of iteration with respect to a). Hence, $\int \cos\vartheta\, \chi' dS = 0$. Thus, in the comoving reference system there is no electric current and, therefore, the electrical conductivity is $\sigma = neuE^{-1}$.

To determine the drift velocity u we have to solve Eq. (3.11) together with the condition (3.12). Far from a hot spot the second term on the left-hand side of (3.11) can be ignored, so that we obtain

$$\chi' = \frac{eEp_F^2}{2D}\left[\cos\vartheta - \frac{1}{3}\beta(\vartheta)\right] \qquad \beta(\vartheta) = \ln\frac{1+\cos\vartheta}{1-\cos\vartheta}.$$

It is impossible to solve Eq. (3.11) exactly within a hot spot. However, the nature of the solution can be understood from the following considerations. A logarithmic rise of the function χ' on approach to the boundary of a hot spot is associated with a static diffusion flux $2\pi\vartheta D d\chi'/d\vartheta \approx (2/3)eE\pi p_F^2$. Inside a hot spot the diffusion flux decreases since part of the flux is carried away by electrons crossing over to the next sphere. Therefore, in the region of a hot spot the variation of the function χ' is slower than logarithmic (with finite value at the center of the hot spot). Consequently, in the determination of the drift velocity u by means of Eq. (3.12), we can ignore the variation of χ' within a hot spot [if the inequality (3.10) is obeyed].

Simple calculations give

$$u = \frac{1}{6}eEp_F(R_d + R_u), \tag{3.13}$$

where

$$R_d = \frac{1}{D}\ln\frac{p_F}{q}, \quad R_u = \frac{1}{DF},$$

$$F = \frac{1}{4\pi}\int A(\vartheta)\, dS = \frac{p_F}{60\zeta(5)q}\int_{T_u/T}^\infty \frac{y^3 e^y}{(e^y-1)^2}\left(y - \frac{\Delta p}{q}\right) dy.$$

We have introduced here the quantities R_d and R_u which are the resistances in the case of the diffusion and umklapp processes, respectively. The interpolation formula for F, which is asymptotically correct when $q \gg \Delta p$ or $q \ll \Delta p$, is of the form

$$F = \frac{1}{60\zeta(5)}\frac{p_F}{q}\left[24\zeta(3) + \left(\frac{\Delta p}{q}\right)^3\right]\exp\left(-\frac{\Delta p}{q}\right).$$

We shall now consider the physical meaning of Eq. (3.13). The electrical conductivity is proportional to the total relaxation time in which an electron completes a closed cycle in p space: this involves diffusion across the Fermi surface and a jump as a result of an umklapp process. It should be stressed that the corresponding relaxation times are additive under the phonon drag conditions.

The first term, R_d, in Eq. (3.13) is proportional to the diffusion time of an electron from the central region of the Fermi surface to a hot spot and it depends on temperature in accordance with the law $T^5 \ln(p_F s/T)$ [a similar result for a short neck is represented by Eq. (3.4)]. The second term, R_u, in Eq. (3.13) is proportional to the umklapp time and it depends on temperature as $T^{-1}\exp(T_0/T)$ if $T \ll T_0$.

It follows from Eq. (3.13) that the temperature T_u below which the Peierls exponential dependence is observed is considerably less than T_0 if $\Delta p \ll p_F$. This important result is related to the following two circumstances. 1) Although the probability of an umklapp process is exponentially small even at $T < T_0$, an electron is transferred to the neighboring sphere as a result of one collision, whereas the diffusion across the whole Fermi surface requires a large number of steps. 2) Each of the three quantities—the square of the matrix element of the interaction, the area of the part of the Fermi surface where the transition is possible ($r_1^2 \approx \Delta pq$), and the layer of energies in which the transition is possible—is $\Delta p/q$ times greater for phonons with momenta of the order of Δp than for thermal phnons. Therefore, the probability of an umklapp process contains an additional factor $(\Delta p/q)^3$ which is large is $T \ll T_0$.

By way of illustration, Fig. 3 shows the dependence, on the parameter of the problem $p_F/\Delta p$, of the temperature T_{u1} at which $R_u/R_d = 0.25$ and of the temperature T_{u2} at which $R_d/R_u = 0.25$. We can see that the temperature range of the transition from the law $\rho \propto T^5/\ln p_F s/T$ to the law $\rho \propto T\exp(-T_0/T)$ is fairly wide and the ratio T_0/T_u increases on increase in $p_F/\Delta p$.

In qualitative estimates and in generalizing the results obtained to other Fermi surface models, it is convenient to write the conductivity in the form

$$\sigma = \frac{ne^2}{m}(\tau_d + \tau_u), \quad \tau_u = \tau_U \frac{S_F}{S_s}.$$

Here, the umklapp time for an electron in a hot spot is $\tau_U \approx [v^U(\vartheta = 0)]^{-1}$; S_s is the hot spot area; S_F is the area of the whole Fermi surface. The factor S_F/S_s in the expression for τ_u allows for the fact that when $\tau_u \gg \tau_d$ an electron wanders for a long time over the whole Fermi surface before spin flipping and it is in a hot spot only for a small proportion (amounting to S_s/S_F) of the total umklapp time τ_u.

It is natural to assume that the value of τ_u depends mainly on the relationship between the parameters q and Δp and it is not very sensitive to the hot spot geometry; the order of magnitude is given by

$$\tau_U \approx \tau' \begin{cases} 1, & T \gg T_0, \\ (T/T_0)^3 e^{T_0/T}, & T \ll T_0, \end{cases}$$

where $\tau' \propto T^{-3}$ is the collision time between an electron and a phonon.

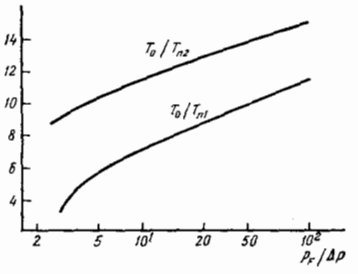

FIG. 3. Position and width of the intermediate temperature range.

In the adopted model a typical hot spot area is $S_s \approx r_0^2 \propto T$. If the surfaces approach one another because of narrow side branches (as is the case for a broken long neck of diameter $d \lesssim \Delta p$), then $S_s \approx dq$, i.e., the hot spot area is again proportional to T. Finally, in the case of elongated hot spots, which appear when surfaces with very different approach principal radii of curvature $\mathcal{R}_1 \gg \Delta p$ and $\mathcal{R}_2 \lesssim \Delta p$ one another, the hot spot area is $S_s \approx \mathcal{R}_1^{1/2} q^{3/2} \propto T^{3/2}$. The temperature dependence of the diffusion time is also to some extent related to the hot spot model: when regions with a large radius of curvature approach one another we have $\tau_d \propto T^{-5}\ln(\theta/T)$, whereas in the presence of a broken long neck we find that $\tau_d \propto T^{-5}$ or $\tau_d \propto T^{-4}$ [Eq. (3.7)].

It is important to note that the conclusion that the temperature T_u is much lower than T_0 (approximately as shown in Fig. 3) is valid for any closed Fermi surface if the characteristic dimensions of the electron or hole pockets are considerably greater than the distances between them.

We shall now return to the hypothesis that the function χ is independent of the energy, which is important in the derivation of the diffusion equation. If the umklapp processes are negligible, this hypothesis can be justified rigorously (see Appendix I) and it is physically related to the fact that the energy relaxation requires many fewer collisions than the momentum relaxation. The situation is different in the region of a hot spot on a closed Fermi surface at temperatures $T \lesssim T_u$ when the momentum relaxation is roughly due to just one umklapp collision. An electron acquires an energy Δps as a result of the umklapp collision and, therefore, it is then in a nonthermal layer. Nevertheless, in this case again the energy relaxation time is shorter than the umklapp time and, therefore, the dependence of χ on ε is unimportant. In fact, after emitting several phonons with momenta of the order of Δp, a nonequilibrium electron finds itself in its "own" thermal layer in a part of the Fermi surface of area $(\Delta p)^2$; if the same electron undergoes an umklapp collision, it finds itself in the neighboring part of the Fermi surface of area of the order of $r_1^2 \approx q\Delta p$. Thus, the ratio of the corresponding times is of the order of $(r_1/\Delta p)^2$ and the ratio of the energy-dependent and energy-independent parts of χ are also of the same order. In the worst case we find that for $T \approx T_u$ this parameter is of the order of T_u/T_0, i.e., it is considerably larger than outside a hot spot where the smallness of the energy correction is governed by the parameters T/ε_F and $(T/\theta)^2$. An allowance for the energy dependence is made in Ref. 18 within the variational method framework.

2) *Alkali metals.* Some of the simplifying assumptions used in the preceding section are known to be invalid for alkali metals. First of all, we cannot expect

the condition (3.10) to be satisfied and even the weaker inequality $\Delta p \gg p_T$ is satisfied poorly. Under these conditions the very diffusion equation (3.11) provides only a qualitative description of the behavior of the function χ in a hot spot (see Footnote 6).

Nevertheless, we shall attempt to approach a real situation in alkali metals and show that the above qualitative conclusions are still valid. The most important refinements of the result (3.13) are allowances for the dependence of χ on p in the region of a hot spot and for the fact that each Fermi sphere of an alkali metal is closely approached by twelve other Fermi spheres. It is probable that the anisotropy of the phonon spectra is weaker so that we shall use only the Debye model.

In the case of a Fermi sphere with several pairs of hot spots the diffusion approximation, similar to Eq. (3.11), is described by

$$\Delta\chi' - \frac{1}{2} A(\vartheta_k)(\chi_p - \chi_{p^*}) = -\frac{eE}{D}\cos\vartheta; \tag{3.14}$$

here, the angle ϑ_k is measured from the direction of the reciprocal lattice vector g_k corresponding to the hot spot nearest to the point p, and the angle is measured from the direction of the electric field **E**, which can conveniently be regarded as applied along one of the reciprocal lattice vectors g_0 (because in this model the electrical conductivity is isotropic).

We shall seek the solution of Eq. (3.14) in the form of a linear combination of solutions of the "one-dimensional" problem corresponding to one pair of hot spots:

$$\chi_p' = \sum_k \tilde{\chi}(\vartheta_k)\cos\alpha_k + \frac{eEp_F^2}{2D}\cos\vartheta. \tag{3.15}$$

Here, α is the angle between the vectors g_k and g_0 and the function $\tilde{\chi}(\ )$ satisfies the one-dimensional equation

$$\Delta\tilde{\chi} - A(\vartheta)\left(\tilde{\chi} \pm \frac{\tilde{u}g}{2}\right) = 0. \tag{3.16}$$

We can easily show that the function of the (3.15) type satisfies Eq. (3.14) if

$$\tilde{u} = u + \frac{2}{g}\sum_{k\neq 0}\tilde{\chi}(\alpha_k)\cos\alpha_k + \frac{eEp_F^2}{gD}$$

and the quantity $\tilde{\chi}(\vartheta_{k\pm k})$ can be regarded as constant within the kth hot spot.

It is easily shown that the condition analogous to Eq. (3.12) is of the form

$$\int A(\vartheta)\left(\tilde{\chi} + \frac{\tilde{u}g}{2}\right) dS = \frac{eE}{gDn_s} 8\pi p_F^2, \tag{3.17}$$

where n_s is the total number of hot spots.

Far from a hot spot the function $\tilde{\chi}$ can be found from Eqs. (3.16) and (3.17), in the same way as has been done in the preceding subsection:

$$\tilde{\chi} = -\beta(\vartheta)\cdot\frac{2eEp_F^2}{n_s gD}.$$

Inside a hot spot Eq. (3.16) can be solved exactly (in the asymptotic sense) only in two limiting cases: $R_d \ll R_s$ and $R_d \gg R_s$. However, the expression for the conductivity may be obtained more simply by a different method. It is easily shown that Eq. (3.16) corresponds to the variational principle

$$\delta\int \tilde{\chi}[\Delta\tilde{\chi} - A(\vartheta)(\tilde{\chi} + 2\tilde{u}p)]\, dS = 0, \tag{3.18}$$

where $\tilde{u} = \tilde{u}g/g$.

We shall select a trial distribution function in the form

$$\tilde{\chi} = -\frac{2eEp_F^2}{gDn_s}\tilde{\beta}(\vartheta), \quad \tilde{\beta}(\vartheta) = \begin{cases} \beta(\vartheta), & \vartheta > \vartheta_0, \\ \beta(\vartheta_0), & \vartheta < \vartheta_0, \end{cases}$$

where ϑ_0 is a variational parameter which, together with $\tilde{u}$, should be found from Eqs. (3.18) and (3.17).

Calculations give the following result[17]:

$$\sigma \approx \frac{ne^2 p_F}{2D}\left\{1 - 2\frac{p_F}{g}(1-\cos\vartheta_0) + \frac{8p_F^2}{n_s g^2}\left[\sum_k \tilde{\beta}(\alpha_k)\cos\alpha_k + \frac{1}{F} - \frac{C\ln 2}{\ln 2 + F}\right]\right\} + \sigma', \tag{3.19}$$

$$\vartheta_0^2 = \frac{q}{p_F}\ln[2(FC+1)], \quad C = \int_{\ln 2}^{\infty} e^{-x}\frac{dx}{x};$$

here, $\alpha_0 = \vartheta_0$; the function F is given by Eq. (3.13); σ' is the correction to the electrical conductivity due to allowance for the integral term a_p in the attached reference system, given by

$$\sigma' = \frac{8ne^2 p_F^2}{n_s g^3 D}\sum_{l=3}^{\infty}\frac{2l+1}{l(l+1)}\frac{\lambda_l}{1-\lambda_l} \times \sum_k [P_l(\alpha_k)]^2\cos\alpha_k;$$

λ_l is defined in Eq. (3.9).

Figure 4 shows the dependence of the quantity ρT^{-5} (in relative units) on the relative temperature T/T_0, calculated using Eq. (3.19). For comparison, the same figure shows (dashed curve) the results obtained in the diffusion approximation but without allowance for the phonon drag. We can see that the drag plays an important role: in the temperature range $(0.15-0.6)T_0$ the results differ by a factor of almost two, whereas at lower temperatures the curves diverge completely.

In the case of potassium, we have $T_0 \approx 10°$ K and the slope of the curve at intermediate temperatures agrees quite well with the experimental results[15,16] exhibiting dependences close to $\rho_T \propto T^6$ and $\rho_T \propto T^7$ ($\rho_T = \rho - \rho_0$, where ρ_0 is the residual resistivity). In the case of sodium, we have $T_0 \approx 20°$ K and, as is clear from Fig. 4, the change from the law $\rho_T \propto T^5$ to a stronger dependence occurs at about 8 °K, in agreement with the experimental results of Woods.[14]

We shall conclude this subsection with the following comments.

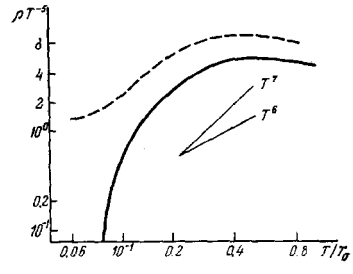

FIG. 4. Theoretical temperature dependence of the resistivity of an alkali metal. The dashed curve represents the temperature dependence in the absence of drag.

1. The crystal symmetry of alkali metals forbids a "horizontal" umklapp transition from the center of a hot spot (i.e., a transition between the nearest points on neighboring Fermi spheres) involving a phonon with a transverse polarization. Therefore, transverse phonons with momenta not parallel to the corresponding reciprocal lattice vector participate in the umklapp processes. These "oblique" transitions result in a somewhat different asymptotic dependence of the conductivity in the limit $T \to 0$ than predicted on the basis of Eq. (3.13): $\sigma \propto T^{-2}\exp(T_0/T)$ instead of $\sigma \propto T^{-1}\exp(T_0/T)$ (the asymptotic behavior is governed by the transverse phonons whose velocity is less than that of the longitudinal phonons). However, this asymptotic dependence was not reached under experimental conditions. An estimate of the characteristic angle between the phonon momentum and the vector g ($\delta\vartheta \leq \sqrt{T/T_0}$) shows that in the temperature range $T_u < T < T_0$ the forbiddenness of the horizontal transitions results in a slight change in the resistivity.

2. The resistivity of alkali metals has been calculated by many authors (in particular, calculations are reported in Refs. 19 and 20) using the variational principle in the one-moment approximation (with a trial function $\chi = \mathbf{u}\cdot\mathbf{p}$). The most detailed calculations of Kaveh and Wiser[21] gave a good agreement with the experimental results. In this connection it should be noted that at temperatures $T > T_u$ the difference $\chi_p - \chi_{p^*}$ is small compared with the reduction in the function χ within the whole Fermi surface. On the other hand, the function $\mathbf{u}\cdot\mathbf{p}$ does not have this property and this results in an overestimate of the contribution of the umklapp processes. However, at the same time the contribution of the diffusion processes is underestimated since the drift function is smoother than the real function χ that varies over distances shorter than p_F. Therefore, the use of the function $\mathbf{u}\cdot\mathbf{p}$ results in a redistribution of the contributions between the diffusion and umklapp processes (for a fixed trial function in the variational method the resistivities are additive: $\rho = \rho_N + \rho_U$). The total resistivity can then change only slightly. This is typical of the variational method: the use of a trial function far from the true one may result in relatively small errors in the resistivity.

Zhernov and Kagan[7] recently reported a calculation of the resistivity of alkali metals in a many-moment approximation. These calculations allow in fuller detail for the characteristics of the electron and phonon spectra, and for the interaction between electrons and phonons than the results reported above and, therefore, they may represent more accurate numerical values. On the other hand, the above discussion allows us to understand the physical situation associated with the competition between the diffusion and umklapp processes, which occurs irrespective of the actual model of the Fermi surface.

4. GALVANOMAGNETIC PROPERTIES OF METALS

In strong magnetic fields the asymptotic behavior of the electrical resistivity of a metal considered as a function of the field is governed entirely by the topological properties of the Fermi surface in the direction of the field. This result can be proved rigorously without any assumptions about the electron scattering mechanism.[22] However, such properties of the magnetoresistance as its dependence on temperature and other parameters characterizing collisions, the dependence on the orientation of the magnetic field, and the field in which asymptotic behavior is established, are determined largely by the nature of the electron scattering mechanism. In general, we can only say that the conductivity is a monotonically decreasing function of the field[23] and this decrease becomes significant for $\Omega\tau_{tr} \gtrsim 1$ (τ_{tr} is the transport or mean free time governing the electrical conductivity in the absence of a magnetic field and Ω is the Larmor frequency).

In the simplest case when a typical electron scattering angle is $\Phi \approx 1$ (as in the case of collisions with local lattice imperfections, such as impurity atoms), satisfactory results can be obtained in the relaxation time approximation. Then, the resistivity depends on just one parameter $\Omega\tau$: in the range $\Omega\tau \ll 1$ the influence of magnetic fields is weak ("weak fields") and the asymptotic dependence is observed for $\Omega\tau \gg 1$ ("strong fields"). If a typical scattering angle is $\Phi \ll 1$ (electron–phonon collisions at low temperatures, scattering by dislocations), the situation becomes much more complex.

Pippard[24] was the first to draw attention to the fact that in a strong magnetic field the effectiveness of small-angle collisions rises strongly in the presence of certain features of the Fermi surface that result in a fast variation of the electron distribution function in momentum space. Then, the asymptotic behavior on increase in the magnetic field is observed in much higher fields and we have to satisfy the inequality $\Omega\tau_{tr} \gg \Phi^{-1}$ or even $\Omega\tau_{tr} \gg \Phi^{-2}$. However, Pippard's treatment ignores a number of specific features of the momentum relaxation in an electron–phonon system of a metal and these features will be used in subsequent discussion.

The unklapp processes in electron–phonon collisions play a role which is as important in galvanomagnetic effects as in the electrical conductivity in zero magnetic fields. In the absence of these processes in a metal with a closed Fermi surface, a drift of electrons and phonons is established.

$$\chi_p = \mathbf{u}_H\mathbf{p}, \quad \Phi_q = \mathbf{u}_H\mathbf{q}, \quad \mathbf{u}_H = cH^{-2}[\mathbf{EH}],$$

and only the Hall components of the transverse conductivity tensor do not vanish:

$$\sigma_{xy} = -\sigma_{yx} = e(n_e - n_h)cH^{-1}, \quad \sigma_{xx} = \sigma_{yy} = 0;$$

here, z is the direction of the magnetic field $\mathbf{H}$ and $\mathbf{E}\cdot\mathbf{H} = 0$.

The role of the umklapp processes in metals with closed Fermi surfaces (or when electrons cross the zone boundaries in metals with open Fermi surfaces) is easily demonstrated by the condition of balance of the quasimomentum perpendicular to H. In the presence of a magnetic field the diffusion equation is (t is

the time of motion along an orbit in a magnetic field)

$$-\frac{1}{v}\frac{\partial\chi}{\partial t}+\operatorname{div}\hat{D}(\nabla\chi-\mathbf{a})+\Pi=-e\mathbf{E}n. \tag{4.1}$$

We shall derive the quasimomentum balance condition by multiplying this equation additionally by $[\mathbf{p}\times\mathbf{H}]$ and integrating with respect to the Fermi surface; after simple transformations using the identity (II.2), we find that

$$\mathbf{j}_\perp=e(n_e-n_h)\mathbf{u}_X+\frac{c}{H^2}\sum_{\mathbf{g},\sigma}[\mathbf{gH}]A_{\mathbf{g},\sigma}, \tag{4.2}$$

$$A_{\mathbf{g},\sigma}=\frac{1}{h^3}\left(\int\limits_{(\mathbf{g},\sigma)}\Pi\,dS+\int\limits_{(\mathbf{g},\sigma)}Q\,dl-e_\sigma^*\mathbf{S}_{\mathbf{g},\sigma}\mathbf{E}\right), \tag{4.3}$$

$$Q=\hat{D}(\nabla\chi-\mathbf{a})+ec^{-1}[\mathbf{Hn}]\chi. \tag{4.4}$$

In the above formulas, $\mathbf{j}_\perp$ is the component of the electric current density perpendicular to the magnetic field; σ is the number of the energy band; the quantities e^* and $\mathbf{S}$ have the same meaning as in Eq. (1.8); summation is carried out over all the reciprocal lattice vectors.

The first term in Eq. (4.2) is clearly the current associated with the Hall drift. The meaning of the second term is easily understood: the quantity $A_{\mathbf{g},\sigma}$ gives the number of transitions per unit time accompanied by a change in the momentum by the vector $\mathbf{g}$ and in each transition the center of an orbit shifts in coordinate space by the vector $ce^{-1}H^{-2}[\mathbf{g}\times\mathbf{H}]$. It should be stressed that under the phonon drag conditions only these processes are responsible for the transverse conductivity in a magnetic field. If the Fermi surface is closed, then in the expression (4.3) for $A_{\mathbf{g},\sigma}$ only the first term (umklapp frequency) differs from zero. In the case of an open Fermi surface there is an additional diffusion flux [the first term in Eq. (4.4)] and the flux associated with the motion of electrons along orbits in a magnetic field [the second term in Eq. (4.4)] across the basic cell boundary. The last term in Eq. (4.3) is associated with the motion of electrons across the boundary of the cell under the action of an electric field.

a) Strong fields. Closed Fermi surfaces

In a strong magnetic field (the criterion of strong fields will be given later) it is natural to employ the method of successive approximations in solving Eq. (4.1). We find that

$$\frac{\partial\chi^{(1)}}{\partial t}=e\mathbf{E}_\perp\mathbf{v},\quad \chi^{(1)}=\mathbf{u}_X\mathbf{p}+f(p_z), \tag{4.5}$$

$$-\frac{\partial\chi^{(2)}}{\partial t}+v\operatorname{div}\hat{D}(\nabla\chi-\mathbf{a}(f))+v\Pi(\chi^{(1)})=-eE_zv_z. \tag{4.6}$$

In the last equation an allowance is made for the invariance of the flux $\nabla\chi-\mathbf{a}$ under the transformation $\chi\to\chi+\mathbf{u}\cdot\mathbf{p}$.

In calculation of the electric current by means of Eq. (4.2) in a strong magnetic field it is sufficient to consider only the first approximation (4.5). However, the function $f(p_z)$ is determined from the conditions of solubility of Eq. (4.6) representing the next approximation:

$$\langle v\operatorname{div}\hat{D}(\nabla f-\mathbf{a}(f))\rangle+\langle v\Pi(\chi^{(1)})\rangle=-eE_z\langle v_z\rangle, \tag{4.7}$$

$$\frac{2}{h^3}\int p_z\Pi(\chi^{(1)})\,dS=-eE_z(n_e-n_h).$$

The angular brackets denote averaging over the revolution period $T(p_z)$:

$$\langle\ldots\rangle=\left|\frac{eH}{c}\right|\int_0^T\ldots dt.$$

We shall make the substitution $f(p_z)=u_cp_z+\psi(p_z)$ and thus go over to the comoving reference system, in which—in accordance with the definition—Eq. (4.7) can be solved by the method of iterations with respect to the integral term a. We shall confine ourselves to the first iteration (i.e., we shall simply omit the term a in the relationships expressed in terms of the comoving system). Simple transformations give the following final equations for the function $\psi(p_z)$ and the velocity u_c:

$$\frac{d}{dp_z}D\frac{d\psi}{dp_z}+\langle v\Pi(\mathbf{up}+\psi)\rangle=-eE_z\langle v_z\rangle, \tag{4.8}$$

$$\frac{2}{h^3}\int p_z\Pi(\mathbf{up}+\psi)\,dS=-eE_z(n_e-n_h), \tag{4.9}$$

where

$$D(p_z)=\left\langle D_{\xi\xi}\frac{v_\perp^2}{v}\right\rangle,\quad \mathbf{u}_\perp=\mathbf{u}_X,\quad u_z=u_c;$$

here, $D_{\xi\xi}$ is the diagonal element of the diffusion tensor $\hat{D}$ at right-angles to an orbit. The term with derivatives in Eq. (4.8) describes the diffusion of electrons on the Fermi surface at right-angles to the orbits; $-D\,d\psi/dp_z$ is the total diffusion flux across the section $p_z=\text{const}$. It should be stressed that in a strong magnetic field the diffusion along an orbit can be ignored and, therefore, the diffusion process is one-dimensional. The term $\langle v\Pi\rangle$ gives transitions of electrons between the orbits passing through equivalent hot spots. Equation (4.9) describes the balance of the quasimomentum in the magnetic field direction.

The nature of the solution of Eqs. (4.8), (4.9) depends strongly on the relative positions of the hot spots on the Fermi surface, on the magnetic field orientation, and on the temperature range. We shall begin by analyzing first a simple physical situation: we shall assume that the layers of orbits passing through each hot spot (belts on the Fermi surface shown in Fig. 5) do not overlap and that the distance between them b is considerably greater than their width.

In the case under consideration the results of solution of Eqs. (4.8), (4.9) can be formulated in terms of the flow of steady-state currents in branched electrical circuits. In view of the one-dimensional nature of diffusion, the analogy with circuits is more complete than

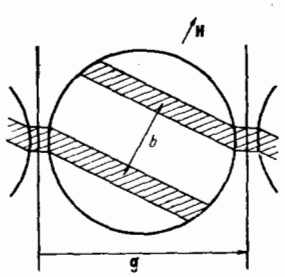

FIG. 5. Hot spots (sinks) and hot-spot belts on a spherical Fermi surface. One pair of hot spots is shown and the belts are shaded.

in the absence of a magnetic field.

The analog of the electric potential is the function $\psi(p_z)$. The diffusion current is

$$J_d = -D\frac{d\psi}{dp_z}. \tag{4.10}$$

We shall consider the case of crossed fields **E** and **H**. Then, the difference between the potentials follows from Eq. (4.8):

$$\delta\psi = J_d R_d, \quad R_d = \int_{p_{z1}}^{p_{z2}} \frac{dp_z}{D(p_z)}. \tag{4.11}$$

Here, integration is carried out within a layer between two neighboring belts and R_d is the resistance of such a layer.

The umklapp current through a given hot spot is

$$J_u = R_u^{-1}(\mathbf{ug} + \psi_s - \psi_s^*), \quad R_u^{-1} = \int \nu_p^U \frac{dS}{v_p}; \tag{4.12}$$

here, ψ_s is the value of the function $\psi(p_z)$ in a given hot spot; ψ_s^* applies to an equivalent hot spot. [The variation of the function $\psi(p_z)$ within a belt can be ignored in the adopted model.] The quantity R_u is the umklapp resistance.

Having integrated Eq. (4.8) over a small region covering one belt, we find that

$$J_{d1} + J_{d2} + J_u = 0, \tag{4.13}$$

where J_{d1} and J_{d2} are the diffusion fluxes on the boundaries of the belt; for the first hot spot corresponding to the minimum value of p_z, we have $J_{d1}=0$, whereas for the second we have $J_{d2}=0$. (It is assumed that the current emerging from a hot spot is positive.)

Equations (4.11)–(4.13) together with the condition of continuity of the function $\psi(p_z)$ are identical with the Kirchhoff laws for a certain electrical circuit. Generalization to the case $E_z \neq 0$ presents no difficulties.[25]

These relationships make it possible to find the function $\psi(p_z)$ and the velocity of the comoving system u_c. In accordance with Eq. (4.2), the electric current density is described by

$$\mathbf{j}_\perp = -\frac{c}{h^3 H^2}\sum_{\mathbf{g}} [\mathbf{gH}]\, J_u + e(n_e - n_h)\,\mathbf{u}_X, \tag{4.14}$$

$$j_z = \frac{2e}{h^3}\int v_z \psi\, dS + e(n_e - n_h)\,u_c. \tag{4.15}$$

We shall now consider the physical meaning of the results obtained. The order of magnitude is given by $J_u \approx \mathbf{u_H g}(R_d + R_u)^{-1}$ and, according to Eq. (4.14), the transverse conductivity is

$$\sigma_{xx} \approx \left(\frac{c^2 g^2}{h^3 H^2}\right)(R_d + R_u)^{-1}. \tag{4.16}$$

It follows from the above expression that the transverse conductivity involves, in the situation considered, both the umklapp and diffusion processes: although σ_{xx} is proportional to the number of the umklapp events [see Eq. (4.14)], if diffusion is ignored it is found that the densities of nonequilibrium electrons at equivalent hot spots are equal and we have $\sigma_{xx}=0$. In other words, in a strong magnetic field the transverse conductivity, like that in $H=0$, is due to the feasilibity of infinite motion of electrons in the reciprocal-lattice space. The characteristic relaxation time in a strong magnetic field consists of the umklapp time and the diffusion time between hot-spot belts: in this time the center of an electron orbit is displaced by the vector **g** (Fig. 5).

It should be noted in this connection that Eq. (4.16) can be represented in the form[7)]

$$\sigma_{xx} \approx \frac{n_{\text{eff}} e^2}{m\Omega^2 \tau_{\text{eff}}}, \qquad n_{\text{eff}} \approx \frac{b}{p_F} n, \quad \tau_{\text{eff}} \approx \tau_d^b + \frac{p_F b}{r_0^2}\tau_U; \tag{4.17}$$

here, n_{eff} is the number of electrons participating in infinite motion; b is the distance between the belts ($b \gg r_0$); $\tau_d^b \approx p_F(b/v)R_d^b \approx \tau_F(b/p_F)^2$ is the time corresponding to a diffusion displacement by a distance b; $\tau_U \approx r_0^2 R_u/v$ is the umklapp time for an electron located in a hot spot; $r_0^2/p_F b$ is the probability of finding an electron in a hot spot. The diffusion resistance of a layer of thickness b is equal to $R_d^b \approx R_F b/p_F$, where $R_F \approx p_F/D \propto T^5$ corresponds to the diffusion displacement across the whole Fermi surface.

As shown in the preceding section, $R_F \gg R_u$ right down to very low temperatures T_u and the conductivity is then governed by the diffusion time. However, in a strong magnetic field, it follows from Eqs. (4.16) and (4.17) that the competition between the diffusion and umklapp times can be avoided by directing the magnetic field in such a way that an electron may be transferred from one hot spot to an equivalent one as a result of its orbital motion. This provides an opportunity for an experimental investigation of the umklapp processes in a wide range of temperatures, and, in particular, of the dependence $\sigma_{xx} \sim \exp(-T_0/T)$ in the range $T < T_\theta$. It should be stressed that this result is independent of the relationship between the numbers of electrons and holes.

It is also clear from the above formulas that in the $T > T_u$ case the transverse conductivity is strongly anisotropic:

$$\frac{\sigma_{xx}^{\max}}{\sigma_{xx}^{\min}} \approx \left(\frac{R_F}{R_u}\right)\left(\frac{b_{\max}}{p_F}\right),$$

where $b_{\max}$ is the greatest distance between the hot-spot belts and the ratio R_F/R_u reaches it maximum at $T \approx T_0$.

It is usual to determine experimentally the resistivity tensor $\hat{\rho} = \hat{\sigma}^{-1}$. The order of magnitude of the transverse resistivity is given by

$$\rho_{yy} \approx \frac{\sigma_{xx}}{\sigma_{xy}^2} \approx \frac{z^2}{e^2\hbar^3(n_e - n_h)}(R_d + R_u)^{-1}, \quad n_e \neq n_h,$$
$$\rho_{yy} \approx \sigma_{xx}^{-1} \approx \frac{h^3 H^2}{c^2 g^2}(R_d + R_u)_2 \quad n_e = n_h.$$

It is clear from these formulas that at temperatures $T > T_u$ and magnetic field directions corresponding to the overlap of the hot spots the resistivity has a maxi-

[7)] It should be noted that the conditions under which the phonon scattering mechanism predominates over the impurity mechanism in strong magnetic fields are generally less stringent than in the absence of a magnetic field. In fact, the impurity-controlled conductivity $\sigma_{xx} \approx ne^2/m\Omega^2\tau_i$ (τ_i is the mean free time for the scattering by impurities) is small compared with that given by Eq. (4.71) if $\tau_i \gg \tau_F b/p_F + \tau_U(p_F/r_0)^2$, whereas in $H=0$ and for $T > T_U$, the contribution of impurities is small when the more stringent condition $\tau_i \gg \tau_F$ is satisfied.

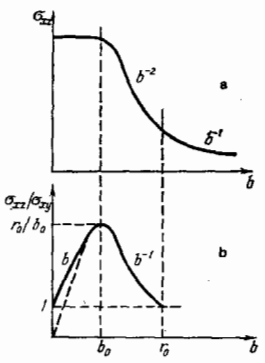

FIG. 6. Anisotropy of the conductivity components σ_{xx} and σ_{xz} in the case of a closed Fermi surface. Here, $b=g\varphi$, where ϑ is the angle of deviation of the magnetic field from the direction corresponding to the exact overlap of hot spots.

mum if $n_e \neq n_h$ and a minimum if $n_e = n_h$.

We shall now discuss the range of validity of our relationships. The Kirchhoff rules (4.11)–(4.13) are essentially based only on the idea that a change in the function $\psi(p_z)$ within one hot-spot belt is small compared with the difference between the values of the distribution function in equivalent hot spots and, therefore, the diffusion and umklapp effects can be separated. Therefore, the results obtained are valid not only for $b \gg r_0$, but also for other directions of the magnetic field if $b \approx r_0$ corresponds to the condition $R_d^b \ll R_u$. The umklapp resistance R_u depends strongly on the structure of the Fermi surface in the region of a hot spot. If in this region at least one of the radii of curvature of the Fermi surface is comparable with the gap $\Delta p \ll p_F$ ("narrow" or "elongated" hot spots), the Kirchhoff rules apply for any direction of the magnetic field.[8)]

The situation is different in the case of metals with "wide" hot spots (both radii of curvature are of the order of p_F and $r_0 \approx \sqrt{qp_F}$). In this case at temperatures $T \gtrsim T_0$ the transition from the "diffusion" to the "umklapp" regime occurs when the overlap of the hot spots is considerable and the dependence $\psi(p_z)$ is very strong.[25] The results of calculations can be described qualitatively by Eq. (4.17), where

$$\tau_{\text{eff}} \approx \tau_d^b + \frac{p_F(r_0+b)}{r_0^2}\tau_U, \quad n_{\text{eff}} \approx n\frac{b+r_0}{p_F}.$$

The angular dependence of the transverse conductivity is shown in Fig. 6a, and the parameter b_0 is

$$b_0^2 \approx r_0 q \begin{cases} 1, & T > T_0, \\ \left(\frac{T}{T_0}\right)^3 \exp\left(\frac{T_0}{T}\right), & T < T_0. \end{cases}$$

When several pairs of hot spots overlap, the pattern naturally becomes much more complex.[25]

In addition to the transverse conductivity, a considerable anisotropy is also exhibited by the value of σ_{xz}. Physically, this is due to the fact that when an electron passes to infinity along a chain of hot-spot orbits, it also is displaced in the direction of the p_z axis (if $b \neq 0$, see Fig. 5). Consequently, part of the function $\psi(p_z)$ becomes anisotropic along p_z and this is also true of the current along the z axis, in accordance with Eq. (4.15). Calculations based on Eqs. (4.8), (4.9), and (4.15) yield the following relationship which gives the right order of magnitude for any model of hot spots:

$$\sigma_{zz}(b) \approx \frac{nec}{H}\left[\frac{\sigma_{xx}(b)}{\sigma_{xx}(p_F)}\frac{b}{g} + \lambda\right], \tag{4.18}$$

where $\lambda \approx 1$ is a smooth function of the angles, that varies considerably over the whole Fermi surface. The corresponding angular dependence is shown in Fig. 6b; the dashed curve corresponds to the case when $\sigma_{xz}=0$ in the case of exact overlap of the hot spots (because of the symmetry of the problem).

It should be stressed that the conclusion about the strong anisotropy of the transverse conductivity applies only to those metals in which, firstly, the size of the hot spot at $T=T_0$ is sufficiently small compared with p_F and, secondly, there are not too many hot spots so that there are directions of H along which they are far from overlapping. In the case of some metals (alkali metals, Al, In, etc.) both these conditions are not obeyed: in such cases it is unlikely that the conductivity anisotropy is strong and, moreover, it is highly improbable that its detailed behavior is in accordance with the above formulas. Favorable conditions from the point of view of this anisotropy may be found in metals such as W and Mn. On the other hand, the conclusion of the possibility of suppression by strong magnetic fields of the competition between the umklapp and diffusion processes is not related to these restrictions and, in particular, it applies to alkali metals.

b) Strong fields. Open Fermi surfaces

In the case of metals with open Fermi surfaces the nature of the conductivity depends strongly on the direction of the magnetic field: in the presence of open orbits, an electron escapes to infinity in p space moving along an orbit in the applied magnetic field; if all the orbits are closed, this escape to infinity results from a displacement along a chain of closed orbits. We shall discuss the diffusion relaxation mechanism in such cases and this will allow us to study the conductivity anisotropy associated with the appearance of open orbits.

1) *No open orbits.* In this case the unit cell in p space can always be selected in such a way that its boundaries intersect the Fermi surface only in the p_z = const cross sections (Fig. 7). The solution of the diffusion equation reduces entirely, as in the case of a

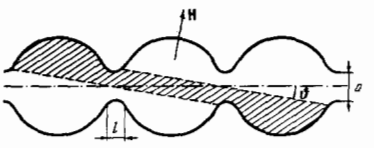

FIG. 7. Selection of a unit cell in p space in the case of an open Fermi surface; $\vartheta = b/g$.

[8)] A special situation occurs under magnetic breakdown conditions, when a hot spot is extremely small ($r_0 \ll q$), whereas the probability of a transition involving crossing a hot spot is of the order of unity.[26]

closed Fermi surface, to the Kirchhoff rules for the equivalent electrical circuit. The density of the electric current deduced from Eq. (4.2) is

$$j_\perp = e(n_e - n_h)u_X - \frac{c}{h^3H^2}\sum_{g,\sigma}[gH](J_{g,\sigma} + e\delta S_{g,\sigma}E_z),$$

where $J_{g,\sigma}$ are the diffusion currents crossing the cell boundaries. The order of magnitude of the transverse conductivity is given by $\sigma_{xx} \approx c^2g^2/h^2H^2R$. As the situation with open orbits is approached, i.e., as H approaches the directions for which layers of open orbits appear, the corresponding diffusion resistance approaches zero.

It is known[22] that there are various types of open orbits differing in respect of the dimensions of the region which they form on a stereographic projection of the magnetic field directions. We shall discuss the behavior of the conductivity in some of the most typical cases of approach to the open orbit regime.

1. The simplest open orbits associated with one open direction of the Fermi surface correspond to a line in the stereographic projection, which is the direction of H in a plane perpendicular to the reciprocal lattice vector g.

Actual calculations can easily be carried out using the following model. The main part of the Fermi surface is assumed to be a sphere of radius p_F; the necks are assumed to be narrow and short: the neck diameter is $d \ll p_F$ and its length is $l \lesssim d$. A magnetic field is inclined at a small angle ϑ to a plane perpendicular to the open direction. The topology of the electron orbits depends on the magnetic field direction. If $b \equiv g\vartheta > d$, there are two types of orbit: those consisting of a single circle and those representing two circles connected by a narrow neck (figures-of-eight). If $b < d$, orbits with a large number of loops appear (Fig. 7).

Elementary calculations based on Eq. (4.11) give the result (it is assumed that $u_c g_z \ll u_H g_y$ for $\vartheta \ll 1$)[27]

$$\sigma_{xx} = \frac{2c^2g^2}{h^3H^2}R^{-1},$$

$$R^{-1} = 2\pi D\frac{g}{b}\left(1+\frac{d}{b}\right)\eta\left(\frac{d}{b}\right),$$

$$\frac{1}{\eta(x)} = 1 + \frac{\Delta(x)(1-\Delta(x))}{(A(x)+1)(A(x)+2)};$$

here, $A(x)$ is the integral part of x; $\Delta(x) = x - A(x)$; the function $\eta(x) \approx 1$ oscillates with a period of unity and with kinks at $x = n$. The angular dependence of σ_{xx} is plotted in Fig. 8. The oscillation period is $\delta(1/b) = 1/d$ and the oscillation amplitude is independent of b.

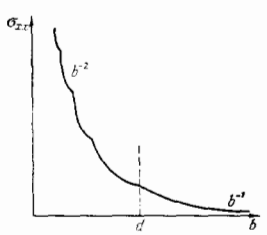

FIG. 8. Anisotropy of the transverse conductivity in the case of an open Fermi surface; $q \ll b, d$.

The physical mechanism resulting in the transverse conductivity is as follows: an electron going over from one trajectory to another is displaced along the p_y axis in momentum space and, consequently, along the x axis in coordinate space. The kinks and oscillations are due to the abrupt appearance of orbits with the number of loops increased by unity. The smooth part of the angular dependence $\sigma_{xx}(b)$ can be interpreted as the result of random wandering of an electron along a chain consisting of extended-orbit layers

$$\sigma_{xx} \approx P_{\text{eff}}e^2\frac{(\delta x)^2}{\tau_{\text{eff}}}; \qquad (4.19)$$

here, $\delta x \approx cg/eh$ is the magnitude of a step in this wandering, $\tau_{\text{eff}} \approx \tau_F(b/p_F)^2 \propto T^{-5}$ is the diffusion time of an electron crossing a layer of thickness b; $P_{\text{eff}} \approx (b+d) \times g(vh^3)^{-1}$ is the effective density of states. It should be noted that the angular dependence $\sigma_{xx} \propto b^{-2}$ obtained for $b \ll d$ by Lifshits and Peschanskiĭ[28] in the relaxation time approximation is of different origin: τ_{eff} is independent of b, but the magnitude of a step is $\delta x \approx (cg/eH)d/b$.

We can easily see how the results obtained are affected when some of the assumptions associated with the Fermi surface model are not invoked. If the neck is wide ($d \approx g$), the dependence $\sigma_{xx} \propto b^{-1}$ may not appear but for $b \ll d$ the law $\sigma_{xx} \propto b^{-2}\eta(d/b)$ remains valid. If the neck is sufficiently long ($l \gg d$), then when $b \gtrsim dg/l$, the orbits with the number of loops greater than unity disappear and, consequently, the dependence $\sigma_{xx} \propto b^{-1}$ is no longer obeyed. However, for lower values of b the results obtained remain valid if we assume that $d = d(b)$, where $d(b)$ is the thickness of a layer of "through" orbits passing through the neck.

We shall now discuss the effects associated with the finite nature of the thermal momentum of phonons $q \approx T/s$. We shall first assume that the momentum q is small compared with the characteristic dimensions of a neck but exceeds the width of one layer of extended orbits: $b < q \ll d, l$. We can easily see that in this case we still have random wandering along a chain or orbits, but now an electron can jump several orbit layers in one step. Therefore, all the results given above remain valid but oscillations are smeared out when $q > b$.

At higher temperatures, when $q \gtrsim d, l$, the failure of the diffusion approximation becomes very significant. To avoid umklapp transitions, i.e., those bypassing the neck, we shall first assume that the neck is long and that $l \gg q \gg d$. The transverse conductivity mechanism is then as follows. During the time of one collision with a phonon $\tau_{\text{eff}} \approx \tau_F(q/p_F)^2$, an electron which is in a layer of extended orbits is most likely to escape from this layer and reach a region of circular orbits or a "cap" (Fig. 7). Then, the electron is displaced by an amount $\delta p_y \approx [(d/b)+1]g$ in reciprocal space and by $\delta x = c\delta p_y/eH$ in coordinate space. We shall estimate the conductivity using Eq. (4.19). An analysis based on the solution of the transport equation on the assumption of constancy of the function $\psi(p_z)$ within the limits of the individual orbit layers gives[27]

$$\sigma_{xx} = \frac{3}{2}n\left(\frac{c}{H}\right)^2\frac{D}{\eta^3}\,d\lambda\left(\frac{d}{b}\right), \qquad (4.20)$$

$$\lambda(x) = x^2 + 3x + 2 + x^{-1}\Delta(x)[1-\Delta(x)][3x+4-2\Delta(x)].$$

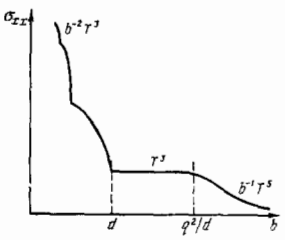

FIG. 9. Anisotropy of the transverse conductivity in the case of an open Fermi surface; $l \gg q \gg d$.

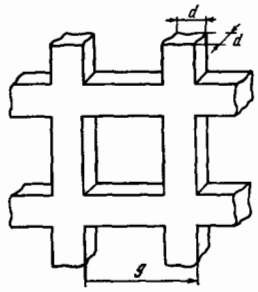

FIG. 10. Fermi surface in the form of a planar grid.

It is clear from the above formula that, in contrast to the $q \ll d$ case, the conductivity obeys $\sigma_{xx} \propto T^3$ and for $b > d$ it is independent of b. A further increase in b results, as is easily shown,[27] in return back to the diffusion approximation: $\sigma_{xx} \propto b^{-1}T^5$. The angular dependence of the conductivity in the $l \gg q \gg d$ case is shown in Fig. 9.

If $q > l$, the umklapp processes appear and their contribution—given by Eq. (4.17)—is simply additive (in the first approximation) with the contribution (4.20) of the extended orbits (it is easily shown that the latter contribution predominates for $b < d\sqrt{d/q}$).

The above ideas on the relationship between σ_{xx} and the potential drop apply also to an open Fermi surface and, therefore, Eq. (4.18) is valid in all the situations discussed here. It follows that the conductivity σ_{xx} also exhibits a considerable anisotropy when the open orbit case is approached.

2. More complex orbits are formed when two or more directions are open. Then, the open orbits may correspond to a two-dimensional region in a stereographic projection.[22] We can easily see that the boundary of this region has a singular structure, which follows already from the qualitative difference between the periodic and aperiodic orbits, which correspond to infinitesimally close directions of the magnetic field. For example, in the case of a Fermi surface of the planar grid type (Fig. 10) a simple analysis gives the following result[9]:

$$\operatorname{tg}\vartheta_{\max} = \frac{d}{g}\left[\left(1-\frac{d}{g}\right)\sqrt{2}\cos\left(\frac{\pi}{4}-\varphi\right) - \frac{1}{\sqrt{m^2+n^2}}\right]^{-1}.$$

Here, $\vartheta_{\max}$ is the angle between H and the normal to the grid plane for which the open orbits disappear; φ is the angle between the projection of H on the grid plane and the reciprocal lattice vector g (of the two vectors the one which satisfies $\varphi < \pi/4$ should be chosen); m and n are integers such that the fraction $m/n = \tan\varphi$ is irreducible.

Thus, the region of open orbits is "bounded" by segments of different length in a hedgehog manner. The longest segments correspond to the symmetric directions of the projection of H with small values of m and n. The size of the region is minimal for irrational values of $\tan\varphi$, when $m, n \to \infty$; for these values of φ the open orbits are aperiodic (Fig. 11).

The problem of determination of the dependence $\sigma(\vartheta, \varphi)$ for arbitrary values of ϑ and φ is difficult. We shall confine ourselves to an analysis of the conductivity in a sufficiently symmetric direction, for example, in the [110] direction ($\varphi = \pi/4$).[10] It should be noted that our earlier discussion[28] of the appearance of open orbits was made in the relaxation time approximation, ignoring the fine structure of the boundary, and it is essentially applicable to this particular direction of the magnetic field.

We can easily see what changes occur in orbits in the $p_z = \text{const}$ section when ϑ increases to $\vartheta_{\max}$ for $\varphi = \pi/4$; the closed orbits, each of them fitted in a unit cell, approach one another and establish contact for $\vartheta = \vartheta_{\max}$ in some particular cross section, giving rise to an infinitesimally thin layer of open orbits. One should also point out a qualitative difference between such an abrupt transition from the situation discussed in the preceding subsection, when the lengths of the orbits increase without limit on approach to the open-orbit case.

The conductivity σ_{xx} can be estimated from Eq. (4.19): a displacement by the vector g results when an electron diffuses across a layer of thickness $\Delta p_z \approx g(\vartheta - \vartheta_{\max})$ (this is a simple geometric consequence of the closer approach of the orbits) and, consequently, we have $\tau_{eff} \approx \tau_F(\vartheta - \vartheta_{\max})^2$ and $P_{eff} \approx \Delta p_z g/vh^3$. Consequently, for $\vartheta - \vartheta_{\max} \ll \vartheta_{\max}$ we obtain

$$\sigma_{xx} \approx \left(\frac{g}{h}\right)^3 \frac{g}{v}\left(\frac{c}{H}\right)^2 \frac{1}{\tau_F(\vartheta - \vartheta_{\max})}. \qquad (4.21)$$

At sufficiently low values of $\vartheta - \vartheta_{\max}$, we can no longer assume approximately that the magnetic fields are strong (see Sec. 4 c below). However, the dependence (4.21) may be violated even in strong magnetic fields: when $\vartheta - \vartheta_{\max}$ (i.e., when $q \gg \Delta p_z$), we cannot use the diffusion approximation. In this case displacement by a reciprocal lattice vector occurs for electrons in a layer of thickness of the order of q as a result of single collisions with phonons. Therefore, $\tau_{eff} \approx \tau_F(q/g)^2$, $P_{eff} \approx qg/vh^3$, and it follows from Eq. (4.19) that

$$\sigma_{xx} \approx \left(\frac{g}{h}\right)^3 \left(\frac{c}{H}\right)^2 \frac{g^2}{vq\tau_F}. \qquad (4.22)$$

We can easily see how the situation changes when the open-orbit case begins to apply along a less symmetric direction ($m, n > 1$). In this case at some angles $\vartheta_1, \vartheta_2, \ldots, \vartheta_k > \vartheta_{\max}$ there is an abrupt increase in the length

[9] A similar formula is obtained in Ref. 29 for a slightly different planar grid model.

[10] In fact, the results obtained are valid in a finite range of angles φ: $\Delta\varphi < \vartheta - \vartheta_{\max}$ (see Sec. 4. c).

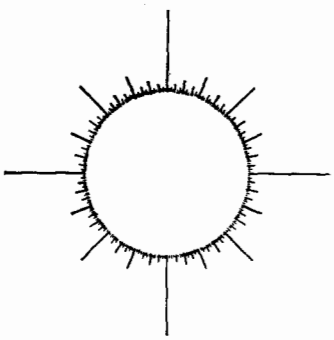

FIG. 11. Region of open orbits on a stereographic projection for a planar grid Fermi surface.

of the closed orbits and the number k increases on increase in m and n. (If $\vartheta = \vartheta_{max}$, an open orbit forms as a result of merging of extended orbits that cannot be fitted into one unit cell.) As in the preceding subsection, each abrupt elongation of an orbit results in a kink in the curve for $\sigma_{xx}(\vartheta)$. The expressions (4.21) and (4.22) are valid when $\vartheta_k > \vartheta > \vartheta_{max}$.

2) *There are open orbits.* In this case the solution of Eq. (4.1) by expansion in terms of the reciprocal magnetic field should be modified somewhat: in the case of an open orbit we have $\langle v_\perp \rangle \neq 0$ and, therefore, the term $e\mathbf{E}v_\perp$ should be used in the second-approximation equation (4.8).

If all the open-orbit layers are narrow ($\Delta \ll g$, where Δ is the thickness of the layer), the contribution of each layer to the conductivity is[27]

$$\sigma_{xx} = \frac{1}{3}\frac{e^2 g_y^2 \Delta^3}{\hbar^3 D}. \qquad (4.23)$$

The result $\sigma_{xx} \propto \Delta^3 T^{-5}$ was obtained earlier by Kaganov, Kadigrobov, and Slutskin.[30] They demonstrated also the limits of its validity: for $q > \Delta$, an electron leaves a layer of this kind as a result of a single collision so that $\sigma_{xx} \propto \Delta T^{-3}$.

In the simplest case when the open orbits are linked to one open direction, the thickness of the layer Δ is governed by the thickness of the neck. In the case of a Fermi surface of the planar grid type when $\vartheta_{max} - \vartheta \ll \vartheta_{max} - \vartheta_{ir}$ $\{\tan\vartheta_{ir} = (d/g)[1 - (d/g)]\sqrt{2}\cos[(\pi/4) - \varphi]$ is the boundary of the region of open orbits for irrational values of $\tan\varphi\}$ we can easily show that the thickness of the layer of open orbits is $\Delta \approx g(\vartheta_{max} - \vartheta)$. Applying $\sigma \approx n_{eff} e^2 \tau_{eff}/m$, we find that when $q \ll \Delta$, then

$$\sigma_{xx} \approx \left(\frac{g}{\hbar}\right)^3 e^2 \frac{v}{g} \tau_F (\vartheta_{max} - \vartheta)^3. \qquad (4.24)$$

If $q \gg \Delta$, we naturally have $\sigma_{xx} \propto \tau_F (q/g)^2 (\vartheta_{max} - \vartheta)$, which corresponds to the results of Ref. 28 obtained in the relaxation time approximation.

c) Intermediate magnetic fields

We have seen above that the characteristic relaxation times governing the electrical conductivity of a metal in strong magnetic fields (τ^∞) and in zero magnetic field (τ^0) may be very different. The reason is that the value of τ^0 is related to the diffusion of an electron over distances of the order of the size of the whole Fermi surface. However, in a strong field the diffusion is required over much shorter distances since the major part of the displacement in momentum space needed in relaxation is attained because of the motion of an electron along an orbit in the applied magnetic field. Consequently, the ranges of weak ($\Omega\tau^0 \ll 1$) and strong ($\Omega\tau^\infty \gg 1$) fields do not abut and a wide region of intermediate magnetic fields exists. The specific relaxation mechanisms which appear in this intermediate region may be identified by considering the two simplest models of a closed Fermi surface: a "one-dimentional" surface with extended (in the direction of the magnetic field) overlapping hot spots, when the diffusion along the p_ε axis can be ignored in the first approximation (subsection 1 below) and a spherical Fermi surface with one pair of hot spots when a complete solution can be achieved (subsection 2 below). In the case of open Fermi surfaces, the conductivity is determined by the same mechanisms (subsection 3 below).

1) *Magnetoumklapp diffusion.* We shall first consider the problem of the traversing of one hot spot by an electron moving along an orbit in a magnetic field. Since the diffusion along p_ε is ignored, the distribution function can be regarded as dependent only on the coordinate p_x measured along the tangent to the trajectories I and II in a hot spot (Fig. 12).

Let us assume that an electron flux $p_F \Omega \psi_1(-\infty)$ enters a hot spot along the orbit I from the left. We shall introduce a "reflection coefficient" W which represents the fraction of electrons that emerge from the hot spot along the orbit II. If the orbits I and II are of the same type (electron or hole), then in the orbit II an electron can escape to the left: $\psi_{II}(-\infty)$, $\psi_{II}(\infty) = 0$; if the orbits are of different type, then it can escape only to the right: $\psi_{II}(\infty) = W\psi_I(-\infty)$, $\psi_{II}(-\infty) = 0$. In any case, we have $\psi_I(\infty) = (1 - W)\psi_I(-\infty)$.

The process of passage across a hot spot can be described by

$$\Omega\psi_I'(x) + \frac{1}{2}\nu_U(x)[\psi_I(x) - \psi_{II}(x)] - \nu_F\tilde{\psi}_I(x) = 0, \qquad (4.25)$$

$$\mp\Omega\psi_{II}'(x) + \frac{1}{2}\nu_U(x)[\psi_{II}(x) - \psi_I(x)] - \nu_F\tilde{\psi}_{II}(x) = 0; \qquad (4.26)$$

here, $x = p_x/p_F$, $\nu_U(x) = 1/\tau_U(x)$, $\nu_F = 1/\tau_F$; the upper sign in (4.26) corresponds to orbits of the same type and the lower to orbits of different type.

In the case of orbits of the same type, we find that Eqs. (4.25) and (4.26) readily yield the following equation for the function $\psi_s = (\psi_I + \psi_{II})/2$:

$$(\Omega^2\nu_U^{-1}\psi_s' + \nu_F\psi_s')' - (\nu_F^2\nu_U^{-1}\psi_s''')' = 0. \qquad (4.27)$$

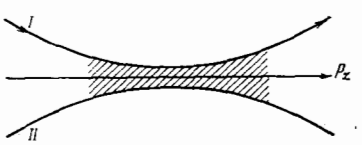

FIG. 12. Electron orbits near a hot spot. The hot spot is shown shaded.

If $\Omega\tau_F r_\perp \gg 1$ ($r_\perp$ is the characteristic size of a hot spot in the x direction) we can drop the last term from Eq. (4.27). Then, the resultant equation describes combined diffusion of electrons inside a hot spot: the usual diffusion characterized by the coefficient $\nu_F p_F^2$ and the "magnetoumklapp" diffusion with the coefficient $\Omega^2 \nu_U^{-1}(x) p_F^2$. The solution of this equation is

$$W = [1 + (\Omega t_0)^{-1}]^{-1}, \quad t_0 = \frac{1}{2\Omega^2} \int_{-\infty}^{+\infty} \frac{dx}{\tau_U(x) + \nu_F \Omega^{-2}}. \tag{4.28}$$

In the opposite limiting case of $\Omega\tau_F r_\perp \ll 1$, the following expressions apply in the case of orbits of the same and different types[31]

$$W = (2 + w^{-1})^{-1}, \quad w = \frac{1}{2\Omega} \int_{-\infty}^{+\infty} \nu_U(x)\, dx.$$

Finally, when $\Omega\tau_F r_\perp \gg 1$ and the orbits are of different types, it follows from Eqs. (4.25) and (4.26) that

$$W = \frac{1}{2}(1 - e^{-2w}).$$

We thus find that in the case of orbits of the same type and in moderately strong fields ($w \gg 1$, but $\Omega\tau_F \gg 1$) the result of crossing a hot spot is determinate ($W \approx 1$) if $\Omega\tau_F r_\perp \gg 1$ and stochastic ($W \approx 1/2$) if $\Omega\tau_F r_\perp \ll 1$. In the case of wide hot spots ($r_\perp \approx 1$) the former is true, whereas in the case of very narrow hot spots ($r_\parallel \approx q/p_F$), only the second possibility applies. In the case of orbits of different types, the result of passage through a hot spot in the $w \gg 1$ case is always stochastic.

The determinate nature of the motion is related to the magnetoumklapp diffusion. During the passage through a hot spot an electron undergoes repeated umklapping ($w \gg 1$). It jumps from one orbit to another and changes each time the direction of its motion along the x axis. Clearly, this process represents diffusion with the length $p_F \Omega\tau_U$ and time τ_U of each single step. An electron begins its path from the left-hand edge of a hot spot along the orbit *I* and it is highly probable that it does not cross the whole hot spot but emerges from the same side as before but along the orbit *II*. This is true for $\Omega\tau_F r_\perp \gg 1$, when the conventional diffusion associated with the normal electron–phonon collisions is not very effective. If $\Omega\tau_F r_\perp \ll 1$, a displacement by distance $r_\perp$ is faster due to the conventional diffusion than due to that under the influence of a magnetic field. Consequently, an electron is equally to arrive at either edge of a hot spot and we now have a stochastic situation.

Clearly, under the conditions when the result of passage through a hot spot is determinate, we can expect the appearance of effectively open electron trajectories. The simplest example of this situation is a chain of

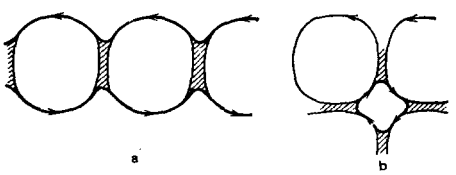

FIG. 13. Magnetoumklapp diffusion producing effective open (a) or hole-like (b) electron orbits.

electron orbits shown in Fig. 13a: the thick curves represent the resultant open trajectories and the magnetoumklapp diffusion regions are shown shaded.

We can easily see that the quantity t_0 [see Eq. (4.28)] represents the relaxation time on an open trajectory: during this time an electron undergoes a transition between open trajectories because the probability of crossing a hot spot right through as a result of combined diffusion is low. Therefore,

$$\sigma_{xx} \simeq \frac{ne^2}{mp_F} \int t_0\, dp_z,$$

and hence

$$\sigma_{xx} \approx \frac{n_{eff} e^2}{m} \tau, \quad n_{eff} = n r_\parallel, \quad \tau^{-1} = \frac{\Omega^2}{\nu_U r_\perp} + \frac{\nu_F}{r_\perp} + \frac{\nu_F}{r_\parallel^2}. \tag{4.29}$$

The first two terms in the last formula are due to the combined diffusion (ν_U is the frequency of umklapp processes at the center of a hot spot). The third term allows for the mechanism of destruction of an open trajectory ignored in Eq. (4.28): an electron escapes to the region of closed orbits because of diffusion along p_z ($r_\parallel$ is the characteristic size of a hot spot in this direction). The results in the case of the resistivity ρ_{yy} are represented by curve 1 in Fig. 14 (it is assumed that $r_\perp \approx r_\parallel \approx 1$; the range of intermediate magnetic fields then corresponds to $\nu < \Omega < \sqrt{\nu_F \nu_U}$). This range is even wider ($\nu_F < \Omega < \nu_U$) if there are two nonparallel directions of effective open orbits [curve 2; the dependence $\rho(H)$ in the range $\nu_F < \Omega < \sqrt{\nu_F \nu_U}$ is governed by the relatively weak dependences of $r_\parallel$ and $r_\perp$ on the magnetic field].[31]

If hot spots are arranged as shown in Fig. 13b, then the magnetoumklapp diffusion forms effective closed orbits which differ from those in a strong field. The behavior of the resistivity in this case is analyzed in Ref. 31.

If the result of crossing a hot spot is stochastic (orbits of different types or relatively narrow hot spots), the transverse conductivity is governed by a random wandering of an electron along a chain of orbits in steps separated by time intervals of the order of Ω^{-1}. We then find that $\sigma_{xx} \approx ne^2 r_\parallel / m\Omega$ and the intermediate range of magnetic fields is defined by $\nu_F < \Omega < \nu_U r_\perp$. Analogous results were obtained earlier by Pippard[24] and Young[32] for the case of very narrow hot spots.

2) *Spherical Fermi surface.* In the preceding subsection we have considered only the case of exact overlap of the hot spots and we have also assumed

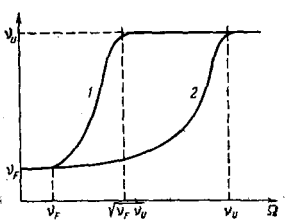

FIG. 14. Dependences of the transverse resistivity ($\rho_{yy} ne^2/m$) on Ω in intermediate magnetic fields: 1) one effective open direction; 2) several nonparallel open directions.

that the diffusion time in the direction p_z is long compared with Ω^{-1}. Both these restrictions can be lifted and an exact solution of the problem can be obtained asymptotically for a chain of spherical Fermi surfaces.[31]

We shall give the final result in the form of diagrams (Figs. 15a and 15b) showing the regions of different behavior of the transverse conductivity σ_{xx}. The first diagram is plotted using the variables $(\Omega\tau_F)^{-1}$, T in the case when $b=0$, whereas the second uses the variables $(\Omega\tau_F)^{-1}$, b at $T>T_0$. The lines 1, 2, . . . , 8 separating the various regions are described by the equations

$$(\Omega\tau_F)^{-1} \approx \sqrt{\alpha\beta},\ \beta^2,\ \frac{\alpha}{\beta},\ \beta,\ \beta e^{\alpha/\beta^2},\ \ln\beta^{-1},\ \alpha\beta^{-2},\ \beta\vartheta,$$

where $\alpha=\tau_U/\tau_F$, $B=r_s/p_F$, $\vartheta=b/p_F$ and the characteristic size of a hot spot r_s differs slightly from r_0 (see Ref. 31). The other boundaries of the regions are straight and their positions are obvious from the marks on the coordinate axes. The conductivity in regions I, II, . . . VIII is described by the formulas labeled in the same way as in Table I.

Regions I, VII, and VIII correspond to strong magnetic fields, whereas regions V and VI correspond to weak fields. The region of intermediate magnetic fields is widest for $T=T_0$ and $b=0$. In region II, the magnetoumklapp diffusion gives rise to open orbits in which the lifetime is governed by the diffusion of an electron in the direction p_z. In region III, the diffusion destroys the open orbits completely and the conductivity is governed by random wandering of an electron along a chain of closed orbits and the jump time in this process is Ω^{-1}.

The situation in region IV can be determined by noting that at distances shorter than $\delta \approx p_F(\Omega\tau_F)^{-1}$ the velocity of diffusion of an electron is higher than the velocity of its motion along an orbit in a magnetic field. The quantity δ, if it does exceed r_s, can be regarded as the size of a hot spot because the probability of an electron being captured by a hot spot at this distance from it is of the order of unity. At the boundaries of regions IV and V we find that $\delta \approx p_F$.

3) *Open Fermi surfaces.* 1. We shall use initially the above surface model with narrow and short necks ($l \leq d \ll p_F$). The results are shown in Fig. 16 (it is assumed that $q \ll d, l$). The dashed curve corresponds to the relationship $p_F(\Omega^*\tau_F)^{-1/2}=b$ where $\Omega^* \approx \Omega b/d$ is

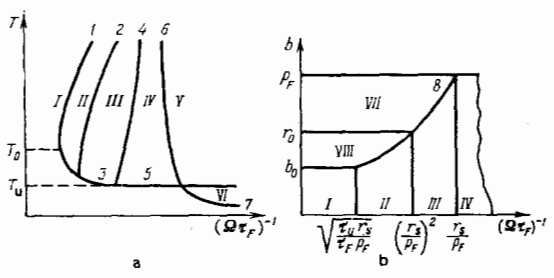

FIG. 15. Diagrams of regions of qualitatively different behavior of conductivity on a spherical Fermi surface expressed in terms of different variables: a) T, $(\Omega\tau_F)^{-1}$ for $b=0$; b) b, $(\Omega\tau_F)^{-1}$ for $T>T_0$ (see Table I).

TABLE I.

	I	II	III	IV
$\frac{\sigma_{xx}m}{ne^2}$	$\frac{\beta^2}{\tau_U\Omega^2}$	$\tau_F\beta^3$	$\frac{\beta}{\Omega}$	$\left[\Omega^2\tau_F \ln\frac{1}{\Omega\tau_F\beta}\right]^{-1}$
	V	VI	VII	VIII
$\frac{\sigma_{xx}m}{ne^2}$	$\tau_F \ln\beta^{-1}$	$\tau_U\beta^{-2}$	$(\Omega^2\tau_F\vartheta^2)^{-1}$	$(\Omega^2\tau_F\vartheta^2)^{-1}\beta\sqrt{\ln\beta\vartheta^2/\alpha}$

the frequency of revolution along an extended orbit ($b \ll d$). This curve divides region II into two parts, in the upper of which the conductivity exhibits oscillations (see Sec. 4b), whereas in the lower part there are no oscillations. All the other lines in Fig. 16 are straight and their positions are identified by the marks on the ordinates.

The conductivity in regions I–VI is described by the expressions listed in Table II, where $\beta_1=d/p_F$.

Strong magnetic fields correspond to region I and, strictly speaking, only to the upper part of region II. Layers of orbits distinguished by the number of loops have a width of the order of b. In the lower part of region II the diffusion displacement of an electron during its period of motion on an extended orbit is $p_F(\Omega^*\tau_F) > b$ and, therefore, there are no oscillations.

In region III after a time $1/\Omega^*$ an electron diffuses out of a layer of extended orbits: $p_F(\Omega^*\tau_F)^{-1/2} \gg d$. Under these conditions the difference between an extended orbit and an open one is unimportant and the electrical conductivity is given by Eq. (4.23). To the right of region III an electron escapes from a layer of extended orbits also in a period Ω^{-1}. It is easy to show that in this case the necks play the same role as the hot spots of size d in a metal with a closed Fermi surface at temperatures $T>T_0$. Therefore, regions IV and V in Fig. 16 are fully equivalent to regions III and IV in Fig. 15. Finally, region VI corresponds to weak fields.

2. A characteristic range of intermediate fields which appears near the boundary of a two-dimensional region of open orbits can be analyzed using the example of a Fermi surface model of the planar grid type. For $\vartheta=\vartheta_{max}$ in a given section p_{z0} there is a layer of open orbits of zero thickness. In any other cross section p_{z0} which may be arbitrarily close to it the orbits are closed but a small diffusion displacement of the order of p_z-p_{z0} is sufficient for an electron to be transferred

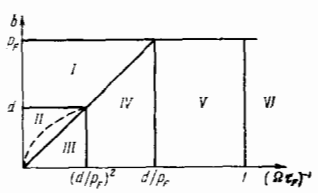

FIG. 16. Diagram showing regions of qualitatively different behavior of the conductivity in terms of the variables b and $(\Omega\tau_F)^{-1}$ (with one open direction); see Table II.

TABLE II.

	I	II	III	IV	V	VI
$\frac{\sigma_{xx}m}{ne^2}$	$(\Omega^2\tau_F\vartheta)^{-1}$	$(\Omega^2\tau_F\vartheta^2)^{-1}\beta_1$	$\tau_F\beta_1^2$	$\Omega^{-1}\beta_1$	$(\Omega^2\tau_F)^{-1}\left(\ln\frac{1}{\Omega\tau_F\beta_1}\right)^{-1}$	$\tau_F\ln\frac{1}{\beta_1}$

from this section to the next cell. Clearly, the main contribution to the conductivity is due to electrons in a layer $|p_z - p_{z0}| \leq g(\tau_F\Omega)^{-1/2} \equiv d^*$ and after each revolution half-period these electrons are equally likely to reach the next cell or to remain in the present one. Following Eq. (4.19), we obtain

$$\sigma_{xx} \approx \frac{e^2g^2P_{\text{eff}}}{m^2\Omega} \approx \left(\frac{g}{h}\right)^3\frac{ec}{H}\frac{1}{\sqrt{\Omega\tau_F}}. \qquad (4.30)$$

It should be noted that the thickness of a layer of open orbits vanishes not only at the boundary of the two-dimensional region but also at an isolated point at the center of this region. The expression (4.30) for the latter case was obtained by Pippard.[24]

We can easily see that Eq. (4.30) is valid not only on the boundary but also within the range of angles $|\vartheta - \vartheta_{\max}| \ll (\Omega\tau_F)^{-1/2}$. In fact, if $\vartheta - \vartheta_{\max} \ll (\Omega\tau_F)^{-1/2}$, the additional diffusion displacement $\Delta p_z = g(\vartheta - \vartheta_{\max})$ necessary if an electron is to reach the next cell is small compared with the displacement d^* during a half-period; if $\vartheta_{\max} - \vartheta \ll (\Omega\tau_F)^{-1/2}$, open orbits essentially do not yet appear because an electron escapes from a layer of infinite sections of thickness $g(\vartheta_{\max} - \vartheta)$ in a time shorter than Ω^{-1}. If $|\vartheta - \vartheta_{\max}| \gg (\Omega\tau_F)^{-1/2}$, the approximation of a strong magnetic field is clearly valid.

It may be found that for $\vartheta = \vartheta_{\max}$ the standard asymptotic behavior of the conductivity is not attained no matter how strong is the field. However, the validity of Eq. (4.30) is limited by the diffusion approximation, which is valid for $d^* \gg q$. Therefore, the intermediate range of magnetic fields on the boundary of the two-dimensional angular region is defined by the inequalities $1 \ll \Omega\tau_F \ll (g/q)^2$. In higher fields a collision with a phonon does not occur during one revolution period and the results of Sec. 4b apply. The dependence $\sigma_{xx}(\vartheta)$ in the angular range $|\vartheta - \vartheta_{\max}| \ll \vartheta_{\max} - \vartheta_{1r}$ is shown in Fig. 17: curve 1 corresponds to $1 \ll \Omega\tau_F \ll (g/q)^2$, whereas curve 2 corresponds to $\Omega\tau_F \gg (g/q)^2$.

Concluding, we shall estimate by how much the orientation of H can deviate from the sufficiently symmetric direction φ ($\tan\varphi = m/n$, where $m \approx n \approx 1$) without invalidating the results (4.21), (4.24), and (4.30). For simplicity, we shall consider the diffusion approximation. If $\vartheta - \vartheta_{\max} > -(\Omega\tau_F)^{-1/2}$, rotation through an angle $\Delta\varphi$ increases the diffusion displacement necessary to transfer an electron to the next cell by an amoung $g\Delta\varphi$. This correction should be less than at least one of the quantities Δp_z and d^* and hence we find $\Delta\varphi \ll \vartheta - \vartheta_{\max} + (\Omega\tau_F)^{-1/2}$. If $\vartheta_{\max} - \vartheta > (\Omega\tau_F)^{-1/2}$, a small rotation by $\Delta\varphi$ has the effect that open orbits become closed but fairly extended with lengths of the order of $g(\vartheta_{\max} - \vartheta)/\Delta\varphi$. A comparison of this quantity with the mean free path on an open orbit gives $\Delta\varphi \ll (\Omega\tau_F)^{-1}(\vartheta - \vartheta_{\max})^{-1}$.

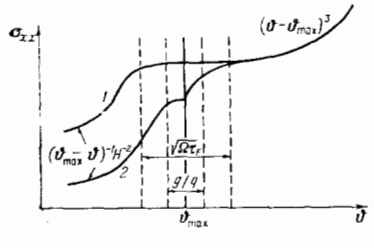

FIG. 17. Anisotropy of the transverse conductivity near the boundary of a two-dimensional region of open orbits: 1) $1 \ll \Omega\tau_F \ll (g/q)^2$; 2) $\Omega\tau_F \gg (g/q)^2$.

CONCLUSIONS

We have stressed already in the Introduction that our aim is to establish primarily the qualitative relationships governing the low-temperature electrical conductivity of pure metals. The results obtained using idealized models can be used, after establishing their physical meaning, to analyze the situation in any specific metal. In a detailed quantitative description, we can use a combined approach: the structure of the distribution function can be found from qualitative physical considerations and the numerical parameters can be deduced by applying the variational principle (as has been done above for the electrical conductivity of alkali metals).

In our opinion, it would be particularly interesting to study experimentally the effects due to the umklapp processes: this would provide an opportunity to observe directly these processes in metals with closed Fermi surfaces (in the range of strong magnetic fields) and to alter qualitatively the topological properties of electron orbits by the magnetoumklapp diffusion (in the range of intermediate magnetic fields).

Calculations of the low-temperature electrical conductivity of a degenerate two-dimensional electron system have yet to be carried out. We have mentioned earlier that the electron–phonon interactions cannot by themselves ensure an equilibrium state of such a system in an external electric field. This should result in qualitatively different behavior of the electrical conductivity of two- and three-dimensional metal systems. Finally, it should be pointed out that the thermal conductivity, as well as the thermoelectric and thermomagnetic effects in metals, may also be analyzed by the method of the diffusion equation (see Appendix I).

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APPENDIX I

As pointed out in Sec. 2, the diffusion equation can be obtained from the transport equation by direct expansion in terms of the parameter q/p_F retaining terms up to the second order inclusively. We shall use the same approach to show that the distribution function of the principal approximation is independent of energy (in the absence of a temperature gradient); moreover, we shall find the form of the function and derive the diffusion equation for heat conduction in a metal.

1. In momentum space we shall introduce a curvilinear coordinate system in which one coordinate is the energy and the other two represent the position of a point on a constant-energy surface [these coordinates will be identified by a vector index p: $\chi_{\mathbf{p}}(\varepsilon)$]. We shall write the result of expansion of the collision operator in powers of the parameter q/p_F in the form $\hat{I}^e = \hat{I}^{(0)} + \hat{I}^{(1)} + \hat{I}^{(2)} + \ldots$.

The structure of the operator $\hat{I}^{(0)}$ is the most important feature for further analysis:

$$\hat{I}^{(0)}\chi_{\mathbf{p}}(\varepsilon) = -\int d\varepsilon' K_{\mathbf{p}}(\varepsilon, \varepsilon')[\chi_{\mathbf{p}}(\varepsilon) - \chi_{\mathbf{p}}(\varepsilon')], \tag{I.1}$$

where $K_{\mathbf{p}}(\varepsilon, \varepsilon') = K_{\mathbf{p}}(\varepsilon', \varepsilon), K_{\mathbf{p}}(\varepsilon, \varepsilon') > 0$. For these reasons the homogeneous equation $\hat{I}^{(0)} = 0$ has just one nontrivial solution $\chi = C_{\mathbf{p}}$ ($C_{\mathbf{p}}$ is an energy-independent arbitrary function of the surface variables) and for any function $\chi_{\mathbf{p}}(\varepsilon)$, we have

$$\int \hat{I}^{(0)}\chi\, d\varepsilon = 0. \tag{I.2}$$

The problem has an additional small parameter: $s/v \ll 1$. We can, in particular, show that allowance for the nonequilibrium state of phonons (i.e., for the function Φ_q) gives rise to a correction to the factor $\hat{I}^{(0)}$ and this correction is proportional to $(s/v)^2$, so that it can be neglected. In this approximation, we have

$$K_{\mathbf{p}}(\varepsilon, \varepsilon') = \frac{4\pi}{\hbar h^3 v}\left[\int_0^{2\pi} d\varphi \frac{M_{\mathbf{p}}(\mathbf{e})}{s^3(\mathbf{e})}\right]_{\mathbf{en}=0}\left(-\frac{dN}{d\omega}\right)\omega^2 |n_\varepsilon - n_{\varepsilon'}|, \tag{I.3}$$

where $\omega = \varepsilon - \varepsilon'$.

We shall report without proof some properties of the operator $\hat{I}^{(1)}$.

If in this operator we distinguish parts of different order in repsect of the parameter s/v: $\hat{I}^{(1)} = I_0^{(1)} + I_1^{(1)}$, $I_1^{(1)} \sim s/v$, then $I_0^{(1)} C_{\mathbf{p}} = 0$ and for any function χ, we have $\int \hat{I}_0^{(1)}\chi d\varepsilon' = 0$. Moreover,

$$\int \hat{I}^{(1)} C_{\mathbf{p}}\, d\varepsilon = 0. \tag{I.4}$$

Application of the method of successive approximations to the transport equation (1.2) gives ($\chi = \chi^{(0)} + \chi^{(1)} + \cdots$)

$$\hat{I}^{(0)}\chi^{(0)} = 0, \tag{I.5}$$

$$\hat{I}^{(0)}\chi^{(1)} + \hat{I}^{(1)}\chi^{(0)} = 0, \tag{I.6}$$

$$\hat{I}^{(0)}\chi^{(2)} + \hat{I}^{(1)}\chi^{(1)} + \hat{I}^{(2)}\chi^{(0)} = e\mathbf{E}\frac{\partial n}{\partial \mathbf{p}}. \tag{I.7}$$

The field term cannot be attributed to Eqs. (I.5) and (I.6) since these equations would then be insoluble as a consequence of Eqs. (I.2) and (I.4).

As pointed out earlier, the only nontrivial solution of Eq. (I.5) is $\chi^{(0)} = C_{\mathbf{p}}$. This demonstrates that the function of the principal approximation is independent of the energy. The dependence of the function $\chi^{(0)}$ on the surface variables can be deduced from the condition of solubility of Eq. (I.7) [the term $\hat{I}^{(1)}\chi^{(1)}$ is small in respect of the parameter s/v]

$$\int \hat{I}^{(2)}\chi^{(0)}\, d\varepsilon = \int e\mathbf{E}\frac{\partial n}{\partial \mathbf{p}}\, d\varepsilon. \tag{I.8}$$

This relationship is equivalent to the diffusion equation (2.5).

2. The main distinction of the low-temperature expansion in the problem of heat conduction in a metal is that the energy dependence of the distribution function χ becomes extremely important.

We shall write the transport equation in the form[11]

$$\hat{I}^e\chi = \mathbf{v}\frac{\partial n}{\partial \mathbf{r}}. \tag{I.9}$$

In the approximation lowest with respect to q/p_F and T/ε_F we have (here, $\mathbf{v}_F$ is the velocity on the Fermi surface)

$$\hat{I}^{(0)}\chi^{(0)} = -\frac{\varepsilon - \varepsilon_F}{T}\mathbf{v}_F\nabla T\frac{dn}{d\varepsilon}. \tag{I.10}$$

This equation is soluble because the integral of its right-hand side with respect to ε vanishes. The solution can be represented in the form $\chi^{(0)} = \varphi_{\mathbf{p}}(\varepsilon) + C_{\mathbf{p}}$, where $\varphi_{\mathbf{p}}(\varepsilon)$ is an odd function of $\varepsilon - \varepsilon_F$ and $C_{\mathbf{p}}$ is an arbitrary function of the surface variables. Calculations in many respects similar to those presented in Sec. 2 give the following diffusion equation for the function $C_{\mathbf{p}}$:

$$\begin{aligned}
&\operatorname{div}\hat{D}(\nabla C - \mathbf{a}\{\nabla C\}) = \nabla T\left(\mathbf{n}\frac{d\mu}{dT} + \frac{\pi^2}{3}T\mathbf{n}\frac{\mathscr{R}_1^{-1} + \mathscr{R}_2^{-1}}{\nu} + \frac{\pi^2}{3}T\mathbf{n}'\right) + \operatorname{div}\mathbf{Q},\\
&\mathbf{Q}_{\mathbf{p}} = \int\left\{\hat{P}_{\mathbf{p}}\psi_{\mathbf{p}}(\mathbf{e}) - \frac{D_{\mathbf{p}}(\mathbf{e})\,\delta(\mathbf{en}_{\mathbf{p}})}{\alpha(\mathbf{e})}\int\psi_{\mathbf{p}'}(\mathbf{e})\, dS_{\mathbf{p}'}\right\} d\Omega_{\mathbf{e}},\\
&\psi_{\mathbf{p}}(v) = \frac{4\pi M_{\mathbf{p}}(\mathbf{e})}{\hbar h^3 v_{\mathbf{p}}^3 s^3(\mathbf{e})}\,\delta'(\mathbf{en}_{\mathbf{p}})\,\mathbf{e}\int_0^\infty\int\omega^3\frac{dN}{d\omega}(n_\varepsilon - n_{\varepsilon+\omega})\,\mathbf{q}_{\mathbf{p}}(\varepsilon)\, d\varepsilon\, d\omega;
\end{aligned} \tag{I.11}$$

here, $\mu(T)$ is the chemical potential; $\mathscr{R}_{1,2}$ are the principal radii of curvature of the Fermi surface at the point p; $\mathbf{n}' = (\partial/\partial\varepsilon)\mathbf{n}_{\mathbf{p}}(\varepsilon)_{\varepsilon=\varepsilon_F}$. All the terms on the right-hand side of Eq. (I.11) are of the same order of magnitude: $\nabla T\, T/\varepsilon_F$.

We shall now consider the meaning of the relationships obtained. The inhomogeneous term $X = \mathbf{v}\,\partial n/\partial \mathbf{r}$ in the transport equation (I.9) can be represented in a natural manner as $X = X_1 + X_2$ by introducing a term X_1 which is antisymmetric with respect to energy: $\int X_1 d\varepsilon/v = 0, X_2 \approx (T/\varepsilon_F)X_1$. The reaction of the system to the perturbation X_1 is described by Eq. (I.10) and the corresponding energy relaxation time obeys $\tau' \propto T^{-3}$. On the other hand, the nonequilibrium associated with X_2 cannot be removed in one collision; the relaxation of this term is described by the diffusion equation (I.11), whose right-hand side contains $\int X d\varepsilon/v$. The term divQ in Eq. (I.11) is related to the fact that a change in the energy causes an electron to travel also along the Fermi surface.

APPENDIX II

We shall consider the equation

$$\eta - \mathbf{a}\{\eta\} = \hat{D}^{-1}\psi, \tag{II.1}$$

where $\eta \equiv \nabla\chi$ and the vector ψ obeys the conditions (2.8). We shall consider two aspects: 1) the uniqueness of the solution of the homogeneous equation corresponding to Eq. (II.1); 2) the convergence of the iteration procedure in the solution of Eq. (II.1).

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[11] This equation does not contain a term with the electric field since the reaction of the system to such a field is discussed above. Moreover, the term $s\partial N/\partial s$ is missing from the phonon transport equation because the drag of electrons by phonons in metals is weak at sufficiently low temperatures.

1. We can easily show that the integral kernel of Eq. (II.1) has the property

$$\int A^{ik}_{pp'}\, dS_{p'} = \delta_{ik}. \qquad \text{(II.2)}$$

We can use this property and write down the homogeneous equation $\eta - a\{\eta\} = 0$ in the form

$$\int \hat{A}_{pp'}(\eta_p - \eta_{p'})\, dS_{p'} = 0. \qquad \text{(II.3)}$$

We shall multiply the last equation by the vector $\hat{D}_p\eta_p$ and integrate over the whole Fermi surface. Bearing in mind that $\hat{D}_p\hat{A}_{pp'} = \hat{D}_{p'}\hat{A}_{p'p}$, we obtain

$$\int \frac{D_p(\mu)\, D_p(\mu)}{\alpha(\mu)\,|\,n_p \times n_{p'}|}\,[(\eta_p - \eta_{p'})\,\mu]^2\, dS_p\, dS_{p'} = 0. \qquad \text{(II.4)}$$

The integrand in Eq. (II.4) is nonnegative. If we assume that for all values of p and μ we have $D_p(\mu) \neq 0$, then

$$(\eta_p - \eta_{p'})\,\mu = 0. \qquad \text{(II.5)}$$

The question is now whether Eq. (II.5) has any nontrivial (apart from drift) solutions? Let us assume that at the points p_1 and p_2 such that $n_1 \neq \pm n_2$ the function η has the values η_1 and η_2. It follows from Eq. (II.5) that out of the four components of the vectors η_1 and η_2, three can be selected independently: the components parallel to the vector $\mu_{12} \sim n_1 \times n_2$ are equal. Clearly, these two vectors can be expressed as projections of a certain three-dimensional vector u on the Fermi surface at the points p_1 and p_2: $\eta_1 = (u_{\parallel})_1$ and $\eta_2 = (u_{\parallel})_2$, where the vector u is defined uniquely by these relationships. We shall consider another point p_3 on the Fermi surface such that the unit vectors n_1, n_2, and n_3 do not lie in the same plane. If we introduce $\eta_3 = (u_{\parallel})_3 + \delta\eta_3$, we find from Eq. (II.5) that $\delta\eta_3$ can be described by $\delta\eta_3\mu_{13} = \delta\eta_3\mu_{23} = 0$ and hence $\delta\eta_3 = 0$, since the vectors μ_{13} and μ_{23} are nonparallel. Finally, if we select the point p_4 so that n_1, n_2, and n_4 lie in the same plane, we can prove the equality $\eta_4 = (u_{\parallel})_4$ using vectors η_1 and η_3 instead of the vectors η_1 and η_2.

We have thus shown that if we can find the three points on the Fermi surface with normals which are not in the same plane, then the homogeneous equation $\eta = a\{\eta\}$ has only the drift solution and, consequently, the inhomogeneous equation (II.1) is soluble for any value of ψ satisfying the condition (2.8).

The exception is thus a cylindrical Fermi surface. In this case the general solution of the homogeneous equation (II.3) is $\eta = u + \eta_\perp$, where u is the drift in the direction of the generators of the cylindrical surface, whereas $\eta_\perp$ is perpendicular to u and depends in an arbitrary manner on the coordinate measured in a direction perpendicular to the generators (an allowance is made for the fact that $\operatorname{curl}\eta = 0$). The condition for solubility of the inhomogeneous equation (II.1) can be represented in the form

$$\int \hat{D}(u + \eta_\perp)\,\hat{D}^{-1}\psi\, dS = \int (u\psi + \eta_\perp\psi)\, dS = 0.$$

In view of the arbitrary nature of the drift u and the function $\eta_\perp$ it follows that $\psi = 0$. However, on the other hand, ψ should satisfy the equation $\operatorname{div}\psi = -e\mathbf{En}$. Therefore, in the case of a cylindrical Fermi surface the diffusion equation (II.1) is insoluble.

2. It should be noted that the eigenfunctions η_i of the integral operator a form a complete set and satisfy the orthogonality conditions

$$\langle \eta_i \mid \eta_k \rangle \equiv \int \eta_i \hat{D}\eta_k\, dS = \delta_{ik}.$$

This follows from the fact that the equation $a\{\eta_i\} = \lambda_i\eta_i$ rewritten for the functions $\varkappa_i = \sqrt{\hat{D}}\eta_i$ in the form $\sqrt{\hat{D}}a\{\sqrt{D^{-1}}\varkappa_i\} = \lambda_i\varkappa_i$ has a symmetric core $\sqrt{\hat{D}_p}\hat{A}_{pp'}\sqrt{\hat{D}^{-1}_{p'}}$.

We can now show easily that the nth term of the iteration series for the solution of Eq. (II.1) has the form

$$\eta^{(n)} = \sum_{i>1} \lambda_i^n C_i \eta_i, \quad C_i = \langle \eta_i \mid \hat{D}^{-1}\psi \rangle.$$

The drift function $\eta_1 = u_{\parallel}$ with $\lambda_1 = 1$ does not occur in the expansion because it follows from the solubility condition (2.9) that $C_1 = 0$. Thus, the iteration procedure converges on condition that for all $i > 1$ the eigenvalues obey $\lambda_i < 1$. This can be proved by noting that the quantity $\langle \eta \mid \eta - a\{\eta\} \rangle$ on the left-hand side of Eq. (II.4) is essentially positive for any value of η which is not a solution of the homogeneous equation (II.3). If we bear in mind that Eq. (II.3) does not have other solutions except η_1, then for $i > 1$, we obtain

$$\langle \eta_i \mid \eta_i - a(\eta_i) \rangle = 1 - \lambda_i > 0.$$

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