Feynman path integrals in a phase space

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Feynman path integrals in a phase space are analyzed in detail. The analysis is based on the theory of operator symbols, in contrast with the traditional approach based on the direct use of canonical commutation relations. Particular attention is paid to the Weyl and Wick symbols, which are the most important in applications. The set of paths on which the integral is concentrated is studied. It is found that these paths are always discontinuous. This discontinuity is responsible for errors in certain papers on path integrals in a phase space. The most important properties of the Weyl and Wick symbols are reviewed.

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CONTENTS

INTRODUCTION

1) Method of path integrals

The method of path integrals plays a central role in the mathematics of theoretical physics, primarily because of two circumstances: First, the path integral is unusually graphic and versatile. With it, any quantum-mechanical quantity can be written as the sum of the effects of virtual classical paths. The simple dependence on the Planck constant h makes it simple to see that in the limit $h\to 0$ the quantum-mechanical quantity is determined primarily by a real classical path, i.e., by a path which satisfies the principle of least action. Second, the method of path integrals is very convenient from the technical standpoint. It is a simple matter to use this method to construct perturbation-theory series and semiclassical asymptotic expressions.

In this paper we shall work from the theory of operator symbols to analyze path integrals systematically.

2) Path integral in a phase space, operator symbols, and quantization

The Feynman path integral in a phase space[1] is

$$J=\int e^{\frac{1}{ih}\int_{t_1}^{t_2} L\,dt}\prod dp(t)\,dq(t). \tag{1}$$

where L is the Lagrangian

$$L=H(p,q)-\sum p_k\dot{q}_k$$

(within a total derivative), and H is the Hamiltonian of of the system. The integral in (1) is evaluated along one set of paths or another, depending on the problem at hand. [For example, to find the matrix element $\langle q_2|e^{(t/ih)\hat{H}}|q_1\rangle$ we would evaluate the integral in (1) along paths satisfying the boundary conditions $q(0)=q_1$, $q(t)=q_2$.]

What is the mathematical meaning of this integral? In other words, how do we evaluate it?

According to one point of view, the integral in (1) is simply a convenient hieroglyphic, shorthand for an algorithm and perturbation theories.[2] This point of view is too restrictive. On the other hand, the traditional (Wiener) view of the path integral as an integral along a measure in a function space runs into excessive complexities here and is thus also imperfect.[1)]

Another reason that the interpretation of integral (1) from the standpoint of measure theory is unsatisfactory is that it must also be suitable for systems of fermions, and in this case the integration is carried out over "anticommuting variables." This approach is also algebraic in nature and clearly is related to no measure of any kind. Our own point of view is that the integral in (1) should be understood as the limit of a finite number of approximations. But which approximations? It has been shown previously[4] that the integral in (1) is very sensitive to the choice of its approximations; the resulting ambiguity is of the same nature as the uncertainty in quantization. This is a fundamental point, and we shall pursue it in more detail.

To specify the quantization means to establish a common rule according to which each classically observable quantity $f(p,q)$ is associated with a quantum-mechanical observable $\hat{f}$, i.e., an operator in a Hilbert space.[2)] The correspondence $f\to\hat{f}$ should be linear. In this situation the function f is called the "symbol of operator" $\hat{f}$. Once this correspondence has been established, an operation arises in the function space which duplicates the product of operators: If f, f_1, f_2 are the symbols of the operators $\hat{f}, \hat{f}_1, \hat{f}_2$, and $\hat{f}=\hat{f}_1\hat{f}_2$, then $f=f_1 * f_2$. The operation denoted by the asterisk, being bilinear in f_1 and f_2, is specified by the integral[3)]

$$f(p,q)=(f_1*f_2)(p,q)=\int K_h(p,q;\,p_1,q_1;\,p_2,q_2)\,f_1(p_1,q_1)\,f_2(p_2,q_2)\,dp_1\,dp_2\,dq_1\,dq_2. \tag{2}$$

The Weyl and Wick quantizations are the most important for practical applications. The Wick quantization is an extremely important tool for studying systems with an infinite number of degrees of freedom.

3) Use of symbols of operators to construct a path integral

We denote by $\hat{H}$ some Hamiltonian and by $H(p,q)$ its symbol. At small values of t we have

$$\hat{G}(t)=\exp\left(\frac{t}{ih}\hat{H}\right)=1+\frac{t}{ih}\hat{H}+O(t^2).$$

The symbol of the operator $\hat{G}(t)$ is thus

$$G(t)=1+\frac{t}{ih}H+O(t^2)=e^{(t/ih)H}(1+O(t^2)). \tag{3}$$

We denote by $\hat{U}(t)$ the operator whose symbol is $U(t)=\exp[(t/ih)H]$. It follows from (3) that

$$\hat{G}(t)=\hat{U}(t)(1+O(t^2)). \tag{4}$$

Let us consider the operator identity $\hat{G}(t)=(\hat{G}(t/N))^N$. In it, we replace the operator $\hat{G}(t/N)$ by the operator $\hat{U}(t/N)$. As a result we find an approximate expression for $\hat{G}$:

$$\hat{G}_N(t)=\left(\hat{U}\left(\frac{t}{N}\right)\right)^N. \tag{5}$$

It follows from (4) that $\lim_{N\to\infty}\hat{G}_N(t)=\hat{G}(t)$. Since the symbol of the operator $\hat{U}(t/N)$ is known, we can transform from operators to symbols in (5):

$$G_N(t)=U\left(\frac{t}{N}\right)*\ldots*U\left(\frac{t}{N}\right). \tag{6}$$

Noting that the operation asterisk is specified by integral (2), we find an expression G_N as a multiple integral, $G=\lim_{N\to\infty}G_N$. An analogous construction can be

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1) We note, however, that an approach to path integrals is being developed at the present time on the basis of far-reaching generalizations of the concept of a measure. See Ref. 3 for a detailed discussion of the questions involved here.

2) For example, with the product $\hat{p}\hat{q}$ we can associate the operator $\hat{p}\hat{q}$ or the operator $\hat{q}\hat{p}$ or also the operator $(\hat{p}\hat{q}+\hat{q}\hat{p})/2$, where $\hat{p}$ and $\hat{q}$ are the ordinary momentum and coordinate operators.

3) The correspondence $f\to\hat{f}$ is not completely arbitrary. The principal requirement imposed on it is the correspondence principle:

$$(1)\ \lim_{h\to 0}(f_1*f_2)(p,q)=f_1(p,q)f_2(p,q),\quad (2)\ \lim_{h\to 0}\frac{1}{h}(f_1*f_2-f_2*f_1)=\frac{1}{i}[f_1,f_2],$$

where f_1, f_2 is the ordinary product of functions, and $[f_1,f_2]$ are the Poisson brackets. All the other requirements imposed on the quantization stem from technical convenience. (This definition of quantization is a particular case of the more general definition proposed in Ref. 5. See also Refs. 6 and 7.)

carried out in the case in which the Hamiltonian H depends on t. Tobocman's paper[8] is based on similar considerations.

4) Scope of this paper

Expression (6) is the initial finite approximation of the path integral (1). The functions K_h in (2) are different, depending on the nature of the symbols, i.e., on the quantization method, so that the integrals in (6) are also different. It has been shown[4] that to take the limit in a simple-minded way in (6) for the basic types of symbols would be to erase the differences among them, so that the result would always be the same (and this result would thus be correct only under certain special circumstances).

In this paper we shall analyze the behavior of integral (6) in the limit $N\to\infty$. We shall see that the limiting expression has the original form in (1) only in the case of Weyl symbols; in other cases, the expression has certain added details, the most typical of which is the appearance of integrals with a displaced argument in the exponential function in (1). For example, the integral

$$\int_{t_1}^{t_2} H(p(t+0),\ q(t))\,dt \tag{7}$$

appears in place of the integral

$$\int_{t_1}^{t_2} H(p(t),\ q(t))\,dt. \tag{8}$$

Nonintegral terms also arise in the exponential function in (1), taking different forms for different kinds of symbols. After the correct expression for integral (1) has been written, it is found that this integral is the limit of finite approximations of a general kind, one of which is (6). The approximation is constructed in the following manner. We consider the Hilbert space $\mathscr{H}(t_1, t_2)$ of paths with the scalar product

$$(x,\ x)=\int_{t_1}^{t_2} x^2(t)\,dt,\quad x=(p_1,\ \ldots,\ p_n,\ q_1,\ \ldots,\ q_n),\quad x^2=\sum(p_i^2+q_i^2). \tag{9}$$

In $\mathscr{H}(t_1, t_2)$ we consider a sequence of nested finite-dimensional subspaces $\mathscr{H}_N\subset\mathscr{H}_{N+1}$ which consist of differentiable paths. The integral

$$\int_{t_1}^{t_2}(H(p,\ q)-p\dot{q})\,dt\quad \text{for}\quad p(t),\ q(t)\in\mathscr{H}_N$$

is a function of a finite number of variables. Replacing the path integration in (1) by an integration over $\mathscr{H}_N$, we find a finite approximation of integral (1): $J=\lim_{N\to\infty}J_N$. To a large extent, the limit $N\to\infty$ in integral (6) is based on intuition, so this limit must be justified. A justification "at the physical level" will be given in the present paper. This justification will be less than rigorous because of a free transposition of various limits. The rigor can be restored by ordinary methods, by making natural assumptions regarding the Hamiltonian. We will also determine the paths on which integral (1) is concentrated. We shall show that these paths are necessarily discontinuous. This circumstance clarifies the difference[4)] between integrals (7) and (8). Furthermore, it will be shown that the set of paths on which the integral is concentrated is not determined unambiguously: This set can be varied by increasing the smoothness of the coordinates at the expense of the momenta (or vice versa), but it is impossible to arrange events such that both the coordinates and the momenta are continuous. This circumstance is in agreement with the uncertainty principle.[5)]

In addition to these qualitative results, this paper contains many equations which express various physical entities as path integrals. As a rule, these equations are not new, and their derivation by the theory of symbols is primarily of methodological value (aside from the refinements mentioned above). The advantages of this derivation over the traditional derivation (based on commutation relations[6)]) are that, first, it is simpler and, second, it can be extended to the case in which the phase space is curved and in which there are no natural coordinates in this space with canonical relations (Poisson brackets). An example of this type is given in Ref. 10.

Finally, we note that this method which takes a limit in integral (6). A similar attempt was made in Ref. 2, but the equations derived there are, unfortunately, not exactly correct[7)]: They do not have the argument shifts as in (7).

To make this paper self-contained, we are also furnishing a supplement with a brief review of the properties of the various symbols.

1. PATH INTEGRAL FOR THE EVOLUTION OPERATOR SYMBOL

a: Weyl evolution operator symbol

1) Basic construction

We denote by $\hat{H}$ some operator in $L^2(R^n)$, and we denote by $H(p, q)$ its Weyl symbol:

$$\hat{H}=\int e^{i(\alpha\hat{p}+\beta\hat{q})}\varphi(\alpha,\ \beta)\,d\alpha\,d\beta,\qquad H(p,\ q)=\int e^{i(\alpha p+\beta q)}\varphi(\alpha,\ \beta)\,d\alpha\,d\beta, \tag{1.1}$$

where $\hat{p}=(\hat{p}_1,\ldots,\hat{p}_n)$, $\hat{q}=(\hat{q}_1,\ldots,\hat{q}_n)$, $\alpha=(\alpha_1,\ldots,\alpha_n)$, $\beta=(\beta_1,\ldots,\beta_n)$, $\alpha\hat{p}=\sum\alpha_i\hat{p}_i$, $\beta\hat{q}=\sum\beta_i\hat{q}_i$; $d\alpha=d\alpha_1\cdots d\alpha_n$, $d\beta=d\beta_1\cdots d\beta_n$. We shall be using some similar notation below. Here $\hat{p}_1$ and $\hat{q}_j$ are the ordinary momentum and coordinate operators in R^n:

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4) Incidentally, this difference is unimportant in the case of quantum field theory: It arises only in the quasilocal terms and is thus eliminated by renormalization.

5) It was shown in Ref. 9 that a path integral can be calculated along paths with a continuous coordinate and a discontinuous momentum.

6) Equations for the basic physical quantities in terms of path integrals based on canonical relations were derived in the 1950's and 1960's in Refs. 12-16.

7) Incidentally, Ref. 2 reproduces the basic part of Ref. 4, which contains a construction of the path integral by means of symbols, with an erroneous reference to a paper[11] by one of the authors of Ref. 2 instead of Ref. 4.

$$(\hat{p}_k f)(s)=\frac{h}{i}\frac{\partial f}{\partial s_k},\quad (\hat{q}_k f)(s)=s_k f(s). \tag{1.2}$$

The functions $\varphi(\alpha,\beta)$ may be either ordinary functions or generalized functions; furthermore, they may depend on the Planck constant h as a parameter. Accordingly, the symbol $H(p,q)$ may also depend on h as a parameter. Where necessary we specify this dependence by a subscript: $H_h(p,q)$, $\varphi_h(\alpha,\beta)$.

For convenience we set

$$x=(p,\ q),\quad \omega=\begin{pmatrix} 0 & I_n \\ -I_n & 0\end{pmatrix}.$$

If $y=(\tilde{p},\tilde{q})$, then

$$x\omega y=(p,\ q)\,\omega\begin{pmatrix}\tilde{p}'\\ \tilde{q}'\end{pmatrix},\quad \tilde{p}'=\begin{pmatrix}\tilde{p}_1\\ \vdots\\ \tilde{p}_n\end{pmatrix},\quad \tilde{q}'=\begin{pmatrix}\tilde{q}_1\\ \vdots\\ \tilde{q}_n\end{pmatrix}$$

(I_n is the unit matrix of order n). We denote by $f_i(x)$ ($i=0,\ldots,N$) the Weyl symbols of certain operators, and we denote by $G_N(x)$ the Weyl symbol of their product. Using the formula for the composition of Weyl symbols (see the Supplement), we immediately find

$$G_N(x)=\frac{1}{(\pi h)^{2nN}}\int f_0(y_1)\ldots f_N(y_{N+1})\,e^{\frac{2}{ih}K_N}\,d^{N+1}y\,d^{N-1}x,$$
$$K_N=\sum_1^{N-1}(x_i\omega y_{i+2}+y_{i+2}\omega x_{i+1}+x_{i+1}\omega x_i)+y_1\omega y_2+y_2\omega x_1+x_1\omega y_1,$$

where $x_N=x$. This equation takes a simpler form if we set $f_0\equiv 1$ and change the notation of the integration variables, $y_1\to x_1$, $y_k\to y_{k-1}$ in the case $k>1$, $x_k\to x_{k+1}$:

$$G_N(x)=\frac{1}{(\pi h)^{2nN}}\int f_1(y_1)\ldots f_N(y_N)\,e^{\frac{2}{ih}K_N}\,d^N y\,d^N x,$$
$$K_N=\sum_1^{N}(x_k\omega y_k+y_k\omega x_{k+1}+x_{k+1}\omega x_k),\quad x_{N+1}=x. \tag{1.3}$$

The integration can be carried out over the variables x_i. Since these variables appear in a power no higher than the second in the exponent, we can use the statistical-phase method here.

We transform K_N:

$$\begin{aligned}K_N=x_1\omega y_1+x_2\omega(y_2-y_1)+\ldots+x_N\omega(y_N-y_{N-1})+y_N\omega x_{N+1}\\ +x_2\omega(x_1-x_3)+x_4\omega(x_3-x_5)\\ \ldots+\begin{cases}x_{N+1}\omega x_N & \text{for odd } N,\\ x_N\omega(x_{N-1}-x_{N+1}) & \text{for even } N.\end{cases}\end{aligned} \tag{1.4}$$

Equating the derivatives with respect to x_i to zero, we find

$$y_1-x_2=0,\ y_k-y_{k-1}+x_{k-1}-x_{k+1}=0,\quad k=2,\ldots,N. \tag{1.5}$$

Now the cases of even and odd N diverge slightly. Let us first consider the case of even N. In this case Eqs. (1.5) can be solved unambiguously for the x_i:

$$\begin{aligned}x_{2k}&=y_1+\sum_1^{k-1}(y_{2l+1}-y_{2l}),\quad 1\leqslant k\leqslant\frac{N}{2},\\ x_{2k-1}&=x+\sum_k^{\frac{N}{2}}(y_{2l-1}-y_{2l}),\quad 1\leqslant k\leqslant\frac{N}{2}.\end{aligned} \tag{1.6}$$

The fact that Eqs. (1.5) can be solved unambiguously means that the quadratic form on the second row in (1.4) is nondegenerate. We can thus apply the statistical-phase method in its simplest version.

We now introduce the continuous parameter τ, such that $t_1\leqslant\tau\leqslant t_2$, and we assume $x_{2k}=x(\tau_{2k})$, $x_{2k+1}=\tilde{x}(\tau_{2k+1})$, $y_k=y(\tau_k)$:

$$f_k(x)=f(\tau_k;\ x)=e^{\frac{1}{ih}\frac{t_2-t_1}{N}g(\tau_k;\,x)},$$

where $\tau_k=t_2-k/N(t_2-t_1)$; $x(\tau)$, $\tilde{x}(\tau)$, and $y(\tau)$ are continuously differentiable functions of τ; and $g(\tau;x)$ is a continuous function of the variables τ and x.

Under these conditions we can take the limit $N\to\infty$ in (1.6):

$$\begin{aligned}x(\tau)&=y(t_2)-\frac{1}{2}\int_\tau^{t_2}\dot{y}(s)\,ds=\frac{y(\tau)+y(t_2)}{2},\\ \tilde{x}(\tau)&=x+\frac{1}{2}\int_{t_1}^{\tau}\dot{y}(s)\,ds=x+\frac{1}{2}(y(\tau)-y(t_1)).\end{aligned} \tag{1.7}$$

Expression (1.4) for K_N is an integral sum, except for the first and last terms on the first row. In the limit $N\to\infty$ these terms become equal to $\tilde{x}(t_2)\omega y(t_2)$ and $y(t_1)\omega x(t_1)$, respectively. Also using (1.7), we find the following expression for $K=\lim K_N$:

$$\begin{aligned}K&=\tilde{x}(t_2)\,\omega y(t_2)-\frac{1}{2}\int_{t_1}^{t_2}\tilde{x}(\tau)\,\omega\dot{y}(\tau)\,d\tau\\ &\quad-\frac{1}{2}\int_{t_1}^{t_2}x(\tau)\,\omega\dot{y}(\tau)\,d\tau+y(t_1)\,\omega x+\int_{t_1}^{t_2}x\omega\dot{\tilde{x}}\,d\tau\\ &=-\frac{1}{4}\int_{t_1}^{t_2}y(\tau)\,\omega\dot{y}(\tau)\,d\tau+\frac{1}{2}(x\omega y(t_2)+y(t_1)\,\omega x)+\frac{1}{4}y(t_2)\,\omega y(t_1).\end{aligned} \tag{1.8}$$

We now incorporate the factor $(1/\pi h)^{2nN}$ in the normalization of the differentials in front of integral (1.3), and we introduce the notation $G=\lim G_N$. For G we find the final expression

$$\begin{aligned}G(t_2,\ t_1\mid x)=\int\exp\Big\{\frac{1}{ih}\Big[\int_{t_1}^{t_2}g(\tau;\ y(\tau))\,d\tau\\ -\frac{1}{2}\int_{t_1}^{t_2}y\omega\dot{y}\,d\tau+x\omega y(t_2)+y(t_1)\,\omega x+\frac{1}{2}y(t_2)\,\omega y(t_1)\Big]\Big\}\prod dy(\tau).\end{aligned} \tag{1.9}$$

The integral is evaluated over all paths.

We turn now to the case of odd N. The solution of (1.5) in this case is

$$\begin{aligned}x_{2k}&=y_1+\sum_1^{k-1}(y_{2l+1}-y_{2l}),\quad 1\leqslant k\leqslant\frac{N+1}{2},\\ x_{2k-1}&=x_N+\sum_k^{\frac{N-1}{2}}(y_{2l-1}-y_{2l}),\quad 1\leqslant k\leqslant\frac{N+1}{2}.\end{aligned} \tag{1.10}$$

The equations are thus not always solvable and the solutions not always unambiguous. The condition under which the equations are solvable follows from the first group of equations in (1.10) with $k=(N+1)/2$ (we recall that $x_{N+1}=x$):

$$x=y_1-\sum_1^{\frac{N-1}{2}}(y_{2l+1}-y_{2l})=0. \tag{1.11}$$

Where the equations are solvable, the solution depends on x_N as a parameter. It follows that the stationary-phase method can be used to evaluate the integral over $x_2,\ldots,x_{N-1}$; the remaining integral, written as a function of x_N, is necessarily

$$\int e^{ic_N x_N \varphi_N}\,dx_N = (2\pi)^n \delta(c_N\varphi_N) = (2\pi)^n c_N^{-1}\delta(\varphi_N),$$

where φ_N is the left side of (1.11), and $c_N \neq 0$ is some constant.

As before, we examine the continuous limit, and as before, we introduce the functions $x(\tau)$, $\tilde{x}(\tau)$, and $y(\tau)$. From (1.10) we find

$$x(\tau)=\frac{y(\tau)+y(t_2)}{2}, \quad \tilde{x}(\tau)=\tilde{x}(t_2)+\frac{1}{2}(y(\tau)-y(t_1)).$$

For $K=\lim K_N$ we find an expression like (1.8):

$$K=\tilde{x}(t_2)\,y(t_2)-\frac{1}{2}\int_{t_1}^{t_2} x(\tau)\,\omega\dot{y}(\tau)\,d\tau-\frac{1}{2}\int_{t_1}^{t_2}\tilde{x}(\tau)\,\omega y(\tau)\,d\tau+y(t_1)\,\omega x$$
$$+\int_{t_1}^{t_2} x(\tau)\,\omega\dot{\tilde{x}}(\tau)\,d\tau+x\omega\tilde{x}(t_1)=-\frac{1}{4}\int_{t_1}^{t_2} y(\tau)\,\omega\dot{y}(\tau)\,d\tau$$
$$+\left[x-\frac{1}{2}(y(t_2)+y(t_1))\right]\omega\tilde{x}(t_1)+y(t_1)\,\omega x+\frac{1}{4}y(t_2)\,\omega y(t_1). \quad (1.12)$$

From (1.12) we see that the integration over $\tilde{x}(t_1)$ leads to a factor

$$\delta\left(x-\frac{y(t_1)+y(t_2)}{2}\right).$$

This result is in complete agreement with the circumstance that in the limit $N\to\infty$ the condition (1.11) becomes

$$x=\frac{y(t_1)+y(t_2)}{2}. \quad (1.13)$$

Using (1.13), we can write the final expression for G as

$$G(t_2, t_1\,|\,x)=\int\limits_{\frac{y(t_1)+y(t_2)}{2}=x} \exp\left\{\frac{1}{i\hbar}\left[\int_{t_1}^{t_2} g(\tau;\,y(\tau))\,d\tau\right.\right.$$
$$\left.\left.-\frac{1}{2}\int_{t_1}^{t_2} y(\tau)\,\omega\dot{y}(\tau)\,d\tau+\frac{1}{2}y(t_1)\,\omega y(t_2)\right]\right\}\prod dy(\tau). \quad (1.14)$$

We have a few final comments on this topic.

a) For a given quantity we find two integral representations, (1.9) and (1.14). One representation can be transformed into the other in the following manner: We denote the integrands of the path integrals in (1.9) and (1.14) by $F_1(x,y)$ and $F_2(x,y)$, respectively; $F_i(x,y)$ is a function of the point in phase space, x, and a functional of the path $y=y(\tau)$. Here F_2 contains the factor $\delta(y(t_1)-y(t_2)/2)-x$. It is easy to see that

$$F_\alpha(x,y)=\frac{1}{(\pi\hbar)^n}\int F_\beta(\tilde{x},y)\,e^{\frac{2}{i\hbar}(\tilde{x}\omega y(t_2)+y(t_2)\omega x+x\omega\tilde{x})}\,d\tilde{x} \quad (\alpha\neq\beta).$$

This relation corresponds to multiplication of the given operator by the unit operator

$$G_N(t_2,t_1\,|\,x)=\frac{1}{(\pi\hbar)^n}\int G_N(t_2,t_1\,|\,x)\,e^{\frac{2}{i\hbar}(\tilde{x}\omega\tilde{\tilde{x}}+\tilde{\tilde{x}}\omega x+x\omega\tilde{x})}\,d\tilde{x}\,d\tilde{\tilde{x}}$$

before the limit is taken.

b) Equations (1.9) and (1.14) can be simplified slightly by setting

$$y(t)=x+u(t).$$

Equation (1.9) becomes

$$G(t_2,t_1\,|\,x)=\int\exp\left\{\frac{1}{i\hbar}\left[\int_{t_1}^{t_2} g(\tau;\,x+u(\tau))\,d\tau-\frac{1}{2}\int_{t_1}^{t_2} u\omega\dot{u}\,d\tau\right.\right.$$
$$\left.\left.-\frac{1}{2}u(t_1)\,\omega u(t_2)\right]\right\}\prod_\tau du(\tau). \quad (1.9')$$

Equation (1.14) becomes

$$G(t_1,t_2;\,x)=\int\exp\left\{\frac{1}{i\hbar}\left[\int_{t_1}^{t_2} g(\tau;\,x+u(\tau))\,d\tau\right.\right.$$
$$\left.\left.-\frac{1}{2}\int_{t_1}^{t_2} u\omega\dot{u}\,d\tau\right]\right\}\prod_\tau du(\tau). \quad (1.14')$$

c) We can transform integral (1.9′) by substituting $\delta(a-\frac{1}{2}[u(t_1)+u(t_2)])$ into the integrand and then carrying out a further integration over a. This is an identity transformation, since $\int\delta(a-\frac{1}{2}[u(t_1)+u(t_2)])du=1$.

As we see in Sec. c below, the result will not be affected if we do not complete the integration over a in (1.9):

$$G(t_2,t_1;\,x)=\int\exp\left\{\frac{1}{i\hbar}\left[\int_{t_1}^{t_2}\left(g(\tau;\,x+u(\tau))-\frac{1}{2}u\omega\dot{u}\right)d\tau\right]\right.$$
$$\left.-\frac{1}{2}u(t_1)\,\omega u(t_2)\right\}\delta\left(a-\frac{1}{2}\left[u(t_1)+u(t_2)\right]\right). \quad (1.15)$$

In other words, the integral is not changed if the integration is carried out over only those paths which satisfy the following condition instead of over all paths:

$$\frac{1}{2}(u(t_1)+u(t_2))=a, \quad (1.16)$$

where a is an arbitrary point in phase space. Although the integrand in (1.15) depends on a as a parameter, the integral does not depend on a.

This property of integral (1.9′) is analogous to the properties of the path integrals which arise in gauge field theories. It might naturally be called the "gauge singularity."

From this standpoint, the transformations $u(t)\to u(t)+c$, where c is a constant, can be treated as guage transformations; relation (1.16) can be thought of as the condition which fixes the gauge; and the integral in (1.14′) can be found from (1.15) by choosing a special gauge, $a=0$.

At first glance, the gauge property of integral (1.9′) seems contradictory: It follows formally from (1.9′) and (1.15) that

$$G(t_2,t_1;\,x)=G(t_2,t_1;\,x)\int da=G(t_2,t_1;\,x)\cdot\infty.$$

Contradictions do not arise because of the definition of the path integral: Roughly speaking, this integral is a fraction in which both the numerator and denominator contain a factor $\int da$. This point is explained in more detail in the following section.

2) Approximations of a general type

The path integrals in (1.9), (1.9′), (1.14), (1.14′), and (1.5) were introduced in the preceding section on the basis of intuitive considerations. In the present section we shall give them a more exact meaning; we shall essentially be giving them a mathematical definition.

We first consider the integral in (1.14′). We denote by $\mathscr{H}(t_1, t_2)$ the Hilbert space which consists of the functions $x(t)$, $t_1 \le t \le t_2$, which take on values in the phase space [i.e., this Hilbert space consists of classical paths with a scalar product

$$(x, x) = \int_{t_1}^{t_2} x^2(t)\,\mathrm{d}t, \tag{1.17}$$

where

$$x(t) = (p_1(t), \ldots, p_n(t), q_1(t), \ldots, q_n(t)),$$
$$x^2(t) = \sum_1^n (p_i^2(t) + q_i^2(t))].$$

We denote by P_N the family of orthogonal-projection operators with the properties

$$\begin{gathered}\dim P_N\mathscr{H}(t_1, t_2) = \mathrm{d}(N) < \infty,\\ P_N P_{N+1} = P_{N+1}P_N = P_N,\\ \lim_{N\to\infty} P_N = \mathrm{I},\end{gathered} \tag{1.18}$$

where I is the unit operator in $\mathscr{H}_{t_1,t_2}$, and the limit is understood in the strong sense. The second condition in (1.18) means that the subspaces $\mathscr{H}_N = P_N\mathscr{H}_{t_1,t_2}$ are nested: $\mathscr{H}_N \subset \mathscr{H}_{N+1}$.

We denote by $\tilde{\mathscr{H}}(t_1, t_2) \subset \mathscr{H}(t_1, t_2)$ a set of continuously differentiable paths, and we denote by $\tilde{\mathscr{H}}^0(t_1, t_2) \subset \mathscr{H}(t_1, t_2)$ that subset of $\mathscr{H}$ which consists of paths satisfying the condition

$$x(t_1) + x(t_2) = 0. \tag{1.19}$$

We set $\tilde{\mathscr{H}}^0_N = P_N\tilde{\mathscr{H}}^0(t_1, t_2)$. In the space $\mathscr{H}(t_1, t_2)$ we consider the operator B with a region of definition consisting of the space $\tilde{\mathscr{H}}^0(t_1, t_2)$

$$Bu = \omega\frac{\mathrm{d}u}{\mathrm{d}t}. \tag{1.20}$$

In the spaces $\tilde{\mathscr{H}}^0_N$ we now consider the operators B_N which satisfy the conditions

$$\det B_N \ne 0, \quad B_N = B_N^*, \quad \lim_{N\to\infty} B_N P_N f = Bf. \tag{1.21}$$

The latter condition must hold for any $f \in \mathscr{H}^0(t_1, t_2)$. [In (1.21), as in the similar limiting expressions which we will see below, we mean the convergence in the sense of the metric of the Hilbert space $\mathscr{H}(t_1, t_2)$.] Finally, we denote by $\mathrm{d}^L u$, $L = \dim P_N\tilde{\mathscr{H}}^0(t_1, t_2)$ the Lebesque measure in the space $\tilde{\mathscr{H}}^0_N$. We set

$$G_N = \frac{\int e^{\frac{1}{ih}\left\{\int_{t_1}^{t_2} g(\tau;\, x+u(\tau))\mathrm{d}\tau - \frac{1}{2}(u,\, B_N u)\right\}}\mathrm{d}^L u}{\int e^{-\frac{1}{2ih}(u,\, B_N u)}\mathrm{d}^L u}. \tag{1.22}$$

Obviously, expression (1.3), which has the same notation, transforms into the particular case in (1.22) after an integration over x_i and the substitution $f_k(y) = \exp\{1/ih[t_2 - t_1/N]g(\tau_k; y)\}$, for odd N. Case (1.22) corresponds to a special choice of subspaces $\mathscr{H}_N = P_N\mathscr{H}(t_1, t_2)$ and operators B_N. The integrals in (1.14) and (1.15) are determined in a completely analogous way. The only difference is that instead of the functional $\int_{t_1}^{t_2} g(\tau; x + u(\tau))\mathrm{d}\tau$ in the first case we consider the functional $\int_{t_1}^{t_2} g((\tau; y(\tau))\mathrm{d}\tau + \frac{1}{2}y(t_1)\omega y(t_2)$, and in the second case we consider the functional $\int_{t_1}^{t_2} g(\tau; x + a + \tilde{u}(\tau))\mathrm{d}\tau - a\omega(\tilde{u}(t_2) - \tilde{u}(t_1))$.

We turn now to integral (1.9′). We consider the Hilbert space $\hat{\mathscr{H}}(t_1, t_2) = \mathscr{H}(t_1, t_2) \oplus R^{2n}$, where $\mathscr{H}(t_1, t_2)$ is the space of paths considered previously, R^{2n} is a real Euclidean space of dimensionality $2n$ with an ordinary scalar product, and w is the number of degrees of freedom. The elements of $\hat{\mathscr{H}}(t_1, t_2)$ are the pairs $\{x, \alpha\}$, $x \in \mathscr{H}(t_1, t_2)$, $\alpha \in R^{2n}$, and the scalar product is defined by

$$(\{x, \alpha\}, \{y, \beta\}) = (x, y) + (\alpha, \beta). \tag{1.23}$$

With each path $u(t) \in \tilde{\mathscr{H}}(t_1, t_2)$ we associate an element $\tilde{u}$ of the space $\hat{\mathscr{H}}(t_1, t_2)$:

$$\hat{u} = \{u, \alpha\}, \quad \alpha = \frac{1}{2}(u(t_1) + u(t_2)). \tag{1.24}$$

We denote the set of elements $\hat{u}$ in (1.24) by $\hat{\tilde{\mathscr{H}}}(t_1, t_2)$. In $\hat{\mathscr{H}}$ we consider the operator B whose region of definition is $\hat{\tilde{\mathscr{H}}}$:

$$\tilde{B}\hat{u} = \left\{\omega\frac{\mathrm{d}u}{\mathrm{d}t},\ \omega(u(t_2) - u(t_1))\right\}. \tag{1.25}$$

We see that

$$u(t_1)\,\omega u(t_2) = \frac{u(t_1) + u(t_2)}{2}\,\omega(u(t_2) - u(t_1)).$$

It follows that the quadratic form in the exponential function in (1.9′) is equal to $-(1/2ih)(\hat{u}, \tilde{B}\hat{u})$. We now consider, in the space $\hat{\mathscr{H}}(t_1, t_2)$, the projection operators P_N with the properties in (1.18) and the operators $\hat{B}_N$ in the spaces $\hat{\mathscr{H}}^N_N = P_N\hat{\mathscr{H}}(t_1, t_2)$ with properties analogous to (1.21):

$$\det \tilde{B}_N \ne 0, \quad \lim \tilde{B}_N P_N f = \tilde{B}f, \quad \lim \tilde{B}_N^* P_N f = \tilde{B}^* f$$

for any $f \in \hat{\tilde{\mathscr{H}}}(t_1, t_2)$. (In contrast with the operator B_N, the operator $\tilde{B}$ is not self-adjoint and in fact not even symmetric.[8]) We define the function $\tilde{G}_N$ by an expression like (1.22):

$$\tilde{G}_N = \frac{\int e^{\frac{1}{ih}\left\{\int_{t_1}^{t_2} g(\tau,\, x+u(\tau))\mathrm{d}\tau - \frac{1}{2}(\hat{u},\, \tilde{B}_N\hat{u})\right\}}\mathrm{d}^L\hat{u}}{\int e^{-\frac{1}{2ih}(\hat{u},\, \tilde{B}_N\hat{u})}\mathrm{d}^L\hat{u}}, \tag{1.26}$$

where $\hat{u} = \{u(t), \frac{1}{2}(u(t_1) + u(t_2))\} \in \hat{\mathscr{H}}(t_1, t_2) u(t) \in \mathscr{H}(t_1, t_2)$.

In the following section we shall show that the functions G_N and $\tilde{G}_N$ defined by Eqs. (1.22) and (1.26) converge to the function $G(t_1, t_2 | x)$ in the limit $N \to \infty$. This function is the Weyl evolution operator symbol. The operator with the Weyl symbol $g(\tau; x)$ serves as the infinitesimal operator of this evolution.

3) Derivation

We should check to see that the functions G_N and $\tilde{G}_N$ defined by (1.22) and (1.26) converge to the function $G(t_1, t_2 | x)$, which is the Weyl symbol of the operator $\hat{G}(t_1, t_2)$, which in turn satisfies the conditions

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[8] In the space $\hat{\mathscr{H}}(t_1, t_2)$ we consider the operator J defined by

$$J\{x, \alpha\} = \{x, -\alpha\}.$$

It is simple to show that the operator $J\tilde{B}$ is self-adjoint. In this connection it is natural to single out a class of approximations for which the condition $(P_N J\tilde{B}_N P_N)^* = P_N J\tilde{B}_N P_N$ holds in addition to (1.26).

$$ih\,\frac{\partial \hat{G}}{\partial t^2} = \hat{g}(t_2)\,\hat{G}(t_2, t_1), \quad \hat{G}(t, t) = I,$$

where $\hat{g}(t)$ is the operator with Weyl symbol $g(t; x)$.

We first use approximations (1.22) and (1.26) to calculate the path integral in the case $g(t; x) = u(t)x$, where $u(t)$ is some arbitrary function. The integrals in (1.22) and (1.26) are evaluated by the stationary-phase method. These integrals should be evaluated at a fixed N, and then the limit $N \to \infty$ should be taken. Under our assumptions regarding the operators B_N and $\tilde{B}_N$, the result will be the same as if we had applied the stationary-phase method directly to the limiting exponential function. The calculations are very simple and can be omitted. They are slightly different in the cases of approximations (1.22) and (1.26), but, as expected, they lead to a common result[9]:

$$G(t_2, t_1 \mid x) = \exp\left\{\frac{1}{ih}\left[x\int_{t_1}^{t_2} u(\,)\,\mathrm{d}t - \frac{1}{4}\int_{t_1}^{t_2}\int_{t_1}^{t_2} \operatorname{sign}(t-s)\,u(t)\,\omega u(s)\,\mathrm{d}t\,\mathrm{d}s\right]\right\}. \tag{1.27}$$

The same result is found by evaluating integral (1.16). The parameter a drops out of the answer. We now assume that $g(\tau; x)$ is an arbitrary function which can be represented by a Fourier transformation:

$$g(\tau; x) = \int e^{-ixv}\tilde{g}(\tau; v)\,\mathrm{d}v.$$

We note that

$$\int_{t_1}^{t_2} g(\sigma; y(\sigma))\,\mathrm{d}\sigma = \int_{t_1}^{t_2}\left(\int e^{\frac{1}{ih}\int_{t_1}^{t_2} y(\tau)u(\tau)\,\mathrm{d}\tau}\,\tilde{g}(\sigma; v)\,\mathrm{d}v\right)\mathrm{d}\sigma, \tag{1.28}$$

where $u(\tau) = h\delta(\tau - \sigma)v$. We need to find the path integral

$$F_1 = \int\left(\int_{t_1}^{t_2} g(\sigma; y(\sigma))\,\mathrm{d}\sigma\, e^{\frac{1}{2ih}K}\right)\prod \mathrm{d}y. \tag{1.29}$$

Substituting the value of the internal integral in (1.28) into (1.29), transposing the integration over dv and $d\sigma$ with the path integration, and using Eq. (1.27), we find

$$F_1 = \int_{t_1}^{t_2}\left(\int e^{-ixv}\tilde{g}(\sigma; v)\,\mathrm{d}v\right)\mathrm{d}\sigma = \int_{t_1}^{t_2} g(\sigma; x)\,\mathrm{d}\sigma. \tag{1.30}$$

[The second term in the exponential function in (1.27) is equal to $-(h/4i)v\omega v\,\operatorname{sign}(0) = 0$ in the case $u(\tau) = hv\delta(\tau - \sigma)$ since $v\omega v = 0$. The uncertainty in sign 0 is unimportant here.]

Let us consider the more general integral

$$F_n = \int\left(\int_{t_1}^{t_2} g(\sigma; y(\sigma))\,d\sigma\right)^n e^{\frac{1}{2ih}K}\prod \mathrm{d}y, \quad n > 1. \tag{1.31}$$

Working in a similar way, we find

$$F_n = \int \tilde{g}(\sigma_1; v_1)\ldots\tilde{g}(\sigma_n; v_n)\,\mathrm{d}^n v\,\mathrm{d}^n\sigma \int e^{-i\int_{t_1}^{t_2} y(u_1+\ldots+u_n)\mathrm{d}\tau + \frac{1}{2ih}K}\prod \mathrm{d}y$$
$$= \int \tilde{g}(\sigma_1; v_1)\ldots\tilde{g}(\sigma_n; v_n)\,e^{-ix(v_1+\ldots+v_n)+\frac{ih}{4}\sum v_i\omega v_j \operatorname{sign}(\sigma_i-\sigma_j)}\,\mathrm{d}^n v\,\mathrm{d}^n\sigma. \tag{1.32}$$

It follows in an obvious way from (1.32) that

$$F_n = O((t_2 - t_1)^n) \quad \text{as} \quad t_2 \to t_1.$$

We now consider the integrals in (1.9) and (1.14). Expanding $\exp(1/ih \int g(y(\tau), \tau)d\tau)$ in a series, and using the results, we find

$$G = \sum_0^\infty \left(\frac{1}{ih}\right)^n \frac{1}{n!}F_n = 1 + \frac{1}{ih}\int_{t_1}^{t_2} g(\tau; x)\,\mathrm{d}\tau + O((t_2 - t_1)^2), \tag{1.33}$$

where F_N is given by (1.32) for $n > 0$, $F_0 = 1$, and the second equation holds in the limit $t_2 \to t_1$. We can now prove the composition formula

$$\frac{1}{(\pi h)^{2n}}\int G(t_2, t' \mid x_1)\,G(t', t_1 \mid x_2)\,e^{\frac{2}{ih}(x_1\omega x_2 + x_2\omega x + x\omega x_1)}\,\mathrm{d}x_1\,\mathrm{d}x_2 = G(t_2, t_1 \mid x). \tag{1.34}$$

For this purpose we use the expansion (1.33). We transpose the summation with the integration over x_1 and x_2; then the integral in (1.34) transforms into a sum of terms of the type

$$\frac{1}{(\pi h)^n}\frac{1}{m!n!}\int \tilde{g}(\sigma_1; v_1)\ldots\tilde{g}(\sigma_m; v_m)\,\tilde{g}(\tau_1; u_1)\ldots\tilde{g}(\tau_n; u_n)$$
$$\times e^{-ix_1\sum v_k + ix_2\sum u_k + \frac{ih}{4}\left[\sum v_l\omega v_k \operatorname{sign}(\sigma_l-\sigma_k) + \sum u_l\omega u_k \operatorname{sign}(\tau_l-\tau_k)\right]}$$
$$\times e^{\frac{2}{ih}(x_1\omega x_2 + x_2\omega x + x\omega x_1)}\prod \mathrm{d}v_l \prod \mathrm{d}u_k \prod \mathrm{d}\sigma_l \prod \mathrm{d}\tau_k\,\mathrm{d}x_1\,\mathrm{d}x_2, \tag{1.35}$$

where the integration over σ_i is between the limits $t_1 \leqslant \sigma_i \leqslant t'$ and that over τ_i is between the limits $t' \leqslant \tau_i \leqslant t_2$.

Integrating over x_1 and x_2 in (1.35) (the stationary-phase method is convenient here), we find a new expression for (1.35):

$$\frac{1}{m!n!}\int \tilde{g}(\sigma_1; v_1)\ldots\tilde{g}(\sigma_m; v_m)\,\tilde{g}(\tau_1; u_1)\ldots\tilde{g}(\tau_n; u_n)$$
$$\times e^{-ix\left(\sum v_l + \sum u_k\right) + \frac{ih}{4}\left[\sum v_i\omega v_k \operatorname{sign}(\sigma_l-\sigma_k) + \sum u_l\omega u_k \operatorname{sign}(\tau_l-\tau_k)\right]}$$
$$\times e^{\frac{ih}{2}\sum u_i\omega v_k}\prod \mathrm{d}u_i \prod \mathrm{d}v_k \prod \mathrm{d}\sigma_i \prod \mathrm{d}\tau_k. \tag{1.36}$$

We now change the notation, setting $u_k = v_{m+k}$, $\tau_k = \sigma_{m+k}$. Then (1.36) can be rewritten in the more compact form

$$\frac{1}{m!n!}\int \tilde{g}(\sigma_1; v_1)\ldots\tilde{g}(\tau_{m+n}; v_{m+n})\,\theta(t' - \sigma_1)$$
$$\ldots\theta(t' - \sigma_m)\,\theta(\sigma_{m+1} - t')\ldots\theta(\sigma_{m+n} - t')$$
$$\times e^{-ix\sum v_k + \frac{ih}{4}\sum v_i\omega v_k \operatorname{sign}(\sigma_i-\sigma_k)}\prod \mathrm{d}v_k \prod \mathrm{d}\sigma_k, \tag{1.37}$$

and the integral over all σ_i is between the limits $t_1 \leqslant \sigma_i \leqslant t_2$. The sum of expressions (1.37) over all m and n satisfying $m + n = N$ can be written

$$\frac{1}{N!}\int \tilde{g}(\sigma_1; v_1)\ldots\tilde{g}(\sigma_N; v_N)\,(\theta(t' - \sigma_1)$$
$$+ \theta(\sigma_1 - t'))\ldots(\theta(t' - \sigma_N) + \theta(\sigma_N - t'))$$
$$\times e^{-ix\sum v_k + \frac{ih}{4}\sum v_i\omega v_k \operatorname{sign}(\sigma_i-\sigma_k)}\prod \mathrm{d}v_k \prod \mathrm{d}\sigma_k$$
$$= \frac{1}{N!}\int \tilde{g}(\sigma_1; v_1)\ldots\tilde{g}(\sigma_N; v_N)\,e^{-ix\sum v_k + \frac{ih}{4}\sum v_i\omega v_k \operatorname{sign}(\sigma_i-\sigma_k)}\prod \mathrm{d}v_k \prod \mathrm{d}\sigma_k. \tag{1.38}$$

By virtue of (1.33), the sum of the right sides of (1.38)

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[9] The role of the denominators inEqs. (1.22) and (1.26) is that they are sued to cancel out det P_N and det $\tilde{B}_N$. We wish to call attention to the fact that the limiting operator $B = \lim B_N$ is not degenerate: It follows from the condition $Bf = 0$, $f\,\mathscr{H}^0(t_1, t_2)$ [i.e., $f(t_1) + f(t_2) = 0$] that $f \equiv 0$. On the other hand, the operator $B = \lim \tilde{B}_N$ is degenrate: $\tilde{B}\hat{u}_0 = 0$, where $\hat{u}_0 = \{u_0, \alpha\}$, $u_0(x) \equiv \text{const} = \alpha$.

over all $N \geq 0$ is equal to $G(t_2, t_1|x)$. This proves Eq. (1.34).

Differentiating identity (1.34) with respect to t_2, setting $t' = t_2$, and using the second equation from (1.33), we find

$$\frac{\partial G(t_2, t_1|x)}{\partial t_2} = \frac{1}{(\pi h)^{2n}} \int \frac{1}{ih} g(t_2; x_1) G(t_2, t_1 | x_2) e^{\frac{2}{ih}(x_1 \omega x_2 + x_2 \omega x + x \omega x_1)} \, dx_1 \, dx_2. \tag{1.39}$$

We adopt the notation $\hat{G}(t_2, t_1)$, $\hat{g}(t)$ for the operators whose Weyl symbols are $G(t_2, t_2|x)$ and $g(t; x)$, respectively. Equation (1.39) is equivalent to the operator equation

$$\frac{\partial \hat{G}(t_2, t_1)}{\partial t_2} = \frac{1}{ih} \hat{g}(t_2) \hat{G}(t_2, t_1).$$

4) Paths on which the path integral is concentrated

The path integrals $G = \lim G_N$ and $G = \lim \tilde{G}_N$ defined above, where G_N and $\tilde{G}_N$ are determined from (1.22) or (1.26), are superficially reminiscent of integrals over a measure in a function space with a density

$$\delta\left(\frac{u(t_1) + u(t_2)}{2}\right) \exp\left\{-\frac{1}{2ih} \int_{t_1}^{t_2} u(\tau) \omega \dot{u}(\tau) \, d\tau\right\}$$

or

$$\exp\left\{-\frac{1}{2ih}\left[\int_{t_1}^{t_2} u(\tau) \omega \dot{u}(\tau) \, d\tau + u(t_1) \omega u(t_2)\right]\right\}$$

respectively. Obviously, however, neither of these expressions could be interpreted as the density of any measure in any sense. We must therefore refine the formulation of the problem of determining the particular paths on which these path integrals are concentrated.[10] We can formulate the problem as follows: We assume that $\mathscr{H}$ is some Hilbert space consisting of classical trajectories $x(t)$, $t_1 \leq t \leq t_2$. We denote by $S(r)$ a sphere of radius r in $\mathscr{H}$. We now consider a sequence of orthogonal projection operators with properties (1.18); we denote by $\mathscr{H}_N$ the space $P_N \mathscr{H}$ and by $S_N(r)$ the intersection of $S(r)$ with $\mathscr{H}_N$. We consider the integral

$$J_N(r) = \int_{y \in S_N(r)} e^{\frac{1}{2ih}(y, B_N y)} \, d^L y, \tag{1.40}$$

where $d^L y$ is the Lebesque measure in $P_N \mathscr{H}$, $L = \dim P_N \mathscr{H}$, and B_N are operators with properties (1.21). We denote by $J_N(\infty)$ the integral analogous to (1.40) but extended to the entire space $P_N \mathscr{H}$, and we set

$$\mathscr{Y}_N(r) = \frac{J_N(r)}{J_N(\infty)}. \tag{1.41}$$

Obviously, $\lim_{r \to \infty} \mathscr{Z}_N(r) = 1$.

We now assume that there is some sequence of projection operators P_N with property (1.18) such that

1) for each $r \geq 0$ there exists a limit $\mathscr{Z}(r) = \lim_{r \to \infty} \mathscr{Z}_N(r)$ and

2) there exists a limit $\lim_{r \to \infty} \mathscr{Z}_N(r)$, and $\lim_{r \to \infty} \mathscr{Z}(r) = 1$.

In this case we say that the path integral in (1.14) is concentrated in the space $\mathscr{H}$. Analogously, we are defining what we mean when we say that the integral (1.9) is concentrated in the space $\mathscr{H}$.

If the function $e^{1/2ih(y, By)}$ generated a measure in any function space K, then any Hilbert space containing K would have these properties. Our definition of the space in which the integral is concentrated is thus a natural generalization of the definition used in the situation in which the integral is generated by a measure.

The definition could of course be altered in such a manner that the role of the Hilbert space and $\mathscr{H}$ would be played by a Banach space or a linear topological space with suitable properties. This approach can apparently yield the same exhaustive description of the trajectories to which the integral is concentrated as is currently done, for example, for a Wiener measure. We shall not pursue this point further.

We now show that the integral in (1.14) is concentrated in the space $\mathscr{H}(t_1, t_2)$ with scalar product (1.17). For simplicity we assume the case of one degree of freedom. We introduce an orthonormal basis in $\mathscr{H}(t_1, t_2)$ consisting of vectors which satisfy the condition $x(t_1) + x(t_2) = 0$: $(e^{\pi i/T(2n+1)t}/0), (0/e^{\pi i/T(2n+1)})$. The expansion of a trajectory in this basis is an expansion in the Fourier series

$$x = \sum_{-\infty}^{\infty} x(2n+1) e^{(2n+1)\frac{\pi i}{T}t},$$
$$x(m) = \begin{pmatrix} \alpha(m) \\ \beta(m) \end{pmatrix}, \quad m = 2n+1, \ \alpha(m), \ \beta(m) - \tag{1.42}$$

where $\alpha(m)$ and $\beta(m)$ are the Fourier coefficients of the momentum and coordinate, respectively, and $\overline{x(m)} = x(-m)$. From (1.42) we find

$$\int_{t_1}^{t_2} x \omega \dot{x} \, d\tau = \pi i \sum m (\bar{\alpha}(m) \beta(m) - \bar{\beta}(m) \alpha(m)).$$

We denote by $\mathscr{H}_N$ the subspace of $\mathscr{H}(t_1, t_2)$ which consists of trajectories whose Fourier-series expansion contains those terms in (1.42) for which $|n| \leq N$. Looking ahead to the calculations below, we shall abandon the sphere in $\mathscr{H}_N$ and instead consider the ellipsoid $S_N(\sigma, \mu, r)$ defined by the inequality

$$\sum_{\left|\frac{m-1}{2}\right| \leq N} (\sigma(m) |\alpha(m)|^2 + \mu(m) |\beta(m)|^2) \leq r^2. \tag{1.43}$$

We can find the integral J_N over ellipsoid (1.43):

$$J_N(r) = \int_{S_N(\sigma, \mu, r)} e^{-\frac{1}{2ih}\int_{t_1}^{t_2} x \omega \dot{x} \, d\tau} \prod d\alpha \, d\bar{\alpha} \, d\beta \, d\bar{\beta}$$

$$= \int_{-r^2}^{r^2} ds \, \delta\left(s - \sum_{\left|\frac{m-1}{2}\right| \leq N} (\sigma(m) |\alpha(m)|^2 + \mu(m) |\beta(m)|^2)\right) e^{-\frac{1}{2ih}\int_{t_1}^{t_2} x \omega \dot{x} \, d\tau} \prod d\alpha \, d\bar{\alpha} \, d\beta \, d\bar{\beta}$$

$$= \frac{1}{2\pi} \int_{-r^2}^{r^2} ds \int \exp\left\{\frac{\pi}{2h} \sum_{\left|\frac{m-1}{2}\right| \leq N} m (\bar{\alpha}(m) \beta(m) - \bar{\beta}(m) \alpha(m)) + ip\left[s - \sum_{\left|\frac{m-1}{2}\right| \leq N} (\alpha(m) \bar{\alpha}(m) \sigma(m) + \beta(m) \bar{\beta}(m) \mu(m))\right]\right] \prod d\alpha \, d\bar{\alpha} \, d\beta \, d\bar{\beta} \cdot dp$$

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[10] If these integrals were integrals over a measure, the problem would be one of describing a space in which this measure was concentrated.

$$= \int \frac{e^{ipr^2}-e^{-ipr^2}}{2\pi ip}\left(\frac{h^2}{\pi^2}\right)^{2N+1}\frac{1}{[(2N+1)!!]^4}\frac{1}{\prod\limits_{\left|\frac{m-1}{2}\right|\le N}\left(1-\frac{p^2h^2\sigma(m)\mu(m)}{\pi^2m^2}\right)}\,dp$$

(the integration contour swings below the poles).

The integral $J_N(\infty)$ is equal to $J_N(\infty)=(h^2/\pi^2)^{2N+1}1/[(2N+1)!!]^4$. Hence

$$\frac{J_N(r)}{J_N(\infty)} = \int \frac{e^{ipr^2}-e^{-ipr^2}}{2\pi ip}\frac{1}{\prod\limits_{\left|\frac{m-1}{2}\right|\le N}\left(1-\frac{p^2h^2\sigma(m)\mu(m)}{\pi^2m^2}\right)}\,dp.$$

Furthermore,

$$\mathscr{I}(r)=\lim_{N\to\infty}\frac{J_N(r)}{J_N(\infty)}=\int \frac{e^{ipr^2}-e^{-ipr^2}}{2\pi ip}F^{-1}(p)\,dp, \qquad (1.44)$$

where

$$F(p)=\prod_{\substack{m=2n+1\\ -\infty<n<\infty}}\left(1-\frac{p^2h^2\sigma(m)\mu(m)}{\pi^2m^2}\right). \qquad (1.45)$$

The infinite product in (1.45) converges if

$$\sum\frac{\sigma(2n+1)\mu(2n+1)}{(2n+1)^2}<\infty. \qquad (1.46)$$

In the case $\sigma(m)=\mu(m)=1$, in which we are interested, condition (1.46) is obviously satisfied. The first factor in the integrand in (1.44) assumes the value $\delta(p)$ in the limit $r\to\infty$. The function $F(p)$ is obviously regular at $p=0$ under condition (1.46), and $F(0)=1$. We thus have $\lim_{r\to\infty}\mathscr{I}(r)=1$.

We have proved our assertion. This discussion leads, however, to another, extremely curious result: It turns out that it is possible to specify a space of paths on which the integral in which we are interested is concentrated and for which the smoothness of the momenta is improved at the expense of the smoothness of the coordinates; alternatively, it is possible to specify a different space, in which the smoothness of the coordinates is improved at the expense of the smoothness of the momenta. We denote by $\mathscr{H}^{\sigma,\mu}$ the Hilbert space of paths with the scalar product

$$(x,x)=\sum_{-\infty}^{\infty}|\alpha(2n+1)|^2\sigma(2n+1)+\sum_{1}^{\infty}|\beta(2n+1)|^2\mu(2n+1). \qquad (1.47)$$

Here $\mathscr{H}_N^{\sigma,\mu}$ denotes the subspace of paths for which $\alpha(2n+1)=\beta(2n+1)=0$ with $\eta>N$. Ellipsoid (1.43) in the space $\mathscr{H}_N$ is a sphere in $\mathscr{H}_N(\sigma,\mu)$. We now assume

$$\sigma(2n+1)=(2n+1)^{1+\varepsilon},\quad \mu(2n+1)=(2n+1)^{-2\varepsilon},\quad \varepsilon>0. \qquad (1.48)$$

Condition (1.46) obviously holds. We can show that condition (1.48) guarantees the continuity of the momenta. It follows from (1.47) that

$$\alpha(2n+1)=\frac{\gamma(n)}{\sqrt{\sigma(2n+1)}},\quad \sum|\gamma(n)|^2<\infty. \qquad (1.49)$$

On the other hand, from (1.48) we have $\sum 1/\sigma(2n+1)<\infty$, so that

$$\sum_{1}^{\infty}|\alpha(2n+1)|<\infty. \qquad (1.50)$$

Condition (1.50) implies that the momenta are continuous. With regard to the coordinates, we see from (1.48) that their differential properties are worse at any $\varepsilon>0$ than those of functions which are square-summable. (They may have singularities which are not square-integrable.) For $\varepsilon>1/2$ the coordinates may in fact be generalized functions. By changing the roles of $\sigma(n)$ and $\mu(n)$, we are improving the smoothness of the coordinates at the expense of the momenta.

This result agrees well with the uncertainty principle: The value of any function at a point can be measured only if this function is continuous at that point. Consequently, the coordinate and momentum cannot be simultaneously continuous. When we try to get more-detailed information on the coordinate (i.e., when the smoothness of the coordinate is improved) we find we can do so only by degrading the situation for the momenta (conversely, the smoother the momentum, the poorer the differential properties of the coordinate).

Which space should we use to evaluate the integral? The answer apparently runs as follows: The space $\mathscr{H}(t_1,t_2)$ is suitable in all cases. In some cases, better results (i.e., a faster convergence of the finite-dimensional approximations) can be obtained with a different space $\mathscr{H}^{\sigma,\mu}$. The particular situation determines which space should be used. For example, if $g=(\frac{1}{2})p^2+v(q)$, then it is natural to consider a space which guarantees the continuity of q but not its differentiability. It seems obvious that in this case the integral will be determined primarily by paths that are similar to Brownian trajectories.

5) Perturbation theory

Equation (1.33) (more precisely, the first equation there) can be thought of as a perturbation-theory series for the evolution operator symbol. The corresponding unperturbed infinitesimal operator is $\hat{g}=0$. The terms in this series are given in (1.32); they are expressed explicitly in terms of the Fourier transformation of the symbol representing the infinitesimal evolution operator. This circumstance is frequently disadvantageous. We shall accordingly put the perturbation-theory series in a form which explicitly involves the symbol representing the infinitesimal operator itself.

We consider an operator in the functional space

$$L=-\frac{1}{4}\int_{\tilde{t}_1}^{\tilde{t}_2}\int_{\tilde{t}_1}^{\tilde{t}_2}\operatorname{sign}(t-s)\frac{\delta}{\delta x(t)}\,\omega\,\frac{\delta}{\delta x(s)}\,dt\,ds, \qquad (1.51)$$

where $(\tilde{t}_1,\tilde{t}_2)$ is an arbitrary interval which includes the interval (t_1,t_2). [In practice, the cases $(\tilde{t}_1,\tilde{t}_2)=(t_1,t_2)$ and $\tilde{t}_2=-\infty,\tilde{t}_2=+\infty$ are convenient.] We note that the functional

$$\tilde{F}_n=\int_{t_1\le\sigma_i\le t_2}\tilde{g}(\sigma_1;v_1)\ldots\tilde{g}(\sigma_n;v_n)\,e^{-i(x(\sigma_1)v_1+\ldots+x(\sigma_n)v_n)}\,d\sigma^n \qquad (1.52)$$

is an eigenfunctional for L with the eigenvalue

$$\frac{1}{4}\sum\operatorname{sign}(\sigma_k-\sigma_e)\,v_k\omega v_e.$$

The function in (1.32) can thus be written

$$F_n=\int e^{ihL}\tilde{F}_n|_{x(\tau)=x}\,d^nv\,d^n\sigma=e^{ihL}\int\tilde{F}_n\,d^nv\,d^n\sigma|_{x(\tau)=x}$$
$$=e^{ihL}\left(\int_{t_1}^{t_2}g(\sigma;x(\sigma))\,d\sigma\right)^n\Big|_{x(\sigma)=x}.$$

Using (1.33), we finally find

$$G(t_2, t_1 | x) = e^{ihL} e^{\frac{i}{ih}\int_{t_1}^{t_2} g(\tau;\, x(\tau))d\tau}\Big|_{x(\tau)=x}. \quad (1.53)$$

b: Wick evolution operator symbol

1) Basic construction

We denote by $\hat{a}^*(i)$, and $\hat{a}(i)$ $(i=1, \ldots, n)$ the Bose or Fermi creation and annihilation operators:

$$[\hat{a}(i),\ \hat{a}^*(j)]_\pm = h\delta_{ij},$$

where $\hat{H}$ is some operator in the Fock space, and $H(a^*, a)$ is its Wick symbol,

$$\hat{H} = \sum H(i_1, \ldots, i_k | j_1 \ldots j_l)\, \hat{a}^*(i_1) \ldots a^*(i_k)\, \hat{a}(j_l) \ldots \hat{a}(j_1),$$
$$H(a^*, a) = \sum H(i_1 \ldots i_k | j_1 \ldots j_l)\, a^*(i_1) \ldots a^*(i_k)\, a(j_l) \ldots a(j_1).$$

In the Bose case, $a(j) = (1/\sqrt{2})(q(j) + ip(j))$ are holomorphic coordinates of the phase space, and $a^*(j) = (1/\sqrt{2})(q(j) - ip(j))$ are antiholomorphic coordinates of the phase space. In the Fermi case, $a(j)$ are the generators of a Grassman algebra with involution. In both cases, we can use the notation $a^* = (a^*(j), \ldots, a^*(j))$, $ab = \sum a(j)b(j)$. When applied to a number or a function, the asterisk means complex conjugation (unless otherwise stipulated); when applied to an element of the Grassmann algebra, it means an involution; $a(j)$ and $a^*(j)$ are independent variables in the Bose case and independent generators of the Grassmann algebra in the Fermi case.[11]

The basic equations and their derivations are essentially the same in the Bose and Fermi cases, so we shall discuss only the Bose case in detail. The Fermi case will be covered in the final subsection of this section.

We denote by $f_i(a^*, a)$ the Wick symbols of the operators f_i $(i=1, \ldots, N)$, $\hat{G}_N = \hat{f}_1 \cdots \hat{f}_N$; also, G_N is the Wick symbol of $\hat{G}_N$. Using the composition formula for the Wick symbols, we find

$$G_N(a^*, a) = \left(\frac{1}{2\pi h}\right)^{n(N-1)} \int f_1(a^*, a_1)\, f_2(a_1^*, a_2) \ldots f_N(a_{N-1}^*, a) \times e^{\frac{1}{h}K_N}\, da_1^*\, da_1 \ldots da_{N-1}^*\, da_{N-1},$$

$$K_N = (a^* - a_1^*)\, a_1 + \sum_1^{N-2} (a_k^* - a_{k+1}^*)\, a_{k+1} + (a_{N-1}^* - a^*)\, a. \quad (2.1)$$

As before, we introduce the continuous parameter t, $(t_1 < t < t_2)$, and we set

$$a_k = a(\tau_k), \quad a_k^* = a^*(\tau_k),$$
$$f_k(a^*, a) = e^{\frac{\Delta}{ih} g(\tau_k;\, a^*,\, a)},$$

where $a(\tau) = \{a(\tau; 1), \ldots, a(\tau; n)\}$ is the phase path in the complex coordinates, $a(t;k) = (1/\sqrt{2})(q(t;k) + ip(t;k))$, $a^*(t) = \{a^*(\tau; 1), \ldots, a^*(\tau; n)\}$, $a(\tau; k)$, and $g(\tau; a^*, a)$ are continuously differentiable functions of τ, $\tau_k = t_2 - (k/N)(t_2 - t_1)$, and $\Delta = (1/N)(t_2 - t_1)$. Furthermore, the form of the integrand in (2.1) suggests that the notation $a_0^* = a^*$, $a_N = a$ is natural; equivalently, it suggests the boundary conditions

$$a^*(t_2) = a^*, \quad a(t_1) = a \quad (2.2)$$

for the phase paths $a(\tau)$. Under these assumptions, K_N has the following value in the limit $N \to \infty$

$$K = \lim K_N = \int_{t_1+0}^{t_2} \frac{da^*(\tau)}{d\tau}\, a(\tau - 0)\, d\tau + (a^*(t_1 + 0) - a^*)\, a \quad (2.3)$$

[the last term for K_N in (2.1) does not appear in the integral sum]. Incorporating the factor $(2\pi h)^{-(N-1)n}$ in the normalization of the differentials, we find the following final expression for $G = \lim G_N$:

$$G(t_2, t_1 | a^*, a) = \int_{\substack{a^*(t_2)=a^*,\\ a(t_1)=a}} \exp\left\{\frac{1}{ih}\int_{t_1+0}^{t_2} g(\tau;\, a^*(\tau),\, a(\tau - 0))\, d\tau + \frac{1}{h}\int_{t_1+0}^{t_2} \frac{da^*(\tau)}{d\tau}\, a(\tau - 0)\, d\tau + \frac{1}{h}(a^*(t_1 + 0) - a^*)\, a\right\} \times \prod_\tau da^*(\tau)\, da(\tau). \quad (2.4)$$

We wish to call attention to the following circumstances.

1. Equations (2.3) and (2.4) contain the expressions $\tau - 0$ and $t_1 + 0$. These expressions remind us of the analogous shift of the arguments in the expressions before the limit is taken:

$$\sum_0^{N-1} \Delta g(\tau_k;\, a^*(\tau_k),\, a(\tau_{k+1})) \to \int_{t_1+0}^{t_2} g(\tau;\, a^*(\tau),\, a(\tau - 0))\, d\tau,$$

$$\sum_0^{N-2} (a^*(\tau_k) - a^*(\tau_{k+1}))\, a(\tau_{k+1}) = \sum \Delta \frac{da^*}{d\tau}\Big|_{\tau=\tilde{\tau}_k} a(\tau_{k+1}) \to \int_{t_1+0}^{t_2} \frac{da^*}{d\tau}\, a(\tau - 0)\, d\tau,$$

$$(a^*(\tau_{N-1}) - a^*)\, a \to (a^*(t_1 + 0) - a^*)\, a. \quad (2.5)$$

[We recall that $\tau_k = t_2 - k\Delta$, $\Delta = (t_2 - t_1)/N$; the first equation in the second row was found with the help of the Lagrange equation, and $\tau_{K_H} \leq \tilde{\tau}_k < \tau_k$.] If the trajectory $\alpha(\tau)$ is continuously differentiable, then we can of course replace $\tau - 0$ by τ_N and $t_1 + 0$ by t_2. Actually, however, as in the Weyl case, the integral (2.4) is concentrated on discontinuous paths. The shift of the arguments is therefore important. Also important is the subtle point that the shift of the argument $a(\tau)$ in the expression for K is smaller than in the first term in the exponential function in (2.4). This can be seen from Eq. (2.5); in the expression for the first term before the limit is taken the shift is $\Delta = \tau_k - \tau_{k+1}$, while in the expression for K before the limit is taken the shift is $\tau_k - \tilde{\tau}_{k+1} \leq \Delta$. This circumstance could be incorporated in Eq. (2.4), but we shall not do this since we shall be discussing below a refinement of Eq. (2.4) which ex-

[11] The relationship between the Wick symbols and the operators in the Bose case can also be described by equations analogous to the Weyl equations in (1.1). We denote by $f(a^*, a)$ a function in the phase space which can be represented as a Fourier transform:

$$f(a^*, a) = \int \tilde{f}(z^*, z)\, e^{i(a^*z + z^*a)} \prod dz^*\, dz.$$

We set

$$\hat{f} = \int \tilde{f}(z^*, z)\, e^{i\hat{a}^*z} e^{iz^*\hat{a}} \prod dz^*\, dz.$$

The function f is the Wick symbol of the operator $\hat{f}$. The important distinction between the Wick and Weyl symbols is that the Weyl equations in (1.1) are meaningful for essentially any functions f, while the analogous Wick equations are meaningful for only entire analytic functions of p_k, q_k of a certain class. (see the Supplement).

plicitly incorporates the difference in the shift of the arguments [see Eq. (2.19) below].

2. We also note that in the absence of an argument shift the first term in the exponential function in (2.4) would be

$$\int_{t_1}^{t_2} g(\tau; a^*(\tau), a(\tau))\, d\tau. \tag{2.6}$$

The integrand in (2.6) is meaningful if the function $g(\tau; a^*, a)$ is a continuous function of τ and of the phase variables p_k, q_k. At the same time, the expression in (2.1) before the limit is taken contains the sum

$$\Delta \sum g(\tau_k; a_k^*, a_{k+1})$$

in an exponential function. Since the arguments a_k, a_{k+1} have different numbers, they are independent complex variables. Expression (2.1) thus assumes that it is possible to continue the function $g(\tau; a^*, a)$ analytically into the complex region in the phase variables p_k, q_k. This is a point of fundamental importance: According to the meaning of the problem at hand, the function $g(\tau; a^*, a)$ is the Wick symbol of an infinitesimal evolution operator, but we know that the Wick symbol $A(a^*, a)$ of any operator $\hat{A}$ is an entire function of phase variables.

3. We mentioned earlier that Eq. (2.1) suggests that the notation $a_0^* = a^*$, $a_N = a$ would be natural; this notation is equivalent to (2.2). Let us adopt this notation; we note that a_0 and a_N^* are not present in (2.1). We are thus justified in setting $a_0 = a_N = a$, $a_N^* = a_0^* = a^*$. With this notation we can now rewrite Eq. (2.1) in the more compact form

$$F_N(a^*, a) = \int e^{\frac{1}{h}\left(\sum_0^{N-1} i\Delta g(\tau_k; a_k^*, a_{k+1}) + (a_k^* - a_{k+1}^*) a_{k+1}\right)} \prod_1^{N-1} da_k^*\, da_k. \tag{2.7}$$

The formal limit $N \to \infty$ in (2.7) would lead us to Eq. (2.4) without the term in the exponential function which is outside the integrals. The same result could be found by ignoring the shift of the argument in the last term in the exponential function in the integrand in (2.4). Adding to Eqs. (2.2) their complex conjugates, we find

$$a(t_2) = a, \quad a^*(t_1) = a^*. \tag{2.8}$$

In the absence of an argument shift, Eq. (2.8) cause the term outside the integrals in the exponential function in (2.4) to vanish.

4. Equations (2.2) and (2.8) show that integral (2.4) is to be evaluated along closed phase paths. These equations, however, play very different roles. The most obvious difference between them can be seen by examining an approximation of the general type for the integral (2.4). We see that, because of the argument shift, the holomorphic coordinates of the phase path, $a(\tau)$, $a^*(\tau)$, participate in the construction of the approximation on the interval $t_1 \leqslant \tau \leqslant t_2 - \sigma$, while the antiholomorphic coordinates participate on the interval $t_1 + \sigma \leqslant \tau \leqslant t_2$, where $\sigma > 0$ is some number. Equations (2.8) thus hold on intervals of the paths $a(t)$ and $a^*(t)$ which are not involved in the construction of the finite-dimensional approximations. At the same time, Eqs. (2.2) hold on important parts of the paths. Although these differences are not so obvious, they can also be seen in the original approximation [(2.1), (2.7)] of integral (2.4).

There is yet another important circumstance related to these differences. When the method of steepest descent is applied to integral (2.4) on a stationary trajectory, Eqs. (2.2) always hold. At the same time, if this trajectory is complex, it may not satisfy (2.8).

Taking all these circumstances into account, we incorporate (2.2) in the description of the integration range for integral (2.4), and we omit (2.8).

2) Approximations of a general type

We set

$$a(t) = a + b(t), \quad a^*(t) = a^* + b^*(t). \tag{2.9}$$

Using conditions (2.2), we can transform (2.3) for K to

$$K = \int_{t_1+0}^{t_2} \frac{db^*}{d\tau} b(\tau - 0)\, d\tau. \tag{2.10}$$

We adopt (2.10) as the basis for the further constructions. We denote by $\mathscr{H}(t_1, t_2)$ the Hilbert space of the trajectories $a(t) = 1/\sqrt{2}\,(q(t) + p(t))$ with the scalar product

$$(a_1^*, a_2) = \int_{t_1}^{t_2} a_1^*(t)\, a_2(t)\, dt. \tag{2.11}$$

We denote by $\tilde{\mathscr{H}}(t_1, t_2) \subset \mathscr{H}(t_1, t_2)$ the subset of $\mathscr{H}(t_1, t_2)$ which consists of continuously differentiable paths, and we denote by $\tilde{\mathscr{H}}^0(t_1, t_2) \subset \tilde{\mathscr{H}}(t_1, t_2)$ the subset of $\tilde{\mathscr{H}}(t_1, t_2)$ which consists of paths satisfying the condition

$$a(t_1) = a(t_2) = 0. \tag{2.12}$$

We consider the operator B_σ, which is defined on $\tilde{\mathscr{H}}^0(t_1, t_2)$:

$$(B_\sigma f)(\tau) = \theta(t_2 - \sigma - \tau)\, \theta(\tau - t_1) \frac{d}{d\tau} f(\tau + \sigma). \tag{2.13}$$

We note that the operator B_σ is nilpotent: It follows from (2.13) that $B^n = 0$ for $n > (t_2 - t_1)/\sigma$. It follows that for any $\delta > 0$ and $f \neq 0$, $f \in \tilde{\mathscr{H}}^0(t_1, t_2)$, we have

$$\operatorname{Re}(\delta(f^*, f) + (B_\sigma f^*, f)) > 0. \tag{2.14}$$

We turn now to a description of the approximations. We assume that P_N are orthogonal projection operators in $\mathscr{H}(t_1, t_2)$ with properties (1.18), $\mathscr{H}_N = P_N \mathscr{H}(t_1, t_2)$, $\tilde{\mathscr{H}}_N^0 = P_N \tilde{\mathscr{H}}^0(t_1, t_2)$. We denote by $B_{N,\sigma}$ operators in $\mathscr{H}_N$ which have the following properties:

1) $\lim B_{N,\sigma} P_N f = B_\sigma f$ for any path $f \in \tilde{\mathscr{H}}^0(t_1, t_2)$.

2) For each $\delta > 0$ there exists a number N_0 such that at $N > N_0$ a relation analogous to (2.14) holds,

$$\operatorname{Re}(\delta(f^*, f) + (B_{N,\sigma} f^*, f)) > 0 \tag{2.15}$$

for $f \neq 0$, $f \in \tilde{\mathscr{H}}_N^0$.

We define $F_{N,\varepsilon,\sigma}(\delta \mid t_2, t_1; a^0, a)$ as follows:

$$F_{N,\varepsilon,\sigma}(\delta \mid t_2, t_1 \mid a^*, a) = \int_{\tilde{\mathscr{H}}_N^0} \exp\left\{\frac{1}{h}\left[\frac{1}{i} \int_{t_1+\varepsilon}^{t_2} g(\tau; a^*(\tau), a(\tau - \varepsilon))\, d\tau + (B_{N,\sigma} b^*, b) - \delta(b^*, b)\right]\right\} \prod db\, db^*. \tag{2.16}$$

where $b \in \mathscr{H}^0(t_1, t_2)$, $a(\tau) = a + b(\tau)$, $\varepsilon > \sigma > 0$, and $\delta > 0$ is a number which satisfies condition (2.15). We denote by $F^{(0)}_{N,\varepsilon,\sigma}(\delta\,|t_2, t_1|a^*, a)$ the analogous function obtained from (2.16) in the case $g(\tau; a^*, a) \equiv 0$.

We then set

$$G_{N,\varepsilon,\sigma}(\delta\,|\,t_1, t_2\,|\,a^*, a) = \frac{F_{N,\varepsilon,\sigma}(\delta\,|\,t_1, t_2\,|\,a^*, a)}{F^0_{N,\varepsilon,\sigma}(\delta\,|\,t_1, t_2\,|\,a^*, a)}, \tag{2.17}$$

$$G_{\varepsilon,\sigma} = \lim_{\delta\to 0}\lim_{N\to\infty} G_{N,\varepsilon,\sigma}(\delta). \tag{2.18}$$

Equation (2.18) serves as a definition of the path integral:

$$G_{\varepsilon,\sigma}(t_2, t_1\,|\,a^*, a) = \int_{\substack{a^*(t_2)=a,\\ a(t_1)=a}} \exp\left\{\frac{1}{h}\left[\frac{1}{i}\int_{t_1+\varepsilon}^{t_2} g(\tau; a^*(\tau), a(\tau-\varepsilon))\,d\tau + \int_{t_1+\sigma}^{t_2}\frac{da^*}{d\tau}a(\tau-\sigma)\,d\tau + (a^*(t_1+\sigma) - a^*)\,a\right]\right\}\prod_\tau da^*(\tau)\,da(\tau). \tag{2.19}$$

The evolution operator symbol G is related to $G_{\varepsilon,\sigma}$ by

$$G(t_1, t_2\,|\,a^*, a) = \lim_{\substack{\varepsilon,\sigma\to 0\\ \varepsilon>\sigma}} G_{\varepsilon,\sigma}(t_1, t_2\,|\,a^*, a). \tag{2.20}$$

We note a characteristic feature of integral (2.19): It involves holomorphic path coordinates $a(\tau)$, $a^*(\tau)$ on the interval $t_1 + \sigma \leqslant \tau \leqslant t_2$ and antiholomorphic coordinates of this path on the interval $t_1 \leqslant \tau \leqslant t_2 - \sigma$ (see footnote 4 at the end of Sec. 1).

Equations (2.16)–(2.20) contain a description of a scheme of finite approximations of the path integral for the evolution operator symbol. These approximations are quite general and seem to be the most convenient for applications.

It should be kept in mind, however, that this scheme is still not the most general scheme possible; it is easy to see that it does not include the original approximation, (2.1). This omission can be remedied by generalizing the scheme: We "permit" the numbers σ, ε, and δ to depend on N.

3) Derivation; paths on which the integral is concentrated

As in the Weyl case, the generalization of Eqs. (2.18)–(2.20) is based on an evaluation of an integral for the function $g(t; a^*, a)$, which is a linear function of a and a^*. We assume $g(t; a^*, a) = f(t)a^* + f^*(t)a$. It is convenient to assume $f(t) = f^*(t) = 0$ for $t < t_1$ and $t > t_2$. We set

$$\left.\begin{aligned} f_1(\tau) &= \begin{cases} f(\tau) & \text{for}\quad t_1+\varepsilon < \tau < t_2,\\ 0 & \text{for}\quad \tau > t_2,\ \tau < t_1+\varepsilon,\end{cases}\\ f_2(\tau) &= \begin{cases} f^*(\tau+\varepsilon) & \text{for}\quad t_1 < \tau < t_2-\varepsilon,\\ 0 & \text{for}\quad \tau > t_2-\varepsilon,\ \tau < t_1,\end{cases}\\ \bar{f}_i &= \int_{t_1}^{t_2} f_i(\tau)\,d\tau. \end{aligned}\right\} \tag{2.21}$$

In terms of this notation, the exponential function in (2.16) becomes

$$\frac{1}{i}(a^*\bar{f}_1 + \bar{f}_2 a) + \frac{1}{i}((b^*, f_1) + (f_2, b)) + (B_{N,\sigma}b^*, b) - \delta(b^*, b). \tag{2.22}$$

Using the method of steepest descent, we can evalute the integral in (2.16); for $G_{N,\varepsilon,\sigma}(\delta)$ we find the following expression:

$$G_{N,\varepsilon,\sigma}(\delta) = e^{\frac{1}{h}\left[\frac{1}{i}(\bar{f}_1 a^* + \bar{f}_2 a) - ((\delta - B_{N,\sigma})^{-1}P_N f_2,\, P_N f_1)\right]}. \tag{2.23}$$

To find the operator $(\delta - B_{N,\sigma})^{-1}$ we should solve the equation

$$(\delta - B_{N,\sigma})\,P_N\varphi = P_N\psi.$$

Taking the limit $N\to\infty$ in this equation, and using (2.13), we find

$$\delta\varphi(\tau) - \theta(t_2 - \sigma - \tau)\,\theta(\tau - t_1)\frac{d}{d\tau}\varphi(\tau+\sigma) = \psi(\tau). \tag{2.24}$$

We will have to take the limit $\delta\to 0$ below. Obviously, Eq. (2.24) with $\delta = 0$ can be solved only if $\psi(\tau) = 0$ for $\tau > t_2 - \sigma$. Under this condition, this equation has a solution with the boundary condition $\varphi(t_1) = \varphi(t_2) = 0$, which is defined uniquely for $t_1 + \sigma < \tau < t_2$:

$$\varphi(\tau) = \lim_{\delta\to 0}(\delta - B_\sigma)^{-1}\psi = \int_{\tau-\sigma}^{t_2-\sigma}\psi(s)\,ds.$$

For $t_1 < \tau < t_1 + \sigma$, the function φ is not defined on this interval; it is arbitrary, except for the single requirement[12] $\varphi(t_1) = 0^*$.

We now recall that $\varepsilon > \sigma$, so it follows from (2.21) that the function f_2 has the necessary property (i.e., $f_2 = 0$ for $\tau > t_2 - \sigma$). We finally find

$$\lim_{\delta\to 0}\lim_{N\to\infty}(\delta - B_{N,\sigma})^{-1}P_N f_2 = \int_{\tau-\sigma}^{t_2-\sigma} f_2(s)\,ds = \int_{\tau+\varepsilon-\sigma}^{t_2} f^*(s)\,ds.$$

Hence,

$$\lim_{\delta\to 0}\lim_{N\to\infty}((\delta - B_{N,\sigma})^{-1}P_N f_2,\ P_N f_1) = \iint_{\substack{t_1+\sigma<\tau<t_2,\\ t_1+\varepsilon<s<t_2}} \theta(s - \tau - \varepsilon + \sigma)\,f^*(s)\,f(\tau)\,ds\,d\tau.$$

Thus

$$G(t_1,\ t_2;\ a^*,\ a) = \exp\left[\frac{1}{ih}\int_{t_1}^{t_2}(a^* f(\tau) + f^*(\tau)\,a)\,d\tau - \frac{1}{h}\iint_{t_1\leqslant\tau,\ s\leqslant t_2}\theta(s - \tau - 0)\,f^*(s)\,f(\tau)\,ds\,d\tau\right]. \tag{2.25}$$

The term -0 in (2.25), which supplements the definition of $\theta(\tau - s)$ at $\tau = s$, is not important if the functions $f(s)$ and $f(\tau)$ are ordinary functions, but does become important for generalized functions.

Let us assume that the function $g(\sigma; a^*, a)$ can be represented as a Fourier transformation:

$$g(\tau;\ a^*,\ a) = \int \tilde{g}(\tau;\ v^*,\ v)\,e^{-i(a^*v + av^*)}\,dv^*\,dv. \tag{2.26}$$

Using (2.25), we can evaluate the integral

$$G^{(n)}(t_2,\ t_1;\ a^*,\ a) = \lim_{\substack{\varepsilon,\sigma\to 0\\ \varepsilon>\sigma}}\int\left(\int_{t_1+\varepsilon}^{t_2} g(\tau;\ a^*(\tau),\ a(\tau-\varepsilon))\right)^n e^{\frac{1}{h}(B_\sigma b^*,\, b)}\prod db^*\,db. \tag{2.27}$$

As in the Weyl case, we substitute (2.26) into (2.27), and we transpose the path integration with the integration over τ, v, and v^*. We then use integral (2.25) with

$$f(t) = h\sum_1^n v_k\,\delta(t - \tau_k),\quad f^*(t) = h\sum_1^n v_k^*\,\delta(t - \tau_k).$$

---

[12] We recall that the region of definition of the operator $B\sigma$ is the space $\mathscr{H}^0(t_1, t_2)$, which consists of paths with homogeneous boundary conditions.

As a result we find

$$G^{(n)}(t_2, t_1 | a^*, a) = \int \tilde{g}(\tau_1; v_1^*, v_1) \ldots \tilde{g}(\tau_n; v_n^*, v_n) e^{\frac{1}{i}\left(a^*\left(\sum v_k\right) + \left(\sum v_k^*\right) a\right) - h \sum v_k^* v_l \theta(\tau_k - \tau_l - 0)} \times d^n v \, d^n v \, d^n \tau. \quad (2.28)$$

The integral in (2.4) can be expressed in terms of the integrals in (2.28) by a series

$$G(t_2, t_1 | a^*, a) = \sum \left(\frac{1}{ih}\right)^n \frac{1}{n!} G^{(n)}(t_2, t_1 | a^*, a). \quad (2.29)$$

It follows from Eqs. (2.28) and (2.29) that $G(t_2, t_1 | a^*, a)$ is a Wick evolution operator symbol; the infinitesimal operator corresponding to this evolution has the symbol $g(t; a^*, a)$. To derive this result from (2.28) and (2.29) would be to reproduce the corresponding part of the derivation completely for the case of Weyl symbols, and we shall accordingly skip this step.

As in the case of Weyl symbols, we are defining what we mean when we say that the integral in (2.4) is concentrated on some set of functions or other and when we say that is concentrated on a set of paths $\mathscr{H}(t_1, t_2)$ which are square-summable. However, there is hardly any point in a further refinement of the set of paths on which the integral in (2.4) is concentrated (this refinement would be similar to that which was carried out for the case of Weyl symbols).

4) Perturbation theory

We can put the perturbation-theory equations in (2.28), (2.29) in a form which is at once more compact and more convenient for applications. We consider the operator L in the functional space

$$L = -i \int\limits_{\tilde{t}_1 < t,\, s < \tilde{t}_2} \theta(s - t - 0) \frac{\delta}{\delta a(s)} \frac{\delta}{\delta a^*(t)} \, ds \, dt, \quad \tilde{t}_1 \leqslant t_1, \ \tilde{t}_2 \leqslant t_2. \quad (2.30)$$

We note that the functional $u = e^{1/ih \int_{t_1}^{t_2} g(\tau; a^*(\tau), a(\tau)) d\tau}$ is an eigenfunctional for L if $g(\tau; a^*(\tau), a(\tau)) = a^*(\tau) f(\tau) + f^*(\tau) a(\tau)$:

$$Lu = \frac{i}{h^2} \int \theta(s - t - 0) f^*(s) f(t) \, dt \, ds u. \quad (2.31)$$

Equation (2.25) for the evolution operator symbol can thus be rewritten

$$G(t_2, t_1 | a^*, a) = e^{ihL} e^{\frac{1}{ih} \int_{t_1}^{t_2} g(\tau; a(\tau), a^*(\tau)) d\tau} \Bigg|_{\substack{a(\tau) = a, \\ a^*(\tau) = a^*}}. \quad (2.32)$$

We now note that series (2.29) is none other than an expansion in the eigenfunctions of the operator L. It then follows from (2.28) and (2.29) that Eq. (2.32) remains valid for functionals $g(\tau; a^*, a)$ which are Wick symbols of operators of a general kind. We note that all the argument shifts in the preceding steps are concentrated in the term -0 in Eq. (2.30). It is not difficult to see that the presence of this term is equivalent to the following equation (at least from the standpoint of the perturbation theory):

$$Lg = 0. \quad (2.33)$$

In using the perturbation theory we are thus justified in ignoring the term -0 in (2.30), replacing it with (2.33).

5) Method of steepest descent

The equation for the stationary path of the integral, (2.16), transforms in the limit $N \to \infty$, $\delta \to 0$ into an equation for the stationary path of integral (2.19):

$$\frac{d}{d\tau} a^*(\tau + \sigma) + \frac{1}{i} \frac{\partial}{\partial a} g(\tau; a^*(\tau + \varepsilon), a(\tau)) = 0, \quad t_1 < \tau < t_2 - \sigma,$$

$$-\frac{d}{d\tau} a(\tau - \sigma) + \frac{1}{i} \frac{\partial}{\partial a^*} g(\tau; a^*(\tau), a(\tau - \varepsilon)) = 0, \quad t_1 + \sigma < \tau < t_2. \quad (2.34)$$

The boundary conditions on these equations are $a(t_1) = a(t_2) = a$, $a^*(t_1) = a^*(t_2) = a^*$. We note, however, that $a(\tau)$ appears in Eq. (2.37) only for $t_1 < \tau < t_2 - \sigma$, and $a^*(\tau)$ appears only for $t_1 - \sigma < \tau < t_2$. Accordingly, only the following relations are important:

$$a^*(t_2) = a^*, \quad a(t_1) = a. \quad (2.35)$$

The functions $a(t)$ and $a^*(t)$, which serve as the solution of Eqs. (2.34) with conditions (2.35), are not necessarily complex conjugates. If they are not, this means that the trajectory is complex, $a(t) = 1/\sqrt{2}(g(t) + ip(t))$, $a^*(t) = 1/\sqrt{2}(q(t) - ip(t))$: $q(t) = (q(t; 1), \ldots, q(t; n))$, $p(t) = (p(t; 1), \ldots, p(t; n))$, where $q(t, k)$ and $p(t; k)$ are complex functions. The functions $a(t)$ and $a^*(t)$, which are the solution of system (2.34) with condition (2.35), may also be other than complex conjugates in the limit $\varepsilon = \sigma = 0$. In this case the relations $a(t_2) = a^*$, $a^*(t_1) = a^*$ can of course not be satisfied. Accordingly, when we use the method of steepest descent we should treat the functions $a(t)$ and $a^*(t)$ as independent functions satisfying Eqs. (2.34) and condition (2.35). The reason that the functions $a(t)$ and $a^*(t)$ are not complex conjugates is that the operator B_σ, defined in Eq. (2.13), and its limit $B = \lim_{\sigma \to 0} B_\sigma$ are not self-adjoint.

Let us consider some simple examples.

1) $g(\tau; a^*, a) = f^*(\tau) a + a^* f(\tau)$. (This example was discussed in the preceding section.) With $\varepsilon = \sigma = 0$, Eqs. (2.30) yield

$$-\frac{da^*(\tau)}{d\tau} + if^* = 0, \quad \frac{da(\tau)}{d\tau} + if = 0.$$

Also using (2.35), we find

$$a(\tau) = a - i \int_{t_1}^{\tau} f(s) \, ds, \quad a^*(\tau) = a^* - i \int_{\tau}^{t_2} f^*(s) \, ds.$$

If f and f^* are complex conjugates, then the functions $a(\tau)$ and $a^*(\tau)$ may also be complex conjugates only if $\int_{t_1}^{t_2} f \, ds = 0$.

2) $g(\tau; a^*, a) = \omega a^* a$ (a harmonic oscillator). With $\varepsilon = \sigma = 0$, Eqs. (2.34) yield

$$-\frac{da^*(\tau)}{d\tau} + i\omega a^*(\tau) = 0, \quad \frac{da(\tau)}{d\tau} + i\omega a(\tau) = 0.$$

Hence, using (2.35), we find

$$a^*(\tau) = a^* e^{i(\tau - t_1)}, \quad a(\tau) = a e^{i(t_1 - \tau)\omega}.$$

In this case the path integral is Gaussian, so it is equal (within a factor) to the value of the functional in the integrand on the stationary path:

$$G(t_2, t_1 | a^*, a) = c(t_2, t_1) e^{\frac{1}{h}(i\omega(t_1 - t_2) - 1) a^* a}.$$

The simplest way to find the constant c is to use Eq. (2.29).

In this case we have $\tilde{g}(\tau; v^*, v) = -\omega\delta'(v)\delta'(v^*)$, so that

$$G^{(n)}(t_2, t_1 \mid 0, 0) = (-1)^n \omega^n \int \frac{\partial^{2n}}{\partial v_1\, \partial v_1^* \ldots \partial v_n\, \partial v_n^*} e^{-h\sum v_k^* v_l \theta(\tau_k - \tau_l - 0)} \Big|_{v=v^*=0} d^n\tau.$$

Because of the term -0 in the argument of θ we have

$$\frac{\partial^{2n}}{\partial v_1 \ldots \partial^* v_n} e^{-h\sum v_k^* v_l \theta(\tau_k - \tau_l - 0)} \Big|_{v=v^*=0} = 0,$$

for $n > 0$. We thus have $c(t_1, t_2) = G(t_1, t_2; 0, 0) \equiv 1$, which is the same as the result found previously.[13)]

6) Fermi case

Equation (2.4) and its refinements—Eqs. (2.17)–(2.20)—remain in force; it is simply necessary to replace the ordinary integration by an integration over anticommuting variables. The heuristic derivation and justification of these equations remain essentially the same; the only difference is that the functions $f(\tau)$ and $f^*(\tau)$ now do not take on numerical values but anticommute with each other, with the operators $\hat{a}(i)$ and $\hat{a}^*(i)$ and with their symbols $a(i)$ and $a^*(i)$.

The perturbation-theory equations, (2.32) and (2.35), also remain valid. The only point to note is that the variational derivatives should be understood as left-hand products, and it should be kept in mind that they do not commute, in contrast with the Bose case, so that their order is rigidly fixed.

The situation is slightly different with the set of paths on which the integral is concentrated. Instead of the space $\mathscr{H}(t_1, t_2)$ of square-summable paths we should consider the space of paths satisfying

$$\int_{t_1}^{t_2} \| a^*(t)\, a(t) \| \, dt < \infty,$$

where $\| \ \|$ denotes the norm in the Grassman algebra.

c: Symbols of other types

1) Definition of p-q and q-p symbols

The definition of $p-q$ symbols, like that of the $q-p$ symbols, which we shall discuss below, is analogous to the definition of the Weyl symbols.

We assume that $f(p, q)$ is a function in the phase space which can be represented as a Fourier transformation:

$$f(p, q) = \int \tilde{f}(u, v)\, e^{i(pu+qv)} du\, dv.$$

We associate with this function the operator

$$\hat{f} = \int \tilde{f}(u, v)\, e^{i\hat{p}u} e^{i\hat{q}v}\, du\, dv.$$

The function f is called the "$p-q$ symbol" of the operator $\hat{f}$. If the function f is associated with another operator,

$$\hat{f} = \int \tilde{f}(u, v)\, e^{i\hat{q}v} e^{i\hat{p}v}\, du\, dv,$$

then the function f is called the "$q-p$ symbol" of the operator f.

2) Basic construction

The original finite-dimensional approximation of the $p-q$ symbol of the evolution operator is

$$G_N(p, q \mid t_2, t_1) = \left(\frac{1}{2\pi h}\right)^{n(N-1)} \int e^{\frac{1}{ih}\left[\Delta \sum_1^N g(\tau_k;\, p_{k-1},\, q_k) + K_N\right]} \prod_1^{N-1} dp\, dq, \tag{3.1}$$

where

$$p_0 = p, \quad q_N = q, \quad \Delta = \frac{t_2 - t_1}{N},$$
$$K_N = p(q_1 - q) + p_1(q_2 - q_1) + \ldots + p_{N-1}(q - q_{N-1}). \tag{3.2}$$

By analogy with the preceding results, introducing the continuous parameter τ, $(t_1 \leqslant \tau \leqslant t_2)$, we set $p_k = p(\tau_k)$, $q_k = q(\tau_k)$, $\tau_k = t_2 - K/N(t_2 - t_1)$, and we combine all the terms in (3.2) other than the first into an integral sum. Then we take the limit $N \to \infty$ in (3.2):

$$K = \lim_{N\to\infty} K_N = p(q(t_2) - q) - \int_{t_1}^{t_2} p(\tau)\, \dot{q}(\tau)\, d\tau. \tag{3.3}$$

Incorporating the coefficient of the integral in (3.1) in the normalization of the differential, and taking the formal limit $N \to \infty$, we find the final expression for G:

$$G(t_2, t_1 \mid p, q) = \int_{p(t_2)=p,\ q(t_1)=q} e^{\frac{1}{ih}\left[\int_{t_1+0}^{t} (g(\tau;\, p(\tau),\, q(\tau-0)) - p\dot{q})\, d\tau + p(q(t_2) - q)\right]} \prod dp(\tau)\, dq(\tau). \tag{3.4}$$

The initial finite-dimensional approximation of the $q-p$ symbol of the evolution operator is

$$G_N(t_2, t_1 \mid p, q) = \frac{1}{(2\pi h)^{n(N-1)}} \int e^{\frac{1}{ih}\left(\Delta \sum_1^N g(\tau;\, p_k,\, q_{k-1}) + K_N\right)} \sum_1^{N-1} dp\, dq, \tag{3.5}$$

where $q_0 = q$, $p_N = p$, $\Delta = t_2 - t_1/N$, and

$$K_N = p_1(q_1 - q) + p_2(q_2 - q_1) + \ldots + p_{N-1}(q_{N-1} - q_{N-2}) + p(q - q_{N-1}). \tag{3.6}$$

As before, we introduce the continuous parameter τ, and we group all the terms except the last in an integral sum. Then we take the *limit* $N \to \infty$ in (3.6):

$$K = \lim K_N = -\int_{t_1}^{t_2} p\dot{q}\, d\tau + p(q - q(t_1)). \tag{3.7}$$

[13)] We note that the series

$$U = \int \sum_{n=0}^{\infty} \frac{(-\omega)^n}{n!} \frac{\partial^{2n}}{\partial v_1 \ldots \partial v_n^*} e^{\sum K(\tau_i, \tau_j) v_i v_j} \Big|_{v=v^*=0} d^n\tau,$$

is easily summed for any function $K(\tau_1, \tau_2)$:

$$(-\omega)^n \int \frac{\partial^{2n}}{\partial v_1 \ldots \partial v_n^*} e^{\sum K(\tau_i, \tau_j) v_i^* v_j} \Big|_{v=v^*=0} d^n\tau = \int \frac{\delta^{2n}}{\delta v(\tau_1)\, \delta v^*(\tau_1) \ldots \delta v(\tau_n)\, \delta v^*(\tau_n)} e^{-\omega \int K(t, s) v^*(t) v(s)\, dt\, ds} \Big|_{v=v^*=0} d^n\tau.$$

Hence

$$U = e^{\int_{t_1}^{t_2} \frac{\delta_2}{\delta v(\tau)\delta v^*(\tau)} d\tau} e^{-\omega \int K(t, s) v^*(t) v(s)\, dt\, ds} \Big|_{v=v^*=0} = \int e^{-\int v^* v\, d\tau - \omega \int K(t, s) v^*(t) v(s)\, dt\, ds} \prod_t dv(t)\, dv^*(t) = \det(1 + \omega\hat{K})^{-1},$$

where $\hat{K}$ is an operator in $L^2(t_1, t_2)$ defined by the function K, $(\hat{K}f)(\tau) = \int_{t_1}^{t_2} K(\tau, s) f(s)\, ds$. In our case we have $K(\tau, s) = h\theta(\tau - s - 0)$, i.e., the operator $\hat{K}$ is a triangle with a zero diagonal. Accordingly, $\det(1 + \omega\hat{K}) = 1$.

The final expression for G is

$$G(t_2,\ t_1 \mid p,\ q) = \int\limits_{q(t_2)=q,\ p(t_1)=p} e^{\frac{1}{ih}\left[\int\limits_{t_1+0}^{t_2} (g(\tau;\ p(\tau-0),\ q(\tau))-p\dot{q})\,d\tau + p(q-q(t_1))\right]} \prod \mathrm{d}p\,\mathrm{d}q. \quad (3.8)$$

As in the Weyl and Wick cases, the integrals in (3.4) and (3.8) are the limits of finite-dimensional approximations of a more general type than the original. These approximations are constructed by analogy with the Weyl and Wick symbols. The sign of $\tau - 0$ in (3.4) and (3.8) has the same meaning as the analogous sign in the case of Wick symbols. Precisely as in the Wick case, we must not replace $\tau - 0$ by τ in any of these equations, but in contrast with the Wick case the shift of the argument is introduced in only one place in Eqs. (3.4) and (3.8).

Approximations of a general type can be constructed and justified on the basis of the same considerations as in the Weyl and Wick cases. In particular, to justify approximations of the general type of integrals (3.4) and (3.8) we would naturally use, respectively, the $p-q$ and $q-p$ symbols of the operator of the evolution generated by the infinitesimal operator $\hat{q}(t) = u(t)\hat{p} + v(t)\hat{q}$. Here the $p-q$ symbol is

$$G(t_2,\ t_1;\ p,\ q) = \exp\left\{\frac{1}{ih}\left[\int\limits_{t_1}^{t_2} (u(\tau)\,p + v(\tau)\,q)\,d\tau + \int\limits_{t_1\leqslant s,\ t\leqslant t_2} \theta(s-t-0)\,v(s)\,u(t)\,ds\,dt\right\}\right], \quad (3.9)$$

and the $q-p$ symbol is

$$G(t_2,\ t_1;\ p,\ q) = \exp\left\{\frac{1}{ih}\left[\int\limits_{t_1}^{t_2} (u(\tau)\,p + v(\tau)\,q)\,d\tau - \int\limits_{t_1\leqslant s,\ t\leqslant t_2} \theta(s-t-0)\,u(s)\,v(t)\,ds\,dt\right]\right\}. \quad (3.10)$$

Equations (3.9) and (3.10), like the analogous equations in the theory of Weyl and Wick symbols, can be used as a basis for a perturbation theory for calculating the evolution operator symbol for the case of an arbitrary infinitesimal operator. The perturbation-theory series is given by an equation analogous to the corresponding equation in the theory of Weyl and Wick symbols:

$$G(t_2,\ t_1 \mid p,\ q) = e^{ihL} e^{\frac{1}{ih}\int\limits_{t_1}^{t_2} g(\tau;\ p(\tau),\ q(\tau))\,d\tau}\Big|_{\substack{p(\tau)=p,\\ q(\tau)=q}}, \quad (3.11)$$

where $g(\tau; p, q)$ is the $p-q$ or $q-p$ symbol of the infinitesimal evolution operator, and the operator L is

$$L = \int \theta(s-t-0)\,\frac{\delta^2}{\delta q(s)\,\delta p(t)}\,ds\,dt \quad (3.12)$$

for the $p-q$ symbols and

$$L = -\int \theta(s-t-0)\,\frac{\delta^2}{\delta p(s)\,\delta q(t)}\,ds\,dt \quad (3.13)$$

for the $q-p$ symbols.

The perturbation-theory series is constructed by expanding the second exponential function in (3.11) in powers of its argument. Each term of this series is expressed explicitly in terms of the Fourier transform of the function $g(\tau; p, q)$ with respect to p, q. The resulting equations are analogous to Eqs. (1.33) and (2.29), and we shall not dwell on them here.

To study the set of paths on which the integrals in (3.4) and (3.8) are concentrated we proceed in the same manner as in the Weyl case, and we find the same results.

3) Matrix elements of the evolution operator

We shall make use of the relationship between the $q-p$ operator symbols and their matrix elements in the x representation. We assume that $\hat{A}$ is the same operator in $L^2(R^n)$, that $A(p, q)$ is its $q-p$ symbol, and that $\langle x|\hat{A}|y\rangle$ is a matrix element. Then (see the Supplement)

$$\langle x \mid \hat{A} \mid y\rangle = \frac{1}{(2\pi h)^n}\int f(p,\ x)\,e^{\frac{1}{ih}p(y-x)}\,d^n p. \quad (3.14)$$

As the operator $\hat{A}$ we consider the $q-p$ symbol of the evolution operator. Substituting the expression for G from (3.8) into (3.14), integrating first over p and then over $p(t_1)$, we find

$$\begin{aligned} G(t_2,\ t_1 \mid x,\ y) &= \langle x \mid \hat{G} \mid y\rangle \\ &= \frac{1}{(2\pi h)^n}\int e^{\frac{1}{ih}\left\{\int\limits_{t_1}^{t_2} (g(\tau;\ p(\tau-0),\ q(\tau))-p\dot{q})\,d\tau + p(x-q(t_1))\right\}}\,\delta(q(t_2)-x) \\ &\quad\times \delta(p(t_1)-p)\,e^{\frac{1}{ih}p(y-x)}\prod dp(t)\,dq(t)\,d^n p \\ &= \frac{1}{(2\pi h)^n}\int e^{\frac{1}{ih}\left\{\int\limits_{t_1}^{t_2} (g(\tau;\ p(\tau-0),\ q(\tau))-p\dot{q})\,d\tau + p(t_1)(y-q(t_1))\right\}} \\ &\quad\times \delta(q(t_2)-x)\prod dp(t)\,dq(t) = \int e^{\frac{1}{ih}\int\limits_{t_1}^{t_2} (g(\tau;\ p(\tau-0),\ q(\tau))-p\dot{q})\,d\tau}\,\delta(q(t_2)-x) \\ &\quad\times \delta(q(t_1)-y)\prod dp(t)\,dq(t). \end{aligned}$$

We finally find

$$G(t_2,\ t_1) \mid x,\ y) = \langle x \mid \hat{G} \mid y\rangle = \int\limits_{\substack{q(t_1)=y,\\ q(t_2)=x}} e^{\frac{1}{ih}\int [g(\tau;\ p(\tau-0),\ q(\tau))-p\dot{q}]\,d\tau}\prod dp(\tau)\,dq(\tau). \quad (3.15)$$

A few comments are in order regarding this derivation.

1) The integral over p is evaluated by making use of the function in the integrand.

2) In integrating over $p(t_1)$, we take into account the dependence of the term outside the integral on $p(t_1)$, and we ignore the dependence of the first term in the argument on $p(t_1)$. Our defense for this approach runs as follows: Let us consider a finite approximation of the integral in (3.8) given by Eqs. (3.5) and (3.6). Integrating first over p and then over $p_N = p(t_2)$, we find a finite approximation of the last integral in (3.15); it is easy to see that $\delta(q(t_1) - y)$ is replaced by the integral

$$\frac{1}{(2\pi h)^n}\int e^{\frac{1}{ih}[\Delta g(\tau_N;\ p_N,\ q_{N-1}) - p_N(y-q_{N-1})]}\,d^n p_N, \quad (3.16)$$

where $\Delta = (t_2 - t_1)/N$. Setting $q_{N-1} = q(t_2)$, we find that in the limit $N \to \infty$ the integrals in (3.16) become $\delta(q(t_2) - y)$.

For the integral in (3.15) we can construct finite approximations of a general type by working from the approximations of the symbols. The approximations are of the form

$$G_N(t_2, t_1 \mid x, y) = \frac{F_N(t_2, t_1 \mid x, y)}{F_N^{(0)}(t_2, t_1 \mid x, y)}.$$

In contrast with the case of the symbols, the denominator is not found from the numerator in the case $g(\tau; p, q) \equiv 0$; it remains the same as for the operator symbol $\hat{G}$. The reason for this difference is that in the case $g(\tau; p, q) \equiv 0$ the evolution operator is a unit operator, and its symbol is identically 1, while the matrix element is $\delta(x-y)$.

We are naturally interested in what would happen if we replaced the $q-p$ symbols by $p-q$ symbols or Weyl symbols in this construction. The answer, easily seen, is that Eq. (3.15) is altered in the following manner: In the case of $p-q$ symbols, $p(\tau-0)$ is replaced by $p(\tau)$, while $q(\tau)$ is replaced by $q(\tau-0)$. In the case of Weyl symbols, the only change is that $p(\tau-0)$ is replaced by $p(\tau)$; there is no argument shift of $p(\tau)$ or $q(\tau)$. These distinctions are important: In choosing the symbol for the infinitesimal operator, we should use the formula corresponding to this symbol to derive the matrix element of the evolution operator. If the infinitesimal operator is of the form $\hat{g}(t) = A(t, \hat{p}) + B(t, \hat{q})$, of course, then the $p-q$, $q-p$ and Weyl symbols of this operator are the same, so it makes no difference which version of Eq. (3.15) is used. In general, this is not so; a very simple example is an operator of the type $\hat{g} = \hat{q}\hat{p}$ in $L^2(R^1)$. Its $q-p$ symbol is pq, while its Weyl symbol is $pq - (h/2i)$. Therefore, if we use the version of Eq. (3.15), in which the shift of the argument is ignored, and if we substitute $g = pq$ into it, in order to calculate $\langle x | e^{t\hat{g}/ih} | y \rangle$, we obtain a wrong answer.[14)]

2. PATH INTEGRAL FOR THE SCATTERING OPERATOR SYMBOL AND FOR A PARTITION FUNCTION

a: Path integral for the scattering operator symbol

1) Formal definition of the scattering operator

We assume that some system evolves over time in accordance with the Schrödinger equation

$$\left(ih\frac{\partial}{\partial t} - \hat{H}(t)\right)\psi = 0. \quad (4.1)$$

We write the solution of Eq. (4.1) with the initial condition $\psi(t') = \varphi$ as

$$\psi(t) = \hat{G}(t, t')\varphi.$$

The operator $\hat{G}(t, t')$ satisfies the operator equation and initial condition

$$\left(ih\frac{\partial}{\partial t} - \hat{H}(t)\right)\hat{G}(t, t') = 0, \quad \hat{G}(t', t') = I, \quad (4.2)$$

i.e., it is an evolution operator. We now assume that the solutions of (4.1) with the initial data $\psi(0) = \varphi \in D$, where D is some region in the state space, have in the limits $t \to \pm\infty$ the asymptotic form

$$e^{t\hat{H}_0/ih}\psi_+ \underset{t\to+\infty}{\sim} \psi(t) \underset{t\to-\infty}{\sim} e^{t\hat{H}_0/ih}\psi_-, \quad (4.3)$$

where $\hat{H}_0$ is some operator different from $\hat{H}(t)$. It follows from (4.3) that ψ_- and ψ_+ are linear functions of φ:

$$\psi_+ = \hat{V}_+\varphi, \quad \psi_- = \hat{V}_-\varphi, \quad (4.4)$$

where $\hat{V}_\pm = \lim_{t\to\pm\infty} \hat{V}(t)$,

$$\hat{V}(t) = e^{-t\hat{H}_0/ih}\hat{G}(t, 0). \quad (4.5)$$

According to (4.4),

$$\psi_+ = \hat{V}_+\hat{V}_-^{-1}\psi_-$$

the operator

$$\hat{S} = \hat{V}_+\hat{V}_-^{-1}$$

is called the (formal) "scattering operator." Clearly,

$$\hat{S} = \lim_{\substack{t\to+\infty,\\ t'\to-\infty}} \hat{S}(t, t'), \quad (4.6)$$

where

$$\hat{S}(t, t') = \hat{V}(t)\hat{V}^{-1}(t') = e^{-t\hat{H}_0/ih}\hat{G}(t, 0)(\hat{G}(t', 0))^{-1}e^{t'\hat{H}_0/ih} = e^{-t\hat{H}_0/ih}\hat{G}(t, t')e^{t'\hat{H}_0/ih}. \quad (4.7)$$

From (4.7) we find the evolution equation for $\hat{S}(t, t')$ which is analogous to (4.1) and an initial condition:

$$ih\frac{\partial\hat{S}(t, t')}{\partial t} = \hat{K}(t)\hat{S}(t, t'), \quad \hat{S}(t', t') = I, \quad (4.8)$$

where

$$\hat{K}(t) = e^{-t\hat{H}_0/ih}\hat{H}_1(t)e^{t\hat{H}_0/ih}, \quad \hat{H}_1(t) = \hat{H}(t) - \hat{H}_0. \quad (4.9)$$

Equations (4.7) and (4.8) (in the interaction picture) are the basis for writing the scattering operator symbol as a path integral.

In most applications,

$$\hat{H}_1(t) = \hat{H}_{\text{int}}e^{-\alpha|t|}, \quad \alpha > 0, \quad (4.10)$$

where H_{int} is independent of t. The scattering operator in this case depends on α as a parameter; $\hat{S} = \hat{S}_\alpha$ and is called "adiabatic." In some particularly simple cases, the adiabatic scattering operator has values in the limit $\alpha \to 0$. This case includes, in particular, the case of nonrelativistic quantum mechanics, with

$$\hat{H}_0 = \frac{1}{2m}\hat{P}^2 = -\frac{1}{2mh^2}\Delta \text{ and } H_{\text{int}}\psi = v(x)\psi,$$

where $v(x)$ is a rapidly decaying potential.

2) Weyl symbol of the scattering operator

We consider the Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{H}_1, \quad \hat{H}_0 = \frac{\hat{p}^2}{2m}, \quad \hat{H}_1 = v(t; q). \quad (4.11)$$

We assume that the potential $v(t; q)$ tends rapidly toward zero as $|t| \to \infty$, at a fixed q. A typical example is

$$v(t; q) = e^{-\alpha|t|}u(q). \quad (4.12)$$

The operator $e^{t\hat{H}_0/ih}$ has a Weyl symbol $e^{tp^2/2mih}$. Using the composition formula for Weyl symbols and (4.7), and carrying out some obvious transformations, we find a relationship between the operator symbols $\hat{G}(t_2, t_1)$ and $\hat{S}(t_2, t_1)$:

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14) We find $\langle x | e^{tg_1/ih} | y \rangle$, where g_1 is the operator whose Weyl symbol is pq. Obviously, $g = g_1 + (h/2i)$; consequently, $\langle x | e^{tg/ih} | y \rangle = e^{-t/2}\langle x | e^{tg_1/ih} | y \rangle$.

$$S(t_2, t_1 \mid p, q) = \frac{1}{(\pi h)^n} \int G(t_2, t_1 \mid p, q_1)\, e^{-\frac{t_2}{2imh} p_2^2 + \frac{t_1}{2imh} p_1^2 + \frac{2}{ih}(p_1 - p)(q - q_1)} \times \delta(2p - p_1 - p_2)\, \mathrm{d}p_1\, \mathrm{d}p_2\, \mathrm{d}q_1. \tag{4.13}$$

Substituting in $G(t_2, t_1 | p, q)$ from (1.9), we find a path-integral expression for $\hat{S}(t_2, t_1)$:

$$S(t_2, t_1 \mid p, q) = \int \exp\left\{\frac{1}{ih}\left[\int_{t_1}^{t_2}\left(g(\tau; y(\tau)) - \frac{1}{2} y\omega\dot{y}\right)\mathrm{d}\tau + x\omega y(t_2) + y(t_1)\omega x + \frac{1}{2} y(t_2)\omega y(t_1) + \frac{1}{2m} t_1\left(p - \frac{\Delta p}{2}\right)^2 - \frac{1}{2m} t_2\left(p + \frac{\Delta p}{2}\right)^2\right]\right\} \prod \mathrm{d}y(t), \tag{4.14}$$

where $\Delta p = p(t_2) - p(t_1)$.

The integral in (4.14) has the same gauge singularity as the original integral in (1.9). This property can be proved by precisely the same arguments as in Subsection a3; we shall skip the proof.

The operator $\hat{S}(t_2, t_1)$ has a value in the limit $t_1 \to -\infty$, $t_2 \to +\infty$. Its symbol should have the same property. We can transform (4.14) to make this circumstance immediately obvious. We fix a number $T > 0$ and set

$$v_T(t; q) = \theta(T - |t|)\, v(t; q).$$

We denote by $\hat{S}_T(t_2, t_1)$ the operator found from $\hat{S}(t_2, t_1)$ by replacing $v(t; q)$ by $v_T(t; q)$, and we denote by $S_T(t_2, t_1 | p, q)$ the symbol of the operator $\hat{S}_T(t_2, t_1)$. Clearly, $\hat{S}(t_2, t_1) = \lim_{T\to\infty} \hat{S}_T(t_2, t_1)$, and the symbols of these operators are related by an analogous expression. We now note that for $t_2 > T$, $t_1 < -T$ we have

$$\hat{S}_T(t_2, t_1) = \hat{S}_T(t_2, T)\, \hat{S}_T(T, -T)\, \hat{S}_T(-T, t_1). \tag{4.15}$$

Furthermore, the operators $\hat{S}_T(t_2, T)$ and $\hat{S}_T(-T, t_1)$ are found from $\hat{S}(t_2, T)$ and $\hat{S}(-T, t_1)$, respectively, in the case $v(t; q) \equiv 0$. Accordingly, the path integrals for their symbols can be calculated by the stationary-phase method, i.e., are concentrated on classical paths. Taking into account the form of the operator $\hat{H}_0$, we find that these paths are

$$q(t) = a + \frac{b}{m} t, \quad p(t) = b, \quad a,\ b = \mathrm{const}. \tag{4.16}$$

We now transform from operators to symbols in (4.15), using the composition formula for Weyl symbols. We write $S_T(t_2, t_1 | p, q)$ as a single path integral, noting that the path integrals for the symbols of the operators $\hat{S}_T(t_2, T)$ and $\hat{S}_T(-T, t_1)$ are concentrated on functions of the type in (4.16). As a result we find that the path integral for the symbol of the operator $\hat{S}_T(t_2, t_1)$ is concentrated on functions of the type

$$q(t) = \begin{cases} \tilde{q}(t), & |t| < T, \\ a_+ + \frac{p_+}{m} t & t > T, \\ a_- + \frac{p_-}{m} t & t < -T, \end{cases} \qquad p(t) = \begin{cases} \tilde{p}(t), & |t| < T, \\ p_+ & t > T, \\ p_- & t < -T, \end{cases}$$

where $\tilde{q}(t)$, and $\tilde{p}(t)$ are arbitrary functions, and $a_\pm$ and $p_\pm$ are constants.

We now let T go to ∞, and we thereby transform from the operator $\hat{S}_T(t_2, t_1)$ to the operator $\hat{S}(t_2, t_1)$.

We would naturally expect on the basis of the considerations above that the path integral for $S(t_2, t_1 | p, q)$ would be concentrated on paths having functions of the type in (4.16) as their asymptotic expressions in the limit $|t| \to \infty$:

$$\begin{aligned} p(t) &= \tilde{p}(t) + p_{\mathrm{out}} r_+(t) + p_{\mathrm{in}} r_-(t), \\ q(t) &= \tilde{q}(t) + \frac{t}{m}\left(p_{\mathrm{out}} s_+(t) + p_{\mathrm{in}} s_-(t)\right), \end{aligned} \tag{4.17}$$

where $\tilde{p}(t) \to 0$ in the limit $|t| \to \infty$, $\tilde{q}(t)$ has finite limits at $t \to \pm\infty$, $\lim_{t\to+\infty} \tilde{q}(t) = \tilde{q}_{\mathrm{out}}$, $\lim_{t\to-\infty} q(t) = \tilde{q}_{\mathrm{in}}$, and $r_\pm(t)$ and $s_\pm(t)$ are fixed functions with the properties

$$\lim_{t\to\pm\infty} r_\pm(t) = \lim_{t\to\pm\infty} s_\pm(t) = 1, \quad \lim_{t\to\mp\infty} r_\pm(t) = \lim_{t\to\mp\infty} s_\pm(t) = 0. \tag{4.18}$$

These functions are otherwise arbitrary. In particular, we can set

$$r_+(t) = s_+(t) = \theta(t), \quad r_-(t) = s_-(t) = \theta(-t). \tag{4.19}$$

All the functions in (4.17) which have limits at $t \to \pm\infty$ will be assumed to converge rapidly enough to satisfy limiting relations of the type $\lim_{t\to\pm\infty} t\tilde{p}(t) = 0$ and such that integrals of the type $\int_{-\infty}^{\infty} |t\tilde{p}(t)|\, dt$, etc., exist.

Under the assumption that the integral in (4.14) is concentrated at paths of the type (4.17), we transform the terms outside the integral in the argument in (4.14):

$$\begin{aligned} &x\omega(y(t_2) - y(t_1)) + \frac{1}{2} y(t_2)\omega y(t_1) + \frac{1}{2m} t_1\left(p - \frac{\Delta p}{2}\right)^2 - \frac{1}{2m} t_2\left(p + \frac{\Delta p}{2}\right)^2 \\ &\quad = -q(p(t_2) - p(t_1)) + \frac{1}{2}(p(t_2)\tilde{q}(t_2) - p(t_1)\tilde{q}(t_1)) \\ &\quad + \left(p - \frac{1}{2}(p(t_2) + p(t_1))\right)\tilde{q} + \frac{t_1 - t_2}{2m}\left(p - \frac{1}{2}(p(t_2) + p(t_1))\right)^2. \end{aligned} \tag{4.20}$$

All the terms on the right side of (4.20) except the last clearly have limits as $t_2 \to +\infty$, $t_1 \to -\infty$. With regard to the integral term in the argument in (4.14), we easily see that for functions of the type in (4.17) it also has limits as $t_2 \to +\infty$, $t_1 \to -\infty$. The argument thus has limits as $t_2 \to +\infty$, $t_1 \to -\infty$ only if

$$\frac{1}{2}(p_{\mathrm{out}} + p_{\mathrm{in}}) = p. \tag{4.21}$$

Proceeding on the basis that the integral in (4.14) clearly has a limit as $t_2 \to +\infty$, $t_1 \to -\infty$, we would naturally assume that it is concentrated at functions of the type (4.17), which satisfy the gauge condition (4.21). Equation (4.21) only partially eliminates the gauge singularity of the integral in (4.14). To eliminate it completely, we must supplement it with an analogous equation concerning the coordinates. In contrast with the momenta, this relation is not dictated by any sort of necessary condition. We choose it in the form

$$\frac{1}{2}(\tilde{q}_{\mathrm{out}} + \tilde{q}_{\mathrm{in}}) = r, \tag{4.22}$$

where r is an arbitrary constant. Using conditions (4.21) and (4.22), we can write the scattering operator symbol as

$$\begin{aligned} S(p, q) = \int \exp\Big\{\frac{1}{ih}\Big[\int_{-\infty}^{\infty}\left(\frac{p^2(t)}{2m} + v(t; q(t)) - \frac{1}{2}(p\dot{q} - q\dot{p})\right)\mathrm{d}t \\ + \frac{1}{2}(p_{\mathrm{out}}\tilde{q}_{\mathrm{out}} - p_{\mathrm{in}}\tilde{q}_{\mathrm{in}}) - q(p_{\mathrm{out}} - p_{\mathrm{in}})\Big]\Big\}\, \delta\left(p - \frac{1}{2}(p_{\mathrm{out}} + p_{\mathrm{in}})\right) \\ \times \delta\left(r - \frac{1}{2}(\tilde{q}_{\mathrm{out}} + \tilde{q}_{\mathrm{in}})\right) \prod \mathrm{d}\tilde{p}(t)\, \mathrm{d}\tilde{q}(t)\, \mathrm{d}p_{\mathrm{out}}\, \mathrm{d}p_{\mathrm{in}}. \end{aligned} \tag{4.23}$$

[The functions $\tilde{p}(t)$ and $\tilde{q}(t)$ are related to the functions $p(t)$ and $q(t)$ by (4.17).] Although the right side of (4.23) depends on the parameter r, the left side does not. The arguments leading to the integral in (4.23) have been somewhat intuitive. We shall derive Eq.

(4.23) below at the level of rigor of this paper.

In (4.23) we can carry out the integration over $\tilde{p}$. For this purpose we use the stationary-phase method. Varying the argument with respect to $\tilde{p}$, we find

$$\frac{p(t)}{m}=\dot{q}(t). \tag{4.24}$$

Before we substitute the expression for $p(t)$ from (4.24) into the argument in (4.23), we transform it:

$$\begin{aligned}\frac{1}{2}\int_{t_1}^{t_2} q\dot{p}\,\mathrm{d}t&=\frac{1}{2}qp\Big|_{t_1}^{t_2}-\frac{1}{2}\int_{t_1}^{t_2}\dot{q}p\,\mathrm{d}t=\frac{1}{2m}(t_2p_{\mathrm{out}}^2-t_1p_{\mathrm{in}}^2)\\&+\frac{1}{2}(\tilde{q}_{\mathrm{out}}p_{\mathrm{out}}-\tilde{q}_{\mathrm{in}}p_{\mathrm{in}})-\frac{1}{2}\int_{t_1}^{t_2}\dot{q}p\,\mathrm{d}t+\ldots=\frac{1}{2m}\int_{t_1}^{t_2}(\theta(t)\,p_{\mathrm{out}}+\theta(-t)\,p_{\mathrm{in}})^2\,\mathrm{d}t\\&+\frac{1}{2}(\tilde{q}_{\mathrm{out}}p_{\mathrm{out}}-\tilde{q}_{\mathrm{in}}p_{\mathrm{in}})-\frac{1}{2}\int_{t_1}^{t_2}\dot{q}p\,\mathrm{d}t+\ldots,\end{aligned} \tag{4.25}$$

where the ellipsis denotes terms which tend toward zero in the limits $t_2\to+\infty$, $t_2\to-\infty$.

Using (4.24) and (4.25), we find

$$\begin{aligned}\int_{t_1}^{t_2}\left[\frac{p^2}{2m}-\frac{1}{2}(p\dot{q}-q\dot{p})\right]\mathrm{d}t&=-\frac{m}{2}\int_{t_1}^{t_2}\left[\dot{q}^2-\frac{1}{m^2}(\theta(t)\,p_{\mathrm{out}}+\theta(-t)\,p_{\mathrm{in}})^2\right]\mathrm{d}t\\&+\frac{1}{2}(\tilde{q}_{\mathrm{out}}p_{\mathrm{out}}-\tilde{q}_{\mathrm{in}}p_{\mathrm{in}})+\ldots.\end{aligned} \tag{4.26}$$

A summable function stands within the integral, so we take the limit $t_2\to+\infty$, $t_1\to-\infty$ in (4.26). We finally find

$$\begin{aligned}S(p,q)=\int\exp\Big\{\frac{1}{i\hbar}\Big[\int_{-\infty}^{\infty}\Big[-\frac{m}{2}\dot{q}^2+\frac{1}{2m}(\theta(t)\,p_{\mathrm{out}}+\theta(-t)\,p_{\mathrm{in}})+v(t;q(t))\Big]\mathrm{d}t\\-q(p_{\mathrm{out}}-p_{\mathrm{in}})+p_{\mathrm{out}}\tilde{q}_{\mathrm{out}}-p_{\mathrm{in}}\tilde{q}_{\mathrm{in}}\Big]\Big\}\\\times\delta\left(r-\frac{1}{2}(\tilde{q}_{\mathrm{out}}+\tilde{q}_{\mathrm{in}})\right)\delta\left(p-\frac{1}{2}(p_{\mathrm{out}}+p_{\mathrm{in}})\right)\prod\mathrm{d}\tilde{q}(t)\,\mathrm{d}p_{\mathrm{out}}\,\mathrm{d}p_{\mathrm{in}}.\end{aligned} \tag{4.27}$$

Comment. In evaluating the integral over $\bar{p}$, we carried out the transformations in the argument required by the stationary-phase method, but we ignored the determinant which arose. This procedure is justified on the basis of the definition of the path integral,

$$S(p,q)=\lim_{N\to\infty}\frac{S_N(p,q)}{S_N^{(0)}(p,q)},$$

where $S_N(p,q)$ is a finite integral constructed in the manner of Sec. 1, and $S_N^{(0)}(p,q)$ is the integral found from $S_N(p,q)$ in the case $v(t;q)\equiv 0$. The determinants which arise in the integration over $\tilde{p}$ are the same in the numerator and denominator and thus cancel out. We apply Eq. (4.27) to the case

$$v(t;q)=f(t)\,q. \tag{4.28}$$

Using the stationary-phase method, we find

$$m\ddot{q}+f(t)=0. \tag{4.29}$$

Hence,

$$q(t)=-\frac{1}{2m}\int_{-\infty}^{\infty}|t-s|\,f(s)\,\mathrm{d}s+a+\frac{b}{m}t. \tag{4.30}$$

The asymptotic behavior of $q(t)$ in the limits $t\to\pm\infty$ is

$$q(t)=\frac{t}{m}\left(b-\frac{1}{2}\int_{-\infty}^{\infty}f(s)\,\mathrm{d}s\right)+\frac{1}{2m}\int_{-\infty}^{\infty}sf(s)\,\mathrm{d}s+a+O(1)\quad t\to+\infty,$$

$$q(t)=\frac{t}{m}\left(b+\frac{1}{2}\int_{-\infty}^{\infty}f(s)\,\mathrm{d}s\right)-\frac{1}{2m}\int_{-\infty}^{\infty}sf(s)\,\mathrm{d}s+a+O(1)\quad t\to-\infty.$$

Thus

$$\begin{aligned}p_{\mathrm{out}}&=b-\frac{1}{2}\int_{-\infty}^{\infty}f(s)\,\mathrm{d}s,\quad \tilde{q}_{\mathrm{out}}=a+\frac{1}{2m}\int_{-\infty}^{\infty}sf(s)\,\mathrm{d}s,\\p_{\mathrm{in}}&=b+\frac{1}{2}\int_{-\infty}^{\infty}f(s)\,\mathrm{d}s,\quad \tilde{q}_{\mathrm{in}}=a-\frac{1}{2m}\int_{-\infty}^{\infty}sf(s)\,\mathrm{d}s.\end{aligned} \tag{4.31}$$

The appearance of relations involving the integration variables means that corresponding δ-functions appear. Taking account of these functions and also of the factor $\delta(p-\frac{1}{2}(p_{\mathrm{out}}+p_{\mathrm{in}}))$ in (4.27), we can evaluate the integral over p_{out}, p_{in}, and b. As a result we find as a supplement to (4.31)

$$b=p. \tag{4.32}$$

We should now substitute expression (4.30) for $q(t)$ into the argument of the exponential function in the path integral in (4.27) and then integrate the result over a. As a result of the obvious calculations, using (4.31) and (4.32), we find[15]

$$S(p,q)=\exp\left\{\frac{1}{i\hbar}\left[\int_{-\infty}^{\infty}\left(q+\frac{p}{m}t\right)f(t)\,\mathrm{d}t-\frac{1}{4m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|t-s|\,f(t)\,f(s)\,\mathrm{d}s\,\mathrm{d}t\right]\right\}. \tag{4.33}$$

As expected, the result is independent of the parameter r which appears on the right side of (4.27). Equation (4.23) must be justified; this can be done by means of arguments which are the same as those used for Eqs. (1.9) and (1.4), and we shall omit this justification.

3) Wick symbol of the scattering operator symbol

We assume $\hat{H}=\hat{H}_0+\hat{H}_1(t)$,

$$\left.\begin{aligned}\hat{H}_0&=\int\omega(p)\,\hat{a}^*(p)\,\hat{a}(p)\,\mathrm{d}p,\\\hat{H}_1(t)&=\sum\int V_{mn}(t\mid p_1,\ldots,p_m\mid q_1,\ldots,q_n)\,a^*(p_1)\ldots\\&\qquad\ldots a^*(p_m)\,a(q_n)\ldots a(q_1)\,\mathrm{d}^mp\,\mathrm{d}^nq.\end{aligned}\right\} \tag{4.34}$$

Using the interaction picture, (4.8), we note that the path integral for $\hat{s}(t_2,t_1)$ is the same as the path integral for the evolution operator which is generated by the infinitesimal operator $\hat{g}$ with the symbol $g(t;a^*,a)=H_1(t;e^{it\omega}a^*,e^{-it\omega}a)$. Applying Eq. (2.32), we find

$$S(t_2,t_1\mid a^*,a)=e^{i\hbar L}\exp\left[\frac{1}{i\hbar}\int_{t_1}^{t_2}H_1(\tau;e^{i\tau\omega}\alpha^*(\tau),e^{-i\tau\omega}\alpha(\tau))\,\mathrm{d}\tau\right]\Big|_{\alpha(t_1)=a,\ \alpha(t_2)=a^*}. \tag{4.35}$$

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[15] To streamline the calculations, we first integrate by parts:

$$\int_{t_1}^{t_2}\dot{q}^2\,\mathrm{d}q=\dot{q}q\Big|_{t_1}^{t_2}-\int_{t_1}^{t_2}\ddot{q}q\,\mathrm{d}t.$$

It is easy to see that

$$\lim_{\substack{t_1\to-\infty\\t_2\to+\infty}}\left[-\frac{m}{2}\dot{q}q\Big|_{t_1}^{t_2}+\frac{1}{2m}\int_{t_1}^{t_2}(\theta(t)\,p_{\mathrm{out}}+\theta(-t)\,p_{\mathrm{in}})^2\right]=-\frac{1}{2}(p_{\mathrm{out}}q_{\mathrm{out}}-p_{\mathrm{in}}q_{\mathrm{in}}).$$

Hence, using (4.29), we find that the argument is

$$\frac{1}{2}\int_{-\infty}^{\infty}f(t)\,q(t)\,\mathrm{d}t+\frac{1}{2}(p_{\mathrm{out}}q_{\mathrm{out}}-p_{\mathrm{in}}q_{\mathrm{in}})-q(p_{\mathrm{out}}-p_{\mathrm{in}}).$$

Substituting in the expression for $q(t)$ from (4.30), and using (4.31) and (4.32), we find (4.33).

Equation (4.35) is the basis for the diagram technique in perturbation theory. In the relativistically invariant case, it is more convenient to consider the field variables than the integration variables $\alpha(\tau)$ and $\alpha^*(\tau)$. As a result, Eq. (4.35) is transformed into the Hori equation, which is the most convenient starting point for the development of the Feynman-diagram technique. The corresponding formal transformations can be found in Ref. 2.

b: Path integral for a partition function

1) Expression of the partition function as a path integral

We denote by $\hat{A}$ some operator in $L^2(R^n)$, and we denote by $A(p, q)$ its Weyl symbol. In this case

$$\mathrm{Sp}\,\hat{A} = \frac{1}{(2\pi h)^n}\int A(p, q)\,\mathrm{d}p\,\mathrm{d}q. \tag{5.1}$$

If $\hat{A}$ is an operator in a Fock space, its trace in the Bose and Fermi cases is expressed in terms of the Wick symbol in slightly different ways:

$$\mathrm{Sp}\,\hat{A} = \frac{1}{h^{\varepsilon n}}\int A(a^*, a)\,e^{\frac{1}{h}(1-\varepsilon)a^*a}\prod \mathrm{d}a^*\,\mathrm{d}a, \tag{5.2}$$

where $\varepsilon = 1$ in the Bose case and $\varepsilon = -1$ in the Fermi case. We assume that $\hat{G}(\beta)$ is an evolution operator with an imaginary time $t = -ih\beta$. Setting $g = -ihH$, $t_1 = 0$, $t_2 = \beta$, in Eqs. (1.9), (1.15), and (2.4), we find the Weyl and Wick symbols, respectively, for the operator $\hat{G}(\beta)$. Then using Eqs. (4.36) and (4.37), we find the partition function. Equations (1.9) and (1.15) yield

$$\Xi(\beta) = \mathrm{Sp}\,\hat{G}(\beta) = \int \exp\left\{-\int_0^\beta \left[H(y(\tau)) + \frac{1}{2ih}\,y(\tau)\,\omega\dot{y}(\tau)\right]\mathrm{d}\tau\right\}\prod \mathrm{d}y(\tau) \tag{5.3}$$

and

$$\Xi(\beta) = \mathrm{Sp}\,\hat{G}(\beta) = \frac{1}{(2\pi h)^n}\int \exp\left\{-\int_0^\beta \left[H(y(\tau)) + \frac{1}{2ih}\,y(\tau)\,\omega\dot{y}(\tau)\right]\mathrm{d}\tau + \frac{1}{ih}\,a\omega\,(y(0) - y(\beta)) + \frac{1}{2ih}\,y(0)\,\omega y(\beta)\right\}\prod \mathrm{d}y(\tau), \tag{5.4}$$

respectively. The integral in (5.3) is evaluated along closed paths, while that in (5.4) is evaluated over all paths. As expected, Eq. (5.3) can be found from (5.4) by carrying out an additional integration over a.

We turn now to the Wick symbols. Here it is more convenient to consider the expression in (2.1) before the limit is taken. To find the corresponding approximation of the partition function, we should multiply (2.1) by $e^{(1-\varepsilon)a^*a}$ and then integrate over $\mathrm{d}a^*\mathrm{d}a$. We introduce the notation $a^* = a_0^*$, $a = a_N$. We note that the integrand in terms of this notation does not contain a_0 and a_N^*, so we are justified in writing $a_0 = a_N$ and $a_N^* = a_0^*$. With this notation in mind, we find the final expression for the partition function:

$$\Xi(\beta) = \mathrm{Sp}\,\hat{G}(\beta) = \int \exp\left\{\int_0^\beta \left[-H(a^*(\tau), a(\tau - 0)) + \frac{1}{h}\frac{\mathrm{d}a^*(\tau)}{\mathrm{d}\tau}\,a(\tau)\right]\mathrm{d}\tau\right\}\prod \mathrm{d}a^*\mathrm{d}a. \tag{5.5}$$

The integral is taken over periodic paths with a period β in the Bose case and over antiperiodic paths in the Fermi case:

$$a(\beta) = \varepsilon a(0), \quad a^*(\beta) = \varepsilon a^*(0). \tag{5.6}$$

A few comments are in order regarding this derivation.

1) According to the derivation, the path integrals in (5.3)–(5.5), like the corresponding integrals for the evolution operators, should be understood as the limiting values of the ratios

$$\Xi(\beta) = \lim_{N\to\infty} \frac{\Xi_N(\beta)}{F_N^{(0)}}, \tag{5.6$'$}$$

where $\Xi_N(\beta)$ is the direct finite approximation of integrals (5.3)–(5.5), which can be found from the numerator in the corresponding equation for the evolution operators [see (1.22) and (1.20)] by an additional integration over $\mathrm{d}p\mathrm{d}q$. The denominator $F_N^{(0)}$ is the same as in (1.22) or (1.26). This method of defining the path integral is not convenient, because of the different properties of the differential term in the numerator and denominator in (5.6). In both cases the term is of the form (y, By), where $B = \omega(\mathrm{d}/\mathrm{d}t)$, but in Eq. (5.3) the boundary conditions are periodic, while in (1.26) there are no boundary conditions at all. There is a corresponding difference in Eqs. (5.4) and (1.22). This inconvenience can be avoided in the following manner: We consider the operator

$$\hat{H}_0(\lambda) = \frac{\lambda}{2}\sum\left(\hat{p}^2(k) + \hat{q}^2(k) - \frac{h}{2}\right), \tag{5.7}$$

and we denote by $\Xi_\lambda^{(0)}(\beta)$ the corresponding partition function, $\Xi_\lambda^{(0)}(\beta) = \mathrm{Sp}\,e^{-\beta\hat{H}_0}$. Finally, we set

$$\Xi(\beta) = \lim_{\lambda\to\infty}\lim_{N\to\infty}\frac{\Xi_N(\beta)}{\Xi_{N,\lambda}^{(0)}(\beta)}, \tag{5.8}$$

where $\Xi_N(\beta)$, $\Xi_{N,\lambda}^{(0)}(\beta)$ are finite-dimensional approximations of the corresponding integrals, constructed from the same system of finite-dimensional projection operators P_N.

The role of the operator $\hat{H}_0(\lambda)$ is as follows: It has a single eigenvalue $E_0(\lambda) = 0$, while all its other eigenvalues have the behavior $E_n(\lambda) \to +\infty$ in the limit $\lambda \to \infty$. Accordingly, $\lim_{\lambda\to\infty}\Xi_\lambda(\beta) = 1$. Instead of (5.7) we could consider any other operator having the same property.

In the case of the Wick symbols, an approximation of the general kind for the integral in (4.39) can be constructed with the help of an equation analogous to (4.43):

$$\Xi(\beta) = \lim_{\lambda\to\infty}\lim_{\varepsilon\to 0}\lim_{N\to\infty}\frac{\Xi_N(\beta)}{\Xi_{N,\lambda}^{(0)}(\beta)}.$$

The parameter $\varepsilon > 0$ appears in the numerator and denominator of the right side because of the argument shift in Eq. (5.5). As $\hat{H}_0(\lambda)$ in the Wick case it is natural to consider the operator

$$\hat{H}_0(\lambda) = \lambda\sum\omega(k)\,\hat{a}^*(k)\,\hat{a}(k). \tag{5.9}$$

2) We wish to call attention to the fact that the path integral for the partition function in the Wick version, in contrast with the path integral for the evolution operator symbol, contains an argument shift only in the first term in the argument of the exponential function. The reason lies in the different properties of the differential term in these integrals: In both cases it is of the form (Ba^*, a), where $B = \mathrm{d}/\mathrm{d}\tau$, but in the case of the partition function the operator B has periodic or

antiperiodic boundary conditions and is thus self-adjoint, while in the case of the evolution operator it has homogeneous boundary conditions and is thus symmetric, but it is definitely not self-adjoint.

2) Example

We consider a system with a single degree of freedom: the Bose version of second quantization. We set

$$\hat H = \omega \hat a^* \hat a \tag{5.9'}$$

(corresponding to a harmonic oscillator).

Taking into account the periodic boundary conditions, we can expand the path in a Fourier series:

$$a(\tau) = \sum_{-\infty}^{\infty} a_n e^{2\pi i n\tau/\beta}, \quad a^*(\tau) = \sum_{-\infty}^{\infty} a_n^* e^{-2\pi i n\tau/\beta}. \tag{5.10}$$

As $\mathscr{H}_N$ we consider the subspace of paths for which the Fourier coefficients are nonvanishing only for $|n| \leqslant N$. In this case,

$$\Xi_N(\beta) = \int \exp\left[-\sum_{-N}^{N}\left(\beta\omega e^{-2\pi i n/\beta} + \frac{2\pi i n}{h}\right) a_n^* a_n\right] \prod_{-N}^{N} da_n^*\, da_n = \prod_{-N}^{N}\left(\beta\omega e^{-2\pi i n\varepsilon/\beta} + \frac{2\pi i n}{h}\right)^{-1}.$$

Analogously,

$$\Xi_{N,\lambda}(\beta) = \sum_{-N}^{N}\left(\beta\lambda e^{-2\pi i n\varepsilon/\beta} + \frac{2\pi i n}{h}\right)^{-1}.$$

Hence

$$\frac{\Xi_N(\beta)}{\Xi_{N,\lambda}(\beta)} = \frac{\lambda}{\omega}\,\frac{\prod\limits_{|n|=1}^{N}\left(1 + \frac{h\beta\lambda}{2\pi i n} e^{-2\pi i n\varepsilon/\beta}\right)}{\prod\limits_{|n|=1}^{N}\left(1 + \frac{h\beta\omega}{2\pi i n} e^{-2\pi i n\varepsilon/\beta}\right)}.$$

We find

$$\Phi(\beta,\lambda) = \lim_{\varepsilon\to 0}\lim_{N\to\infty}\prod_{|n|=1}^{N}\left(1 + \frac{h\beta\lambda}{2\pi i n} e^{-2\pi i n\varepsilon/\beta}\right).$$

We denote by $\Phi(\beta, \lambda)$ the function within the first limit. We have

$$\ln \Phi_\varepsilon = \sum_{1}^{\infty}\left[\ln\left(1 + \frac{h\beta\lambda}{2\pi i n} e^{-2\pi i n\varepsilon/\beta}\right) + \ln\left(1 - \frac{h\beta\lambda}{2\pi i n} e^{2\pi i n\varepsilon/\beta}\right)\right] = \sum_{1}^{\infty}\frac{h\beta\lambda}{2\pi i n}\left(1 - e^{-2\pi i n\varepsilon/\beta} - e^{2\pi i n\varepsilon/\beta}\right) + R_\varepsilon(\beta,\lambda), \tag{5.11}$$

where $R_\varepsilon(\beta, \lambda)$ is an absolutely converging series, in which the limit $\varepsilon \to 0$ can be taken term by term. Obviously,

$$\lim_{\varepsilon\to 0} R_\varepsilon(\beta,\lambda) = \sum_{1}^{\infty}\ln\left[1 + \left(\frac{h\beta\lambda}{2\pi n}\right)^2\right].$$

We denote by $T_\varepsilon(\beta, \lambda)$ the first term on the right side of (5.11). We use an identity which holds on the interval $-\pi < x < \pi$:

$$\theta(x)(\pi - x) - \theta(-x)(\pi + x) = 2\sum_{1}^{\infty}\frac{\sin nx}{n}. \tag{5.12}$$

From (5.12) we find

$$T_\varepsilon(\beta,\lambda) = \frac{h\beta\lambda}{2\pi}\left[\theta\left(-\frac{2\pi\varepsilon}{\beta}\right)\left(\pi + \frac{2\pi\varepsilon}{\beta}\right) - \theta\left(\frac{2\pi\varepsilon}{\beta}\right)\left(\pi - \frac{2\pi\varepsilon}{\beta}\right)\right].$$

We thus have $\lim_{\varepsilon\to\infty} T_\varepsilon(\beta, \lambda) = -h\beta\lambda/2$,

$$\lim_{\varepsilon\to 0}\Phi_\varepsilon(\beta,\lambda) = e^{-h\beta\lambda/2}\prod_{1}^{\infty}\left[1 + \left(\frac{h\beta\lambda}{2\pi n}\right)^2\right] = \frac{e^{-h\beta\lambda/2}}{h\beta\lambda}\left(e^{h\beta\lambda/2} - e^{-h\beta\lambda/2}\right) = \frac{1 - e^{-h\beta\lambda}}{h\beta\lambda}.$$

We finally find

$$\Xi(\beta) = \lim_{\lambda\to\infty}\lim_{\varepsilon\to 0}\lim_{N\to\infty}\frac{\Xi_N(\beta)}{\Xi_{N,\lambda}(\beta)} = \lim_{\lambda\to\infty}\frac{\lambda}{\omega}\frac{\Phi(\beta,\lambda)}{\Phi(\beta,\omega)} = \frac{1}{1 - e^{-h\beta\omega}}.$$

SUPPLEMENT

PROPERTIES OF THE SYMBOLS

a: Weyl symbols

1) Definition

We assume $p = (p_1, \ldots, p_n)$, $q = (q_1, \ldots, q_n)$, where p_i and q_j are the canonical coordinates in the phase space R_{2n}; H is an operator in the space $L^2(R^n)$; and $H(p, q)$ is its Weyl symbol,

$$H(p,q) = \int e^{i(\alpha p + \beta q)}\varphi(\alpha,\beta)\, d\alpha\, d\beta, \qquad \hat H(p,q) = \int e^{i(\alpha\hat p + \beta\hat q)}\varphi(\alpha,\beta)\, d\alpha\, d\beta, \tag{S.1}$$

where $\hat p = (\hat p_1, \ldots, \hat p_n)$, $\hat q = (\hat q_1, \ldots, \hat q_n)$, and $\hat p_i$ and $\hat q_j$ are the ordinary momentum and coordinate operators. The function φ in (S.1) may be either an ordinary function or a generalized function. In particular, we do not rule out the case in which $H(p, q)$ is a polynomial. In this case, $\varphi(\alpha, \beta)$ is a linear combination of derivatives of a δ-function, and the relationship between an operator and its symbol is purely algebraic. If

$$H(p,q) = \sum h_{m,n} p^m q^n,$$

then

$$\hat H = \sum h_{m,n}(\hat p^m \hat q^n), \tag{S.2}$$

where m and n are multiple indices [$m = (m_1, m_2, \ldots)$, $n = (n_1, n_2, \ldots)$]; $p^m = p_1^{m_1}p_2^{m_2}$, $q^n = q_1^{n_1}q_2^{n_2}, \ldots$; and $(\hat p^m \hat q^n)$ denotes the symmetric product. [The "symmetric product" of noncommuting operators $A_1^{k_1}, \ldots A_N^{k_N}$ is the operator $(A_1^{k_1}, \ldots, A_N^{k_N})$, defined by

$$(\alpha_1 A_1 + \ldots + \alpha_N A_N)^M = \sum\frac{m!}{k_1!\ldots k_N!}\alpha_1^{k_1}\ldots\alpha_N^{k_N}(A_1^{k_1}\ldots A_N^{k_N}), \tag{S.3}$$

where α_i are numbers and $M = (k_1 + \cdots + k_N)$.] In particular, if $H(p, q) = (p^2/2m) + v(q)$, then $\hat H = (\hat p^2/2m) + v(\hat q)$; if $H(p, q) = pq$, then $\hat H = (1/2)(\hat p\hat q + \hat q\hat p) = \hat q\hat p + (h/2i)$.

2) Relationship with the matrix elements

We consider the $\hat q$ representation. Solving the Cauchy problem for the equation

$$\frac{\partial f}{\partial t} = i(\alpha\hat p + \beta\hat q) f$$

and then setting $t = 1$, we find

$$\left(e^{i(\alpha\hat p + \beta\hat q)} f\right)(s) = f(s + \alpha h)\, e^{i\left(\beta s + \frac{\alpha\beta h}{2}\right)}. \tag{S.4}$$

Hence

$$(\hat H f)(s) = \int\varphi(\alpha,\beta)\, e^{i\left(\beta s + \frac{\alpha\beta h}{2}\right)} f(s + \alpha h)\, d\alpha\, d\beta = \int K(s, s') f(s')\, ds',$$

where $K(s, s') = \langle s|H|s'\rangle$ is the matrix element of the operator $\hat H$, given by

$$K(s, s') = \frac{1}{h^n} \int \varphi\left(\frac{s'-s}{h}, \beta\right) e^{\frac{i\beta}{2}(s+s')} d\beta$$

$$= \frac{1}{(2\pi)^{2n} h^n} \int H(p, q) e^{-i\left(\frac{s'-s}{h} p + \beta q\right) + \frac{i\beta}{2}(s'+s)} dp\, dq\, d\beta$$

$$= \frac{1}{(2\pi h)^n} \int H(p, q)\, \delta\left(q - \frac{s+s'}{2}\right) e^{-i \frac{s'-s}{h} p} dp\, dq$$

$$= \frac{1}{(2\pi h)^n} \int H\left(p, \frac{s+s'}{2}\right) e^{\frac{i}{h} p(s-s')} dp. \qquad \text{(S.5)}$$

Equation (S.5) can be inverted:

$$H(p, q) = \int K\left(q - \frac{\xi}{2}, q + \frac{\xi}{2}\right) e^{ip\xi/h} d\xi. \qquad \text{(S.6)}$$

3) Multiplication law

If $\hat{H} = \hat{H}_1 \hat{H}_2$, then the matrix elements for these operators are related by

$$K(x, y) = \int K_1(x, s) K_2(s, y)\, ds. \qquad \text{(S.7)}$$

Hence, using (S.5) and (S.6), we find the relationship between their symbols:

$$H(p, q) = (H_1 * H_2)(p, q)$$

$$= \frac{1}{(\pi h)^{2n}} \int H_1(p_1, q_1) H_2(p_2, q_2) e^{\frac{2i}{h} s(p, q; p_1, q_1; p_2, q_2)} dp_1\, dq_1\, dp_2\, dq_2, \qquad \text{(S.8)}$$

where

$$s = p_1(q - q_1) + p_2(q_2 - q) + p(q_2 - q_1) = -2 \oint_{A_1 \overset{A}{\triangle} A_2} p\, dq, \qquad \text{(S.9)}$$

and $A_1 \overset{A}{\triangle} A_2$ is a rectilinear triangle in phase space with the vertices $A = (p, q)$, $A_1 = (p_1, q_1)$, $A_2 = (p_2, q_2)$. In particular, with $n = 1$, this triangle has an area $|s/2|$. We consider the following operator in $L^2(R^{4n})$:

$$L = \frac{1}{2}\left(\frac{\partial^2}{\partial p_1 \partial q_2} - \frac{\partial^2}{\partial p_2 \partial q_1}\right). \qquad \text{(S.10)}$$

Setting $U(t) = e^{itL}$, we easily find

$$(U(t) f)(p_1, p_2, q_1, q_2) = \frac{1}{(\pi t)^{2n}} \int e^{2i/t\, [(p_1 - p_1')(q_2 - q_2') - (p_2 - p_2')(q_1 - q_1')]} f(p_1', p_2', q_1', q_2')\, dp_1'\, dp_2'\, dq_1'\, dq_2'. \qquad \text{(S.11)}$$

Comparing (S.11) with (S.7) and (S.8), we find

$$(H_1 * H_2)(p, q) = (U(t) H_1 H_2)(p_1, p_2, q_1, q_2)\big|_{\substack{p_1 = p_2 = p,\\ q_1 = q_2 = q}} \qquad \text{(S.12)}$$

where $H_1 H_2 = H_1(p_1', q_1') H_2(p_2', q_2')$.

From (S.12) we find a power-series expansion of $H_1 * H_2$:

$$(H_1 * H_2)(p, q) = \sum \frac{(ih)^n}{n!} L^n H_1(p_1, q_1) H_2(p_2, q_2)\Big|_{\substack{p_1 = p_2 = p,\\ q_1 = q_2 = q}}. \qquad \text{(S.13)}$$

This equation can be rewritten

$$(H_1 * H_2)(p, q) = H_1\left(p + \frac{ih}{2} \frac{\partial}{\partial \tilde{q}}, q - \frac{ih}{2} \frac{\partial}{\partial \tilde{p}}\right) H_2(\tilde{p}, \tilde{q})\Big|_{\substack{\tilde{p} = p,\\ \tilde{q} = q}}$$

$$= H_2\left(p - \frac{ih}{2} \frac{\partial}{\partial \tilde{q}}, q + \frac{ih}{2} \frac{\partial}{\partial \tilde{p}}\right) H_1(\tilde{p}, \tilde{q})\Big|_{\substack{\tilde{p} = p,\\ \tilde{q} = q}}. \qquad \text{(S.14)}$$

The first terms of the expansion in powers of H yield

$$(H_1 * H_2)(p, q) = H_1(p, q) H_2(p, q) + \frac{ih}{2}\left(\frac{\partial H_1}{\partial p} \frac{\partial H_2}{\partial q} - \frac{\partial H_1}{\partial q} \frac{\partial H_2}{\partial p}\right),$$

in accordance with the general definition of quantization.

In conclusion, we write the composition formula in terms of the Fourier transform of the symbols:

$$\varphi(\alpha, \beta) = \int \varphi_1(\alpha - \alpha', \beta - \beta') \varphi_2(\alpha', \beta') e^{-\frac{ih}{2}(\alpha\beta' - \beta\alpha')} d\alpha'\, d\beta'. \qquad \text{(S.15)}$$

[Equation (S.15) can be derived most simply from Eq. (S.12).]

4) Hermitian-adjoint operator. Trace

The matrix elements of the Hermitian-adjoint operators $\hat{H}$ and $\hat{H}_1 = \hat{H}^*$ are related by $K_1(s_1, s_2) = \overline{K(s_2, s_1)}$. Using (S.5) we find that the symbols for these operators are complex conjugates:

$$H_1(p, q) = \overline{H(p, q)}. \qquad \text{(S.16)}$$

Furthermore, $\mathrm{Sp}\hat{H} = \int K(s, s) ds$. Hence, using (S.5), we find

$$\mathrm{Sp}\, \hat{H} = \frac{1}{(2\pi h)^n} \int H(p, q)\, dp\, dq. \qquad \text{(S.17)}$$

Analogously,

$$\mathrm{Sp}(\hat{H}_1 \hat{H}_2^*) = \frac{1}{(2\pi h)^n} \int H_1(p, q) \overline{H_2(p, q)}\, dp\, dq. \qquad \text{(S.18)}$$

5) Linear canonical transformations

We denote by $\hat{U}$ a unitary operator in $L^2(R^n)$ which performs a linear canonical transformation,

$$\hat{U}\hat{p}\hat{U}^{-1} = \hat{p}_1 = A\hat{p} + B\hat{q} + a,$$
$$\hat{U}\hat{q}\hat{U}^{-1} = \hat{q}_1 = C\hat{p} + D\hat{q} + b, \qquad \text{(S.19)}$$

where $\binom{AB}{CD}$ is a real simplicial matrix, and a and b are arbitrary vectors. It follows from (S.1) that the symbols of the operators $\hat{H}$ and $\hat{H}_1 = \hat{U}\hat{H}\hat{H}^{-1}$ are related by

$$H_1(p, q) = H(Ap + Bq + a, Cp + Dq + b). \qquad \text{(S.20)}$$

In other words, a linear transformation of the symbols reduces to a change of variables. It can be shown[5,17] that this property is an unambiguous characteristic of Weyl symbols. We note the relationship between the quantum-mechanical and classical linear canonical transformations. Ignoring the first equations in (S.13), and eliminating the caret (^) on $\hat{p}$ and $\hat{q}$, we find a classical canonical transformation, which may naturally be called the classical version of transformation (S.13). Inversely, putting a caret on p and q in the equations for the classical linear canonical transformation, we find the quantum-mechanical canonical transformation which may naturally be called the quantum-mechanical version of the classical transformation.

6) Reflections

We associate with each classical linear canonical transformation a unitary operator $\hat{U}$, defined within a factor, which performs the quantum-mechanical version of this operation in accordance with Eq. (S.19). We thus have a projection representation of the group of classical linear canonical transformations. In general, it is not possible to reduce the ambiguity in the association of an operator with a canonical transformation,[16)] but for certain families of canonical transfor-

16) Within a factor, the operator corresponding to a parallel translation by a vector $x + (\alpha, \beta)$ is $\hat{T}_x = \exp\{i/h(\beta\hat{p} - \alpha q)\}$. If

mations this can be done. One such family is the family of reflections

$$p_1 = -p + 2p_0, \quad q_1 = -q + 2q_0. \tag{S.21}$$

The reflection in (S.21) at the point (p_0, q_0) is naturally associated with the operator $\hat{U}_{p_0 q_0}$ which has the Weyl symbol

$$U_{p_0, q_0}(p, q) = (\pi h)^n \delta(p - p_0)\delta(q - q_0). \tag{S.22}$$

From (S.8), (S.14), and (S.22) we find

$$\hat{U}_{p_0, q_0}\hat{p}\hat{U}^{-1}_{p_0, q_0} = -\hat{p} + 2p_0, \quad \hat{U}_{p_0, q_0}\hat{q}\hat{U}^{-1}_{p_0, q_0} = -\hat{q} + 2q_0, \quad \hat{U}^2_{p_0, q_0} = 1, \quad \hat{U}^* = \hat{U}. \tag{S.23}$$

We consider a Hilbert space of operators with a scalar product $(A, B) = \mathrm{Sp}(AB^*)$. Equations (S.18) and (S.22) show that in this space the reflection operators $\hat{U}_{pq}$ play the role of generalized orthonormal system analogous to e^{ipx} in ordinary space, L^2. The Weyl symbols serve as a Fourier transformation:

$$(\hat{U}_{p_1, q_1}, \hat{U}_{p_2, q_2}) = \left(\frac{\pi h}{2}\right)^n \delta(p_1 - p_2)\delta(q_1 - q_2), \tag{S.24}$$
$$\hat{H} = \frac{1}{(\pi h)^n}\int H(p, q)\hat{U}_{p, q}\,\mathrm{d}p\,\mathrm{d}q, \quad H(p, q) = 2^n \mathrm{Sp}(\hat{H}\hat{U}_{p, q}).$$

This interpretation of the Weyl symbols may serve as the basis for some far-reaching generalizations.[18] The Weyl symbols were introduced by Weyl in his well-known book,[19] but he gave no equations of operator calculus there. The equations given in the present paper are scattered over a large number of papers, some of which have been speculative or philosophical in nature.[20] At this point it does not seem possible to say just who should be credited with each particular equation. One of the first papers using the Weyl symbols in the modern manner was that by Grünewold.[21] (A construction based on a path integral is carried out in that paper, but for a corresponding automorphism in operator algebra rather than for an evolution operator in a state space.) The analytic properties of the Weyl symbols were the subject of Ref. 22.

b: Wick symbols. Bose version

1) Definition and basic properties

We denote by F_2 the Fock space of entire antianalytic functions $f(z)$ of n variables with the scalar product

$$(f, g) = \frac{1}{h^n}\int f(z)g(z)e^{-\frac{1}{h}z\bar{z}}\prod \mathrm{d}z\,\mathrm{d}\bar{z}. \tag{S.25}$$

The product of differentials incorporates a normalization factor which ensures that

$$(f, f) = 1 \quad \text{for} \quad f(\bar{z}) \equiv 1. \tag{S.26}$$

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there existed a factor $c(x) \neq 0$ which made the correspondence unambiguous, this factor would have to satisfy the equation

$$c(x)c(y) = c(x+y)e^{-\frac{i}{2h}x\omega y}, \quad \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{*}$$

Equation (*) is contradictory, since its left side is symmetric with respect to x and y but its right side is not. Consequently, the ambiguity in the representation of the group of linear canonical transformations is a fundamental property of quantization. [Equation(*) follows from the equation $\hat{T}_x\hat{T}_y = e^{i/2h\, x\omega y}T_{x+y}$, which in turn follows from (S.15).]

In F_2 there are creation and annihilation operators $\hat{a}^*_k$ and $\hat{a}_k$:

$$(\hat{a}^*_k f)(\bar{z}) = \bar{z}_k f(\bar{z}), \quad (\hat{a}_k f)(\bar{z}) = h\frac{\partial}{\partial \bar{z}_k}f(\bar{z}). \tag{S.27}$$

With each operator written in the normal Wick form,

$$\hat{A} = \sum A_{mn}(\hat{a}^*)^m \hat{a}^n, \tag{S.28}$$

we associate a corresponding Wick symbol,

$$A(\bar{z}, z) = \sum A_{mn}\bar{z}^m z^n. \tag{S.29}$$

[The coefficients A_{mn} in Eqs. (S.28) and (S.29) are the same; here m and n are the multiple indices.] The symbol $A(z, \bar{z})$ is a contraction in R^{2n} of the entire analytic function of two invariables,

$$A(v, z) = \sum A_{mn}v^m z^n$$

(see below). The effect of the operator on a vector is defined by

$$(\hat{A}f)(\bar{z}) = \frac{1}{h^n}\int A(\bar{z}, v)f(\bar{v})e^{\frac{1}{h}(\bar{z}-\bar{v})v}\prod \mathrm{d}v\,\mathrm{d}\bar{v}. \tag{S.30}$$

The symbol of the operator $\hat{A} = \hat{A}_1\hat{A}_2$ is expressed in terms of the symbols of the operators $\hat{A}_1$ and $\hat{A}_2$ by

$$A(\bar{z}, z)(A_1 * A_2)(z, \bar{z}) = \frac{1}{h^n}\int A_1(\bar{z}, v)A_2(\bar{v}, z)e^{-\frac{1}{h}(\bar{z}-\bar{v})(z-v)}\prod \mathrm{d}v\,\mathrm{d}\bar{v}. \tag{S.31}$$

The functions $A(\bar{z}, v)A_1(\bar{z}, v), A_2(\bar{v}, z)$ in Eqs. (S.30) and (S.31) are analytic continuations of the corresponding symbols. The symbol for the operator $\hat{A}_1 = \hat{A}^*$, the Hermitian adjoint of $\hat{A}$, is the complex conjugate of the symbol of the operator $\hat{A}$:

$$A_1(\bar{z}, z) = \overline{A(\bar{z}, z)}. \tag{S.32}$$

In particular, the symbol for the self-adjoint operator is real.

The trace of the operator can be expressed in terms of its Wick symbol by means of

$$\mathrm{Sp}\,\hat{A} = \frac{1}{h^n}\int A(\bar{z}, z)\prod \mathrm{d}\bar{z}\,\mathrm{d}z. \tag{S.33}$$

In F^2 we consider a family of vectors which depend on the parameter v:

$$\Phi_v(\bar{z}) = e^{\frac{1}{h}v\bar{z}}. \tag{S.34}$$

(These vectors are "coherent states.") It turns out that the Wick symbol of the operator $\hat{A}$ is its average value over this state:

$$A(\bar{v}, v) = \frac{(\Phi_v, \hat{A}\Phi_{\bar{v}})}{(\Phi_{\bar{v}}, \Phi_{\bar{v}})}. \tag{S.35}$$

From (S.35) we see that it is possible to continue the symbol analytically to C^{2n}:

$$A(z, v) = \frac{(\Phi_z, \hat{A}\Phi_{\bar{v}})}{(\Phi_z, \Phi_{\bar{v}})}. \tag{S.36}$$

[From (S.25) it follows that $(\Phi_z, \Phi_{\bar{v}}) = e^{1/h\, zv}$.] The Wick symbols have yet another important property: In F_2 there is an orthonormal basis consisting of the vectors

$$e_k = \frac{z^k}{\sqrt{k!h^{|k|}}}, \tag{S.27}$$

where $k = (k_1, \ldots, k_n)$ is a multiple index, and $|k| = \sum k_i$ (k_i are the so-called occupation numbers).

We denote by $\|\tilde{A}_{kl}\|$ the matrix of the operator $\hat{A}$ in the basis (S.37), where k and l are multiple indices. We consider the generating function for the matrix elements $\tilde{A}_{kl}$

$$\tilde{A}(\bar{z}, z) = \sum \frac{\tilde{A}_{k,\,l}}{\sqrt{k!\,l!\,h^{|k|+|l|}}} \bar{z}^k z^l. \tag{S.38}$$

The function $\tilde{A}$ is closely related to the Wick symbol of the operator $\hat{A}$:

$$\tilde{A}(\bar{z}, z) = A(\bar{z}, z)\, e^{\frac{1}{h}\bar{z}z}. \tag{S.39}$$

To conclude this section, we write the expansion for the composition of symbols in powers of h. We consider the operator

$$\Delta = \sum \frac{\partial^2}{\partial z_k \partial \bar{z}_k} = \frac{1}{4} \sum \left(\frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial y_k^2} \right),$$

where $z_k = x_k + iy_k$. We recall that the solution of the Cauchy problem for the heat-conduction equation,

$$\frac{\partial u}{\partial t} = \Delta u, \tag{S.40}$$

is given by the Poisson kernel. In complex coordinates, it can be written

$$u(t; z, \bar{z}) = \int K_t(z - v, \bar{z} - \bar{v})\, f(v, \bar{v}) \prod dv\, d\bar{v}, \tag{S.41}$$

where

$$K_t(z, \bar{z}) = \frac{1}{t^n} e^{-\frac{1}{t}\bar{z}z}. \tag{S.42}$$

[Equation (S.38) has the same normalization of differentials as in the preceding integrals.] Comparing Eqs. (S.41), (S.42), and (S.31), we find

$$(A_1 * A_2)(\bar{z}, z) = e^{h\Delta_{v\bar{v}}} A_1(\bar{z}, v)\, A_2(\bar{v}, z)|_{v=z, \bar{v}=\bar{z}}. \tag{S.43}$$

(The operator $\Delta_{v\bar{v}}$ acts on A_1, A_2 as on the function $v, \bar{v}$. The arguments $z, \bar{z}$ serve as parameters.) Equation (S.43) can be rewritten in the equivalent form

$$(A_1 * A_2)(\bar{z}, z) = A_1\left(\bar{z}, z + h\frac{\partial}{\partial \bar{v}}\right) A_2(\bar{v}, z)|_{\bar{v}=\bar{z}} = A_2\left(\bar{z} + h\frac{\partial}{\partial v}, z\right) A_1(\bar{z}, v)|_{v=z}. \tag{S.44}$$

The first few terms of the expansion in powers of h are

$$(A_1 * A_2)(\bar{z}, z) = A_1(\bar{z}, z)\, A_2(\bar{z}, z) + h\frac{\partial A_1}{\partial z} \frac{\partial A_2}{\partial \bar{z}},$$

in agreement with the general concept of quantization.

2) *Relationship between the Wick and Weyl symbols*

In the space F_2 we consider the operators

$$\hat{q}_k = \frac{1}{\sqrt{2}}(\hat{a}_k + \hat{a}_k^*), \quad \hat{p}_k = \frac{1}{i\sqrt{2}}(\hat{a}_k - \hat{a}_k^*). \tag{S.45}$$

It is easy to see that these operators satisfy the same commutation relations as the ordinary coordinate and momentum operators. Furthermore, it can be shown that by diagonalizing the operators $\hat{q}_k$ of the type in (S.45) we find the ordinary $\hat{q}$ representation, in which the operators $\hat{p}_k$ and $\hat{q}_k$ have the form in (1.2).[17] We thus find it possible to consider both the Wick and Weyl symbols, $A_{\mathrm{Wick}}(z, \bar{z})$ and $A_{\mathrm{Weyl}}(p, q)$, for the same operator $\hat{A}$. The arguments of these functions are related by (S.45):

$$q_k = \frac{1}{\sqrt{2}}(z_k + \bar{z}_k), \quad p_k = \frac{1}{i\sqrt{2}}(z_k - \bar{z}_k). \tag{S.46}$$

To relate the Wick and Weyl symbols we first give the Wick symbol of the reflection operator U_{p_0, q_0}; it turns out to be

$$K_{p_0, q_0}(\bar{z}, z) = e^{-\frac{2}{h}(\bar{z}-\bar{z}_0)(z-z_0)}, \tag{S.47}$$

where $z_0, \bar{z}_0$ are related to p_0, q_0 as in (S.46).[18] It follows from the second equation in (S.24) and from (S.35) that

$$A_{\mathrm{Wick}}(\bar{z}, z) = \frac{1}{(\pi h)^n} \int A_{\mathrm{Weyl}}(p, q)\, K_{p,q}(\bar{z}, z)\, dp\, dq$$
$$= \left(\frac{2}{h}\right)^n \int A_{\mathrm{Weyl}}(p, q)\, e^{-\frac{2}{h}(\bar{z}-\bar{v})(z-v)} dv\, d\bar{v}. \tag{S.48}$$

The Wick symbol is thus a solution of the Cauchy problem of the heat-conduction equation (S.40), for the time $t = h/2$; the Weyl symbol is the initial condition for this problem.

3) *Anti-Wick symbols*

We write the operator A in the anti-Wick normal form:

$$\hat{A} = \sum \mathring{A}_{mn} \hat{a}^m (\hat{a}^*)^n. \tag{S.49}$$

The generating function $\mathring{A}(z, \bar{z})$ for the coefficients $\mathring{A}_{mn}$ is called the "anti-Wick symbol" of the operator $\hat{A}$:

$$\mathring{A}(\bar{z}, z) = \sum \mathring{A}_{mn} z^m \bar{z}^n. \tag{S.50}$$

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[17] We set $\hat{b}_k = 1/\sqrt{2}(\hat{q}_k + i\hat{p}_k)$, $\hat{b}_k^* = 1/\sqrt{2}(q_k - ip_k)$, where $\hat{q}_k$ and $\hat{p}_k$ are the coordinate and momentum operators in the $\hat{q}$ representation and have the form in (1.2). We set

$$\Phi_0(s) = e^{-\frac{s^2}{2h}}, \quad \Phi_{k_1 \ldots k_n}(s) = \hat{b}_1^{*k_1} \ldots \hat{b}_n^{*k_n} \Phi_0(s).$$

The functions $\Phi_{k_1, \ldots, k_n}(s)$ form an orthogonal (but not normal) basis in $L^2(R^n)$ [$\Phi_{k_1, \ldots, k_n} = h^{1/2\Sigma k_i} H_{k_1}(s/\sqrt{h} \ldots H_{k_n}(s/\sqrt{h})$ where H_{k_i} are the Hermite functions]. We consider the mapping of L, $L^2(R^n) \to F_2$, defined by

$$L\Phi_{k_1 \ldots k_n} = \bar{z}_1^{k_1} \ldots \bar{z}_n^{k_n}.$$

It is easy to see that L is an isomorphism of spaces and that $\hat{L} b_k L^{-1} = \hat{a}_k$, $L\hat{b}_k^* L^{-1} = a_k^*$, where $\hat{a}_k$ and $\hat{a}_k^*$ are operators of the type in (S.27). Accordingly, $L\hat{q}_k L^{-1} = 1/\sqrt{2}\,(a_k + a_k^*)$, $\hat{L}_{p_k} L^{-1} = 1/\sqrt{2}\,(\hat{a}_k - \hat{a}_k^*)$. Finally, we note that the isomorphism L can be specified in integral form by means of the generating function for Hermite polynomials:

$$(Lf)(\bar{z}) = \frac{1}{(\pi h)^{n/4}} \int \exp\left[-\frac{1}{2n}(s^2 - 2\sqrt{2}\,\bar{z}s + \bar{z}^2)\right] f(s)\, ds,$$

(Ref. 23).

[18] Equation (S.47) is found in the following manner: Using (S.44), we find that if $\hat{V}_{p_0 q_0}$ is the Equation with the Wick symbol in (S.47), then

$$\hat{a}_k \hat{V}_{p_0, q_0} = \hat{V}_{p_0, q_0}(-\hat{a}_k + 2z_0), \quad \hat{a}_k^* \hat{V}_{p_0, q_0} = \hat{V}_{p_0, q_0}(-\hat{a}_k^* + 2\bar{z}_0),$$

where $z_0 = 1/\sqrt{2}\,(p_0 + ip_0)$. It follows that the operator $\hat{V}_{p_0 q_0}$ differs from the operator $\hat{U}_{p_0 q_0}$ by a factor: $\hat{U}_{p_0 q_0} = c(p_0, q_0)\, \hat{V}_{p_0 q_0}$. Hence, using (S.24) and (S.235), we find a relationship between the Wick and Weyl symbols which differs from (S.47) in the presence of a factor c on the right-hand side. We also note that both the Wick and Weyl symbols of the unit operator are identically 1. Consequently, $c = 1$.

The Wick and anti-Wick symbols of a given operator are related by

$$\mathring{A}(\bar{z}, z)=\frac{1}{h^n}\int \mathring{A}(\bar{v}, v)\, e^{-\frac{1}{h}(\bar{z}-\bar{v})(z-v)}\prod dv\, d\bar{v}. \tag{S.51}$$

In other words, the Wick symbol is the solution of the Cauchy problem for the heat-conduction equation, (S.40), at the time $t=h$; the anti-Wick symbol is the initial condition. Equation (S.51) permits a generalization of the definition of the anti-Wick symbol: We assume that the Wick symbol $\mathring{A}(\bar{z}, z)$ of the operator $\hat{A}$ permits integral representation (S.51) with some function $\mathring{A}(\bar{v}, v)$; then $\mathring{A}(\bar{v}, v)$ is called the anti-Wick symbol of the operator $\hat{A}$.

There are several important duality relations between the Wick and anti-Wick symbols. Let us examine the most important of them:

1) $$\mathrm{Sp}\,\hat{A}\hat{B}^{*}=\frac{1}{h^n}\int \mathring{A}(\bar{z}, z)\,\mathring{B}(\bar{z}, z)\prod dz\, d\bar{z}. \tag{S.52}$$

2) We assume that $D(\hat{A})$ is a set of values of the quadratic form $(\hat{A}f, f)$, with f running over the unit sphere, $\|f\|=1$. [It can be shown that $D(\mathring{A})$ is a convex set on the complex plane; see, for example, Ref. 24.] We furthermore assume that $D(A)$ is a set of values of the Wick symbol of the operator $\hat{A}$ and that $D(\mathring{A})$ is a convex shell of the set of values of the anti-Wick symbol. Then

$$D(A)\subset D(\hat{A})\subset D(\mathring{A}). \tag{S.53}$$

3) We assume that $\hat{A}$ is a self-adjoint operator and that $\varphi(x)$ is a function of a real variable which is convex downward.

$$\frac{1}{h^n}\int \varphi(A(\bar{z}, z))\prod dz\, d\bar{z}\leqslant \mathrm{Sp}\,\varphi(\hat{A})\leqslant \frac{1}{h^n}\int \varphi(\mathring{A}(\bar{z}, z))\prod dz\, d\bar{z}. \tag{S.54}$$

Equations (S.53) and (S.54) are useful for studying the spectrum of the operator $\hat{A}$.

We note in conclusion that the anti-Wick symbols, in contrast with the Weyl and Wick symbols, exist for only a comparatively narrow set of operators. For example, the Schrödinger operator $\hat{H}=(\hat{p}^2/2m)+v(\hat{q})$ has an anti-Wick symbol only if the potential is consistent with the integral representation

$$v(q)=\int e^{-\frac{1}{h}(q-q')^2}\, u(q')\, dq.$$

It necessarily follows that the potential is analytic. Wick symbols were introduced in Ref. 25 (for both the Bose and Fermi versions). Their properties were studied in detail in Ref. 26. The anti-Wick symbols first appeared in papers on quantum optics.[27,28] The properties of these symbols which we have seen here have been taken from Ref. 29. The relationship between the Weyl and Wick symbols was established in Ref. 30.

c: Wick symbols. Fermi case

The definitions of the Wick symbol [Eqs. (S.28) and (S.29)] remains the same; the only difference is that now z_k and $\bar{z}_k$ are not complex variables but the generators of a Grassmann algebra Λ with involution. The superior bar denotes the involution. Consequently, $A(\bar{z}, z)\in\Lambda$. Equations (S.32) and (S.39), which relate the symbols of the adjoint operators and which relate the symbol of an operator and the generating function for its matrix elements in the basis of occupation numbers, are completely conserved. The definition of the scalar product, (S.25), in the Fock space and also Eqs. (S.30) and (S.31) for the action of an operator on a vector and for the product of operators remain the same, with one change: The factor $1/h^n$ in front of the integral should be replaced by h^n (the sign $\int$ means, of course, an integral over the anticommuting variables). Equations (S.27) and (S.43) remain valid, but with refinements: $\partial/\partial\bar{z}_k$ in (S.27) means the left-hand derivative, and the operator Δ_{vv} in (S.43) is

$$\Delta_{v\bar{v}}=(\partial/\partial v_r)\,\partial/\partial\bar{v}_l, \tag{S.55}$$

where the indices r and l denote the right-hand and left-hand derivatives. Equation (S.44) can be modified:

$$(A_1 * A_2)(\bar{z}, z)=A_1\left(\bar{z}, z+h\frac{\vec{\partial}}{\partial\bar{v}}\right)A_2(\bar{v}, z)\Big|_{\bar{v}=\bar{z}}=A_1(\bar{z}, v)\,A_2\left(\bar{z}+h\frac{\overleftarrow{\partial}}{\partial v}, z\right)\Big|_{v=z} \tag{S.56}$$

($\vec{\partial}/\partial\bar{v}$ denotes the left-hand derivative and $\overleftarrow{\partial}/\partial v$ the right-hand derivative). The equation for the trace changes substantially:

$$\mathrm{Sp}\,\hat{A}=h^n\int A(\bar{z}, z)\, e^{\frac{2}{h}\bar{z}z}\prod dz\, d\bar{z}. \tag{S.57}$$

The correct order of the factors must be observed in all the equations. (This is unimportant in the Bose case. However, the equations of the preceding section which are cited here have been written in a form such that they remain applicable in the Fermi case.)

In the Fermi case we can also examine analogies between the Weyl and anti-Wick symbols. The Weyl symbols and the associated path integrals were studied in detail in Ref. 31, and we will not discuss them here.

The anti-Wick symbols in the Fermi case lose all their remarkable properties, and there is no particular point in studying them here.

d: *p-q* and *q-p* symbols

1) Definition and basic properties

We associate the following operators with the function $H(p, q)$ of the type in (S.1)

$$\hat{H}_1=\int e^{i\alpha\hat{p}}e^{i\beta\hat{q}}\varphi(\alpha, \beta)\, d\alpha\, d\beta, \quad \hat{H}_2=\int e^{i\beta\hat{q}}e^{i\alpha\hat{p}}\varphi(\alpha, \beta)\, d\alpha\, d\beta, \tag{S.58}$$

where $\hat{p}_i$, $\hat{q}_j$ are the momentum and coordinate operators. The function $H(p, q)$ is called the $p-q$ symbol of the operator $\hat{H}_1$ and the $q-p$ symbol of the operator $\hat{H}_2$. If the function $H(p, q)$ is a polynomial,

$$H(p, q)=\sum h_{m,n}p^m q^n,$$

the operators $\hat{H}_1$ and $\hat{H}_2$ can be defined in a purely algebraic manner:

$$\hat{H}_1=\sum h_{mn}\hat{p}^m\hat{q}^n, \quad \hat{H}_2=\sum h_{mn}\hat{q}^n\hat{p}^m.$$

We assume that H is some operator and that $H_{\hat{p}\hat{q}}(p, q)$ and $H_{\hat{q}\hat{p}}(p, q)$ are its $p-q$ and $q-p$ symbols, respectively; also, $K(s_1, s_2)=\langle s_1|\hat{H}|s_2\rangle$ is the matrix element in the $\hat{q}$ representation. The symbols H_1 and H_2 are related to $K(s_1, s_2)$ by

$$H_{\hat p \hat q}(p, q) = \int K(x, q)\, e^{\frac{p}{ih}(x-q)}\, dx, \qquad H_{\hat q \hat p}(p, q) = \int K(q, y)\, e^{\frac{p}{ih}(q-y)}\, dy, \tag{S.59}$$

$$K(s_1, s_2) = \frac{1}{(2\pi h)^n} \int H_{\hat p \hat q}(p, s_2)\, e^{-\frac{p}{ih}(s_1 - s_2)}\, dp = \frac{1}{(2\pi h)^n} \int H_{\hat q \hat p}(p, s_1)\, e^{-\frac{p}{ih}(s_1 - s_2)}\, dp.$$

Equations (S.59) are derived in the same way as the analogous equations for the Weyl symbols. From them we find multiplication laws and equations for the trace:

$$(A_{\hat p \hat q} * B_{\hat p \hat q})(p, q) = \frac{1}{(2\pi h)^n} \int A_{\hat p \hat q}(p, q_1)\, B_{\hat p \hat q}(p_1, q)\, e^{-\frac{1}{ih}(q-q_1)(p-p_1)}\, dq_1\, dp_1, \tag{S.60}$$

$$(A_{\hat q \hat p} * B_{\hat q \hat p})(p, q) = \frac{1}{(2\pi h)^n} \int A_{\hat q \hat p}(p_1, q)\, B_{\hat q \hat p}(p, q_1)\, e^{\frac{1}{ih}(q-q_1)(p-p_1)}\, dp_1\, dq_1,$$

$$\mathrm{Sp}\, \hat A = \frac{1}{(2\pi h)^n} \int A_{\hat q \hat p}(p, q)\, dp\, dq = \frac{1}{(2\pi h)^n} \int A_{\hat p \hat q}(p, q)\, dp\, dq, \tag{S.61}$$

$$\mathrm{Sp}\, \hat A \hat B^* = \frac{1}{(2\pi h)^n} \int A_{\hat p \hat q}(p, q)\, \overline{B_{\hat p \hat q}(p, q)}\, dp\, dq = \frac{1}{(2\pi h)^n} \int A_{\hat q \hat p}(p, q)\, \overline{B_{\hat q \hat p}(p, q)}\, dp\, dq.$$

We also find a relationship between the $p-q$ and $q-p$ symbols of a given operator,

$$H_{\hat p \hat q}(p, q) = \frac{1}{(2\pi h)^n} \int H_{\hat q \hat p}(p_1, q_1)\, e^{-\frac{1}{ih}(q-q_1)(p-p_1)}\, dp_1\, dq_1,$$
$$H_{\hat q \hat p}(p, q) = \frac{1}{(2\pi h)^n} \int H_{\hat p \hat q}(p_1, q_1)\, e^{\frac{1}{ih}(q-q_1)(p-p_1)}\, dp_1\, dq_1. \tag{S.62}$$

In contrast with the Weyl and Wick symbols, the relationship between the symbols of adjoint operators is complicated. If $A = \hat B^*$, then

$$A_{\hat q \hat p}(p, q) = \overline{B_{\hat q \hat p}(p, q)}. \tag{S.63}$$

We note that the function $(1/2\pi h)\bar e^{1/ih(q-q_1)(p-p_1)}$ serves as the Green's function for the Cauchy problem for the equation $\partial u/\partial h = i\partial^2/\partial p \partial q u$. Equations (S.62) can thus be rewritten

$$H_{\hat p \hat q}(p, q) = e^{ih \frac{\partial^2}{\partial p\, \partial q}} H_{\hat q \hat p}(p, q). \tag{S.64}$$

It also follows that the expansion of compositions (S.60) in powers of h is

$$(A_{\hat p \hat q} * B_{\hat p \hat q})(p, q) = e^{ih \frac{\partial^2}{\partial p_1\, \partial q_1}} A_{\hat p \hat q}(p, q_1)\, B_{\hat p \hat q}(p_1, q)\, |_{\substack{q_1 = q,\\ p_1 = p}}, \tag{S.65}$$

$$(A_{\hat q \hat p} * B_{\hat q \hat p})(p, q) = e^{-ih \frac{\partial^2}{\partial p_1\, \partial q_1}} A_{\hat q \hat p}(p_1, q)\, B_{\hat q \hat p}(p, q_1)\, |_{\substack{q_1 = q,\\ p_1 = p}}. \tag{S.66}$$

2) Relation with the Weyl symbols

Comparing the third equation in (S.59) with (S.6), we find a relationship between the Weyl symbol $H(p, q)$ of the operator $\hat H$ and its $q-p$ symbol:

$$H(p, q) = \frac{1}{\pi h} \int H_{\hat q \hat p}(p_1, q_1)\, e^{-\frac{2}{ih}(q-q_1)(p-p_1)}\, dp_1\, dq_1 = e^{\frac{ih}{2} \frac{\partial^2}{\partial p\, \partial q}} H_{\hat q \hat p}(p, q). \tag{S.67}$$

Comparing this equation with Eq. (S.64), we find

$$H(p, q) = e^{-\frac{ih}{2} \frac{\partial^2}{\partial p\, \partial q}} H_{\hat p \hat q}(p, q). \tag{S.68}$$

The $\hat p \hat q$ and $\hat q \hat p$ symbols are the basis for the theory of pseudodifferential operators; the modern state of this theory is presented in Ref. 32.

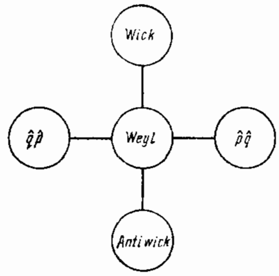

FIG. 1.

Comment. The equations relating the symbols in (S.64), (S.67), and (S.68) and also (S.48) and (S.51) suggest an interpolation. We set

$$H_t(p, q) = e^{it \frac{\partial^2}{\partial p_1\, \partial q}} H_{\hat q \hat p}(p, q),$$

$$A_s(\bar z, z) = e^{-s \frac{\partial^2}{\partial z\, \partial \bar z}} \mathring{A}(\bar z, z).$$

The mutual relations between the interpolation symbols H_t, A_s and the four types of symbols discussed above are conveniently described by the diagram in Fig. 1.

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