Magnetic properties of ³He–⁴He quantum solutions

E. P. Bashkin and A. É. Meĭerovich

Institute of Physical Problems of the Academy of Sciences of the USSR
Usp. Fiz. Nauk **130**, 279–313 (February 1980)

A description is presented of all the thermo- and hydrodynamic, high-frequency, kinetic, and magnetic properties of ³He–⁴He solutions. The treatment does not resort to any model representations on the nature of the interaction of the ³He quasiparticles. It is based on the fact that an expansion in terms of the interaction for a solution at low temperatures formally coincides with an expansion in terms of the ³He concentration. All of the experimental data on the low-temperature properties of the solution known to us and analyzed in this study agree well with the presented theory up to the very greatest concentrations. An entire set of new phenomena is predicted, among which the most striking are the magnetokinetic effects–an enormous increase in the kinetic coefficients upon magnetizing the solution. The transition of ³He in solution to the superfluid state is discussed. Detection of this effect is apparently possible even now.

PACS numbers: 67.60.Dm

CONTENTS

1. INTRODUCTION

A considerable fraction of the low-temperature phenomena in the physics of the condensed state can be explained well within the framework of the theory of the Fermi fluid. This includes the electron fluids in metals and semiconductors, superconductive systems, normal and superfluid ³He, etc. However, as a rule, the description of these phenomena by the theory of the Fermi fluid is phenomenological or qualitative in nature because a microscopical calculation of the quantities governing the Fermi-fluid interaction is generally impossible. Practically the only system in which one can perform such calculations rigorously and completely is the weakly nonideal Fermi gas. Yet, strictly speaking, one does not encounter a degenerate Fermi gas in nature, since all real Fermi systems condense before manifesting effects involving quantum degeneracy.

The sole exception is the degenerate solution of ³He in superfluid ⁴He. Study of the properties of superfluid ³He–He II solutions both offers independent interest, since they possess an entire set of new, unusual properties, and also allows one to trace the origin of a large number of the laws common to all Fermi systems.

A superfluid ³He–⁴He solution amounts to a quantum fluid in which excitations with a Fermi-type spectrum corresponding to the impurity ³He atoms exist along with the phonons and rotons characteristic of pure ⁴He. With decreasing temperature the number of phonons and rotons declines rapidly, and the characteristics of the solution are governed exclusively by the Fermi component (³He quasiparticles). Although the literature has already repeatedly thrown light on the properties of ³He–⁴He solutions (see, e.g., the books[1-4] and reviews[5-10]), a full, systematic description of the system in the low-temperature region has thus far been lacking, since the law of interaction of the ³He quasiparticles has not yet been established. However, the systematic quantum-mechanical approach presented below makes possible a detailed microscopical analysis of all the properties without resorting to any model concepts, with prediction of a set of interesting effects not previously known.[11-15,75,82]

Interest in low-temperature phenomena in superfluid solutions has considerably heightened since the experimental discovery (Edwards *et al.*[16]) of the finite solubility of ³He in superfluid ⁴He at absolute zero. As $T \to 0$, the solution does not separate into layers up to a concentration of 6.5% (one can raise the separation concentration to 10% by increasing the pressure). As is known, temperature decrease leads to degeneracy of the Fermi component. An experimental confirmation of this fact was the discovery of a linear temperature dependence of the heat capacity of the solution and of

temperature dependences of the spin diffusion coefficients and of the heat conductivity characteristic of degenerate Fermi systems (Anderson et al.,[17,18] Abel et al.[19]). Measurements of the heat capacity revealed a certain dependence of the effective mass of the impurity particles on the concentration of the solution. This indicated the need of accounting for the Fermi-fluid interaction of the [3]He quasiparticles even at very low concentrations of the solution and the impossibility of treating the impurity [3]He atoms as an ideal fermion gas.

However, one cannot reconstruct the exact form of the interaction of the impurity atoms in the medium, since the interaction involves the polarization of the superfluid [4]He background and is generally nonlocal. In this regard, by analogy with the well-known study of Bardeen, Baym, and Pines,[20,21] a number of authors have employed model interaction potentials of various types, whose parameters were chosen by comparison with the experimental data.[22-34] These potentials have no direct physical meaning, and it is hard to justify a choice among them from first principles. Hence it is difficult to interpret all the experimental data simultaneously with equal success by using a model potential. To gain good enough agreement with experiment, one must resort to a theory containing a rather large number of adjustable parameters. A more detailed discussion of these studies and a presentation of the results is contained in the reviews of Radebaugh,[9] Ebner and Edwards,[7] Esel'son et al.,[2] and Baym and Pethick.[10]

Khalatnikov[1,35,36] has devised a phenomenological theory of the [3]He–He II solution in the spirit of the theory of the Fermi fluid of Landau.[37-39] According to this theory, all the thermodynamic properties of the solution are determined by fixing a Fermifluid function whose explicit form is not given by the theory, but must be established by comparison with the experimental data. However, it has proved possible experimentally to find the values of only the first two harmonics in the expansion of the f function in Legendre polynomials. This situation hinders concrete application of the results of the theory to real dilute solutions.

Nevertheless, one can conduct a complete and rather exact microscopical study of the properties of dilute solutions that corresponds to studying the effects of nonideality of the Fermi gas of impurity quasiparticles in the Landau-Pomeranchuk theory.[40,41] Here all the effects of interaction up to the highest concentrations (of the order of several percent) are governed by only one microscopical constant that has a pictorial physical meaning.

The treatment is based on the fact that [3]He impurity atoms dissolved in a superfluid Bose background form a dilute, nonideal Fermi gas of slow quasiparticles when at low enough concentrations and temperatures. As we know (see, e.g., Ref. 42), if the interaction energy of two particles declines rapidly enough with increasing distance between them, the interaction of the slow particles is reduced fundamentally to s scattering. Here the scattering amplitude does not depend on the momenta, and is the above-mentioned exact microscopical characteristic of the two-particle interaction

that exhaustively fixes all the properties of the solution. The small parameter in which one performs the expansion proves to be $k|a| \ll 1$. Here k is the characteristic wave vector of the impurity fermion, while a is the s scattering length. In this situation the properties of the solution are analogous[11,12] to the well-known case of a system of Fermi particles having a small radius of interaction (as compared with the mean de Broglie wavelength), which had been studied by Huang and Yang,[43] Lee and Yang,[44] and by Abrikosov and Khalatnikov.[45] Calculations by this scheme in the Born approximation would correspond to an ordinary hard-sphere model. Generally they should yield the same results as with employment of model potentials in the limit of zero momenta.[46] The fundamental differences in the method used below are the lack of model concepts as to the structure of the interaction, and the explanation of all the experimentally observed concentration-dependences of the various quantities by s scattering alone. To interpret the same relationships by model potentials, one employs the momentum component of the interaction potential, i.e., actually, the scattering with the higher moments.

The properties of superfluid solutions in a magnetic field have hardly been studied thus far. The existing theory of the Fermi fluid describes the behavior of fermions only in a weak magnetic field.[37,47] The theory developed below permits a generalization to the case of an arbitrary external magnetic field, since the employed procedure for calculating the Fermi-fluid function can also be applied in the presence of a field.[13-15] A uniform magnetic field does not affect the motion of an isolated, uncharged fermion, nor, in the nonrelativistic approximation, the two-particle interaction. Applying the field alters the distribution function of the particles, and hence also alters their energy spectrum, which is a functional of their distribution function. Thus the magnetic field affects not only the occupation numbers of quasiparticles having different spin orientations, but also the Fermi-fluid interaction. Here the occupation numbers and the energy spectrum in the magnetic field must be calculated in self-consistent fashion. As before, the Fermi-fluid interaction in the magnetic field is described well by s scattering of the [3]He quasiparticles. However, owing to the quantum-mechanical indistinguishability of the fermions, only collisions of particles having opposing spins are essential in s-scattering. Therefore, in strong magnetic fields, in which practically all the spins have the same orientation, the Fermi-fluid interaction is now determined by p scattering.[13] The amplitudes of p- and s-scattering of slow particles differ by a small factor proportional to the square of the momentum of relative motion. This considerably weakens the interaction in strong magnetic fields and substantially alters the concentration-dependence of all the thermodynamic quantities. This is manifested most strikingly in the distinctive magnetokinetic effects—the enormous increase in the mean free path of excitations and in the kinetic coefficients of the solution in strong magnetic fields.[13-15]

Thus far, in discussing the properties of a dilute [3]He–He II solution, we have assumed that the impurity

E. P. Bashkin and A. É. Meǐerovich   157

atoms form an isotropic, normal Fermi fluid of low density. Understandably, however, a phase transition of the impurity component to the superfluid state can occur at low enough temperature in the solution, involving Cooper pairing of the ^{3}He Fermi particles. Interest in this possibility has especially increased recently in line with the experimental discovery of a transition of pure ^{3}He to the superfluid state. Our last chapter discusses this problem. Estimates show that the transition temperature of ^{3}He in solution to the superfluid state may prove not to be so low as had been assumed previously[8,21] and the detection of this phenomenon is apparently practical and of experimental interest even at present.

2. THERMODYNAMICS OF SOLUTIONS

a) Fermi-fluid interaction

The state of the solution is fully determined by fixing the distribution function $n_\sigma(\mathbf{p})$ of the impurity excitations ($\mathbf{p}$ is the momentum of the excitation, and σ is the spin index), the density of the Bose component of the liquid $m_4 N_4$ (m_4 is the mass of a ^{4}He atom, and N_4 is the number of ^{4}He atoms per unit volume of the solution), the velocity of $\mathbf{v}_s$ of superfluid motion, and the temperature T. We shall be interested in the region of temperatures and concentrations in which we can neglect the contribution of phonons and rotons (at $T \lesssim 0.7$ K, the contribution of phonons to the normal density proves to be less than 1% of the impurity contribution, even at a concentration of the solution of the order of 0.5%). Another independent variable can be the magnetic field intensity $\mathbf{H}$.

First let us study the solution in the absence of superfluid motion, with $\mathbf{v}_s = 0$. According to the theory of Landau and Pomeranchuk,[40,41] an isolated ^{3}He impurity atom in superfluid ^{4}He constitutes a delocalized quasiparticle with a large de Broglie wavelength. The states of the impurity are classified by using the continuous energy spectrum $\mathscr{E}_{\alpha\beta}(\mathbf{p})$ (α and β are the spinor indices). The data on the density of the normal component of dilute solutions (Lynton and Fairbank,[48] Pellam[49]; see also the book of Esel'son et al.[2]) have permitted the conclusion that the energy spectrum $\mathscr{E}_{\alpha\beta}(\mathbf{p})$ reaches its minimum at zero momentum. In an isotropic liquid, the spectrum $\mathscr{E}_{\alpha\beta}(\mathbf{p})$ near the minimum can be represented as a series in even powers of the momentum $\mathbf{p}$. The parameter of the expansion is the ratio of the velocity of the quasiparticle to the velocity of sound in helium. At low temperatures and concentrations, at which the characteristic velocities of the bare quasiparticles are small, we can restrict the treatment to the first terms of the expansion of $\mathscr{E}_{\alpha\beta}$ in powers of p^2:

$$\mathscr{E}_{\alpha\beta}(\mathbf{p}) = \left\{ -\Delta + \frac{p^2}{2M}\left[1 - \gamma\left(\frac{p}{p_c}\right)^2\right]\right\}\delta_{\alpha\beta} - \beta\sigma_{\alpha\beta}\mathbf{H}. \quad (2.1)$$

Here the binding energy Δ and the effective mass M of a single impurity atom are $\Delta \approx 2.8$ K,[50,51] $M \approx 2.3\, m_3$, $p_c = m_4 s_0$ ($p_c/\hbar \approx 1.5$ Å^{-1}), where m_3 is the mass of a ^{3}He atom, and s_0 is the velocity of sound in pure ^{4}He at zero pressure, while the dimensionless parameter γ is very small. The experimental results[52] give $\gamma = 0.14 \pm 0.05$. The data of Ref. 53 give $\gamma = 0 \pm 0.01$. In contrast to Refs. 40 and 41, one must still retain the p^4 term in the dis-

persion law (2.1). As will be elucidated below, this is because its contribution has the same degree of smallness in terms of the concentration as the Fermi-fluid interaction in which we are interested. The parameters Δ, M, and γ in the spectrum of a single bare quasiparticle of (2.1) are functions of the density N_4 of ^{4}He atoms. In (2.1) we define $\sigma_{\alpha\beta}$ as the Pauli matrix, $\mathbf{H}$ is the magnetic field, and β is the magnetic moment of the ^{3}He nucleus.

With increasing concentration of the solution, the interaction of the impurity fermions begins to play a substantial role, and the energy spectrum of the excitations of the Fermi fluid differs from the spectrum of the bare quasiparticles of (2.1).

In the presence of an external magnetic field, the distribution of the quasiparticles of the Fermi fluid is described by the one-particle density matrix $n_{\alpha\beta}(\mathbf{p})$. The equilibrium one-particle density matrix for a polarized Fermi system has the form

$$n_{\alpha\beta}^{(0)}(\mathbf{p}) = \frac{1}{2}n(\mathbf{p})\,\delta_{\alpha\beta} + \frac{1}{2}\rho(\mathbf{p})\,\sigma_{\alpha\beta}\,. \quad (2.2)$$

The values $n(\mathbf{p}) \equiv n_{\alpha\alpha}^{(0)}$ and $\rho(\mathbf{p}) \equiv \sigma_{\alpha\beta}n_{\beta\alpha}^{(0)}$ (here and below, summation is to be performed over repeated indices) are determined from the condition that at equilibrium the occupation numbers $n_0^\pm(\mathbf{p})$ of particles having a projected spin of $\pm 1/2$ on the axis $\mathfrak{M} = \mathbf{H}/H$ are given by the Fermi distribution function:

$$n_0^\pm(\mathbf{p}) = \frac{1}{2}\left[1 - \tanh\frac{\varepsilon_\pm(\mathbf{p}) - \mu_3}{2T}\right],$$
$$n(\mathbf{p}) = n_0^+(\mathbf{p}) + n_0^-(\mathbf{p}), \quad \rho(\mathbf{p}) = [n_0^+(\mathbf{p}) - n_0^-(\mathbf{p})]\,\mathfrak{M}. \quad (2.3)$$

Here $\varepsilon_\pm$ is the energy of the fermions having different spin orientations, and μ_3 is the chemical potential of ^{3}He in the solution. This distribution function maximizes the entropy under the extra condition of constant total energy and number of impurity atoms.

The energy of an excitation depends substantially on the value of the projection of the spin on the field direction. In the absence of superfluid motion it can always be represented in the form

$$\varepsilon_{\alpha\beta}^{(0)} = \varepsilon_0(\mathbf{p})\,\delta_{\alpha\beta} - B(\mathbf{p})\,\sigma_{\alpha\beta}\mathbf{H}, \quad \varepsilon_\pm = \varepsilon_0 \mp BH. \quad (2.4)$$

The functions $\varepsilon_0(\mathbf{p})$ and $B(\mathbf{p})$ generally depend on H^2. The possibility of writing expressions for the density matrix and the Hamiltonian in the form of (2.2) and (2.4) stems from the fact that any one-particle operator for particles of spin $1/2$ reduces to a linear function of the Pauli matrix $\sigma_{\alpha\beta}$.

The total energy E and momentum $\mathbf{P}$ of the system are functionals of the basis variables $n_{\alpha\beta}(\mathbf{p})$, $\mathbf{v}_s$, N_4, and $\mathbf{H}$. According to the theory of the Fermi fluid,[1,36,37] the one-particle energy spectrum $\varepsilon_{\alpha\beta}(\mathbf{p})$ and the momentum $\mathbf{p}$ of an excitation in the solution are determined by the following variational relationships:

$$\varepsilon_{\alpha\beta}(\mathbf{p}) = \left[\frac{\delta E}{\delta n_{\beta\alpha}(\mathbf{p})}\right]_{\mathbf{v}_s, N_4, \mathbf{H}}, \quad \mathbf{P} = m_4 N_4 \mathbf{v}_s + \sum \mathbf{p}\,n_{\alpha\alpha}(\mathbf{p}). \quad (2.5)$$

The excitation energy $\varepsilon_{\alpha\beta}(\mathbf{p})$ is also a functional of $n_{\alpha\beta}(\mathbf{p})$, $\mathbf{v}_s$, N_4, and $\mathbf{H}$, and its first variational derivation determines the Fermi-fluid function $f_{\alpha\beta,\,\mu\nu}(\mathbf{p}, \mathbf{p}')$ of the system:

$$\delta \varepsilon_{\alpha\beta}(\mathbf{p}) = -\beta\,(\sigma_{\alpha\beta}\delta H) + \int f_{\alpha\beta,\ \mu\nu}(\mathbf{p},\ \mathbf{p}')\,\delta n_{\nu\mu}(\mathbf{p}')\,d\Gamma'. \qquad (2.6)$$

Here we have $d\Gamma = d^3p/(2\pi\hbar)^3$, and $\delta n_{\alpha\beta}$ is the deviation of the density matrix from the equilibrium value of (2.3). The Fermi-fluid function $f_{\alpha\beta,\ \mu\nu}(\mathbf{p},\mathbf{p}')$ in the exchange approximation can always be written in the form

$$f_{\alpha\beta,\ \mu\nu}(\mathbf{p},\ \mathbf{p}') = \psi\,(\mathbf{p},\ \mathbf{p}')\,\delta_{\alpha\beta}\delta_{\mu\nu} + \zeta\,(\mathbf{p},\ \mathbf{p}')\,\sigma_{\alpha\beta}\sigma_{\mu\nu}$$
$$+ \,[\varphi\,(\mathbf{p},\ \mathbf{p}')\,\sigma_{\alpha\beta}\delta_{\mu\nu} + \varphi\,(\mathbf{p}',\ \mathbf{p})\,\sigma_{\mu\nu}\delta_{\alpha\beta}]\,\mathfrak{M}$$
$$+ \,\xi\,(\mathbf{p},\ \mathbf{p}')\,(\sigma_{\alpha\beta}\mathfrak{M})\,(\sigma_{\mu\nu}\mathfrak{M}).$$
$$(2.7)$$

By analogy to Ref. 54, we can use Eqs. (2.3)–(2.7) to express the function $B(\mathbf{p})$ of (2.4) in terms of the component of the Fermi-fluid function $\xi(\mathbf{p},\mathbf{p}')$:

$$B\,(\mathbf{p}) = \beta - \frac{1}{H}\int \zeta\,(\mathbf{p},\ \mathbf{p}')\,[n_0^+(\mathbf{p}') - n_0^-(\mathbf{p}')]\,d\Gamma'. \qquad (2.8)$$

To determine all the thermodynamic characteristics of the solution, we must know the expression for the Fermi-fluid function. We shall calculate the f-function by following the method of Refs. 12, 14, and 43–45 (see also the books[55,56]). We can write the total energy of the system to an accuracy of the third-order terms of perturbation theory in the following form ($E_4^{(0)}$ is the contribution of pure ^{4}He):

$$E = E_4^{(0)} + \sum_p \mathscr{E}_{\alpha\beta}(\mathbf{p})\,n_{\beta\alpha}^{(0)}(\mathbf{p}) + \frac{\pi a \hbar^2}{M}\sum_{\mathbf{p},\ \mathbf{p}'}[n_{\alpha\alpha}^{(0)}(\mathbf{p})\,n_{\beta\beta}^{(0)}(\mathbf{p}')$$
$$- n_{\alpha\beta}^{(0)}(\mathbf{p})\,n_{\mu\nu}^{(0)}(\mathbf{p}')\,\sigma_{\beta\alpha}\sigma_{\nu\mu}] - \frac{16\pi^2 a^2 \hbar^4}{M}\sum_{\mathbf{p}_1,\ \mathbf{p}_2,\ \mathbf{p}_3,\ \mathbf{p}_4}$$
$$\times (\delta_{\beta\alpha}\delta_{\nu\mu} - \sigma_{\beta\alpha}\sigma_{\nu\mu})\frac{n_{\alpha\beta}^{(0)}(\mathbf{p}_1)\,n_{\mu\nu}^{(0)}(\mathbf{p}_2)\,n_{\gamma\gamma}^{(0)}(\mathbf{p}_3)}{p_1^2 + p_2^2 - p_3^2 - p_4^2}. \qquad (2.9)$$

Equation (2.9) contains only one quantity characterizing the interaction: the s-scattering length a. In the Born approximation, the total energy of the system of (2.9) is a bilinear form of one-particle statistical operators. In the first order in the interaction, the f-function does not depend on the momenta and the magnetic field:

$$f_{\alpha\beta,\ \mu\nu}^{(1)}(\mathbf{p},\ \mathbf{p}') = \left[\frac{\delta^2 E^{(1)}}{\delta n_{\beta\alpha}(\mathbf{p})\,\delta n_{\nu\mu}(\mathbf{p}')}\right]_{V_s,\ N_4,\ H} = \frac{2\pi a \hbar^2}{M}\,(\delta_{\alpha\beta}\delta_{\mu\nu} - \sigma_{\alpha\beta}\sigma_{\mu\nu}).$$
$$(2.10)$$

Here $E^{(1)}$ is given by the first three terms in Eq. (2.9). The first variational derivative of $E^{(1)}$ fixes to the same accuracy the excitation energy of the Fermi fluid:

$$\varepsilon_{\alpha\beta}^{(1)}(\mathbf{p}) = \mathscr{E}_{\alpha\beta}(\mathbf{p}) + \frac{2\pi a \hbar^2}{M}\left(\delta_{\alpha\beta} - \frac{N_+ - N_-}{N_3}\sigma_{\alpha\beta}\mathfrak{M}\right)N_3. \qquad (2.11)$$

Here we have

$$N_\pm = \int n^\pm\,d\Gamma = \frac{p_\pm^3}{6\pi^2\hbar^3},\ \ N_+ + N_- = N_3.$$

The quantities $N_\pm$ and $p_\pm$ correspond to the number of particles per unit volume and to the radius of the Fermi sphere for quasiparticles having spins lying parallel and antiparallel to the field. In Eq. (2.11), we do not take into account the term containing γ in $\mathscr{E}_{\alpha\beta}$ in (2.1), since it is substantial in s-scattering only in the higher orders of perturbation theory.

We can derive the second-order perturbation-theory correction to the Fermi-fluid function of (2.10) by double variation of the corresponding term in (2.9).[14] The final expression for the f-function is very unwieldy (see Ref. 14). Hence we shall give only its value in the absence of a magnetic field, which coincides the known result[45]:

$$f_{\alpha\beta,\ \mu\nu}(\theta) = \frac{2\pi a \hbar^2}{M}\left[1 + 2\lambda\left(2 + \frac{1-2w^2}{w}\tanh^{-1}w\right)\right]\delta_{\alpha\beta}\delta_{\mu\nu}$$
$$- \frac{2\pi a \hbar^2}{M}\,[1 + 2\lambda\,(1 - w\,\tanh^{-1}w)]\,\sigma_{\alpha\beta}\sigma_{\mu\nu}.$$
$$(2.12)$$

Here θ is the angle between the vectors $\mathbf{p}$ and $\mathbf{p}'$, and we have $p = p' = p_0$, $p_0 = (3\pi^2 N_3)^{1/3}\hbar$ is the limiting Fermi momentum in the absence of a field, $\lambda = p_0 a/\pi\hbar \ll 1$ is the small parameter of the theory, and we have $w = \sin(\theta/2)$. In weak magnetic fields, in the approximation linear in H, the functions $\psi(\theta)$ and $\zeta(\theta)$ in (2.12) do not change. Here the relationship of the f-function to the magnetic field is contained only in the function $\varphi(\theta)$[12]:

$$\varphi\,(\theta) = 16\,\frac{a^2\hbar}{p_0 w}\,\beta H\,\tanh^{-1}w.$$

Although the Fermi-fluid function of (2.10) does not depend formally on the field, the contribution of s-scattering to all the thermodynamic quantities will approach zero with increasing field, owing to decrease in the number of particles having spins directed against the field. This involves the fact that only interaction between particles of opposite spins is effective in s-scattering. Upon almost complete polarization of the spin system, the properties of a low-density Fermi fluid are determined mainly by p-scattering. Here the only essential interaction is that between the quasiparticles at the Fermi surface of radius

$$p_+ \approx (6\pi^2 N_3)^{1/3}\hbar = 2^{1/3}p_0 \equiv p_F.$$

The scattering amplitude of two slow, bare particles of momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ ($p_1 = p_2 = p_F$) in the center-of-inertia system is determined by the angle of rotation φ with respect to the momentum $\mathbf{p} = (\mathbf{p}_1 - \mathbf{p}_2)/2$:

$$\mathrm{Re}\,f\,(\mathbf{p}',\ \mathbf{p}) = \frac{4\pi\hbar^2}{M}\left(-a + \frac{3bp^2}{\hbar^2}\cos\varphi\right). \qquad (2.13)$$

Here $\mathbf{p}' = (\mathbf{p}_1' - \mathbf{p}_2')/2$ is the relative momentum of the scattered particles ($p_1' = p_2' = p_F, p' = p$). In order of magnitude, the constant b is equal to the gas-kinetic volume of an atom: $b \sim a^3$.

Upon accounting for the indistinguishability of fermions, we find the vertex component in the first order to be equal to

$$\Gamma\,(\mathbf{p}',\ \mathbf{p}) = \mathrm{Re}\,[f\,(\varphi) - f\,(\pi - \varphi)] = \frac{24\pi b p^2}{M}\cos\varphi. \qquad (2.14)$$

The Fermi-fluid function $f(\theta)$ (θ is the angle between the vectors $\mathbf{p}_1$ and $\mathbf{p}_2$) is determined to the same accuracy by the forward-scattering amplitude $\Gamma(\mathbf{p},\mathbf{p})$[13]:

$$f\,(\theta) = \frac{24\pi b p_F^2}{M}\sin^2\frac{\theta}{2}. \qquad (2.15)$$

In the presence of a Bose background, one cannot obtain the corrections of the next orders in the interaction in an analogous way, owing to retardation effects. The contribution to the f-function of (2.10) and (2.12) from retardation effects $f_{\mathrm{ret}}^{(s)} \sim (p_0/p_c)^2$ is of the same order of smallness as the third-order term of perturbation theory in s-scattering and the main term corresponding to p-scattering in (2.15). Consequently one can account for p-scattering for fermions in a medium only for systems having a high enough degree of polarization, for which s-scattering is inconsequential. The contribution of retardation to p-scattering $f_{\mathrm{ret}}^{(p)} \sim (p_F/p_c)^4$ already exceeds appreciably the second-order correction of perturbation theory to (2.15): $f_2^{(p)} \sim (p_F/p_c)^5$.

In principle, the fundamental results of the theory of the Fermi fluid are rigorously grounded only for degenerate Fermi systems in which all the quasiparticles lie near the Fermi surface. However, the ^{3}He–He II solution constitutes a Fermi fluid of low density. For dilute Fermi systems, the decay of the quasiparticles is small in terms of the characteristic gas parameter $p|a|/\hbar \ll 1$ even far from the Fermi surface, while in the Born approximation the quasiparticles do not decay at all. Hence the expressions derived above for the f-function in first-order perturbation theory can be used also to determine the properties of nondegenerate solutions, i.e., at arbitrary temperatures. Thus it is interesting to trace the transition to the case of a fully polarized degenerate solution via a state of the system in which the subsystem of particles with spins parallel to the field is degenerate, while the number of particles with spins oriented against the field is small, and they obey Boltzmann statistics. Nevertheless, the region of concentrations and temperatures in which the effect of phonons and rotons is insignificant pertains mainly to degenerate solutions. For this reason, we shall henceforth be studying the properties of degenerate solutions, unless it is expressly stipulated otherwise.

The motion of the superfluid He II background at the velocity $\mathbf{v}$ alters the energy spectrum (2.1) of the bare ^{3}He quasiparticles. The dependence of the Hamiltonian of a bare quasiparticle on $\mathbf{v_s}$ in the absence of a magnetic field has been determined by Bardeen, Baym, and Pines[21] by using a Galilean transformation. The change in the bare-particle spectrum with the onset of superfluid motion alters the distribution function of the impurity excitations, and hence causes all the Fermi-fluid characteristics to depend on $\mathbf{v_s}$. Thus, after cumbersome calculations,[14] by employing the results of Refs. 1, 21, and 36, one can determine the dispersion law and the one-particle density matrix for the quasiparticles of the Fermi fluid (not bare particles!) in an approximation linear in $\mathbf{v_s}$ in arbitrary magnetic fields:

$$\varepsilon_{\alpha\beta} = \varepsilon_0 \delta_{\alpha\beta} - B\sigma_{\alpha\beta}\mathbf{H} + (\delta m)_{\alpha\gamma}\left(\frac{1}{m}\right)_{\gamma\beta}\mathbf{p}\cdot\mathbf{v_s},$$
$$n_{\alpha\beta} = n^{(0)}_{\alpha\beta} + \left(\frac{(\delta m)_+}{m_+}\frac{\partial n^{(0)}_+}{\partial \varepsilon_+} + \frac{(\delta m)_-}{m_-}\frac{\partial n^{(0)}_{\alpha\beta}}{\partial \varepsilon_-}\right)\mathbf{p}\cdot\mathbf{v_s}. \tag{2.16}$$

Here $n^{(0)}_{\alpha\beta}$ is determined by Eqs. (2.2) and (2.3), and we have $(\delta m)_{\alpha\beta} = M_{\alpha\beta} - m_3\delta_{\alpha\beta}$; $(\delta m)_\pm = M_\pm - m_3$, $M_{\alpha\beta}$ is the spinor of the effective mass of a single bare quasiparticle having the energy spectrum $\mathscr{E}_{\alpha\beta}(\mathbf{p})$ in superfluid ^{4}He at rest; and we have

$$M_{\alpha\beta} = \tfrac{1}{2}(M_+ + M_-)\delta_{\alpha\beta} + \tfrac{1}{2}(M_+ - M_-)\sigma_{\alpha\beta}\mathfrak{M},$$
$$\mathscr{E}_{\alpha\beta} = \tfrac{1}{2}(\mathscr{E}_+ + \mathscr{E}_-)\delta_{\alpha\beta} + \tfrac{1}{2}(\mathscr{E}_+ - \mathscr{E}_-)\sigma_{\alpha\beta}\mathfrak{M}.$$

Here the eigenvalues $\mathscr{E}_\pm$ and $M_\pm$,

$$M_\pm^{-1} = \frac{1}{p_\pm}\frac{\partial \mathscr{E}_\pm}{\partial p_\pm}, \tag{2.17}$$

have the meaning of the energy and effective mass of a bare quasiparticle having a defined projection of the spin on the direction of the field. The eigenvalues of the effective-mass spinor $(1/m)_{\alpha\beta}$ of a quasiparticle of the Fermi fluid are

$$\left(\frac{1}{m}\right)_{\alpha\beta} = \tfrac{1}{2}\left(\frac{1}{m_+} + \frac{1}{m_-}\right)\delta_{\alpha\beta} + \tfrac{1}{2}\left(\frac{1}{m_+} - \frac{1}{m_-}\right)\sigma_{\alpha\beta}\mathfrak{M}.$$

$$\frac{1}{m_\pm} = \frac{1}{p_\pm}\frac{\partial \varepsilon_\pm}{\partial p_\pm}.$$

In the first order in $\mathbf{v_s}$, they prove to be equal to

$$m_\pm = \frac{M_\pm}{A^\pm}. \tag{2.18}$$

The formulas (2.18) are exact for isotropic Fermi fluids in the absence of a superfluid background, given a quadratic dispersion law of bare particles. In this case we must understand $M_+ = M_-$ to be the true mass of the real particles forming the fluid.

As usual, the effective-mass spinor $m_{\alpha\beta}$ (i.e., actually the quantities $A^\pm$) is expressed in terms of the first harmonics of the f-function. Since the Fermi-fluid function does not depend on the momenta of (2.10) in the Born approximation, in this approximation we have $A^\pm = 1$, and the total effective mass $m_\pm$ coincides with the effective mass of the bare quasiparticles. Exact expressions were derived[14] for the quantities $A^\pm$ in terms of the Fermi-fluid function and the value of the f-function for a solution in a magnetic field in second-order perturbation theory. These expressions determine the dependence of the effective mass on the concentration of the solution and the field intensity.

The effective mass $M_\pm$ of the bare quasiparticles differs from its value in the absence of a magnetic field, and it exhibits a weak concentration-dependence owing to the fourth-order term in the momentum in the Hamiltonian of (2.1):

$$M_\pm = \frac{M}{1 - (2\gamma p_\pm^2/p_0^2)}.$$

The dependence of the Fermi-fluid properties on $\mathbf{v_s}$ and H in weak magnetic fields in which the difference in radii of the Fermi spheres p_+ and p_- is insignificant has also been studied in Ref. 12.

b) Thermodynamic properties

Currently there is a large number of experimental results on the properties of degenerate solutions in the absence of a magnetic field. It is of interest to compare these data with the values of the thermodynamic functions of a ^{3}He–He II solution calculated by the above-derived relationships. This enables one to determine the fundamental characteristic of the interaction of the impurity ^{3}He atoms in the solution: the s-scattering length a.

A direct calculation of the free energy in the absence of a magnetic field while neglecting the terms quadratic in $\mathbf{v_s}$ and T/T_0 (T_0 is the degeneracy temperature of the Fermi component of the solution at H = 0) by using Eq. (2.9) yields the result[12]

$$F = F_4^{(0)} - N_3\Delta + \frac{3}{10}\frac{p_0^2}{M}N_3\left[1 + \frac{10}{9}\lambda + \frac{4}{21}\lambda^2(11 - 2\ln 2) - \frac{5}{7}\gamma\left(\frac{p_0}{p_c}\right)^2\right]. \tag{2.19}$$

Upon differentiating (2.19), we directly obtain the chemical potentials of ^{3}He and ^{4}He in the superfluid solution:

$$\mu_3 = -\Delta + \frac{p_0^2}{2M}\left[1 + \frac{4}{3}\lambda + \frac{4}{15}\lambda^2(11 - 2\ln 2) - \gamma\left(\frac{p_0}{p_c}\right)^2\right], \tag{2.20}$$

$$\mu_4 = \mu_4^{(0)} + \alpha_1 m_4 s_0^2 c - \frac{3}{10}\beta_1\frac{p_0^2}{M}c. \tag{2.21}$$
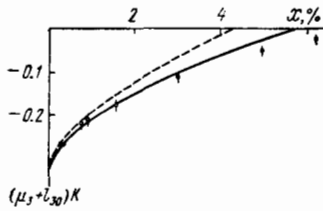
FIG. 1. Concentration-dependence of the chemical potential of ^{3}He. The dotted line corresponds to an ideal gas. We define $x = N_3/(N_3 + N_4)$; the circles show the experimental results.[58]

Here $\mu_4^{(0)}$ is the chemical potential of pure ^{4}He, $c = N_3/N_4$ is the atomic concentration of the impurity, and we have

$$\alpha_1 = \frac{N_4}{m_4 s_0^2} \frac{\partial(-\Delta)}{\partial N_4}, \qquad \beta_1 = \frac{\partial \ln M}{\partial \ln N_4}.$$

Reference 57 gives the values $\alpha_1 = 1.28$, $\beta = 1.25$. We emphasize that the given value of α_1 was not obtained from direct measurements of $\Delta(N_4)$, but from data on the molar volumes of ^{3}He and ^{4}He. The ratio of the molar volumes actually characterizes the binding energy Δ. Yet the stated method of calculating α_1 can hardly be considered fully warranted, as we can see even from the fact that it leads to a change in sign of Δ at a pressure of 6–8 atm,[7] which is physically implausible. One can determine the correct value of α_1, e.g., from data on the velocity of second sound in the solution.

We can represent the change in the chemical potential of ^{3}He, upon change in the number of particles by δN_3, in the following form[37]:

$$\delta\mu_3 = \frac{p_0}{m^*}\delta p_0 + \frac{1}{2}\delta N_3 \int_{-1}^{1} \psi(\theta)\, d(\cos\theta). \qquad (2.22)$$

Equation (2.22) contains the function $\psi(\theta)$ of the Fermi-fluid function in the absence of the field of (2.12).

Upon substituting (2.20) in Eq. (2.22), we find the total effective mass of an excitation in the absence of a field:

$$\frac{m^*}{M} = 1 + \frac{8}{15}\lambda^2(7\ln 2 - 1) + 2\gamma\left(\frac{p_0}{p_c}\right)^2. \qquad (2.23)$$

The osmotic pressure Π in a system having a finely porous membrane through which only the superfluid component can flow (superleak) can be calculated from the condition of equality of the chemical potentials of the solvent on both sides of the membrane[12]:

$$\Pi = \frac{2}{5}\frac{p_0^2}{2M}N_3\left[1 + \frac{5}{3}\lambda + \frac{8}{21}\lambda^2(11 - 2\ln 2) - \frac{10}{7}\gamma\left(\frac{p_0}{p_c}\right)^2\right]. \qquad (2.24)$$

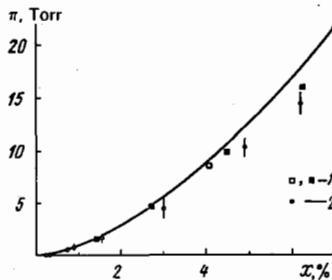Let us now define the excess enthalpy of the system W^E

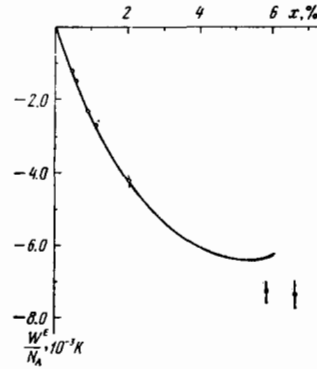FIG. 2. Osmotic pressure in a degenerate solution. 1—data of Refs. 59, 60; 2—results of Ref. 58.

FIG. 3. Excess enthalpy of a degenerate solution. N_A is Avogadro's number; the circles show the experimental data of Ref. 58.

by the following relationship:

$$W = W^E - l_{30}N_3 - l_{40}N_4.$$

Here $-l_{30}$ and $-l_{40}$ are the latent heats of vaporization of pure ^{3}He and ^{4}He per atom at $T = 0$, and W is the enthalpy per unit volume of the solution. To an accuracy of terms of the order of $(T/T_0)^2$, we have $W = \mu_3 N_3 + \mu_4 N_4$ and

$$W^E = (\mu_3 + l_{30})N_3 - \Pi. \qquad (2.25)$$

The magnetic susceptibility χ of the solution with account for the exchange Fermi-fluid interaction in the system of dissolved ^{3}He atoms is[37]

$$\frac{\chi_{1d}}{\chi - \chi_4^{(0)}} = \frac{1 + Z_0}{1 + F_1/3} = 1 - 2\lambda - \frac{16}{15}\lambda^2(\ln 2 + 2). \qquad (2.26)$$

Here χ_{1d} is the susceptibility of an ideal Fermi gas with the energy spectrum of (2.1) with $H = 0$, F_1 and Z_0 are the Fermi-fluid harmonics of the functions ψ and ζ of (2.12) defined in the usual way, and $\chi_4^{(0)}$ is the diamagnetic susceptibility of pure ^{4}He.

In a superfluid solution, the total momentum $\mathbf{P}$ of the fluid can always be represented in the form $\mathbf{P} = \rho^{(n)}\mathbf{v}_n + \rho^{(s)}\mathbf{v}_s$, where $\mathbf{v}_n$ is the velocity of the normal motion, and $\rho^{(n)}$ and $\rho^{(s)}$ are the densities of the normal and superfluid components, respectively. Calculation by the definitions (2.5) and (2.16) yields[12]

$$\rho^{(n)} = m_4 N_4 + m_3 N_3 - \rho^{(s)} = M N_3\left[1 + 2\gamma\left(\frac{p_0}{p_c}\right)^2\right]. \qquad (2.27)$$

This agrees with Ref. 1 with Eq. (2.23) taken into account.

Figures 1–5 compare the theoretical curves calculated by Eqs. (2.20)–(2.27) with the experimental data of Refs. 16, 18, 58–63. The value of the s-scattering
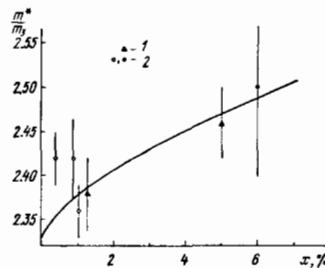

FIG. 4. Total effective mass of impurity quasiparticles. 1—data of Ref. 18; 2—experimental results of Refs. 58 and 16, respectively.
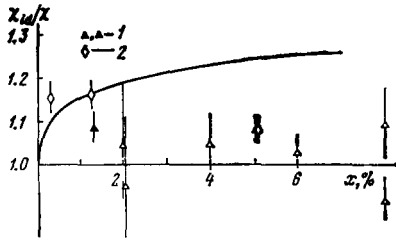
FIG. 5. Magnetic susceptibility of the solution. 1—experimental data[61-62,18]; 2—results of relative measurements of the magnetic susceptibility[63] normalized to the value for $x = 1.3\%$ as calculated by Eq. (2.26).

length a was determined by the requirement of best fit simultaneously with all the results of measuring both the thermodynamic and kinetic properties (see below) of the solution. It turned out that $a = -1.5$ Å. The sign of the scattering length corresponds to attraction between the impurity ^{3}He atoms.

As we see from Figs. 1–4, the presented method gives a good description of the interaction in a real solution up to a concentration of the order of 3%. For comparison, the dotted line in Fig. 1 shows the concentration-dependence of the chemical potential of ^{3}He in the solution in the approximation of a noninteracting degenerate ideal Fermi gas (in all the diagrams x is the molar concentration of ^{3}He in percent). Figure 5 shows all the experimental data on the magnetic susceptibility known to us. The large scatter of points in this diagram and the poor agreement of the experimental results with one another, even as to the sign of the interaction of the impurity ^{3}He quasiparticles, casts doubt on the accuracy of a part of the data. Moreover, a certain decline in the susceptibility with increasing concentration apparently indicates a weakening of the effective attraction with increasing momentum of the fermions. This does not agree qualitatively with the results of the rest of the thermodynamic measurements (see Figs. 1–3).

One can perform calculations by the scheme presented above for nondegenerate solutions only in first-order perturbation theory. However, this diminishes the accuracy only insignificantly, since the regions of applicability of Boltzmann statistics to the impurity quasiparticles correspond to low concentrations of the solution for which the corrections for nonideality are small.

In the Boltzmann region $T \gg T_0$, the principal concentration correction δF_{int} to the free energy, which is due to the interaction of the impurity quasiparticles, has the following form for s-scattering[55]:

$$\delta F_{\text{int}} = -2TN_3^2 \left(\frac{\pi \hbar^2}{MT}\right)^{3/2} \frac{1}{\pi} \int_0^\infty \frac{d\delta_0}{dp} e^{-p^2/MT} \, dp.$$

The phase of the amplitude of s-scattering of slow particles is $\delta_0 = -pa/\hbar$, and we have

$$\delta F_{\text{int}} = \frac{\pi a \hbar^2}{M} N_3^2. \qquad (2.28)$$

The result (2.28) coincides with the corresponding term in (2.19). This reflects the fact that the Fermi-fluid function of (2.10) in the Born approximation does

not depend on the momenta. The free energy of the solution and the chemical potential of ^{3}He in the Boltzmann region $T \gg T_0$ are then given by the following formulas:

$$F = F_4^{(0)} - N_3 \Delta - N_3 T \ln\left[\frac{2e}{N_3}\left(\frac{MT}{2\pi\hbar^2}\right)^{3/2}\right] + \frac{\pi a \hbar^2}{M} N_3^2, \qquad (2.29)$$

$$\mu_3 = -\Delta + T \ln\left[\frac{N_3}{2}\left(\frac{2\pi\hbar^2}{MT}\right)^{3/2}\right] + \frac{2\pi a \hbar^2}{M} N_3. \qquad (2.30)$$

Here the temperature dependence of all the thermodynamic quantities is determined only by the terms in (2.29) and (2.30) that correspond to an ideal Boltzmann gas of the impurity excitations.[41] We note that in Eqs. (2.29) and (2.30) the expansion is not performed in $c^{1/3}$ ($\lambda \sim p_0 \sim c^{1/3}$) as in the case of a degenerate solution with $T \ll T_0$, but in the first power of the concentration c.

The effect of the magnetic field on the thermodynamic functions of the solution becomes substantial when the characteristic parameter $\beta H/T_0$ is not too small. In actually attainable fields, this corresponds to solutions with a low concentration of ^{3}He for which the interaction of the impurities is described well even in first-order perturbation theory. This enables one to avoid unwieldy calculations involving the Fermi-fluid function in the second approximation.[14] We note that application of a magnetic field appreciably affects not only the Fermi-fluid interaction, but also the thermodynamic functions of an ideal Fermi gas. This involves the substantial dependence of the radii p_+ and p_- of the Fermi spheres on the field intensity.

Direct calculation of the energy of (2.9) in the Born approximation yields the result:

$$E = E_4^{(0)} - N_3 \Delta + \frac{3}{10M}(p_+^2 N_+ + p_-^2 N_-) - \beta H (N_+ - N_-) + \frac{4\pi a \hbar^2}{M} N_+ N_-. \qquad (2.31)$$

The quantities $N_\pm$ are calculated by using the equation $(\partial E/\partial N_\pm)_{N_3, H} = 0$:

$$N_\pm = \tilde{N}_\pm \pm 2\left(\frac{6}{\pi}\right)^{1/3} a (\tilde{N}_+ \tilde{N}_-)^{1/3} \frac{\tilde{N}_+ - \tilde{N}_-}{\tilde{N}_+^{1/3} + \tilde{N}_-^{1/3}},$$

while the chemical potential of ^{3}He is equal to

$$\mu_3 = \mu_0 + \frac{4\pi a \hbar^2}{M}(\tilde{N}_+ \tilde{N}_-)^{1/3} \frac{\tilde{N}_+^{2/3} + \tilde{N}_-^{2/3}}{\tilde{N}_+^{1/3} + \tilde{N}_-^{1/3}}. \qquad (2.32)$$

Here the $N_\pm(H)$ are the numbers of particles of the different spin orientations per unit volume of an ideal Fermi gas, and $\mu_0(H)$ is the chemical potential of the fermions in the absence of interaction. Henceforth, it will sometimes be convenient to employ the dimensionless variables in place of the quantities $p_\pm$, $N_\pm$, $\tilde{N}_\pm$, and H,

$$\chi_\pm = \frac{p_\pm}{p_0} = \left(\frac{2N_\pm}{N_3}\right)^{1/3}, \quad \tilde{\chi}_\pm = \left(\frac{2\tilde{N}_\pm}{N_3}\right)^{1/3},$$

$$\mathscr{H} = \frac{2\beta H}{T_F}, \qquad T_F = 2^{2/3} T_0.$$

Here T_F is the degeneracy temperature of a fully polarized solution. In these variables, the equations defining the concentrations $\tilde{N}_\pm$ for an ideal Fermi gas have the following form:

$$\tilde{\chi}_+^2 - \tilde{\chi}_-^2 = 2^{2/3} \mathscr{H}, \quad \tilde{\chi}_+^3 + \tilde{\chi}_-^3 = 2, \qquad (2.33)$$

while the chemical potential $\mu_0(H)$ is given by

$$\mu_0(H) = \frac{p_0^2}{M} \frac{1}{\tilde{\chi}_+ + \tilde{\chi}_-} - \beta H \frac{\tilde{\chi}_+ - \tilde{\chi}_-}{\tilde{\chi}_+ + \tilde{\chi}_-}. \qquad (2.34)$$

E. P. Bashkin and A. É. Meĭerovich   162

It is also easy to obtain the value of the chemical potential of ^{4}He in the solution

$$\mu_4 = \mu_4^{(0)} + \alpha_1 m_4 s_0^2 c - \frac{3}{10} \frac{\beta_1}{MN_4} (p_+^2 N_+ + p_-^2 N_-),\qquad (2.35)$$

and the magnetic susceptibility of the solution

$$\chi = \beta \frac{\partial}{\partial H} (N_+ - N_-) + \chi_4^{(0)}.$$

Analogously, we can determine the values of the rest of the thermodynamic quantites in magnetic fields in the usual way.

For nondegenerate solutions, the term in the total energy that corresponds to interaction keeps its previous form (2.31) in first order:

$$\delta E_{int} = \frac{4\pi a \hbar^2}{M} \widetilde{N}_+ \widetilde{N}_-.\qquad (2.36)$$

Thus, in the presence of a magnetic field, in contrast to the energy (2.28) in the absence of a field, δE_{int} of (2.36) depends on the temperature, since the pertinent numbers of particles $N_\pm$ of a definite spin orientation in a nondegenerate ideal Fermi gas, as tabulated by Stoner,[64] are functions of the temperature.

As mentioned above, in strong magnetic fields for which $N_-/N_3 \ll 1$, the thermodynamics of the solutions depends on the amplitude of p-scattering of the impurity excitations of (2.14). Here the pole of the one-particle Green's function $G(p)$ is given by

$$G^{-1}(p) = \varepsilon(p) + \Delta - \frac{p^2}{2M}\left[1 - \gamma\left(\frac{p}{p_c}\right)^2\right] + \beta H - \Sigma,$$

With the aid of Galitskiĭ's relationship[65] for the eigen-energy contribution $\Sigma(p)$ of the fermions in first-order perturbation theory and of Eq. (2.14) $[\mathbf{q} = (\mathbf{p}-\mathbf{p}')/2]$:

$$\Sigma = \int d\Gamma' n_+(\mathbf{p}') \Gamma(\mathbf{q},\mathbf{q}) = -\frac{18 \pi b N_3}{M}\left(\frac{p^2}{3} + \frac{p_F^2}{5}\right),\qquad (2.37)$$

this pole determines the excitation spectrum $\varepsilon(p)$

$$\varepsilon(p) = -\Delta + \frac{p^2}{2M}\left[1 - \gamma\left(\frac{p}{p_c}\right)^2 + 12\pi b N_3\right] + \frac{18}{5}\frac{p_F^2}{M}\pi b N_3 - \beta H.$$

Moreover, it determines the chemical potential

$$\mu_3 = \varepsilon(p_F) = -\Delta + \frac{p_F^2}{2M}\left[1 - \gamma\left(\frac{p_F}{p_c}\right)^2 + \frac{96\pi}{5}b N_3\right] - \beta H\qquad (2.38)$$

and the effective mass of the excitations

$$\frac{1}{m_+} = \frac{1}{p_F}\left(\frac{\partial \varepsilon}{\partial p}\right)_{p=p_F},\qquad \frac{m_+}{M} = 1 + 2\gamma\left(\frac{p_F}{p_c}\right)^2 - 12\pi b N_3.$$

One can also easily derive the same relationships by using the f-function of (2.15) by the ordinary formulas of Landau's theory of the Fermi fluid.

The expressions (2.37) and (2.15) define all the thermodynamic properties of the polarized solutions. Thus we have the following expression for the total energy:

$$E = E_4^{(0)} - N_3\Delta + \frac{3}{10M}p_F^2 N_3\left[1 - \frac{5}{7}\gamma\left(\frac{p_F}{p_c}\right)^2 + 12\pi b N_3\right] - \beta H N_3.\qquad (2.39)$$

The presence of the magnetic field also leads to a distinctive magnetoosmotic effect. Actually, since the radius of the Fermi sphere of the ^{3}He quasiparticles with spins aligned with the field as $\mathscr{H} \to 1$ increases by a factor of $2^{1/3}$ owing to the orienting action of the field, the osmotic pressure in a system having a membrane impermeable to the impurities is increased by a factor of about $2^{2/3}$ as compared with (2.24):

$$\Pi = \frac{2}{5}N_3\frac{p_F^2}{2M}\left[1 - \frac{10}{7}\gamma\left(\frac{p_F}{p_c}\right)^2 + 30\pi b N_3\right].\qquad (2.40)$$

This effect practically disappears with rising temperature and transition to the Boltzmann region, while the magnetoosmotic effect is completely absent if we neglect the interaction for an ideal Boltzmann gas.

3. VIBRATIONAL PROCESSES IN SOLUTION

a) Hydrodynamic vibrations

The propagation of low-frequency sound vibrations in a ^{3}He–He II solution is described by a system of hydrodynamic equations[1] that has the following form in linear approximation:

$$\left.\begin{array}{l} m_4\dfrac{\partial N_4}{\partial t} + m_3\dfrac{\partial N_3}{\partial t} + \rho^{(n)}\,\mathrm{div}\,\mathbf{v_n} + \rho^{(s)}\,\mathrm{div}\,\mathbf{v_s} = 0, \\[2mm] \dfrac{\partial N_3}{\partial t} + N_3\,\mathrm{div}\,\mathbf{v_n} = 0, \quad \dfrac{\partial S}{\partial t} + S\,\mathrm{div}\,\mathbf{v_n} = 0, \\[2mm] \rho^{(n)}\dfrac{\partial \mathbf{v_n}}{\partial t} + \rho^{(s)}\dfrac{\partial \mathbf{v_s}}{\partial t} + \nabla P = 0, \quad \dfrac{\partial \mathbf{v_s}}{\partial t} + \dfrac{1}{m_4}\nabla\mu_4 = 0. \end{array}\right\}\quad (3.1)$$

Here S is the entropy per unit volume of the solution, and P is the pressure.

Although the equations (3.1) are not formally altered by applying an external magnetic field, the thermodynamic quantities figuring in them are functions of the field intensity H, and are defined by the relationships derived above. Lhuillier and Francois[66] have studied the effect of a weak magnetic field on the hydrodynamic properties of the solution. We shall immediately treat below the case of arbitrary magnetic fields.[13-15] We shall assume that the small perturbations of all the equilibrium quantities are varying in the field of a sound wave by the law $\exp(i\omega t - i\mathbf{k}\cdot\mathbf{r})$. Upon neglecting terms of the order of T/T_0, we can employ the thermodynamic identity for P to determine easily the sought dispersion equation for the velocity of propagation of the vibrations $s = \omega/k$[12]:

$$m_4\frac{\rho^{(n)}}{N_3}s^4 - s^2\left[\rho^{(s)}\left(\frac{m_3}{m_4}\frac{\partial\mu_4}{\partial N_4} - \frac{\partial\mu_4}{\partial N_3}\right) + m_4 A_3\right.$$
$$\left. - \left(m_3 - \frac{\rho^{(n)}}{N_3}\right)A_4\right] + \left[A_3\frac{\partial\mu_4}{\partial N_4} - A_4\frac{\partial\mu_4}{\partial N_3}\right]\frac{\rho^{(s)}}{m_4} = 0.\qquad (3.2)$$

Here we have $A_i = N_3(\partial\mu_3/\partial N_i) + N_4(\partial\mu_4/\partial N_i)$, with $i = 3$ or 4.

The velocity of propagation of first sound in the absence of a magnetic field is determined by the large root (of the order of the velocity of sound s_0 in pure ^{4}He) of Eq. (3.2) with account for the expressions for the thermodynamic functions (2.20), (2.21), and (2.27):

$$s_1^2 = s_0^2\left\{1 + c\left[\frac{m_4}{M}\left(\alpha_1 + \frac{\delta m}{m_4}\right)^2 + \alpha_2 - \frac{\delta m}{m_4}\right]\right.\qquad (3.3)$$
$$+ \left(\frac{m_4}{M}\right)^2\left(\frac{p_0}{p_c}\right)^2 c\left[\frac{1}{3}\frac{m_4}{M}\left(\alpha_1 + \frac{\delta m}{m_4}\right)^2 - \beta_1\left(\alpha_1 + \frac{\delta m}{m_4}\right)\right.$$
$$\left.\left. - 2\gamma\frac{M}{m_4}\left(\alpha_1 - \frac{m_3}{m_4}\right)^2 + \frac{3}{10}\beta_2\frac{M}{m_4}\right]\right\} \approx s_0^2(1 - 0.30c - 0.29c^{5/3}).$$

Here we have

$$\delta m = M - m_3,\quad \alpha_2 = -\frac{N_3^2}{m_4 s_0^2}\frac{\partial^2\Delta}{\partial N_3^2},\quad \beta_2 = MN_3^2\frac{\partial^2(1/M)}{\partial N_3^2}.$$

The last term in Eq. (3.3) describes the nonlinear concentration-dependence of the velocity of sound in the solution observed by Abraham et al.[67] Equation (3.3) shows that the parameters characterizing the interac-

tion of the ^{3}He quasiparticles do not enter at all into the expression for the velocity of first sound. Thus the latter coincides with the velocity of sound in a solution of an ideal Fermi gas having the spectrum of (2.1) in superfluid ^{4}He.

Figure 6 shows the experimental data[67] and the curve corresponding to Eq. (3.3). The parameters α_2 and β_2 proved to be equal to $\alpha_2 = -2.26, \beta_2 = -1.38$ for $\gamma = 0.14$[52] (if $\gamma = 0$,[53] then $\beta_2 = -1.64$).

Just as in pure He II, fourth sound can propagate in the degenerate solution[68] when the solution is contained in capillaries so narrow that their diameter proves smaller than the depth of penetration of a viscous wave or the mean free path of excitations. Here the vibrations will propagate only in the superfluid component of the liquid, while its normal component proves immobile. The velocity of fourth sound is determined by the equations (3.1), in which we should set $v_n = 0$:

$$s_4^2 = \frac{\rho^{(s)}}{m_4^2}\frac{\partial \mu_4}{\partial N_4} = s_0^2\left\{1+c\left(\alpha_2-\frac{\delta m}{m_4}\right)\right.$$
$$\left.+\left(\frac{p_0}{p_c}\right)^2\frac{m_4}{M}c\left[\frac{3}{10}\beta_2-2\gamma\left(\frac{M}{m_4}\right)^2\right]\right\} \approx s_0^2(1-3.27c).$$
$$(3.4)$$

Applying an external magnetic field affects the velocities of propagation of first and fourth sounds only slightly. Only the small nonlinear terms proportional to $c^{5/3}$ are altered by polarizing the solution.

In contrast to the velocities of first and fourth sound, the velocity of second sound in the system of impurity excitations is considerably altered upon magnetizing the solution. The nature of this phenomenon is analogous to the magnetoosmotic effect treated above. In an approximation linear in the concentration, according to the dispersion equation (3.2), the velocity of second sound in the solution is determined by the expression

$$s_2^2 = \frac{N_3}{M}\frac{\partial \mu_3}{\partial N_3}-\alpha_1^2\frac{m_4}{M}s_0^2 c. \qquad (3.5)$$

Since the principal term with respect to the concentration in (3.5) is proportional to the square of the Fermi velocity, then upon complete polarization of the solution, the velocity of propagation of second sound in the polarized solution exceeds its value in the absence of the field by a factor of $2^{1/3}$. In the approximation of an ideal Fermi gas, we can find the relationship of the velocity of second sound to the magnetic field by using the formula (2.34) for the chemical potential:

$$\frac{s_2^2(\mathcal{H})}{s_2^2(0)}=\frac{2}{\chi_++\chi_-}, \quad s_2(\mathcal{H}\geqslant 1)=2^{1/3}s_2(0), \quad s_2(0)=\frac{p_0}{\sqrt{3}M}. \qquad (3.6)$$
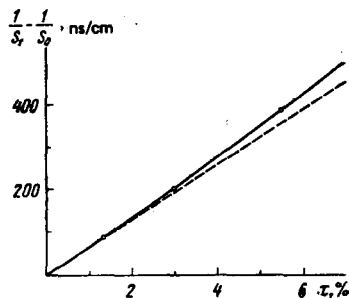
FIG. 6. Deviation from linearity of the concentration-dependence of the velocity of first sound in the solution (linear relation—dotted line). Circles—experimental data.[67]

Figure 7 shows a graph of the function of (3.6). Account of the temperature effects $\sim(T/T_F)^2$ smooths out the break in the curve at $\mathcal{H}=1$.

Account of the Fermi-fluid interaction for the thermodynamic functions in the dispersion equation (3.2) yields more accurate expressions for the velocity of second sound in the solution in the absence of a magnetic field[12]:

$$s_2^2(0)=\frac{p_0^2}{3M^2}\left[1+2\lambda+\frac{8}{15}\lambda^2(11-2\ln 2)-4\gamma\left(\frac{p_0}{p_c}\right)^2\right]-\frac{m_4}{M}\alpha_1^2 s_0^2 c.$$
$$(3.7)$$

In the limiting case of a completely polarized solution, we have[14]

$$s_2^2(\mathcal{H}\geqslant 1)=\frac{p_0^2}{3M^2}\left[1-4\gamma\left(\frac{p_F}{p_c}\right)^2+3\alpha_1\left(\alpha_1+2\alpha_1\gamma\frac{M}{m_4}\right)c\right.$$
$$\left.-\left(\alpha_1+\frac{\delta m}{m_4}\right)^2\frac{m_4}{M}c+48\pi b N_4\right]-\alpha_1^2 s_0^2\frac{m_4}{M}c.$$
$$(3.8)$$

We note that the velocities of first and fourth sound of (3.3) and (3.4) are defined to an accuracy of terms of the order of c^2, whereas the velocity of second sound in the absence of a magnetic field of (3.7) is defined only to an accuracy of $c^{5/3}$. This is explained by the fact that the effects of retardation, which we have neglected, make a contribution to the velocity of second sound of the order of $c^{5/3}$, whereas the corresponding terms in the expressions for the velocities of first and fourth sound completely drop out. In the case of p-scattering of (3.8), one can also account for terms of the order of $c^{5/3}$ in the velocity of second sound.

b) Propagation of high-frequency vibrations

In the last section we have treated low-frequency ($\omega\tau\ll1$) hydrodynamic vibrations in the solution (ω is the frequency of the wave, and τ is the characteristic relaxation time). In the quantum system ^{3}He–He II, various high-frequency modes with $\omega\tau\gg1$ can also propagate. We note that polarization of the solution by a magnetic field sharply increases the characteristic relaxation time τ (see Chap. 4). Consequently, in a polarized dilute solution the condition $\omega\tau\gg1$ can be fulfilled even for infrared frequencies. In principle, high-frequency vibrations that are characteristic of both a Fermi fluid (zero sound, spin waves) and of a Bose system (high-frequency first sound) can propagate in the Fermi-Bose quantum liquid ^{3}He–^{4}He.
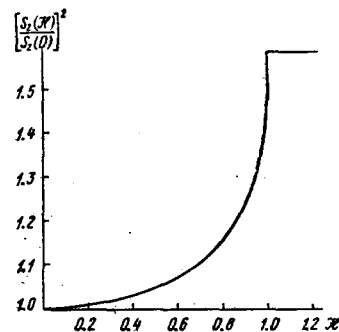
FIG. 7. Relative dependence of the square of the velocity of second sound in a degenerate ^{3}He–^{4}He solution on the magnetic field. We define $\mathcal{H}=2\beta H/T_F$. Account of the temperature effects T/T_F smooths the break in the curve at $\mathcal{H}=1$.

E. P. Bashkin and A. É. Meǐerovich   164

According to Landau's theory of the Fermi fluid,[38] the oscillations of the first type correspond to nonisotropic oscillations of the Fermi surface of the ^{3}He quasiparticles. The high-frequency first-sound in the solution is analogous to the high-frequency vibrations in pure He II (Andreev and Khalatnikov[69]). They amount to vibrations of ^{4}He in which the impurity quasiparticles also participate owing to the interaction of the ^{3}He atoms with the superfluid background.

The high-frequency vibrations of the solution are described by a collisionless kinetic equation for the one-particle density matrix

$$\frac{\partial n_{\alpha\beta}}{\partial t} + \frac{i}{\hbar} [\hat{\varepsilon}, \hat{n}]_{\alpha\beta} + \frac{1}{2} \left(\frac{\partial \varepsilon_{\alpha\gamma}}{\partial p} \nabla n_{\gamma\beta} + \nabla n_{\alpha\gamma} \frac{\partial \varepsilon_{\gamma\beta}}{\partial p} \right) - \frac{1}{2} \left(\nabla \varepsilon_{\alpha\gamma} \frac{\partial n_{\gamma\beta}}{\partial p} + \frac{\partial n_{\alpha\gamma}}{\partial p} \nabla \varepsilon_{\gamma\beta} \right) = 0,$$

(3.9)

by the continuity equation

$$\frac{\partial}{\partial t} \left(m_4 N_4 + m_3 \int n_{\alpha\alpha} \, d\Gamma \right) + \text{div} \left(m_4 N_4 \mathbf{v}_s + \int \mathbf{p} n_{\alpha\alpha} \, d\Gamma \right) = 0$$

(3.10)

and by the equation of superfluid flow

$$\frac{\partial \mathbf{v}_s}{\partial t} + \frac{1}{m_4} \nabla \frac{\partial}{\partial N_4} \left(E_0 + \int \varepsilon_{\alpha\beta} \delta n_{\beta\alpha} \, d\Gamma \right) = 0.$$

(3.11)

Here $[\hat{\varepsilon}, \hat{n}]_{\alpha\beta}$ is the commutator of the spin matrices, and E_0 is the total equilibrium energy of the system.

We shall represent the deviation $\delta n_{\alpha\beta}$ from the equilibrium value in the following form by using (2.16):

$$\delta n_{\alpha\beta} = \nu(\mathbf{p}) \delta_{\alpha\beta} + \lambda(\mathbf{p}) \sigma_{\alpha\beta}; \quad n_{\alpha\beta} = n_{\alpha\beta}^{(0)} + \delta n_{\alpha\beta}$$
$$+ \left[\frac{(\delta m)_+}{m_+} \frac{\partial n_0^+}{\partial \varepsilon_+} (\delta_{\alpha\beta} + \sigma_{\alpha\beta} \mathfrak{M}) + \frac{(\delta m)_-}{m_-} \frac{\partial n_0^-}{\partial \varepsilon_-} (\delta_{\alpha\beta} - \sigma_{\alpha\beta} \mathfrak{M}) \right] \frac{\mathbf{p} \cdot \mathbf{v}_s}{2}.$$

(3.12)

According to (2.4), (2.6), and (2.16), the excitation energy $\varepsilon_{\alpha\beta}$ proves to be

$$\varepsilon_{\alpha\beta} = \varepsilon_{\alpha\beta}^{(0)} + \frac{\partial \varepsilon_{\alpha\beta}^{(0)}}{\partial N_4} \delta N_4 + (\delta m)_{\alpha\gamma} \left(\frac{1}{m} \right)_{\gamma\beta} \mathbf{p} \cdot \mathbf{v}_s + 2\delta_{\alpha\beta} \int \psi \nu \, d\Gamma + 2\sigma_{\alpha\beta} \int \zeta \lambda \, d\Gamma.$$

(3.13)

1) *High-frequency vibrations in the absence of a magnetic field.* In the absence of a magnetic field we can employ the expression (2.12) for the f-function. Then the system of equations (3.9)–(3.11) describing the high-frequency vibrations in the solution is substantially simplified. We shall seek small perturbations of the basis variables ν, λ, $\mathbf{v}_s$, and δN_4 in the form $\exp(i\omega t - i\mathbf{k} \cdot \mathbf{r})$. When $H = 0$, Eqs. (3.9)–(3.11) reduce to two independent equations for the vibrations of the magnetic moment $\sigma_{\alpha\beta} n_{\beta\alpha}$ per unit volume of the solution and for the scalar distribution function $n_{\alpha\alpha}$. Since $a < 0$ and the Fermi-fluid interaction is small, only a symmetric spin wave with the azimuthal number $m = 0$ and high-frequency first sound can propagate in the solution in the absence of a magnetic field, while the existence of undamped waves of the zero-sound type proves impossible.[11,12]

The corresponding dispersion equation for the symmetric spin waves in solution in the Born approximation has the form[38,94]

$$1 + \zeta_0 \int \frac{\mathbf{k} \cdot \mathbf{v}}{\omega - \mathbf{k} \cdot \mathbf{v}} \frac{\partial n_0}{\partial \varepsilon} \cdot 2 \, d\Gamma = 0.$$

(3.14)

Here we have $\zeta_0 = \zeta(\mathbf{p}_0, \mathbf{p}_0) = -2\pi a \hbar^2/M$, and n_0 is the equilibrium Fermi distribution function in the absence of a magnetic field. When $T = 0$, the velocity $u_0 = \omega/k$

of propagation of the vibrations is exponentially close to the Fermi velocity according to (3.14) (cf. Refs. 11, 12):

$$u_0 - v_0 = v_0 \exp(1/\lambda) \ll v_0, \quad \lambda = p_0 a/\pi\hbar.$$

(3.15)

The expression (3.15) for the velocity of spin waves is written with logarithmic accuracy. A calculation of the pre-exponential coefficient would require accounting for the values of all the previous derivatives of the f-function of (2.12) for $\chi = 0$.

The absorption of the high-frequency modes is usually governed by the collisions of the quasiparticles of the Fermi fluid with one another. With declining temperature, the absorption declines in proportion to T^2.

There is also a collisionless mechanism of damping of the high-frequency vibrations that involves Cherenkov absorption of the waves by quasiparticles. The probability of these processes depends on the number of quasiparticles moving in phase with the wave. If the velocity of propagation of the high-frequency modes considerably exceeds the Fermi velocity v_0, then when $T \ll T_0$, the collisionless damping is exponentially small. The described mechanism of Landau damping proves to dominate whenever the velocity of the wave is so close to the Fermi velocity that the energy of the quasiparticles moving with these velocities lies within the region of thermal blurring of the Fermi edge.

Just this situation arises for spin waves in dilute ^{3}He-^{4}He solutions, in which the velocity of propagation of the vibrations of (3.15) is exponentially close to v_0 in terms of the 1/3 power of the ^{3}He concentration.

The fundamental experimental difficulty in studying high-frequency vibrations propagating at a velocity close to the Fermi velocity is the identification of the collective mode on the background of the signal of the free quasiparticles of the Fermi fluid. The most accurate method of determining the velocity of the wave is apparently time-of-flight measurements. Here, in order to resolve the signals of the wave and of the free fermions, the time difference of arrival between the signals $\Delta t \sim L(u - v_0)/v_0^2$ (L is the distance between the receiver and the emitter) must be large in comparison with the width of the pulse of high-frequency vibrations $\Delta t \sim 1/\omega$. That is, we have $L > v_0^2/\omega(u - v_0)$. On the other hand, for the pulse to be detected by the receiver, the signal must not decay in the distance $L:L < v_0/\omega''$. Here $\omega'' = \text{Im}\,\omega$ is the absorption coefficient of the high-frequency mode. Hence we see that experimental resolution of the contribution of the wave requires fulfillment of the condition[94]

$$\frac{\omega''}{\omega} < \frac{u - v_0}{v_0} \ll 1.$$

(3.16)

This condition is stricter than simply that of weak attenuation ($\omega'' \ll \omega$).

Another method is possible in principle for resolving the signal of the collective mode. It involves the fact that the incoherence of the radiation of the free quasiparticles diminishes the amplitude of their signal in proportion to $1/L^2$, even when we neglect relaxation.[95]

The collisionless damping of the symmetric spin

wave is governed by the imaginary component of the dispersion equation (3.14) as derived by the usual rule of bypassing the poles[94]

$$\omega'' = +\pi n_0(z)\, kv_0 \exp\left(-\frac{1}{|\lambda|}\right). \tag{3.17}$$

Here we have $z = \omega/kv_0 = u/v_0$, and $n_0(z)$ is the value of the distribution function for ^{3}He quasiparticles moving at a velocity equal to the velocity of propagation of the spin wave, i.e., with $v = \omega/k$. The real component of the dispersion equation (3.14) determines the temperature corrections to the velocity u_0 of the wave at $T = 0$ of (3.15):

$$u = \frac{\omega}{k} = u_0 - \frac{\pi^2}{24}\left(\frac{T}{T_0}\right)^2 v_0 \exp\left(-\frac{1}{|\lambda|}\right), \quad \frac{T}{T_0} \ll \exp\left(-\frac{1}{|\lambda|}\right) \ll 1. \tag{3.18}$$

We note that the second term in (3.18) is small in comparison with the difference between the true value of u_0 and that given by Eq. (3.15) with logarithmic accuracy. However, this term at $T/T_0 \ll \exp(-1/|\lambda|)$ fully determines the temperature-dependence of the velocity of spin waves.

The condition of small attenuation that was used in deriving Eq. (3.17) implies in this case that $\omega'' \ll \omega - kv_0 \sim kv_0 \exp(-1/|\lambda|)$, owing to the closeness of the velocity of propagation of the spin wave to the Fermi velocity. This means that the region of applicability of the expression (3.17) for ω'' is bounded by the condition

$$n_0(z) \approx \exp\left(-2e^{-1/|\lambda|}\frac{T_0}{T}\right) \ll 1. \tag{3.19}$$

In this case the inequality of (3.19) is equivalent to the condition (3.16). Thus spin waves in a ^{3}He-^{4}He solution can be detected only at low enough temperatures $T \ll T_0 \exp(-1/|\lambda|)$. This corresponds to the numerical inequality

$$T\,[\mu K] \ll 4.2 \cdot 10^2 x^{3/2} \exp\left(-\frac{2.4}{x^{1/3}}\right).$$

Here x is the molar concentration of ^{3}He in the solution.

Let us take up the role of collisional damping in the case in which the wave velocity is close to the Fermi velocity. The coefficient of collisional damping of the high-frequency mode with account for the quantum properties of magnons is expressed in terms of the relaxation time τ as follows[38]:

$$\omega''_{\text{coll}} = \frac{1}{\tau}\left[1 + \left(\frac{\hbar\omega}{2\pi T}\right)^2\right].$$

For collisional absorption the condition (3.16) is fulfilled for the frequencies

$$\omega_1 \ll \omega \ll \omega_2 \ll \frac{T_0}{\hbar}, \quad \omega_{1,2} = 2\alpha_0 \frac{\pi^2 T^2 \tau}{\hbar^2}\left\{1 \pm \left[1 - \left(\frac{\hbar}{\pi T \tau \alpha_0}\right)^2\right]^{1/2}\right\}, \quad \alpha_0 = e^{-1/|\lambda|}.$$

In order that such a frequency region can exist, the temperature must be low enough:

$$T < \frac{\alpha_0 \pi B}{\hbar}, \quad B = \tau T^2 = \text{const.}$$

Here we have accounted for the fact that in a normal Fermi fluid $\tau \sim 1/T^2$.

Upon employing the experimental data,[2] the derived inequalities prove equivalent to the following numerical equations:

$$\omega_{1,2} = 10^{12}\alpha_0\left(1 \pm \sqrt{1 - \frac{T^2\,[K]}{10\alpha_0^2}}\right), \quad T\,[K] \ll \alpha_0.$$

This temperature bound is weaker than the derived

above for Landau damping. By comparing the coefficient of Cherenkov absorption of (3.17) with ω''_{coll} one can easily determine the frequency interval lying between ω_1 and ω_2 and the condition on the temperature $T\,(K) < 10\alpha_0 \exp(-2\alpha_0 T_0/T)$ under which collisionless damping dominates.

At higher temperatures of the solution such that $T \gtrsim T_0$, the damping involving the finite lifetime of the quasiparticles of the Fermi fluid is important. However, the low concentration of the solution causes the damping of excitations of ^{3}He to be small in proportion to the smallness of the gas parameter λ.[65] In the Born approximation, damping of the quasiparticles does not exist at all. This enables one in the first approximation of perturbation theory to solve the problem of propagation of magnetization waves in a dilute ^{3}He-^{4}He solution for arbitrary temperatures.

However, as we can easily convince ourselves, the smallness of the interaction causes the dispersion equation (3.14) in the Boltzmann region to have only a strongly damped solution $|\text{Im}\,\omega| \gg |\text{Re}\,\omega|$.[94] In contrast to the breakdown of the condition (3.16) for resolving the contribution of spin waves, this leads to an impossibility in principle of propagation of a collective mode. This agrees with the result derived above that spin vibrations in the solution in the absence of a field are practically undetectable, even at very low temperatures $T \sim T_0 \exp(-1/|\lambda|) \ll T_0$.

The equation in the scalar distribution function $n_{\alpha\alpha}$ of the impurity excitations obtained by averaging Eqs. (3.9)–(3.11) over the spins describes the dispersion law of first sound near absolute zero, at which the relation time of the ^{3}He quasiparticles rapidly increases and exceeds the period of the sound vibrations (high-frequency first-sound)[12,35]:

$$Q(u) + \frac{u}{v_0} R(u) = \frac{1}{w(u/v_0)}. \tag{3.20}$$

Here $u = \omega/k$ is the velocity of propagation of the vibrations, and we have

$$\begin{aligned}
w(x) &= \frac{x}{2}\ln\frac{x+1}{x-1} - 1, \\
Q(u) &= \frac{3}{p_0^2}\frac{m^*}{m_4}\frac{1}{u^2 - s_4^2}\left[\frac{\rho^{(s)}}{m_4}N_3\left(\frac{\partial\mu_3}{\partial N_4}\right)^2 \right. \\
&\quad \left. - u^2(\rho^{(n)} - m_3 N_3)\frac{m_3}{m_4}\frac{\partial\mu_4}{\partial N_4} - u^2(2m_3 N_3 - \rho^{(n)})\frac{\partial\mu_3}{\partial N_4}\right], \\
R(u) &= \frac{3}{p_0}\frac{m^*}{m_4}\frac{u}{u^2 - s_4^2}\left[(\rho^{(n)} - m_3 N_3)\frac{1}{m_4}\frac{\partial\mu_4}{\partial N_4} + N_3\frac{\partial\mu_3}{\partial N_4}\right].
\end{aligned} \tag{3.21}$$

Also, s_4 is the velocity of fourth sound of (3.4).

The solution of the dispersion equation (3.20) $u \sim s_0$ amounts to the velocity u_1 of high-frequency first sound in the solution:

$$u_1^2 = s_1^2 - c\frac{p_0^2}{3M^2}\frac{m_4}{M}\left(\alpha_1 + \frac{\delta m}{m_4}\right)\left[\alpha_1 + \frac{\delta m}{m_4} + \frac{8}{5}\left(s_0\frac{Ma}{\pi\hbar}\right)^2(7\ln 2 - 1)\left(\alpha_1 - \frac{m_3}{m_4}\right)\right]. \tag{3.22}$$

Thus the velocity of first sound of a given frequency near absolute zero proves smaller than at higher temperatures (at which a low-frequency hydrodynamic limit exists) by an amount of the order of $(p_0/p_c)^5$, as determined by Eq. (3.22). In other words, the tempera-

ture-dependence of the velocity of first sound should show a maximum.

2) High-frequency vibrations in a magnetic field. Application of an external magnetic field changes the pattern of propagation of spin vibrations qualitatively. As always, in determining the dispersion law of spin waves in a magnetic field, we shall restrict the treatment to first-order perturbation theory. Here the expressions (2.16) and (3.12), (3.13) for $n_{\alpha\beta}$ and $\varepsilon_{\alpha\beta}$ are considerably simplified, and the kinetic equation (3.9) breaks down into the scalar component

$$(\omega - \mathbf{k}\cdot\mathbf{v})\nu - \delta_+\mathbf{k}\cdot\mathbf{v}\frac{\omega}{\mathbf{k}\cdot\mathbf{v}}\frac{\delta m}{M}\mathbf{p}\cdot\mathbf{v}_s + \frac{\partial\varepsilon_0}{\partial N_4}\delta N_4 + 2\psi\int\mathbf{v}\,d\Gamma\big]$$
$$+ \mathbf{k}\cdot\mathbf{v}\delta_-\big[H\frac{\partial B}{\partial N_4}\delta N_4 - 2\zeta\mathfrak{M}\int\lambda\,d\Gamma\big] = 0, \tag{3.23}$$

and the vector component

$$(\omega - \mathbf{k}\cdot\mathbf{v})\lambda - (\mathbf{k}\cdot\mathbf{v})\mathfrak{M}\delta\big[\frac{\omega}{\mathbf{k}\cdot\mathbf{v}}\frac{\delta m}{M}\mathbf{p}\cdot\mathbf{v}_s + \frac{\partial\varepsilon_0}{\partial N_4}\delta N_4 + 2\psi\int\mathbf{v}\,d\Gamma\big]$$
$$+ \mathbf{k}\cdot\mathbf{v}\delta_+\big[H\frac{\partial B}{\partial N_4}\delta N_4 - 2\zeta\int\lambda\,d\Gamma\big] - \frac{2i}{\hbar}B\,[\mathbf{H}\lambda] - \frac{2i}{\hbar}\zeta\big[\rho\int\lambda\,d\Gamma\big] = 0. \tag{3.24}$$

Here we have $\mathbf{v} = 2\varepsilon_0/\partial\mathbf{p}$, $\delta_\pm = (1/2)[\delta(\varepsilon_+ - \mu_3) \pm \delta(\varepsilon_- - \mu_3)]$, and we have accounted for the fact that the *f*-function of (2.10) does not depend on the momenta in the first order in the interaction.

We shall take the direction of the magnetic field $\mathfrak{M}$ as the *z* axis. Equation (3.23) for the components of the vector λ perpendicular to the field direction determines the spectrum of transverse spin waves. Thus, for the circular components $\lambda_\pm = \lambda_x \pm i\lambda_y$, the dispersion equation for $T = 0$ leads to the form[14]

$$1 + \frac{p_0 a}{\pi\hbar}\big[w_+(u) + w_-(u)$$
$$+ \frac{p_0}{\hbar k}\big(-\frac{u^2-\chi_+^2}{u}w_+(u) + \frac{u^2-\chi_-^2}{u}w_-(u) + \frac{2}{u}\frac{N_+-N_-}{N_3}\big)\big] = 0. \tag{3.25}$$

Here we have

$$w_\pm(u) = \frac{u}{2}\ln\frac{u+\chi_\pm}{u-\chi_\pm} - \chi_\pm, \quad u = \frac{\omega-(2BH/\hbar)}{v_0 k}, \quad v_0 = p_0/M,$$

with $BH = \beta H - \zeta_0(N_+ - N_-)$. In the region of small wave vectors $u \gg \chi_\pm$ near the energy gap corresponding to free precession of the nuclear spins, the spectrum (3.25) is quadratic in **k**:

$$\omega = \frac{2\beta H}{\hbar} + \frac{\hbar k^2}{2M}\frac{(36\pi)^{1/3}}{10|a|}\frac{N_+^{5/3}-N_-^{5/3}}{(N_+-N_-)^2}. \tag{3.26}$$

Apart from small terms in the concentration, it resembles the spectrum of a ferromagnetic Fermi fluid in the absence of a field as obtained by Abrikosov and Dzyaloshinskiĭ,[54] while in weak fields ($\mathcal{H}\ll 1$), it goes over into the corresponding result of Silin.[47]

Here the Cherenkov damping of the spin wave is exponentially small[94]:

$$\omega'' \sim \frac{\omega_{int}}{kv_0}\exp\big[-\frac{T_0}{T}\big(\frac{\omega_{int}}{kv_0}\big)^2\big], \quad \omega_{int} = \frac{2\zeta_0}{\hbar}(N_+ - N_-).$$

The major dissipative mechanism is collisional relaxation absorption.

We should stress that, in a Fermi fluid that possesses a magnetic moment at equilibrium, the spectrum of transverse spin waves should also contain terms

quadratic in k that are due to the nonlocal nature of the interaction of the quasiparticles and to the deviation of the exact quantum kinetic equation from the quasiclassical equation (3.9). We can easily convince ourselves in the given case of a low-density system that an account for these effects would lead only to corrections to the second term on the right-hand side of Eq. (3.26) that are small in terms of the concentration.

In the short-wave region $kv \gg \zeta_0(N_+ - N_-)/\hbar$, the solution of Eq. (3.25) is exponentially close in terms of the parameter $-\pi\hbar/p_+ a$ to the asymptotic value $\omega - 2BH/\hbar = kv_+$. The region of existence of such a solution is restricted in frequency and magnon wave vector by the condition of applicability of the quasiclassical kinetic equation (3.9). The results derived for the damping of spin waves in the absence of a magnetic field in the previous section of the article can be directly extended to the case of short-wave ($kv \gg \omega_{int}$) transverse spin waves. The dispersion equation of the latter acquires a form analogous to (3.14).

Application of an external magnetic field **H** enables the propagation of long-wave weakly damped transverse spin waves, even in a nondegenerate solution with T at $T > T_0$. After calculating the integrals with the Boltzmann equilibrium distribution functions n_+ and n_-, the corresponding kinetic equation is transformed analogously[96] and yields the following dispersion equation for the transverse magnetization oscillations[94]:

$$\Omega + \big[1 - J\big(\frac{\Omega}{kv_T}\big)\big]\big(\frac{\mu_{int}}{2T}\Omega + \omega_{int}\big) - \omega_{int} = 0, \tag{3.27}$$
$$\mu_{int} = -\frac{2\pi a\hbar^2}{M}N_3.$$

Here we have $\Omega = \omega - 2\beta H/\hbar + \omega_{int}$, $v_T = (T/M)^{1/2}$ is the thermal velocity of the quasiparticles, and we define

$$J(x) = xe^{-x^2/2}\int_{i\infty}^{x}e^{z^2/2}\,dz = i\sqrt{\frac{\pi}{2}}\,x\,\text{erf}\,\frac{x}{\sqrt{2}}. \tag{3.28}$$

We shall seek the long-wave weakly damped solution of Eq. (3.27) in the form of a series expansion in powers of kv_T/ω_{int}. In the zero-order approximation, the solution $\omega = 2\beta H/\hbar \equiv \omega_0$ corresponds to uniform free precession of the nuclear spins of ^{3}He. By using the asymptotic representation of the function $J(x)$ in (3.28), we can find the corrections that are quadratic in kv_T/ω_{int} to the homogeneous solution $\omega = \omega_0$:

$$\omega = \omega_0 + \frac{\omega_{int}}{\omega_0^2}k^2 v_T^2 = \omega_0 + \frac{2\mu_{int}}{\hbar}k^2\frac{T}{M\omega_0^2}\tanh\frac{\beta H}{T}. \tag{3.29}$$

In (3.29) we have accounted for the fact that, in the Born approximation for the population difference $N_+ - N_-$ that enters into ω_{int}, one must employ its value in an ideal Boltzmann gas while neglecting the Fermi-fluid interaction: $N_+ - N_- = N_3\tanh(\beta H/T)$. The absorption coefficient ω'' of the wave is

$$\omega'' = \sqrt{\frac{\pi}{2}}\frac{\omega_{int}}{kv_T}\exp\big[-\frac{1}{2}\big(\frac{\omega_{int}}{kv_T}\big)^2 - \frac{\omega_{int}}{\omega_0}\big]. \tag{3.30}$$

When the wave vectors k are not too small, the Cherenkov damping of (3.30) can prove larger than the collisional damping $\omega''_{coll} \sim N_3 a^2 v_T$.

In the short-wave region in which the existence of a gap in the dispersion law is not essential, strong collisionless damping renders the propagation of a spin

wave in a nondegenerate solution impossible.

Thus the observation of spin waves in a ^{3}He–He II solution in the absence of a magnetic field requires extremely low temperatures. Application of a magnetic field enables the propagation of weakly damped transverse spin modes over a broad range of temperatures. Here an increase in temperature narrows the region of wave vectors in which undamped magnons exist. Observation of spin waves in the Boltzmann region is possible in rather strong magnetic fields.

As was shown above, the transverse spin waves do not involve the vibrations of the superfluid background. The equations (3.24) for the z component of the vector λ and (3.23) determine the velocity of propagation of coupled spin and high-frequency sound waves. We note that, in second-order perturbation theory, the coupling of the spin and sonic modes in a magnetic field also arises from the existence in the Fermi-fluid function (2.7) of a term linear in the spin operators $\phi(\sigma_{\alpha\beta}\delta_{\mu\nu} + \sigma_{\mu\nu}\delta_{\alpha\beta})\mathfrak{M}$.

Let us restrict the treatment to the approximation linear in the concentration. To do this, it suffices to keep in the energy ε_0 only the term independent of the momentum: $\varepsilon_0 \approx -\Delta$. To the same accuracy we have $\partial B/\partial N_4 = 0$. After unwieldy calculations, we find the transcendental dispersion equation for the velocity of propagation of coupled spin-sonic oscillations[14]:

$$1 - \tfrac{1}{2}\Phi(w_+ + w_-) + Z_0(\Phi - Z_0)w_+w_- = 0. \tag{3.31}$$

Here we have $\Phi(u) = Q(u) + uR(u)/v_0$, the functions $Q(u), R(u)$, and $w_\pm(u)$ are defined by expressions (3.21) and (3.25), and $Z_0 = -2\lambda$ is the zero-order harmonic of the spin component of the Fermi-fluid function.

The dispersion equation (3.31) always has a root of the order of the velocity s_0 of sound in pure ^{4}He, which determines the velocity of propagation of the high-frequency sound wave in the solution. In the employed approximation, it coincides with the value s_1 of hydrodynamic first sound of (3.3). The given mode amounts to an oscillation of the density of impurity atoms and an associated oscillation of the projection of the magnetic moment on the direction of the external field. A dependence of the velocity of propagation of the wave u_1 on the magnetic field arises in the higher orders with respect to the concentration of fermions.

In large magnetic fields for which $v_+ \gg v_-$ ($v_\pm = p_\pm/M$ are the velocities of the quasiparticles at the corresponding Fermi surfaces), we can easily convince ourselves that Eq. (3.31) has no real roots differing from $u = u_1$.

On the other hand, in very weak magnetic fields for which

$$\frac{v_+ - v_-}{v_0} \sim \frac{\beta H}{T_0} \ll \exp\left(-\frac{2}{Z_0}\right) \ll 1,$$

the solution u_H of Eq. (3.31) is known to exist. It is close to the velocity u_0 of propagation of spin waves in the solution in the absence of a field in (3.15)[14]:

$$u_H = u_0 - v_0\frac{Z_0}{\Delta_s}\left(\frac{\beta H}{4T_0}\right)^2 \exp\left(-\frac{2}{Z_0}\right), \quad \Delta_s = \frac{2+c\,(3M/Z_0m_4)\,(\alpha_1 p_0/p_0)^2}{1+c\,(3M/Z_0m_4)\,(\alpha_1 p_0/p_0)^2} \approx 1.03.$$

As the magnetic field is increased and attains a certain critical value H_c, the velocity of propagation of a spin wave becomes complex (here the real component of the velocity is smaller than the Fermi velocity v_+), and the wave suffers strong Landau damping involving decay of the magnon into a particle and a hole. The size of the critical field H_c is equal with logarithmic accuracy[14] to

$$\beta H_c \approx T_0\exp\left[-\frac{\Phi_0}{Z_0(\Phi_0 - Z_0)}\right],$$
$$\Phi_0 = -3m_4(s_0\alpha_1)^2\frac{c}{p_0v_0} < 0,$$

This corresponds to the numerical equation

$$H_c(\mathrm{Oe}) \approx 10^7 x^{2/3}\exp\left(-\frac{1.16}{x^{1/3}}\right).$$

Here $x = N_3/(N_3 + N_4) \equiv c/(1 + c)$ is the molar concentration.

A more general approach to the problem of suppression of the high-frequency modes by a magnetic field has been developed by the authors of Refs. 70 and 94 as applied to the case of transverse zero sound in pure ^{3}He. This method allows one to take exact account of all the harmonics of the f-function. This involves the closeness of the velocity of propagation of the wave to the Fermi velocity and the possibility of separating out the logarithmic singularities in the determinant of the dispersion equation. Here the critical field H_c is simply expressed in terms of the value of the velocity of the high-frequency modes in the absence of a field.

4. TRANSPORT PHENOMENA

a) Kinetic phenomena in the absence of a magnetic field

The kinetic properties of degenerate ^{3}He–He II solutions at low temperatures are governed exclusively by the the impurity excitations. The mean free path of the ^{3}He quasiparticles is limited by their mutual collisions. The values of the kinetic coefficients in the Fermi fluid can be determined by solving the kinetic equation by the method of Abrikosov and Khalatnikov.[1,71] The exact expressions for the coefficients of heat conductivity $\varkappa$, viscosity η, and spin diffusion D derived by Brooker and Sykes[72] have the following form in the absence of a field:

$$\left. \begin{aligned}
\varkappa T &= \frac{8}{3}\frac{\pi^2\hbar^2 p_0^3}{m^{*4}}\left\langle\frac{W_\varkappa(\theta,\varphi)}{\cos(\theta/2)}(1-\cos\theta)\right\rangle^{-1} H(\lambda_\varkappa), \\
\eta T^2 &= \frac{64}{45}\frac{\hbar^3 p_0^5}{m^{*4}}\left\langle\frac{W_\eta(\theta,\varphi)}{\cos(\theta/2)}(1-\cos\theta)^2\sin^2\varphi\right\rangle^{-1} C(\lambda_\eta), \\
DT^2 &= \frac{32}{3}\frac{\pi^2\hbar^4 p_0^3}{m^{*6}}(1+Z_0) \\
&\quad\times\left\langle\frac{2W_D(\theta,\varphi)}{\cos(\theta/2)}(1-\cos\theta)(1-\cos\varphi)\right\rangle^{-1} C(\lambda_D).
\end{aligned} \right\} \tag{4.1}$$

Here $W_\varkappa$, W_η, and W_D amount to the phenomenological probabilities of collisions of pairs of quasiparticles near the Fermi surface for the corresponding processes, θ is the angle between the directions of the momenta of the interacting particles, φ is the scattering angle in the center-of-inertia system, while the coefficients $H(\lambda_\varkappa)$, $C(\lambda_\eta)$, and $C(\lambda_D)$ are related to the collision probabilities W by rather unwieldy relationships[72]; $\langle\ldots\rangle$ denotes averaging over the angles.

To an accuracy of terms corresponding to p-scatter-

ing of the ^{3}He quasiparticles, we have[13]

$$W_\eta(\theta, \varphi) = W_\varkappa(\theta, \varphi) = 4W_D(\theta, \varphi)$$

$$= (2\pi\hbar)^3 \left(\frac{a}{M}\right)^2 \left[1 + \lambda\left(7 + \frac{1-5w^2}{w}\tanh^{-1}w\right)\right],$$

$$\lambda_\varkappa = \frac{1}{3} + 4.76\lambda, \quad \lambda_\eta = \frac{1}{5} + 2.23\lambda, \quad \lambda_D = \frac{1}{3} + 1.19\lambda,$$

$$H(\lambda_\varkappa) = 0.52(1 - 0.26\lambda),$$

$$C(\lambda_\eta) = 0.81(1 - 0.18\lambda); \quad C(\lambda_D) = 0.80(1 - 0.10\lambda).$$

The final results for the kinetic coefficients look like this[12]:

$$\left.\begin{array}{l} \varkappa T = \frac{1}{8\pi} \frac{p_0^3}{(Ma)^2} \cdot 0.52(1 - 0.34\lambda), \\[2mm] \eta T^2 = \frac{1}{12\pi^3} \frac{p_0^5}{(Ma)^2} \cdot 0.81(1 + 0.74\lambda), \\[2mm] DT^2 = \frac{1}{\pi}\left(\frac{\hbar}{M}\right)^3 \left(\frac{p_0}{a}\right)^2 \cdot 0.80(1 - 2.18\lambda). \end{array}\right\} \tag{4.2}$$

The characteristic relaxation time in the degenerate Fermi system $\tau = (8\pi^2\hbar^6/m^*T^2)\langle W_\eta/\cos(\theta/2)\rangle^{-1}$ has the following concentration-dependence:

$$\tau = \frac{1}{2\pi}\frac{\hbar}{M}\left(\frac{\hbar}{a}\right)^2 \frac{1}{T^2}(1 - 1.86\lambda). \tag{4.3}$$

A comparison of the calculated kinetic coefficients with the experimental data[18,19,74] is shown in Figs. 8 and 9. It shows satisfactory agreement between the theoretical and experimental results.

b) Magnetokinetic effects

The mean free path of quasiparticles having a spin directed along H increases with increasing magnetic field, since the efficient mode of scattering for these particles is that with particles with spins directed against the field, the number of which declines with increasing H. The corresponding magnetokinetic effects are manifested especially markedly in determining the kinetic coefficients, whose size is proportional to the mean free path of the particles.

In order to determine the kinetic coefficients, one must solve the collisional kinetic equation for the density matrix. As always happens for weakly inhomogeneous systems, one must set the local-equilibrium density matrix on the left-hand side of the kinetic equation (3.9) and linearize the equation with respect to small values of the gradients. When we choose the z axis along the direction of the magnetic field, the equilibrium density matrix and the Hamiltonian of the excitations (2.4)–(2.6), (2.16) are diagonal in spin space. Therefore two equations for the diagonal components of the density matrix $n_\pm$ arise in place of the four equations for $n_{\alpha\beta}$:
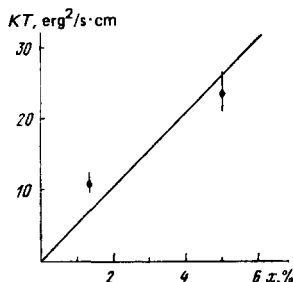

KT, erg^2/s·cm

FIG. 8. Concentration-dependence of the heat conductivity. The points are the experimental results of Ref. 19.

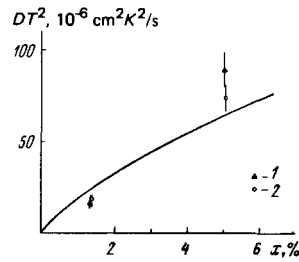DT^2, 10^{-6} cm^2K^2/s

FIG. 9. Concentration-dependence of the spin diffusion coefficient. 1—experimental data[18]; 2—results of Ref. 74.

$$\frac{\partial n^\pm}{\partial t} + \nabla n^\pm \frac{\partial \varepsilon_\pm}{\partial p} - \frac{\partial n^\pm}{\partial p}\nabla\varepsilon_\pm = I^\pm(p). \tag{4.4}$$

The diagonal elements of the density matrix $n_\pm$ and the excitation energies $\varepsilon_\pm$ determine the distribution function and the energy of particles having a definite projection of the spin on the direction of the field. In the primary approximation in terms of the concentration, the collision integrals entering into (4.4) have the usual form:

$$I^\pm(p_1) = -\int d^3p_2\, d\Gamma_1'\, d\Gamma_2'\, \delta(p_1 + p_2 - p_1' - p_2')\,\delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_1' - \varepsilon_2')$$
$$\times W\,[n_1^\pm n_2^\mp (1 - n_{1'}^\pm)(1 - n_{2'}^\mp) - n_{1'}^\pm n_{2'}^\mp (1 - n_1^\pm)(1 - n_2^\mp)]. \tag{4.5}$$

Here $W = (2\pi\hbar)^3(a/M)^2$ is the probability of scattering of the particles having the momenta $p_1^\pm$ and $p_2^\mp$ in the state $p_{1'}^\pm, p_{2'}^\mp$.

The procedure of determining the kinetic coefficients differs from the standard procedures in that, instead of a single kinetic equation for the particles at the Fermi surface $|p| = p_0$, there are the two kinetic equations (4.4), related via the collision integrals of (4.5), for the distribution functions $n_\pm$ of particles lying on Fermi spheres of differing radii $p_\pm$.

After unwieldy calculations for the hydrodynamic viscosity and heat conductivity, one gets the following expressions[14]:

$$\left.\begin{array}{l} \dfrac{\eta(\mathcal{H})}{\eta(0)} = \dfrac{\chi_+^2}{2d^3}\,\dfrac{2 + 5d^8 - 3d^{10}}{5 - 3d^8}, \\[3mm] \dfrac{\varkappa(\mathcal{H})}{\varkappa(0)} = \dfrac{\chi_+^2}{2d}\,\dfrac{4 + 3d^4 + d^6}{3 + d^6}, \quad d = \dfrac{\chi_-}{\chi_+}. \end{array}\right\} \tag{4.6}$$

The viscosity and heat conductivity in the absence of a magnetic field that enter into (4.6) are

$$\eta(0) = \frac{1}{12\pi^3}\frac{p_0^5}{(MaT)^2},$$

$$\varkappa(0) = \frac{1}{8\pi}\frac{p_0^3}{(Ma)^2\,T}.$$

They are calculated in the same way as $\eta(\mathcal{H})$ and $\varkappa(\mathcal{H})$ by the approximate method of Ref. 71, and differ from the exact values of (4.2) by factors of the order of unity. Figure 10 shows a graph of the functions $\eta(\mathcal{H})/\eta(0)$ and $\varkappa(\mathcal{H})/\varkappa(0)$ as given by the formulas of (4.6). In strong magnetic fields $\mathcal{H} \sim 1$, the viscosity and heat conductivity of (4.6) increase without limit:

$$\frac{\eta(\mathcal{H} \to 1)}{\eta(0)} \approx \frac{2^{5/3}}{5(1 - \mathcal{H})^{3/2}},$$

$$\frac{\varkappa(\mathcal{H} \to 1)}{\varkappa(0)} \approx \frac{4}{3(1 - \mathcal{H})^{1/2}}. \tag{4.7}$$

The limiting values of the kinetic coefficients are attained in a completely polarized solution at $\mathcal{H} \geq 1$. In
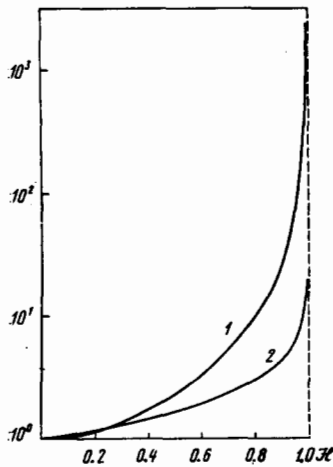
FIG. 10. Relative variation of the viscosity and the heat conductivity in a magnetic field.

FIG. 11. Graph of the function $V(z)$ from (4.10).

this case only one of the equations of (4.4) remains, and one can employ the exact expressions (4.1) to determine the viscosity and the heat conductivity. In these expressions one should substitute the probability $W(\theta, \varphi)$ of collision of the particles, which is related to the amplitude of p-scattering of (2.14):

$$W_\varkappa = W_\eta = \frac{\pi}{2\hbar}|\Gamma(p_F'; p_F)|^3 = \frac{\pi}{2\hbar}\left(\frac{24\pi b}{M}\right)^2 p_F^4 \sin^4\frac{\theta}{2}\cos^2\varphi. \qquad (4.8)$$

In this case, the coefficients $C(\lambda_\eta)$ and $H(\lambda_\varkappa)$ for the functions W of (4.8) prove to be $C(\lambda_\eta) = 0.79$ and $H(\lambda_\varkappa) = 0.55$. Finally we have

$$\eta T^2 = \frac{7}{\pi}\left(\frac{\hbar^3}{24\pi b M}\right)^2 p_F \cdot 0.79 \quad \varkappa T = \frac{35\pi}{6}\left(\frac{\hbar^3}{24\pi b M}\right)^2 \cdot \frac{0.55}{p_F}, \qquad (4.9)$$

That is, they change substantially in their concentration-dependence and differ from their values in the absence of a field $\eta(0)$, $\varkappa(0)$ of (4.2) by a large factor of the order of

$$\frac{a^4}{b^4}(N_3 a^3)^{-4/3} \gg 1.$$

We have assumed above that the entire Fermi component of the solution is degenerate, including the system of excitations of ³He with spins oriented against the direction of the field. Here, in the field $2\beta H = T_F$, the number of particles with spins antiparallel to the field vanishes. This corresponds to an infinite increase in the kinetic coefficients of (4.6)–(4.7) as $\mathscr{H} \to 1$.

Actually, there is always a certain number of fermions with spins directed against the field at nonzero temperature. An account for this fact yields a substantial difference of the kinetic coefficients in strong magnetic fields $\mathscr{H} \gtrsim 1$ from the values given by Eqs. (4.7) and (4.9)[75]:

$$\frac{\eta(\mathscr{H})}{\eta(0)} = \frac{2^{5/3}}{5}\left(\frac{T_F}{T}\right)^{3/2} V(z), \quad z = (1-\mathscr{H})T_F/T. \qquad (4.10)$$

Figure 11 shows a graph of the function $V(z)$:

$$V(z) = \frac{\pi^2}{4}\left[\int_0^\infty \frac{t^{3/2}\,dt}{e^{t-z}+1}\frac{e^{t-z}}{e^{t-z}-1}\ln\frac{1+e^t}{1+e^z}\right]^{-1}.$$

Equation (4.10) holds for a degenerate solution at $T_F \gg T$ for almost complete polarization of the solution $(T_F/T)^{3/2}V(z) \gg 1$.

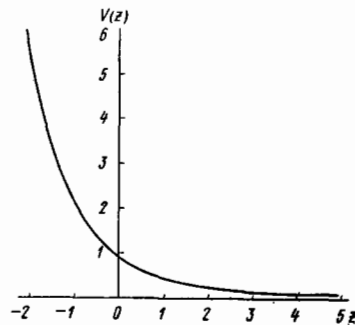The dimensionless parameter $z = (1-\mathscr{H})T_F/T$ actually

characterizes the distribution function $n^-(\mathbf{p})$. When $z \gg 1$, the system of quasiparticles with spins directed against the field is degenerate. Since $V(z \gg 1) = z^{-3/2}$, Eq. (4.10) coincides with (4.7). In stronger fields $(-z) \gg 1$, there is an exponentially small number of particles with reversed spins, and the function n^- is a Boltzmann function. Here the hydrodynamic viscosity of (4.10) increases exponentially with increasing magnetic field: $V(z \to -\infty) \approx 0.70\exp(-z)$. Thus, with increasing temperature, the increase in the kinetic coefficients as functions of the field becomes slower, and the asymptotic value of (4.9) is attained in stronger fields.

The magnetokinetic effects (increase in the kinetic coefficients of the solution with increasing magnetic field) can also be observed in a nondegenerate solution. This requires that the Fermi component should prove to be sufficiently polarized to make s-scattering ineffective, i.e., $\beta H \gg T$. In the Boltzmann region $T \gg T_F$,[75] we have

$$\frac{l_+(H)}{l(0)} = \frac{\exp(2\beta H/T)+1}{2}. \qquad (4.11)$$

In strong magnetic fields with $\beta H \gg T$, the viscosity of the solution increases in the same way as the mean free path l_+ in (4.11) of the quasiparticles having spins directed along the field: $\eta(H) = \eta(0)\exp(2\beta H/T)/2$. The limiting value of l_+ in strong fields and in the Boltzmann region is determined by p-scattering: $l_+(\infty)/l_+(0) \sim (\theta/T)^2(\theta/Ms_0^2)^2$. This value is reached in fields $\beta H \sim T\ln(\theta^2/MTs_0^2)$.[75] (Here θ is the Debye temperature of ⁴He.) Although the mean free path of the impurity excitations increases strongly upon magnetizing a dilute solution, the scattering of the ³He quasiparticles by thermal phonons in the strongly polarized solution is almost always inconsequential.[75-78]

The fundamental experimental difficulty in detecting magnetokinetic effects is that of ensuring a high enough degree of polarization of the spin system, which is governed by the relationship between the degeneracy temperature $(T_F[\mathrm{K}] \approx 4.2 x^{2/3})$, the magnetic field $2\beta H$ $(2\beta = 0.16 \text{ [mk/kOe]})$, and the temperature. A field $H \sim 75$ kOe at a low enough temperature $(2\beta H \sim 12$ mK) can polarize a solution having a concentration down to $x \sim 10^{-4}$ $(T_F \sim 8$ mK). Here the mean free path of the ³He atoms increases by a factor of 10^5 and reaches tens of centimeters. The values of the kinetic coefficients increase by just as great a factor. Thus the hydrodynamic viscosity increases to a value of $\sim 10^{-2}$ Ps, which corresponds to the viscosity of water. For the system of quasiparticles, the fundamental factor may prove to be scattering by the walls. It becomes

possible to observe various effects of nonlocality characteristic of a Knudsen regime. In particular, thermal creep or the radiometric effect can be exhibited, while a tube of centimeter diameter can serve as a superleak that transmits only the superfluid component.

Such a considerable increase in relaxation time implies considerable expansion of the region of existence of weakly damped high-frequency modes $\omega\tau \gg 1$. This corresponds to the condition $\omega \gg 5\,(10x^{1/3})^4 T^2$, where T is the temperature in mK. Owing to the converse condition, undamped hydrodynamic vibrations of a strongly polarized solution are actually quasistatic.

5. SUPERFLUIDITY OF ^{3}He IN SOLUTION

a) Transition temperature of ^{3}He to the superfluid state

Thus far we have been treating the ^{3}He impurity subsystem as a normal isotropic Fermi fluid. In the case of attraction between the dissolved atoms, a possibility exists in principle of rearrangement of the spectrum of Fermi excitations owing to Bardeen-Cooper-Schrieffer (BCS) pairing and of transition of the Fermi component to the superfluid state. By using a model potential for the interaction of the impurity quasiparticles, Bardeen, Baym, and Pines[21,8] have estimated the temperature T_c of transition of ^{3}He in solution to the superfluid state for singlet pairing. They predicted[21] that T_c is of the order of 10^{-6} K at zero pressure, while declining with increasing pressure. Conversely, employment of an empirical potential with five adjustable parameters[60] qualitatively alters this result, while elevating the transition temperature under pressure to $T_c \sim 10^{-4}$ K. The possibility of a superfluid transition in concentrated solutions at high pressures owing to triplet pairing of the impurity fermions has also been discussed.[79,80] Fay[97] has also performed some interesting model calculations of T_c.

According to all the experimental data, the s-scattering length a is negative, which corresponds to attraction between the impurity atoms. Therefore one can easily find the temperature of superfluid transition involving s-pairing of the quasiparticles[11] by using the results of Gor'kov and Melik-Barkhudarov[81]:

$$T_c = \frac{\gamma}{\pi}\left(\frac{2}{e}\right)^{7/3} T_s \exp\left(-\frac{1}{2|\lambda|}\right). \qquad (5.1)$$

Here $\ln\gamma = C$ is Euler's constant. At zero pressure, Eq. (5.1) is equivalent to the following numerical expression[82]:

$$T_c \, [\text{K}] = 0.71 x^{2/3} \exp\left(-\frac{1.20}{x^{1/3}}\right).$$

For a solution of molar concentration $x = 0.03$, this corresponds to a temperature $T_c \approx 1.4$ mK.

In contrast to the electronic Fermi fluid in a metal, the application of the BCS theory to degenerate ^{3}He–^{4}He

TABLE I.

P, atm	M/m_3 (m_3 is the mass of a ^{3}He atom, M is the effective mass)	V, cm^3 · mole^{-1} (molar volume)	a, Å
0	2.33	27.580	−1.5
10	2.57	25.180	−1.7
20	2.85	23.744	−2.4

solutions is not a model approximation. For a sufficiently dilute solution, it gives an exact description of the superfluid transition. Application of the formulas of the theory of superconductivity (see, e.g., Ref. 55) enables us to determine the jump in heat capacity of the solution at the phase-transition point:

$$C_s - C_n = 1.95\, x \exp\left(-\frac{1.20}{x^{1/3}}\right), \quad \frac{C_s(T_c)}{C_n(T_c)} = 2.43.$$

It also yields the value of the energy gap in the spectrum of Bogolyubov excitations $\Delta(T)$ at zero temperature: $\Delta(0) = 1.75\, T_c$. Here the coherence length $\xi = r_0 T_0 / \Delta(0)$ considerably exceeds the mean distance between dissolved atoms $r_0 \sim \hbar/p_0$. In a sufficiently dilute solution, one can expect manifestation of an analog of the Josephson effect. Experimentally, one can observe the transition of ^{3}He to the superfluid state, e.g., by measuring the magnetic susceptibility, in calorimetric experiments, and from resonance absorption of ultrasound at the frequency of the energy gap.

Since the parameter λ enters into the thermodynamic functions in the form of small corrections to the contribution of an ideal Fermi gas, one can determine most accurately the pressure-dependence of the s-scattering length from experimental data on the kinetic coefficients of the solution under pressure.[82] By using the data of Murdock et al.[74] for a solution with $x = 1.3 \times 10^{-2}$, we find the value of a at a pressure $P = 10$ atm (see Table I). At a pressure $P = 20$ atm and concentration $x = 1.3 \times 10^{-2}$, the power-series expansion in λ is only qualitatively valid. Hence the value of a at $P = 20$ atm was determined from the data of Ahonen et al.[63] on the magnetic susceptibility of a solution having $x = 0.27 \times 10^{-2}$ by employing Eq. (2.26). Figure 12 compares the susceptibility calculated by Eq. (2.26) with account of the data of Table I with the experimental results.[63] The dotted lines in Figs. 12 and 13 correspond to calculations in the region where the expansion in the parameter λ is extrapolative. When $P = 10$ atm and $x = 1.32 \times 10^{-2}$ (where the λ-expansion is valid), the osmotic pressure calculated within the framework of the presented scheme (2.24) attains a value of 1.51 Torr. This agrees satisfactorily with the measured value 1.59 Torr.[60] Upon using the data on the susceptibility of the solution for $x = 1.33 \times 10^{-2}$,[63] while accounting for the concentration-dependence of the total effective mass of
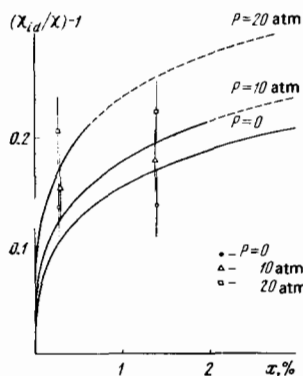

FIG. 12. Comparison of the experimental data on the magnetic susceptibility of dilute solutions under pressure[63] with the results calculated by Eq. (2.26) with the aid of Table I.

$$P = -E_0 + \mu_1 \rho_1 + \mu_2 \rho_2 + TS. \qquad (5.5)$$
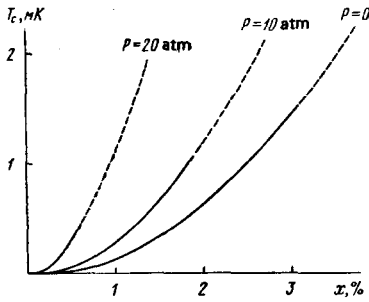
(Figure at top left)

FIG. 13. Concentration-dependence of the transition temperature of ^{3}He in dilute ^{3}He-He II solutions to the superfluid state at pressures of 0, 10, and 20 atm.

the excitation of (2.23), the relationship $\partial(\ln M)/\partial P = 1.5 \times 10^{-2}$ atm^{-1},[57] and the temperature corrections, we find the following relationship for low pressures: $\partial \ln a / \partial \ln P = 0.12$. Figure 13 shows the concentration-dependence of the transition temperature of ^{3}He in a solution to the superfluid state[82] at pressures of 0, 10, and 20 atm [see Eq. (5.1) and Table I]. The authors[63] cooled a solution having $x = 1.33 \times 10^{-2}$ to a temperature of 2.4 mK and observed no phase transition. In line with Fig. 13, the latter should occur at $T_c \approx 0.5$ mK ($P = 10$ atm). Since T_c depends exponentially on a, the accuracy of the obtained results strongly depends on the accuracy of the employed experimental data. Yet we can hope that the calculations give the correct scale of all the quantities for the treated phenomena.

b) Equations of three-velocity hydrodynamics

At low enough temperature $T < T_c$, the solution amounts to a macroscopic system in which two forms of condensate exist simultaneously, and correspondingly two forms of superfluid motion. The properties of such a liquid must be described by the equations of three-velocity hydrodynamics with two superfluid velocities and one normal velocity[83, 84]:

$$\begin{aligned}
&\dot{\rho}_1 + \text{div}(\rho_1 v_n + p_1) = 0, \quad \dot{\rho}_2 + \text{div}(\rho_2 v_n + p_2) = 0, \\
&\dot{j}_i + \frac{\partial \Pi_{ik}}{\partial x_k} = 0, \quad \dot{S} + \text{div} S v_n = 0, \\
&\dot{v}_1 + \nabla\left(\mu_1 - \frac{v_n^2}{2} + v_n v_1\right) = 0, \\
&\dot{v}_2 + \nabla\left(\mu_2 - \frac{v_n^2}{2} + v_n v_2\right) = 0, \\
&\text{curl } v_1 = \text{curl } v_2 = 0.
\end{aligned} \qquad (5.2)$$

Here ρ_1 and ρ_2 are the densities of the particles of each type (the total density of the solution is $\rho = \rho_1 + \rho_2$); v_n, v_1, and v_2 are respectively the velocities of the normal and the two superfluid motions; $j = \rho v_n + p_1 + p_2$ is the momentum per unit volume; and S is the entropy per unit volume. The meaning of the quantities μ_1, μ_2, p_1, and p_2 is defined by the thermodynamic identity for the energy per unit volume of the solution E_0 in a system of coordinates in which $v_n = 0$:

$$dE_0 = TdS + \mu_1 d\rho_1 + \mu_2 d\rho_2 + p_1 d(v_1 - v_n) + p_2 d(v_2 - v_n). \quad (5.3)$$

Here the energy in the laboratory system is

$$E = \tfrac{1}{2} \rho v_n^2 + (p_1 + p_2) v_n + E_0. \qquad (5.4)$$

The momentum flux tensor has the form

All the formulas that we have written thus far differ from the corresponding formulas of two-velocity hydrodynamics only in the trivial doubling of the number of terms describing the relative motion of the superfluid and normal components. However, an essential feature appears upon introducing quantities that play the role of the densities of the superfluid components.[85]

Since the effective mass M of the ^{3}He quasiparticles is 2.3 times as large as the mass m_3 of a ^{3}He atom, the motion of the quasiparticles transports a mass of ^{4}He that is not at all small in addition to the mass of ^{3}He. Cooper pairs are produced by the weak interaction between the fermions. They amount to a bound state of the quasiparticles, whose properties are hardly altered by the onset of superfluidity. Therefore it is evident that the superfluid motion of ^{3}He must also be accompanied by transport of a mass of ^{4}He. If the velocities of motion are not too great, then the relative momenta p_1 and p_2 can be expanded in powers of the small relative velocities $v_1 - v_n$ and $v_2 - v_n$ with constant coefficients:

$$p_1 = \rho_{11}^{(s)} (v_1 - v_n) + \rho_{12}^{(s)} (v_2 - v_n),$$

$$p_2 = \rho_{21}^{(s)} (v_1 - v_n) + \rho_{22}^{(s)} (v_2 - v_n). \qquad (5.6)$$

The set of these coefficients $\rho_{\alpha\beta}^{(s)}$ ($\alpha, \beta = 1, 2$) is an analog of the density of the superfluid component in two-velocity hydrodynamics. As we see from the thermodynamic identity (5.3), the quantities $\rho_{\alpha\beta}^{(s)}$ are equal to the second derivatives of the energy E_0 with respect to the relative velocities. Therefore the matrix $\rho_{\alpha\beta}^{(s)}$ is symmetric, i.e., we have

$$\rho_{12}^{(s)} = \rho_{21}^{(s)}. \qquad (5.7)$$

We can say that three-velocity hydrodynamics manifests three independent quantities $\rho_{11}^{(s)}$, $\rho_{22}^{(s)}$, and $\rho_{12}^{(s)}$ in place of the density of the superfluid component. The last of these describes the above-mentioned effect of entrainment by either of the superfluid motions of both components of the solution.

The relationships (5.3) and (5.6) enable us easily to find the dependence of all the thermo- and hydrodynamic quantities on the relative velocities. Thus, the energy E of the solution in the laboratory system of coordinates is[85]

$$E = \tfrac{1}{2} (\rho^{(n)} v_n^2 + \rho_{11}^{(s)} v_1^2 + 2\rho_{12}^{(s)} v_1 v_2 + \rho_{22}^{(s)} v_2^2) + E^{(0)} (S, \rho_1, \rho_2). \qquad (5.8)$$

Here E^0 is the energy of the solution while at rest, and the quantity $\rho^{(n)} = \rho - \rho_{11}^{(s)} - \rho_{22}^{(s)} - 2\rho_{12}^{(s)}$ plays the role of the density of the normal component. In order that the kinetic component of the energy of the solution be positive definite, as we see from (5.8), we also require the condition $\rho_{12}^{(s)2} < \rho_{11}^{(s)} \rho_{22}^{(s)}$, in addition to the positivity of $\rho_{11}^{(s)}$, $\rho_{22}^{(s)}$, and $\rho^{(n)}$. Yet the quantity $\rho_{12}^{(s)}$ itself can in principle be either positive or negative.

Equation (5.2) together with the conditions (5.6) and (5.8) constitute a complete system of equations of hydrodynamics if we know the thermodynamic functions of

172 Sov. Phys. Usp. 23(2), Feb. 1980

E. P. Bashkin and A. É. Meĭerovich 172

the solution and all three superfluid densities.

Moreover, we must stress the fact that there is an ambiguity in the definition of the corresponding velocities when two superfluid motions exist. In principle, one could define each of the velocities as the ratio of mass flux of particles of a given type to their density. Then the above-discussed entrainment effect would be lacking by definition. However, an essential point is that with such a definition, superfluid motion is not a potential one. Nevertheless the condition of potentiality plays a very important role in formulating the complete system of hydrodynamic equations. In preserving the potentiality condition, we are obliged to account for the possible transport of both components of the solution by each of the superfluid motions.

On the basis of the BCS theory, we can easily give a microscopic derivation of the formulas for the mass fluxes and thus calculate the quantities $\rho_{\alpha\beta}^{(s)}$[85]:

$$
\left.
\begin{array}{c}
\rho_{11}^{(s)} = \frac{m_3^2}{M} N_s, \quad \rho_{12}^{(s)} = \rho_{21}^{(s)} = \frac{m_3}{M}(M - m_3) N_s; \\
\rho_{22}^{(s)} = \rho_2 - (M - m_3) N_n - \frac{m_3}{M}(M - m_3) N_s, \quad \rho^{(n)} = M N_n, \\
N_n = N_3 - N_s, \\
N_s = N_3 \left[1 - \frac{T \partial \ln \Delta(T)}{\partial T} \right]^{-1}.
\end{array}
\right\}
\tag{5.9}
$$

Here N_s is the number of "superfluid" ^{3}He atoms, while the size $\Delta(T)$ of the energy gap is determined by usual formulas of the BCS theory. At zero temperature, the density of the normal component vanishes, while the total mass flux proves to be $\mathbf{j} = \rho_2 \mathbf{v}_2 + m_3 N_3 \mathbf{v}_1$. We note that in dilute solutions the inequality $\rho_{12}^{(s)1/2} < \rho_{11}^{(s)} \rho_{22}^{(s)}$ is satisfied with much room to spare, since $\rho_{12}^{(s)}$ and $\rho_{11}^{(s)}$ are proportional to the concentration, while $\rho_{22}^{(s)}$ does not contain this small factor.

The existence of the effect of entrainment of both components of the solution by each of the superfluid motions is also confirmed by detailed calculations by the diagram method of the properties of a system with two Bose condensates at $T = 0$.[86]

The equations of three-velocity hydrodynamics (5.2) admits the existence of sound waves of three types.[83-85,87] Let us determine the velocities of the sound vibrations of a superfluid solution. One can easily do this by employing the linearized equations of (5.2) upon considering the fact, as we shall see from the result, that the velocities of the three types of sound vibrations satisfy the condition $c_1 \gg c_2 \gg c_3$. Here c_2 has a power degree of smallness with respect to the solution, while c_3 has an exponential degree of smallness (just like the transition temperature of the Fermi component to the superfluid state). Without taking time for the simple calculations, we shall present the results.[85]

The velocity of vibrations of the first type is determined in the usual way in terms of the compressibility of the solution

$$
c_1^2 = \frac{\partial P}{\partial \rho}.
$$

It depends weakly on the temperature and is close to the velocity of first sound in pure ^{4}He.

The vibration of the second type amounts to an oscil-

lation of the concentration of the solution. The corresponding velocity is determined by the formula

$$
c_2^2 = \frac{\rho^{(n)} \rho_{11}^{(s)} + (\rho_1 - \rho_{11}^{(s)} - \rho_{12}^{(s)})^2}{\rho^2 \rho^{(n)}} \left(\frac{\partial^2 E_0}{\partial x^2} \right)_{P,T} \approx \frac{v_3^2}{3}. \tag{5.10}
$$

Here $x \rho_1/(\rho_1 + \rho_2)$ is the concentration. Thus the vibrations of the second type amount to nothing other than the well-known[88-90] collective vibrations of the superfluid Fermi gas of the impurity ^{3}He particles.

The vibrations of the third type are primarily an oscillation of the temperature. In this sense they are analogous to second sound in an ordinary superfluid liquid. The square of the velocity of the vibrations is

$$
c_3^2 = \frac{TS^2}{C} \frac{\rho_{11}^{(s)}}{\rho^{(n)} \rho_{11}^{(s)} + (\rho_1 - \rho_{11}^{(s)} - \rho_{12}^{(s)})^2} = \frac{TS^2}{C} \frac{N_s}{MN_3 N_n}. \tag{5.11}
$$

This also is highly reminiscent of the expression for the velocity of second sound; here $C = T \partial S/\partial T$ is the heat capacity per unit volume of the solution. At the point $T = T_c$ of the transition of the Fermi component to the superfluid state, c_3 falls to zero in proportion to $(T_c - T)^{1/2}$. When $T \ll T_c$ it is proportional to the first power of the temperature: $c_3 = (9/2\pi)^{1/4} (T/P_0)$.

The formulas (5.9) cease to hold at very low temperatures at which the fundamental contribution to the temperature-dependence of the thermodynamic quantities and the superfluid densities does not come from Bogolyubov excitations, but from phonons. In this case there are two types of phonons corresponding to the first two types of sound vibrations of the solution. However, the fundamental contribution to the temperature-dependence of all the quantities comes from the phonons of second sound, since their velocity is much smaller[85] (for more details, see Ref. 91).

Here the waves of third sound amount to sound vibrations in the phonon gas of second sound, and their velocity of propagation is $c_3 = c_2/\sqrt{3} = v_0/3$. We should stress that the frequency of the third-sound vibrations in the studied temperature range should be very low, since the wavelength should considerably exceed the mean free path of the thermal excitations, which is extremely large at such low temperatures.

The hydrodynamic and high-frequency vibrations in a solution of a superfluid Fermi fluid in a superfluid Bose fluid have also been studied by Volovik, Mineev, and Khalatnikov[92] by using the kinetic equation and the phenomenological f-function.

Many of the obtained results can also be employed for other types of Fermi fluids of low density in which the interaction between the bare particles declines rapidly enough with increasing distance between them (e.g., for a screened Coulomb potential).

As we see it, it would be most interesting to study Fermi systems having a high degree of spin polarization. A considerable polarization of the spin system can be attained, not only by using an external magnetic field, but also by many other methods. An example of a fully polarized low-density Fermi fluid is the ferromagnetic phase of crystalline ^{3}He,[93] in which the mean free path of the vacancions proves to be quite considerable. For electronic systems, one can create a non-

uniform occupancy of the spin states, e.g., by methods of injection of spins having a definite orientation. A recently developed method[98-100] of magnetic polarization of liquid ^{3}He is also of considerable interest. For such dense Fermi systems, some of the results obtained above are also rigorous.

[1] I. M. Khalatnikov, Teoriya sverkhtekuchesti (Theory of Superfluidity), Nauka, M., 1971.

[2] B. N. Esel'son, V. N. Grigor'ev, V. G. Ivantsov, É. Ya. Rudavskiĭ, D. G. Sanikidze, and I. A. Serbin, Rastvory kvantovykh zhidkosteĭ He^3–He^4 (Solutions of ^{3}He–^{4}He Quantum Liquids), Nauka, M., 1973.

[3] J. Wilks, Liquid and Solid Helium, University Press, Oxford, 1967.

[4] W. E. Keller, Helium-3 and Helium-4, Plenum Press, New York, 1969.

[5] V. P. Peshkov, Usp. Fiz. Nauk **94**, 607 (1968) [Sov. Phys. Usp. **11**, 209 (1968)].

[6] K. W. Taconis and R. DeBruyn Ouboter, in Progress in Low Temperature Physics, North-Holland, Amsterdam, 1964, Vol. 4, p. 38.

[7] C. Ebner and D. O. Edwards, Phys. Rep. **2C**, 77 (1971).

[8] J. C. Wheatley, Am. J. Phys. **36**, 181 (1968).

[9] R. Radebaugh, NBS Techn. Note **362**, 105 (1968).

[10] G. Baym and C. Pethick, in The Properties of Liquid and Solid Helium, eds. K. H. Bennemann and J. B. Ketterson, Wiley, New York, 1978, Vol. 2, p. 123.

[11] E. P. Bashkin, Pis'ma Zh. Eksp. Teor. Fiz. **25**, 3 (1977) [JETP Lett. **25**, 1 (1977)].

[12] E. P. Bashkin, Zh. Eksp. Teor. Fiz. **73**, 1849 (1977) [Sov. Phys. JETP **46**, 972 (1977)].

[13] E. P. Bashkin and A. É. Meĭerovich, Pis'ma Zh. Eksp. Teor. Fiz. **26**, 696 (1977) [JETP Lett. **26**, 534 (1977)].

[14] E. P. Bashkin and A. É. Meĭerovich, Zh. Eksp. Teor. Fiz. **74**, 1904 (1978) [Sov. Phys. JETP **47**, 992 (1978)].

[15] E. P. Bashkin and A. É. Meĭerovich, Usp. Fiz. Nauk **125**, 360 (1978) [Sov. Phys. Uspekhi **21**, 542 (1978)].

[16] D. O. Edwards, D. F. Brewer, P. Seligmann, M. Skertic, and M. Yaqub, Phys. Rev. Lett. **15**, 773 (1965).

[17] A. C. Anderson, W. R. Roach, R. E. Sarwinski, and J. C. Wheatley, ibid. **16**, 263 (1966).

[18] A. C. Anderson, D. O. Edwards, W. R. Roach, R. E. Sarwinski, and J. C. Wheatley, ibid. **17**, 367 (1966).

[19] W. R. Abel, R. T. Johnson, J. C. Wheatley, and W. Zimmermann, ibid. **18**, 737 (1967).

[20] J. Bardeen, G. Baym, and D. Pines, ibid. **17**, 372 (1966).

[21] J. Bardeen, G. Baym, and D. Pines, Phys. Rev. **156**, 207 (1967).

[22] L. J. Campbell, Phys. Rev. Lett. **19**, 156 (1967).

[23] J. M. J. Van Leeuwen and E. G. D. Cohen, Phys. Rev. **176**, 385 (1968).

[24] V. J. Emery, ibid. **148**, 138 (1966); **161**, 194 (1967); **175**, 251 (1968).

[25] G. Baym, Phys. Rev. Lett. **17**, 952 (1966).

[26] C. W. Woo, H. T. Tan, and W. E. Massey, Phys. Rev. **185**, 287 (1969).

[27] T. B. Davison and E. Feenberg, ibid. **178**, 306 (1969).

[28] J. P. Hansen and D. Schiff, Phys. Rev. Lett. **23**, 1488 (1969).

[29] W. L. McMillan, Phys. Rev. **182**, 299 (1969).

[30] E. Ostgaard, Phys. Rev. **A1**, 1048 (1970).

[31] C. Ebner, Phys. Rev. **156**, 222 (1967); **185**, 392 (1969).

[32] G. Baym and C. Ebner, ibib. **170**, 346 (1968).

[33] S. G. Eckstein, Y. Eckstein, C. G. Kuper, and A. Ron, Phys. Rev. Lett. **25**, 97 (1970).

[34] Y. Disatnik and H. Brucker, J. Low Temp. Phys. **7**, 491 (1972).

[35] I. M. Khalatnikov, Pis'ma Zh. Eksp. Teor. Phys. **5**, 288 (1967) [JETP Lett. **5**, 235 (1967)].

[36] I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. **5**, 1919 (1969) [Sov. Phys. JETP **28**, 1014 (1969)].

[37] L. D. Landau, Zh. Eksp. Teor. Fiz. **30**, 1058 (1956) [Sov. Phys. JETP **3**, 920 (1956)]; Sobranie sochineniĭ (Collected Papers), Nauka, M., 1969, Vol. 2, p. 328.

[38] L. D. Landau, Zh. Eksp. Teor. Fiz. **32**, 59 (1957) [Sov. Phys. JETP **5**, 101 (1957)]; Sobranie sochineniĭ (Collected Papers), Nauka, M., 1969, Vol. 2, p. 337.

[39] L. D. Landau, Zh. Eksp. Teor. Fiz. **35**, 97 (1958) [Sov. Phys. JETP **8**, 70 (1959)]; Sobranie sochineniĭ (Collected Papers), Nauka, M., 1969, Vol. 2, p. 363.

[40] L. D. Landau and I. Ya. Pomeranchuk, Dokl. Akad. Nauk SSSR **59**, 669 (1948); Sobranie sochineniĭ (Collected Papers), Nauka, M., 1969, Vol. 2, p. 35.

[41] I. Ya. Pomeranchuk, Zh. Eksp. Teor. Fiz. **19**, 42 (1949).

[42] L. D. Landau and E. M. Lifshits, Kvantovaya mekhanika (Quantum Mechanics), Nauka, M., 1974 (Engl. Transl., Pergamon, Oxford-New York, 1977).

[43] K. Huang and C. N. Yang, Phys. Rev. **105**, 767 (1957).

[44] T. D. Lee and C. N. Yang, ibid. **105**, 1119 (1957).

[45] A. A. Abrikosov and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. **33**, 1154 (1957) [Sov. Phys. JETP **6**, 888 (1958)].

[46] H. H. Fu and C. J. Pethick, Phys. Rev. **B14**, 3837 (1976).

[47] V. P. Silin, Zh. Eksp. Teor. Fiz. **33**, 1227 (1957) [Sov. Phys. JETP **6**, 945 (1958)].

[48] E. A. Lynton and H. A. Fairbank, Phys. Rev. **80**, 1043 (1950).

[49] J. R. Pellam, ibid. **99**, 1327 (1955).

[50] T. R. Roberts, R. H. Sherman, and S. G. Sydoriak, J. Res. NBS, Ser. A **68**, 567 (1964).

[51] W. E. Massey, C. W. Woo, and H. T. Tan, Phys. Rev. **A1**, 519 (1970).

[52] N. R. Brubaker, D. O. Edwards, R. E. Sarwinski, P. Seligmann, and R. A. Sherlock, Phys. Rev. Lett. **25**, 715 (1970).

[53] B. N. Esel'son, V. A. Slyusarev, V. N. Sobolev, and M. A. Strzhemechnyĭ, Pis'ma Zh. Eksp. Teor. Fiz. **21**, 253 (1975) [JETP Lett. **21**, 115 (1975)].

[54] A. A. Abrikosov and I. E. Dzyaloshinskiĭ, Zh. Eksp. Teor. Fiz. **35**, 771 (1958) [Sov. Phys. JETP **8**, 535 (1959)].

[55] L. D. Landau and E. M. Lifshits, Statisticheskaya fizika (Statistical Physics), Nauka, M., 1964 (Engl. Transl., Addison-Wesley, Reading, Mass., 1969).

[56] A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinskiĭ, Metody kvantovoĭ teorii polya v statisticheskoĭ fizike (Methods of Quantum Field Theory in Statistical Physics), Nauka, M., 1962 (Engl. Transl., Prentice-Hall, Englewood Cliffs, N. J., 1963).

[57] D. O. Edwards, E. M. Ifft, and R. E. Sarwinski, Phys. Rev. **177**, 380 (1969).

[58] P. Seligmann, D. O. Edwards, R. E. Sarwinski, and J. T. Tough, ibid. **181**, 415 (1969).

[59] L. Landau, J. T. Tough, N. R. Brubaker, and D. O. Edwards, Phys. Rev. Lett. **23**, 183 (1969).

[60] J. Landau, J. T. Tough, N. R. Brubaker, and D. O. Edwards, Phys. Rev. **A2**, 2472 (1970).

[61] D. L. Husa, D. O. Edwards, and J. R. Gaines, in: Proc. LT-10.—M.: 1966, p. 365.

[62] D. L. Husa, D. O. Edwards, and J. R. Gaines, Phys. Lett. **21**, 28 (1966).

[63] A. I. Ahonen, M. A. Paalanen, R. C. Richardson, and Y. Takano, J. Low Temp. Phys. **25**, 733 (1976).

[64] J. McDougall and E. C. Stoner, Philos. Trans. R. Soc. London Ser. A **237**, 350 (1938).

[65] V. M. Galitskiĭ, Zh. Eksp. Teor. Fiz. **34**, 151 (1958) [Sov. Phys. JETP **7**, 104 (1958)].

[66] D. Lhuillier and M. Francois, J. Low Temp. Phys. **28**, 473 (1977).

[67] B. M. Abraham, Y. Eckstein, and J. B. Ketterson, Phys.

Rev. Lett. **21**, 422 (1968).

[68] D. G. Sanikidze and A. N. Shaanova, Pis'ma Zh. Eksp. Teor. Fiz. **10**, 482 (1969) [JETP Lett. **10**, 310 (1969)].

[69] A. F. Andreev and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. **44**, 2058 (1963) [Sov. Phys. JETP **17**, 1384 (1963)].

[70] E. P. Bashkin and A. É. Meĭerovich, Pis'ma Zh. Eksp. Teor. Fiz. **27**, 517 (1978) [JETP Lett. **27**, 485 (1978)].

[71] A. A. Abrikosov and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. **32**, 1083 (1957) [Sov. Phys. JETP **5**, 887 (1957)].

[72] G. A. Brooker and J. Sykes, Phys. Rev. Lett. **21**, 279 (1968).

[73] V. P. Silin, Vvedenie v kineticheskuyu teoriyu gazov (Introduction to the Kinetic Theory of Gases), Nauka, M., 1971.

[74] E. S. Murdock, K. R. Mountfield, and L. R. Corruccini, J. Low Temp. Phys. **31**, 581 (1978).

[75] A. E. Meyerovich, Phys. Lett. A **69**, 279 (1978).

[76] I. M. Khalatnikov and V. N. Zharkov, Zh. Eksp. Teor. Fiz. **32**, 1108 (1957) [Sov. Phys. JETP **5**, 905 (1957)].

[77] G. Baym and C. Ebner, Phys. Rev. **164**, 235 (1967).

[78] A. É. Meĭerovich, Zh. Eksp. Teor. Fiz. **71**, 1180 (1976) [Sov. Phys. JETP **44**, 617 (1976)].

[79] M. B. Hoffberg, Phys. Rev. A5, 1963 (1972).

[80] B. R. Patton and A. Zaringhalam, Phys. Lett. A **55**, 95 (1975).

[81] L. P. Gor'kov and T. K. Melik-Barkhudarov, Zh. Eksp. Teor. Fiz. **40**, 1452 (1961) [Sov. Phys. JETP **13**, 1018 (1961)].

[82] E. P. Bashkin, Phys. Lett. A **69**, 282 (1978).

[83] I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. **32**, 653 (1957) [Sov. Phys. JETP **5**, 542 (1957)]; Pis'ma Zh. Eksp. Teor. Fiz. **17**, 534 (1973) [JETP Lett. **17**, 386 (1973)].

[84] Galasiewicz, Phys. Lett. A **43**, 149 (1973); Phys. Kondens. Mater. **18**, 141, 155 (1974).

[85] A. F. Andreev and E. P. Bashkin, Zh. Eksp. Teor. Fiz. **69**, 319 (1975) [Sov. Phys. JETP **42**, 164 (1975)].

[86] Yu. A. Nepomnyashchii, *ibid.* **70**, 1070 (1976) [Sov. Phys. J.

JETP **43**, 559 (1976)].

[87] V. P. Mineev, Zh. Eksp. Teor. Fiz. **67**, 683 (1974) [Sov. Phys. JETP **40**, 338 (1975)].

[88] N. N. Bogolyubov, V. V. Tolmachev, and D. V. Shirkov, Novyĭ metod v teorii sverkhprovodimosti (A New Method in the Theory of Superconductivity), Izd. AN SSSR, M., 1958 (Engl. Transl., Consultants Bureau, New York, 1959).

[89] V. M. Galitskiĭ, Zh. Eksp. Teor. Fiz. **34**, 1011 (1958) [Sov. Phys. JETP **7**, 698 (1958)].

[90] P. W. Anderson, Phys. Rev. **112**, 1900 (1959).

[91] E. P. Bashkin, Candidate's dissertation, Institute of Physical Problems of the Academy of Sciences of the USSR, M., 1978.

[92] G. E. Volovik, V. P. Mineev, and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. **69**, 675 (1975) [Sov. Phys. JETP **42**, 342 (1975)].

[93] A. F. Andreev, V. I. Marchenko, and A. É. Meĭerovich, Pis'ma Zh. Eksp. Teor. Fiz. **26**, 40 (1977) [JETP Lett. **26**, 36 (1977)].

[94] E. P. Bashkin and A. É. Meĭerovich, Zh. Eksp. Teor. Fiz. **77**, 383 (1979) [Sov. Phys. JETP **50**, 196 (1979)].

[95] I. A. Fomin, Zh. Eksp. Teor. Fiz. **54**, 1881 (1968) [Sov. Phys. JETP **27**, 1010 (1968)]; Pis'ma Zh. Eksp. Teor. Fiz. **24**, 90 (1976) [JETP Lett. **24**, 77 (1976)].

[96] V. P. Silin and A. A. Rukhadze, Élektromagnitnye svoĭstva plazmy i plasmopodobnykh sred (Electromagnetic Properties of a Plasma and of Plasma-Like Media), Gosatomizdat, M., 1961.

[97] D. Fay, Phys. Rev. B16, 537 (1977).

[98] C. Lhuillier and F. Laloë, J. Phys. (Paris) **40**, 239 (1979).

[99] B. Castaing and P. Nozieres, *ibid.* **40**, 257 (1979).

[100] M. Chapellier, G. Frosatti, and F. B. Rasmussen, Phys. Rev. Lett. **42**, 904 (1979).

Translated by M. V. King