Symmetry and the macroscopic dynamics of magnetic materials

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A systematic exposition is given of a recently developed theory of the low-frequency spin dynamics of any magnetic material in which the magnetic order is caused by exchange forces that considerably exceed the relativistic interactions. The proposed approach does not use any model representations of the state of the magnet (localized spins, sublattices, etc.). A fundamental concept in the theory is that of approximate exchange symmetry.

CONTENTS

1. INTRODUCTION

The magnetic materials known at present are characterized by a great variety of magnetic structures, but only for ferromagnets is the theoretical approach that is usually used satisfactory. In this simplest case, the microscopic problem of the structure of the ground state and the spectrum of excitations can be solved exactly.[1,2] On the other hand, the macroscopic properties of ferromagnets are systematically described by the equation of Landau and Lifshitz.[3]

In all the more complicated cases, the microscopic problem can be solved only for spins that are large (classical), which they usually are not. In such a situation, a phenomenological theory, of the Landau-Lifshitz type, becomes particularly important. The phenomenological theory usually used[4,5] leads to correct results in a number of cases, but this theory is of the model type. In it, complicated magnetic structures are treated as a collection of a certain number of interpenetrating, interacting ferromagnets, each of which is described by a Landau-Lifshitz equation. For this there is in general no basis. Recently a phenomenological theory has been developed that is based only on general considerations of symmetry and does not use any model representations of the state of the magnet; that is, it is as general a theory as, for example, hydrodynamics or the theory of elasticity. The theory is applicable to any magnetic materials in which the occurrence of a magnetic structure is the result of the action of exchange forces that considerably exceed the relativistic interactions; thus it is applicable with equal success to crystals and to disordered magnetic materials. The present paper is devoted to a systematic exposition of this theory.

Solution of the problem posed includes the question of the classification of magnetic structures. The classification that is usually used is based on introduction of magnetic moments of sublattices or of magnetic vectors that are linear combinations of them. Such a description, although it uses a model representation of localized spins, gives a graphic and full description of the structure, sufficient for explanation of the macroscopic dynamics of a magnet and, in particular, of its behavior in not too strong magnetic fields (ones that are small in comparison with the exchange fields). But if we have in mind precisely this last problem, then a description by means of sublattice moments is in most cases redundant. The number of sublattices may be as large as one pleases or, in general, infinite. But as we shall see, every structure can be characterized by no more than three mutually perpendicular magnetic vectors,[6] specification of which is sufficient for writing down the macroscopic dynamic equations. These magnetic vectors are in many respects analogous to the spin densities considered by Dzyaloshinskii.[7] Their introduction makes no use of localized-spin representations but is based on investigation of the symmetry of the magnetic material. In contrast to the spin densities, which determine the change of symmetry that occurs in a phase transition of the second kind from the paramagnetic state, the magnetic vectors that we have introduced determine the symmetry of the exchange forces of the magnet as such, without any assumptions about the nature and number of the phase transitions that separate the state under consideration from the paramagnetic.

The macroscopic dynamic properties, that is, the properties on temporal and spatial scales that considerably exceed the reciprocals of the exchange frequen-

cies and the interatomic distance, respectively, can be described on the basis of the following general property of states that possess a magnetic exchange structure. The appearance of such states is always accompanied by the phenomenon of spontaneous breaking of the symmetry of exchange interactions: the Hamiltonian of exchange interactions is invariant with respect to an arbitrary rotation of all the spins through the same angle, but no magnetic structure is invariant with respect to all such rotations. There are always rotations that convert the equilibrium state of the body to another state. If the angles of rotation are the same for spins located at different points of space, then the state obtained is still as much an equilibrium state as was the original state. If the angles of rotation are slowly varying functions of the coordinates, then a weak nonstationarity appears in the system. Along with small spatial derivatives of the angles of rotation, there appear small time derivatives. Only those degrees of freedom of the magnet that reduce locally to certain rotations of the structure, and that are described by rotation angles varying slowly in space and in time, are important in a macroscopic description. Such a description is completely analogous to the theory of elasticity. The role of the components of the displacement vector of the medium is played by the angles of rotation of the spins; the roles of the velocity and of the strain are played, respectively, by the time and space derivatives of the angles of rotation. In contrast to spatial displacements, different rotations do not in general commute with each other. Therefore the equations of magnetic dynamics differ from elasticity theory in that they are nonlinear even at small velocities and strains.

The considerations set forth regarding the phenomenon of spontaneous breaking of symmetry in magnets as the physical reason for the occurrence of low-frequency oscillations (Goldstone bosons), and regarding the relation between the noncommutativity of the symmetry group and the nonlinearity of the equations, are well known and have been discussed by many authors.[8-18] Description of the dynamics of spins by means of angles of rotation has been used in investigation of the properties of various magnetic materials[15-18] and of the magnetic properties of the superfluid phases of ^{3}He.[13,14] We mention in particular the papers of Halperin and Hohenberg[15] and of Halperin and Saslow.[16]

In the first part of this paper, the concept of exchange symmetry of magnetic materials is developed, and on the basis of it a classification of the possible types of magnetic structures, in crystalline and in disordered systems, is carried out. Then, for all the essentially different cases, the nonlinear dynamical equations are derived, with allowance for a magnetic field and for relativistic interactions. Expressions are given for the magnetic resonance frequencies and for the spin-wave spectrum. We note here that the phenomenological approach used makes it possible to obtain certain general results very simply that were obtained in the microscopic theory only under very stringent limitations. For example, in all (dielectric) collinear pure-exchange magnets the longitudinal magnetic susceptibility van-

ishes at zero temperature. In the Heisenberg model of an antiferromagnet, this assertion was proved only in the limit of infinite spin, by expansion in inverse powers of the spin; this cannot in any case exclude an exponentially small susceptibility.

2. THE EXCHANGE SYMMETRY OF CRYSTALS

A characteristic property of magnetic materials is the existence in them, along with a scalar charge density $\rho(r)$, of a nonvanishing microscopic magnetic-moment density $m(r)$. The symmetry properties of such a system are described (see Ref. 19, §28) by specification of one of 1651 magnetic space groups. These groups contain as their elements the purely spatial transformations (rotations, reflections, translations) and combinations with the time-reflection transformation R with respect to which the values of $\rho(r)$ and $m(r)$ are invariant. According to the specified magnetic group, one can easily determine the purely spatial symmetry of the system, if in the elements of the magnetic group one formally identifies the transformation R with the identity transformation. The resulting one of 230 space groups contains all the transformations with respect to which the charge density, and in general all the quantities that do not change on time reflection, remain invariant.

Such a symmetry characteristic of the body is exact; but precisely for this reason, important approximate symmetry properties are completely lost in it. The fact is that a fundamental role is played in the properties of the majority of magnetic materials, by exchange forces whose symmetry is higher than the symmetry of the weak relativistic interactions. To indicate only the exact symmetry means to lose completely information about the higher symmetry of the exchange forces.

We neglect all relativistic interactions and consider the exchange symmetry of the magnet. Since the exchange forces depend only on the relative orientations of the spins, in the present case, besides the transformation R, there appears an infinite set of new symmetry transformations, consisting of all rotations U of the spin space; that is, rotations of all the spins through the same angle. In such a situation, the aggregate orientation of the spins with respect to the crystallographic axes becomes arbitrary, and we may consider that the components of the magnetic moment $m(r)$ behave as scalars in all purely spatial transformations and are components of a vector in spin space; that is, they transform like the components of a vector only during rotations of the spin space. The transformation R plays the role of inversion of the spin space. The exchange symmetry is determined[6] by specification of one of the exchange space groups, which contain all those combinations of purely spatial transformations, of rotations of the spin space U, and of the transformation R with respect to which the values of $m(r)$ are invariant. If we formally identify the transformations U and R in the elements of the exchange group with the identity transformation, we get some ordinary space group G. This is the symmetry group of the charge density and generally of all spin scalars in the system. But the values

of $\mathbf{m}(\mathbf{r})$ are in general not invariant with respect to it.

Various generalizations of the magnetic groups, in the set of transformations, along with the element R, by inclusion of arbitrary spin rotations, have been made by a number of authors.[20-24] Classification of such "color" or "spin" groups has been treated, however, without relation to the exchange approximation. The topic was therefore not an approximate, but some exact accidental symmetry, the existence of which can be caused (see Ref. 22) by special properties of the Hamiltonian even in the presence of relativistic interactions.

For physical applications, instead of a formal classification of the exchange space groups, it is convenient to proceed as follows. Like any function of the coordinates, each of the components of the magnetic moment $\mathbf{m}(\mathbf{r})$ can be represented as the sum of functions that transform according to irreducible representations of the pure space group G introduced above. Since these functions are in general different for the three different components of the moment, the expansion in the general case can be written in the form

$$\mathbf{m}(\mathbf{r}) = \sum_{an\alpha} \mathbf{M}_{n\alpha}^{(n)} \varphi_{n\alpha}^{(n)}(\mathbf{r}), \tag{1}$$

where the functions $\varphi_{n\alpha}^{(a)}(\mathbf{r})$ with given a transform according to the nth irreducible representation of the group G, and the index α enumerates the functions belonging to this representation. For each representation in formula (1) there occur, in general, three sets of linearly independent functions, enumerated by the index $a = 1, 2, 3$. The symmetry of the magnetic structure is completely determined by specification of the space group G and of the magnetic vectors $\mathbf{M}_{n\alpha}^{(a)}$. They transform as spin vectors in spin rotations and change sign upon R transformation; and it may be considered that they (and not the functions φ) transform according to the nth irreducible representation of the group G. The quantities $\mathbf{M}_{n\alpha}^{(a)}$ are analogous to the spin densities considered by Dzyaloshinskiĭ.[7] We emphasize, however, that in Ref. 7 the expansion was carried out according to representations of the symmetry group of the paramagnetic phase; this in general differs from the space group G.

The magnetic vectors $\mathbf{M}_{n\alpha}^{(a)}$ satisfy[6] a number of stringent conditions, which greatly restrict the number of different magnetic structures with a prescribed space group G. We consider the scalar products $\mathbf{M}_{n\alpha}^{(a)} \cdot \mathbf{M}_{m\beta}^{(b)}$. They are spin scalars; therefore in the equilibrium state, they must be invariants relative to the group G. On the other hand, these quantities transform according to the direct product of the irreducible representations n and m. Since only unitary representations can participate in the expansion (1), the representation $n \times m$ contains the identity only when n is equivalent to $\bar{m}$, complex conjugate to m. Only in this case can the scalar product under consideration be nonzero. Furthermore, by virtue of the unitarity of the representations we have

$$\mathbf{M}_{n\alpha}^{(a)} \cdot \mathbf{M}_{m\beta}^{(b)} = c(a, n)\,\delta_{ab}\delta_{n\bar{m}}\delta_{\alpha\beta}, \tag{2}$$

where $c(a, n)$ are some constants. In general, instead of $c(a, n)\delta_{ab}$ a more general set of constants $c(a, b, n)$ can occur in formula (2). But it is always possible to redefine the magnetic vectors (by appropriate linear transformation of identically transforming functions $\varphi_{n\alpha}^{(a)}$) in such a way as to reduce the formula to the form (2).

For what follows, it is convenient, as usual, to combine the complex representations n and $\bar{n}$ into a single physcially irreducible representation, and to introduce real magnetic vectors in place of the complex vectors $\mathbf{M}_{m\beta}^{(a)} = \mathbf{M}_{m\beta}^{(a)}*$. They satisfy the relations (2), in which the indices n and m now enumerate the real physically irreducible representations; for this reason we must suppose that $\bar{n} = n$. The relations (2) show that different vectors are perpendicular to each other; those of them that correspond to a single multidimensional representation have the same length. The maximum number of nonvanishing vectors and the dimensionality of the corresponding representations are therefore equal to three.

The magnetic vectors that transform according to nonidentity representations obviously make no contribution to the total magnetization of the body; they are therefore antiferromagnetic vectors. If there are one, two, or three vectors that transform according to the identity representation, then the body possesses a nonzero magnetization; we may suppose (this can be achieved by appropriate orthogonal transformation and normalization of the functions $\varphi^{(a)}$) that one of these vectors coincides with the magnetization (we denote this vector by $\mathbf{M}$). Then the remaining vectors of the identity representation are antiferromagnetic.

We note also that the constants $c(a, n)$ corresponding to the antiferromagnetic vectors may be supposed equal to zero or unity, $i.e.$ that the nonvanishing antiferromagnetic vectors are unit vectors; this can always be achieved by appropriate normalization of the functions $\varphi_{n\alpha}^{(a)}$. Below, we shall number the nonvanishing antiferromagnetic vectors $\mathbf{I}_a$ with a single index a, which now has the meaning simply of a serial number and may take the values $a = 1, 2, 3$.

Thus only four essentially different cases are possible: 1) There exists a single magnetic vector that transforms according to the identity representation, and it coincides with the magnetization $\mathbf{M}$. The body is a ferromagnet or a collinear ferrimagnet. Such a ferrimagnet, with respect to its symmetry and macroscopic dynamics in not too strong magnetic fields, is no different from a ferromagnet. 2) There is a single vector $\mathbf{l}$ that transforms according to a nonidentity (one-dimensional) representation. The body is a collinear antiferromagnet. 3) There are the vector $\mathbf{M}$ and one antiferromagnetic vector $\mathbf{l}$ perpendicular to it, transforming according to a one-dimensional (identity or nonidentity) representation, or two vectors $\mathbf{l}_1$ and $\mathbf{l}_2$ perpendicular to each other and to $\mathbf{M}$, transforming according to the same or different one-dimensional representations or according to a single two-dimensional. The body is a noncollinear ferrimagnet. 4) There are two mutually perpendicular vectors $\mathbf{l}_1$ and $\mathbf{l}_2$, transforming according to the same or different

nonidentity one-dimensional representations or according to a single two-dimensional, or three mutually perpendicular vectors l_1, l_2, and l_3, transforming according to one, two, or three nonidentity one-dimensional representations or according to a nonidentity one-dimensional and a two-dimensional, or, finally, according to a three-dimensional representation. The body is a noncollinear antiferromagnet.

Representations of the space groups are achieved, as is well known (see Ref. 25, §134), with functions of the form

$$u_{\mathbf{k}}(\mathbf{r})\, e^{i\mathbf{k}\cdot\mathbf{r}},$$

where k is the wave vector and where $u_{\mathbf{k}}$ are functions periodic in the crystal lattice. We are interested only in one-, two-, and three-dimensional physically irreducible representations. Such a representation can be characterized by wave vectors occupying a general position in the reciprocal cell only for crystals of the triclinic system (in more symmetric crystals, the combined star of vectors k and −k contains more than three vectors). In crystals of the monoclinic system, there can exist representations with a wave vector occupying a general position in the plane of symmetry. Representations with a wave vector occupying a general position on an axis of symmetry are possible for all noncubic crystals. In all the cases enumerated, the physically irreducible representation is necessarily two-dimensional; the pair of corresponding antiferromagnetic vectors l_1 and l_2 transforms, in translations, like the pair of functions sin k·r and cos k·r. From the form of the expansion (1) it is clear that the magnetic-moment density transforms in translations as

$$\mathbf{m}(\mathbf{r}) \sim l_1 \sin \mathbf{k}\cdot\mathbf{r} + l_2 \cos \mathbf{k}\cdot\mathbf{r},$$

that is, the corresponding structure (since l_1 and l_2 are mutually perpendicular) is helicoidal. Thus we arrive at the conclusion that in a spatially periodic crystal, only helicoidal exchange magnetic structures incommensurable with the periods of the space lattice are possible. The occurrence of any other incommensurable structure necessarily destroys the spatial periodicity of the crystal.

Nonhelicoidal structures can be characterized only by wave vectors corresponding to certain distinct points of the reciprocal cell. In particular, if all the nonvanishing magnetic vectors transform according to representations with k = 0, the magnetic and spatial elementary cells coincide.

But not all the magnetic structures enumerated above, and in principle possible, can actually occur. Many of them do not satisfy the following stability criterion, which is analogous to the well-known Lifshitz criterion in the theory of phase transitions of the second kind (see Ref. 25, §145). Namely, the existence of an expression that is invariant with respect to the group G and of the form

$$K_i^{ab}\left(l_a\cdot\frac{\partial l_b}{\partial x_i} - l_b\cdot\frac{\partial l_a}{\partial x_i}\right), \tag{3}$$

where x_i are the spatial coordinates, leads to instability of the corresponding structure. In fact, consider a small departure from equilibrium of the form

$$\delta l_a = \delta\theta(\mathbf{r})\times l_a,$$

where the $\delta\theta$ are slowly varying functions of the coordinates. Since at each point of space such a departure reduces to a rotation of all the magnetic vectors through the same angle, the local (that is, not containing spatial derivatives) part of the departure of the energy from the equilibrium value vanishes. The main part of the change of energy is therefore determined by the terms of the form (3), linear in the derivatives; and this obviously can always take negative values.

An important remark, apropos of the stability criterion, must be made in the case of helicoidal structures. If the wave vector occupies a general position on an axis of symmetry, then the existence of invariants of the form (3) implies instability of the structure only when the invariants contain derivatives with respect to coordinates perpendicular to the axis of symmetry. The existence of an invariant with a derivative along the axis corresponds, for example, to a temperature dependence of the period of the helicoid, just as the existence of a linear invariant div u (where u is the displacement vector) corresponds, in the theory of elasticity (see Ref. 26, §6), to thermal expansion of the lattice. Similarly, for structures with wave vectors occupying a general position in a plane of symmetry, the "dangerous" invariants are those containing differentiations with respect to coordinates perpendicular to the plane.

It is known from the theory of phase transitions (see Ref. 25, §145) that invariants of the form (3) are absent only for representations characterized by wave vectors whose components are definite fractions ($\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$) of the periods of the reciprocal lattice. In our case this means that the periods of the magnetic cell may be only 2, 3, or 4 times the periods of the spatial cell; as in the theory of phase transitions, tripling of the magnetic periods in comparison with the spatial is possible only in hexagonal space lattices, and quadrupling in face-centered and body-centered cubic lattices. We emphasize that this conclusion was reached here by consideration of the symmetry of the magnet as such, without any assumptions about the nature and number of the phase transitions that separate the state under consideration from the paramagnetic.

By way of example, we shall carry out the classification of all the possible exchange magnetic structures in body-centered tetragonal crystals, whose space symmetry group G is the group C_{4h}^6. The arrangement of the basic translation vectors is shown in Fig. 1. The defining elements of the group are the mirror rotation S_4 and the product tI of inversion I and translation by the vector

$$t = \frac{3}{4}a_1 + \frac{1}{4}a_2 + \frac{1}{2}a_3.$$

In accordance with the general statements formulated above, in the present case helicoidal structures are possible with a wave vector occupying a general position on the fourth-order axis. There are two types of hexagonal structures, H_1 and H_2, to which correspond moment densities of the form
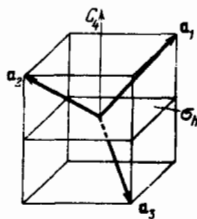
FIG. 1.

H_1: $m(r) \sim l_1 \cos \varkappa (\xi + \eta - \zeta) + l_2 \sin \varkappa (\xi + \eta - \zeta)$,

H_2: $m(r) \sim l_1 \sin 2\pi (\xi - \eta) \cdot \sin 2\pi \zeta \cdot \cos \varkappa (\xi + \eta - \zeta)$
$\qquad\qquad + l_2 \sin 2\pi (\xi - \eta) \cdot \sin 2\pi \zeta \cdot \sin \varkappa (\xi + \eta - \zeta)$,

where $\varkappa$ is an arbitrary parameter, and where ξ, η, and ζ are oblique-angle coordinates along the basic translation vectors: $r = \xi a_1 + \eta a_2 + \zeta a_3$.

Furthermore, it follows from an analysis carried out for this group in Liubarskiĭ's book in connection with the theory of phase transitions,[27] that there are still six nonidentity irreducible representations, with dimensionality not exceeding three, for which invariants of the form (3) are absent. Five of these are representations with $k = 0$. Using for these representations the notation adopted in the book of Landau and Lifshitz (Ref. 28, §95), we give the corresponding expressions for the moment density:

A_u: $m(r) \sim l [\cos 2\pi \zeta - \cos 2\pi (\xi - \eta)]$,

B_g: $m(r) \sim l \sin 2\pi (\xi + \eta - \zeta)$,

B_u: $m(r) \sim l \cos 2\pi (\xi + \eta - \zeta)$,

E_g: $m(r) \sim l_1 \cos 2\pi (\xi - \eta) \cdot \sin 2\pi \zeta + l_2 \cos 2\pi \zeta \cdot \sin 2\pi (\xi - \eta)$,

E_u: $m(r) \sim l_1 \sin 2\pi \zeta + l_2 \sin 2\pi (\xi - \eta)$.

The sixth representation is the two-dimensional representation $\bar{E}$ with $k = (b_1 + b_2 + b_3)/2$, where b_1, b_2, and b_3 are the basic vectors of the reciprocal lattice:

$\bar{E}$: $m(r) \sim l_1 \sin \pi (\xi - \eta) \cdot \cos \pi \zeta + l_2 \cos \pi (\xi - \eta) \cdot \sin \pi \zeta$.

If we add to these the ferromagnetic structure with a moment density that transforms according to the identity representation, A_g, then we get nine different magnetic structures, described by a single irreducible representation of the space group. Also possible are more complicated structures, described by a combination of irreducible representations. Some of these are unstable because of the existence of invariants of the form (3). For example, the instability of the ferrimagnetic structure (A_g, A_u), with moment density

$m(r) \sim M + l [\cos 2\pi \zeta - \cos 2\pi (\xi - \eta)]$

and with two mutually perpendicular vectors M and l, which transform according to the representations A_g and A_u respectively, is due to the existence of the invariant

$$\frac{\partial M}{\partial z} l - M \cdot \frac{\partial l}{\partial z},$$

where z is the coordinate along a principal axis of the crystal. The instability of the noncollinear antiferromagnetic structure $(A_u E_g)$, with three mutually perpendicular vectors l_1, l_2, and l_3, of which l_1 transforms according to A_u and the pair l_2, l_3 according to the two-dimensional representation E_g, i.e., the structure with

moment density

$m(r) \sim l_1 [\cos 2\pi \zeta - \cos 2\pi (\xi - \eta)]$
$\qquad + l_2 \cos 2\pi (\xi - \eta) \cdot \sin 2\pi \zeta + l_3 \cos 2\pi \zeta \cdot \sin 2\pi (\xi - \eta)$

is due to the existence of the invariant

$$l_1 \cdot \frac{\partial l_2}{\partial x} + l_1 \cdot \frac{\partial l_3}{\partial y} - l_2 \cdot \frac{\partial l_1}{\partial x} - l_3 \cdot \frac{\partial l_1}{\partial y},$$

where x and y are Cartesian coordinates in the basal plane.

A simple analysis of all such combinations shows that in crystals with space group C_{4h}^6, only 45 different magnetic exchange structures are possible. One of these is ferromagnetic, A_g; 3 are collinear antiferromagnetic, A_u, B_g, and B_u; 3 are ferrimagnetic with two magnetic vectors, $(A_g A_g)$, $(A_g B_g)$, and $(A_g B_u)$; 7 are ferrimagnetic with three vectors, $(A_g A_g A_g)$, $(A_g A_g B_g)$, $(A_g A_g B_u)$, $(A_g B_g B_g)$, $(A_g B_u B_u)$, $(A_g E_g)$, and $(A_g \bar{E})$; 2 are helicoidal ferrimagnetic, $(A_g H_1)$ and $(A_g H_2)$; 8 are antiferromagnetic with two vectors, E_g, E_u, $\bar{E}$, $(A_u A_u)$, $(A_u B_g)$, $(A_u B_u)$, $(B_g B_g)$, and $(B_u B_u)$; 13 are antiferromagnetic with three vectors, $(A_u A_u A_u)$, $(A_u A_u B_g)$, $(A_u A_u B_u)$, $(A_u B_g B_g)$, $(A_u B_u B_u)$, $(B_g B_g B_g)$, $(B_u B_u B_u)$, $(B_g E_g)$, $(A_u E_u)$, $(B_u E_u)$, $(A_u \bar{E})$, $(B_g \bar{E})$, and $(B_u \bar{E})$; and 8 are helicoidal antiferromagnetic, H_1, H_2, $(A_u H_1)$, $(B_g H_1)$, $(A_u H_2)$, $(B_g H_2)$, and $(B_u H_2)$.

For applications, an important problem is the determination of the number and the transformation character of the magnetic vectors corresponding to some prescribed structure, known from experiment. We shall demonstrate below the solution of this simple problem in several characteristic examples.

In the rhombohedral magnets α-Fe_2O_3 and Cr_2O_3 (space group D_{3d}^6), the magnetic elementary cell coincides with the crystallographic; that is, the magnetic vectors transform according to representations with $k = 0$. Such representations may be regarded as representations of the point group corresponding to the crystal class; that is, in the present case of the group D_{3d}. The relative orientation of the moments of the magnetic atoms of a single cell, located along the third-order axis, is shown for α-Fe_2O_3 and for Cr_2O_3 in Figs. 2a and b respectively.[29] Since both structures are collinear, each of them is described by a single magnetic vector that transforms according to a one-dimensional representation. In order to explain the exchange symmetry, we must consider changes of structure under the action of spatial symmetry elements acting only on the coordinates of the atoms, with constant orientation of their magnetic moments. Rotation about the second-order axis perpendicular to the plane of the figure interchanges atoms with opposite moment directions in

a)   b)

FIG. 2.

both structures; that is, the moment density changes sign in such a rotation. This condition is satisfied only by the representations A_{2g} and A_{2u} of group D_{3d}. Inversion (supplemented in the group D_{3d}^6 by translation along the third-order axis by the distance between the first and third atoms in Fig. 2a and b) changes the sign of the moment density in Cr_2O_3 but does not change the structure in α-Fe_2O_3. Therefore the structure corresponds to the representation A_{2u} in Cr_2O_3 and to A_{2g} in α-Fe_2O_3.

In the six-sublattice noncollinear antiferromagnet $YMnO_3$ (space group C_{6v}^3), the magnetic cell again coincides with the crystallographic. The direction of the moments of the atoms is shown in Fig. 3.[30] The atoms marked with crosses are shifted by a half period with respect to the other atoms along the principal axis perpendicular to the plane of the figure. Since all the moments lie in a single plane but are not collinear, the structure is described by two magnetic vectors. Since the structure changes in the rotation C_3, only the two-dimensional representations E_1 and E_2 may correspond to it. Of them, only E_2 is compatible with invariance of the structure with respect to the rotation C_2 (supplemented by translation by a half period).

The four-sublattice antiferromagnet UO_2 has a face-centered cubic space lattice (group O_h^5). The magnetic atoms (Fig. 4) are located at the points $(0, 0, 0)$, $(0, 1/2, 1/2)$, $(1/2, 0, 1/2)$, and $(1/2, 1/2, 0)$. Their moments are directed[31,32] along space diagonals of the cube; that is, along the directions $[111]$, $[1\bar{1}\bar{1}]$, $[\bar{1}1\bar{1}]$, and $[\bar{1}\bar{1}1]$ respectively. The structure is described by three magnetic vectors $l_1, l_2,$ and l_3. Since the moment density is invariant with respect to translation by twice the elementary translation vectors a_α ($\alpha = 1, 2, 3$) of the space lattice (these vectors are shown by arrows in Fig. 4), a representation of the space group can be characterized only by those nonzero wave vectors whose doubled values are equivalent to zero. Of these, only the vectors

$$k_1 = (b_2 + b_3)/2, \quad k_2 = (b_1 + b_3)/2, \quad k_3 = (b_1 + b_2)/2$$

form a star containing no more than three vectors. In a translation by a_α, the expression $\exp(i k_\alpha \cdot r)$ and hence l_α do not change. But the other two magnetic vectors change sign. Hence it is clear that the directions of the atomic moments indicated above correspond to vectors l_α directed along the cube edges $[100]$, $[010]$, and $[001]$. The elementary cell of the space lattice of UO_2 contains one magnetic atom. In this case, only those representations can be realized in which the periodic Bloch factors may be considered equal to unity. In
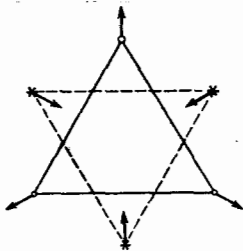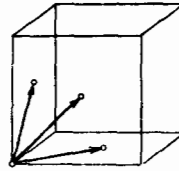

FIG. 4.

other words, only identity representations of a small group of wave vectors are realized. Precisely such a representation, with the star indicated above, is thus realized in UO_2.

If we take into consideration the relativistic interactions that fix the orientation of the magnetic vectors with respect to the crystallographic axes, each of the exchange magnetic structures will be described by some point symmetry group, determined by one of the magnetic space groups. We emphasize that here, in general, the purely spatial symmetry of the system is lowered. The space group G obtained from the exchange group in general differs from the space group obtained from the magnetic. The simplest example of such a change is any ferromagnet that is cubic in the exchange approximation.

3. THE SYMMETRY OF DISORDERED MAGNETIC MATERIALS

This section is devoted to an analysis of the symmetry properties of media that are spatially completely disordered: that is, systems in which the spatial distribution of magnetic atoms is random and, on the average, homogeneous and isotropic. This has to do either with amorphous materials containing magnetic atoms or with dilute solutions of magnetic atoms in nonmagnetic crystals.

A spatially disordered system may be completely ordered in magnetic respects. For example, complete ferromagnetic ordering of the spins of the magnetic atoms is possible. This example, however, is unique. Any other magnetic order is, in an obvious way, incompatible with spatial disorder. The macroscopic properties of a magnetically ordered ferromagnet are no different from the properties of ordinary crystalline ferromagnets and are described by the Landau-Lifshitz equation.

There have recently been extensive investigations[33,34] of disordered systems with magnetic structures of another type. These are, first, spin glasses, in which not only the positions but also the spin directions of different atoms are randomly distributed. In addition, there are systems[34] that possess a finite spontaneous magnetization whose value at zero temperature is nevertheless considerably below the nominal value. One can think of the state of such a disordered ferromagnet as the state of a spin glass in an external magnetic field. There is a partial (spontaneous) ferromagnetic order, superposed on a generally quite random distribution of spin directions.

In order to classify the possible types of such partial ordering, we consider exchange symmetry on the

FIG. 3.

macroscopic level. The macroscopic symmetry of a medium with a microscopic moment density $\mathbf{m}(\mathbf{r})$ can be described by specifying the multipole moments[35,6]

$$\mathbf{M} = \langle \mathbf{m}(\mathbf{r}) \rangle, \quad \mathbf{M}_l = \langle x_i \mathbf{m}(\mathbf{r}) \rangle, \quad \mathbf{M}_{ik} = \left\langle \left(x_i x_h - \frac{1}{3} \delta_{ik} x_i^2 \right) \mathbf{m}(\mathbf{r}) \right\rangle, \ldots,$$

where the brackets denote macroscopic averaging. The exchange symmetry group of the medium consists of all the combinations of spatial and spin rotations, inversion, and the transformation R with respect to which the nonvanishing multipole moments are invariant. The purely spatial symmetry group, as before, is obtained from the exchange group by identifying the spin-rotation elements and the element R, in it, with the identity transformation. For spatially isotropic disordered systems, the group G must be the group of spatial rotations (and inversion). Since the multipole moments $\mathbf{M}, \mathbf{M}_i, \mathbf{M}_{ik}, \ldots$ transform according to irreducible representations of this group, the results of the preceding section are fully applicable to them. In the equilibrium state, no more than three moments can differ from zero; different moments are perpendicular to each other; and those among them that belong to a single multidimensional representation have the same length. It is clear that there are only three possibilities[17]: 1) all moments are zero; 2) the single ferromagnetic moment $\mathbf{M}$ is nonzero; 3) there are three nonvanishing moments $\mathbf{M}_i$, equal in absolute value and mutually perpendicular. The first possibility corresponds to a spin glass, the second to a disordered ferromagnet, the third to a disordered antiferromagnet in which the quantities $\mathbf{M}_i$ play the role of antiferromagnetic vectors.

When relativistic effects are taken into account in a disordered ferromagnet, the spatial isotropy is destroyed. A preferred direction of magnetization appears.

When relativistic effects are taken into account in a disordered antiferromagnet, there should occur some definite orientation of the triplet of mutually perpendicular spin vectors $\mathbf{M}_i = \{M_{i\alpha}\}$ with respect to coordinate space. We shall designate the components of the spin vectors in the exchange case by indices α, β. When relativistic effects are considered, the difference between them and the spatial vector indices i, k disappears. Among all orientations, one is especially distinguished that is described, on suitable normalization, by moments $M_{i\alpha} = \delta_{i\alpha}$. A medium with such moments is, in an obvious way, isotropic even when relativistic interactions are taken into account. Its point symmetry group (magnetic class) is the group $\mathrm{SO}(3) \times \mathrm{IR}$. Any other orientation can be obtained from those discussed above by some rotation of all the spins through an angle $\theta(0 < \theta < \pi)$ about an axis determined by a unit vector $\mathbf{n}$. As a result the moments acquire the following values:

$$M_{i\alpha} = \cos\theta\,\delta_{i\alpha} + (1 - \cos\theta)\,n_i n_\alpha + \sin\theta\,e_{i\alpha\beta} n_\beta,$$

where e_{ikl} is the unit antisymmetric tensor.

The anisotropy energy U_a, dependent on the angle θ and determined by averaging of the Hamiltonian of the relativistic interactions, is in the present case a linear combination of two invariants: $M_{i\alpha}M_{\alpha i}$ and $(M_{\alpha\alpha})^2$. On

substituting the expressions for the moments in terms of θ, we get

$$U_a = \frac{A\varphi^2 - B\varphi^4}{(1 + \varphi^2)^2}. \tag{4}$$

where $\varphi = \mathrm{tg}(\theta/2)$ and where A and B are anisotropy constants proportional to the square of the ratio of the velocity of the electrons to the velocity of light. Here and below, we have in mind amorphous systems in the consideration of relativistic effects in disordered magnets. In dilute solutions of magnetic atoms in crystals, it would be necessary to allow for the anisotropy of the crystal matrix. If $A > 0$, the energy is least when $\varphi = 0$ ($\theta = 0$) or when $\varphi = \infty$ ($\theta = \pi$), depending on the sign of B. If $A < 0$ and $B > A/2$, the minimum is attained at the finite value

$$\varphi = \sqrt{\frac{|A|}{|A| + 2B}}.$$

in the case $A < 0$ and $B < A/2$, again at $\varphi = \infty$.

Thus there are three possible types of states of a disordered antiferromagnet. In addition to the isotropic structure discussed above ($\varphi = 0$), structures are also possible with an exact spatial symmetry of the type of the symmetry of nematic liquid crystals. The unit vector $\mathbf{n}$ along the axis of rotation plays the role of the director; the states $\mathbf{n}$ and $-\mathbf{n}$ are different for finite φ and identical for $\varphi = \infty$.

Since the spin is a pseudovector and changes sign under the time-reflection operation R, a disordered antiferromagnet with moments $M_{\alpha i}$ is always noninvariant with respect to I and R separately but is invariant with respect to the product IR. If such a structure exists in a dielectric, then a magnetoelectric effect should occur (appearance of a magnetization proportional to the external electric field), as is observed, for example, in the crystalline antiferromagnet $\mathrm{Cr_2O_3}$.[29]

4. COLLINEAR STRUCTURES

We shall carry out the derivation of the dynamical equations on the basis of the Lagrangian formalism. This approach is the most convenient for our purposes and has been applied in the investigation of magnetic properties by many authors.[12,14,17,36-38] We shall neglect dissipation; therefore the equations derived below are, strictly speaking, applicable only at zero temperature. But since the equations are nonlinear, use of standard quantization procedures enables one to investigate the interaction of thermal spin waves on the basis of these equations, and to obtain dynamical equations with allowance for dissipation in the spin-wave range of low temperatures.

We consider a collinear antiferromagnet: that is, a structure determined by a single magnetic vector $\mathbf{l}$ that transforms according to some nonidentity one-dimensional representation. Such a structure is characterized by the fact that the spontaneous destruction of the isotropy of spin space is not complete in it. The structure is invariant with respect to spin rotations about an axis parallel to the vector $\mathbf{l}$. There are only two independent angles of rotation about an axis per-

pendicular to 1 that change the state of the system. In accordance with what was said in the introduction, these two angles, as functions of the coordinates and of time, must be treated as independent functions in a macroscopic description. It is convenient to introduce the unit vector $\mathbf{l}(x, t)$ itself in place of them.

The potential energy U due to the inhomogeneity of the system is quadratic in the spatial derivatives of the vector $\mathbf{l}$:

$$U = \int dV \frac{1}{2} a_{ik} \frac{\partial \mathbf{l}}{\partial x_i} \cdot \frac{\partial \mathbf{l}}{\partial x_k} , \tag{5}$$

where a_{ik} is a constant symmetric tensor, invariant with respect to the crystal class of the space group G and satisfying the condition of positive definiteness of the potential energy. For the antiferromagnets considered in Ch. 1, α-Fe_2O_3 and Cr_2O_3, it has two independent positive components, $a_1 = a_{zz}$ and $a_2 = a_{xx} = a_{yy}$.

The kinetic energy T is quadratic in the time derivatives,

$$T = \int dV \frac{1}{2} b \dot{\mathbf{l}}^2, \tag{6}$$

where b is a positive constant.

The Lagrangian function L, which determines the dynamics of the antiferromagnet in the absence of a magnetic field and of relativistic interactions, is thus

$$L = T - U = \frac{1}{2} \int dV \left(b \dot{\mathbf{l}}^2 - a_{ik} \frac{\partial \mathbf{l}}{\partial x_i} \cdot \frac{\partial \mathbf{l}}{\partial x_k} \right). \tag{7}$$

The variation of the action W, considered as a function of the coordinates $\mathbf{l}$[38,39,17] is determined by the formulas

$$\delta W = \int dV b \dot{\mathbf{l}} \cdot \delta \mathbf{l} = \int dV b \, \mathbf{l} \times \dot{\mathbf{l}} \cdot \delta \theta ,$$

where it has been taken into account that an arbitrary variation of the vector $\mathbf{l}$ is an infinitely small rotation $\delta \mathbf{l} = \delta \theta \times \mathbf{l}$ through an angle $\delta \theta$. The derivative $\delta W / \delta \theta$ of the action with respect to the spin-rotation angle is the spin angular momentum, from which, by multiplication by the gyromagnetic ratio γ, we find the magnetic moment of unit volume

$$M = \gamma b \, \mathbf{l} \times \dot{\mathbf{l}}$$

In certain antiferromagnets, for example in Cr_2O_3, in which $\mathbf{l}$ transforms according to a representation that enters into the composition of a vector representation, there is an invariant

$$\mathbf{l} \cdot \mathbf{l} \times \frac{\partial \mathbf{l}}{\partial x_i}$$

It is, as is easily seen, the sum of the total time derivative and a certain divergence. Therefore introduction of it into the Lagrangian function does not change the equations of motion, but does change the action and therefore also the expression for the moment. Instead of the formula given above, we get

$$M = \gamma b \mathbf{l} \times \dot{\mathbf{l}} + \mu_i \frac{\partial \mathbf{l}}{\partial x_i}, \tag{8}$$

where μ_i are some exchange constants. In the presence of an external magnetic field that is small in comparison with the exchange field, the right side of formula (8) contains an additional term, dependent on the magnetic susceptibility $\chi_{\alpha\beta}$ of the antiferromagnet,

$$M_\alpha = \gamma b \, \mathbf{l} \times \dot{\mathbf{l}} \,_\alpha + \mu_i \frac{\partial l_\alpha}{\partial x_i} + \chi_{\alpha\beta} H_\beta. \tag{9}$$

The corresponding additional term in the Lagrangian function is easily determined by use of the fact that the derivative $\partial L / \partial H$ is the magnetic moment $\mathbf{M}$:

$$L = \frac{b}{2} \dot{\mathbf{l}}^2 - \frac{a_{ik}}{2} \frac{\partial \mathbf{l}}{\partial x_i} \cdot \frac{\partial \mathbf{l}}{\partial x_k} + \gamma b \, \mathbf{l} \times \dot{\mathbf{l}} \cdot \mathbf{H} + \frac{1}{2} \chi_{\alpha\beta} H_\alpha H_\beta; \tag{10}$$

here and everywhere below, we understand by L the density of the Lagrangian function, and we omit terms that have no effect on the equations of motion.

From formula (10), by the same method that was used to derive (8) from (7), we find the magnetic moment with allowance for a magnetic field:

$$\mathbf{M} = \gamma b \, \mathbf{l} \times \dot{\mathbf{l}} + \mu_i \frac{\partial \mathbf{l}}{\partial x_i} + \gamma^2 b (\mathbf{H} - \mathbf{l} \, \mathbf{l} \cdot \mathbf{H}).$$

Comparison with (9) shows that the magnetic susceptibility tensor is

$$\chi_{\alpha\beta} = \chi_\perp (\delta_{\alpha\beta} - l_\alpha l_\beta),$$

where $\chi_\perp = \gamma^2 b$ is the susceptibility in directions perpendicular to $\mathbf{l}$. The longitudinal susceptibility is zero. The last statement, as is easily understood, is a consequence of the fact that the kinetic energy defined by formula (6) depends only on the time derivatives of the angles of rotation about axes perpendicular to $\mathbf{l}$ and does not contain the angular velocity of rotation (in spin space) about $\mathbf{l}$. Thus the vanishing of the longitudinal susceptibility is an immediate consequence of the quantum-mechanical indistinguishability principle, known for example from the theory of diatomic molecules (Ref. 28, §82), according to which an axially symmetric body cannot rotate about the axis of symmetry. We emphasize that this deduction is valid only for dielectrics in which there are no other gapless magnetic degrees of freedom. In metals, a longitudinal susceptibility is caused by the contribution of the conduction electrons.

In order to take the relativistic interactions into account, we must introduce the anisotropy energy $U_a = \frac{1}{2} \alpha_{ik} l_i l_k$ into expression (5) for the potential energy; where, α_{ik} is the anistropy-constant tensor, which may be considered to be symmetric and to satisfy the condition $\alpha_{ii} = 0$. Furthermore, in some antiferromagnets the magnetic field must everywhere be supplemented by the Dzyaloshinskiĭ field $\mathbf{H}_D = \mathbf{d} \times \mathbf{l}$, which describes weak ferromagnetism[29]; here $\mathbf{d}$ is a constant, independent of the direction of the magnetic vector $\mathbf{l}$.

The final expression for the Lagrangian function can be written in the form

$$L = \frac{\chi_\perp}{2\gamma^2} \dot{\mathbf{l}}^2 + \frac{\chi_\perp}{\gamma} \mathbf{H} \, \mathbf{l} \times \dot{\mathbf{l}} - U, \tag{11}$$

where

$$U = \frac{1}{2} a_{ik} \frac{\partial \mathbf{l}}{\partial x_i} \cdot \frac{\partial \mathbf{l}}{\partial x_k} + \frac{1}{2} \chi_\perp (\mathbf{H} \cdot \mathbf{l})^2 + \frac{1}{2} \alpha_{ik} l_i l_k + \chi_\perp \mathbf{l} \cdot \mathbf{d} \times \mathbf{H}.$$

In formula (11) we have omitted terms that are total derivatives with respect to the time or are quadratic in the relativistic field $\mathbf{H}_D$.

The magnetization of the antiferromagnet in the general case is

$$\mathbf{M} = \frac{\chi_\perp}{\gamma} \mathbf{l} \times \dot{\mathbf{l}} + \mu_i \frac{\partial \mathbf{l}}{\partial x_i} + \chi_\perp \{\mathbf{H} - \mathbf{l} \, \mathbf{l} \cdot \mathbf{H} + \mathbf{d} \times \mathbf{l}\}. \tag{12}$$

On varying the Lagrangian function (11) with respect to $\mathbf{l}$, with allowance for the equality $\mathbf{l}^2 = 1$, we find the non-

linear equations

$$\frac{\partial}{\partial t} 1 \times \mathbf{i} = \frac{\gamma_2}{\chi_1} 1 \times \mathbf{H}_l + 2\gamma 1 \cdot \mathbf{H} \dot{1} - \gamma (\dot{\mathbf{H}} - 1 1 \cdot \dot{\mathbf{H}}) , \tag{13}$$

where

$$H_{li} = -\frac{\partial U'}{\partial l_i} = a_{kl} \frac{\partial^2 l_i}{\partial x_k \partial x_l} - \alpha_{ik} l_k - \chi_{\perp} 1 \cdot \mathbf{H} H_i - \chi_{\perp} [\mathbf{l} \times \mathbf{H}]_i$$

is the effective field acting on 1.

The magnetic resonance frequencies and the spin-wave spectrum obtained from the linearized equations (13) coincide with the results of the model-type phenomenlogical theory.[4,5,29] In contrast to the latter, equations (13) give a systematic description of nonlinear effects as well.

5. THE SPIN GLASS

The description of spin glasses in the theory set forth is analogous to the theory of elasticity of ordinary glasses.[16-18] As in the theory of elasticity, a glass is a nondissipative system at not too small frequencies, appreciably exceeding a certain characteristic time of dissipation of elastic stresses. We shall consider the dynamics of a spin glass, supposing at first that this condition is satisfied. A discussion of the general case will be given at the end of this section.

In a spin glass, the spontaneous breaking of the isotropy of the spin space is complete; therefore a macroscopic description must involve all three spin-rotation angles. Let the parameters φ^α ($\alpha = 1, 2, 3$) carry out some parametrization of the group of spin rotations. Then these parameters $\varphi^\alpha(x_i, t)$ as functions of the coordinates and the time determine those states that correspond to low-frequency motions. The spin configuration at the point x_i at the instant t differs from the configuration at the same point in the initial equilibrium state by a general rotation $\varphi^\alpha(x_i, t)$.

In order to calculate the potential energy U quadratic in the spatial derivatives of φ^α, we consider the rotations $\varphi^\alpha(x_i)$ and $\varphi^\alpha(x_i + dx_i) = \varphi^\alpha(x_i) + d\varphi^\alpha$ corresponding to two neighboring points. Let

$$\delta\theta = 2\lambda_\alpha (\varphi) \, d\varphi^\alpha$$

(the factor 2 is introduced for future convenience) be an infinitely small rotation angle, determining a rotation such that successive application of the rotations φ^α and $\delta\theta$ gives $\varphi^\alpha + d\varphi^\alpha$. Because of the complete randomness of a spin glass, the only quantity on which U can depend is

$$\left(\frac{\delta\theta}{dx_i}\right)^2 = 4\lambda_\alpha (\varphi) \lambda_\beta (\varphi) \frac{\partial \varphi^\alpha}{\partial x_i} \frac{\partial \varphi^\beta}{\partial x_i} .$$

The quantity $\lambda_\alpha(\varphi)\lambda_\beta(\varphi)$ is, by definition, the metric tensor $g_{\alpha\beta}(\varphi)$ that specifies[40] the metric of the Riemann space corresponding to the group of three-dimensional rotations. We have

$$U = \frac{a}{2} g_{\alpha\beta} (\varphi) \frac{\partial \varphi^\alpha}{\partial x_i} \frac{\partial \varphi^\beta}{\partial x_i} , \tag{14}$$

where $a > 0$ is some constant.

The kinetic energy T is quadratic in the time derivatives of φ^α. In analogy to (14) we get

$$T = \frac{a}{2c^2} g_{\alpha\beta} (\varphi) \, \dot{\varphi}^\alpha \dot{\varphi}^\beta ,$$

where c is a constant, equal, as will be seen below, to the velocity of spin waves.

The Lagrangian function of a spin glass, in the absence of an external magnetic field and of relativistic interactions, is

$$L = T - U = \frac{a}{2} g_{\alpha\beta} (\varphi) \left(\frac{1}{c^2} \dot{\varphi}^\alpha \dot{\varphi}^\beta - \frac{\partial \varphi^\alpha}{\partial x_i} \frac{\partial \varphi^\beta}{\partial x_i} \right) . \tag{15}$$

The corresponding Lagrange equations have the form

$$\ddot{\varphi}^\alpha - c^2 \Delta\varphi^\alpha + \Gamma^\alpha_{\beta\gamma} \left(\dot{\varphi}^\beta \dot{\varphi}^\gamma - c^2 \frac{\partial \varphi^\beta}{\partial x_i} \frac{\partial \varphi^\gamma}{\partial x_i} \right) = 0; \tag{16}$$

here $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols corresponding to the metric $g_{\alpha\beta}$.

We shall now introduce a specific parametrization $\{\varphi^\alpha\} = \boldsymbol{\varphi}$ convenient for what follows. We set $\boldsymbol{\varphi} = \mathbf{n}\varphi$, where $\mathbf{n}$ is the unit vector along the axis of rotation and $\varphi = tg(\theta/2)$; θ is the angle of rotation about $\mathbf{n}$, and $0 < \theta < \pi$. An arbitrary spin vector $\boldsymbol{\sigma}$ transforms, under the action of the spin rotation $\boldsymbol{\varphi}$, according to the formula

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} + \frac{2}{1+\varphi^2} \{ \boldsymbol{\varphi} \times (\boldsymbol{\varphi} \times \boldsymbol{\sigma}) + \boldsymbol{\varphi} \times \boldsymbol{\sigma} \} \tag{17}$$

The advantage of this parametrization manifests itself in the simple law that determines the product of two rotations. If we carry out first the rotation $\boldsymbol{\varphi}_1$ and then $\boldsymbol{\varphi}_2$, we get as a result the rotation corresponding to the parameter

$$\boldsymbol{\varphi}_{21} = \frac{\boldsymbol{\varphi}_2 + \boldsymbol{\varphi}_1 + [\boldsymbol{\varphi}_2 \times \boldsymbol{\varphi}_1]}{1 - \boldsymbol{\varphi}_2 \cdot \boldsymbol{\varphi}_1} \tag{18}$$

The element of "length" dl of group space is easily determined by means of formula (18). We have

$$dl^2 = g_{\alpha\beta} \, d\varphi^\alpha \, d\varphi^\beta = \left(\frac{\delta\theta}{2}\right)^2 ,$$

where $\delta\theta/2$ is obtained from (18) by the substitution $\boldsymbol{\varphi}_1 = -\boldsymbol{\varphi}$, $\boldsymbol{\varphi}_2 = \boldsymbol{\varphi} + d\boldsymbol{\varphi}$:

$$\frac{d\boldsymbol{\varphi} + [\boldsymbol{\varphi} \times d\boldsymbol{\varphi}]}{1 + \varphi^2} . \tag{19}$$

As a result, we find

$$g_{\alpha\beta} = \frac{\delta_{\alpha\beta}}{1+\varphi^2} - \frac{\varphi^\alpha \varphi^\beta}{(1+\varphi^2)^2} . \tag{20}$$

The equation of motion (16), in the chosen parametrization, acquires the following form:

$$\ddot{\varphi} - c^2 \Delta^2\varphi = \frac{\boldsymbol{\varphi}(\dot{q}^2)^\cdot - c^2 q_i (q^2)_i}{1 + q^2} , \tag{21}$$

where

$$(\varphi^2)^\cdot = \frac{\partial \varphi^2}{\partial t} , \quad \varphi_i = \frac{\partial \varphi}{\partial x_i} , \quad (\varphi^2)_i = \frac{\partial \varphi^2}{\partial x_i} .$$

The metric (20) is the metric of a sphere in four-dimensional Euclidean space. In fact, writing dl^2 in variables $\theta/2$, ϑ, and Φ, where (ϑ, Φ) are the polar angle and azimuth of the direction n, we get

$$dl^2 = \left(\frac{d\theta}{2}\right)^2 + \sin^2\left(\frac{\theta}{2}\right) (d\vartheta^2 + \sin^2\vartheta \, d\Phi^2),$$

which coincides (see Ref. 41, §104) with the metric of a three-dimensional sphere of unit radius. It is therefore clear that the equations of motion are invariant with respect to the six-parameter group O(4) of four-dimensional rotations. The infinitely small transformations of the coordinates φ that conserve the metric

are determined by the formulas

$$\delta\varphi = \frac{\varphi + \varepsilon_1 - \varphi \times \varepsilon_1}{1 - \varphi \cdot \varepsilon_1} - \varphi \approx \varepsilon_1 + \varphi\varphi \cdot \varepsilon_1 + \varepsilon_1\varphi,$$
$$\delta\varphi = \varepsilon_2 \cdot \varphi. \tag{22}$$

The first of these is an infinitely small spin rotation through angle $\delta\theta = 2\varepsilon_1$.

The variation of the action density W as a function of the coordinates, in arbitrary $\delta\varphi$, is

$$\delta W = \frac{a}{c^2} g_{\alpha\beta}\dot{\varphi}^\beta \delta\varphi^\alpha.$$

On substituting (22) in this, we find the density of the quantities that are conserved by virtue of the symmetry $O(4)$:

$$\mathbf{S} = \frac{\delta W}{2\varepsilon_1} = \frac{a}{2c^2}\frac{\dot{\varphi} + \varphi \times \dot{\varphi}}{1 + \varphi^2},$$
$$\mathbf{N} = \frac{\delta W}{\varepsilon_2} = \frac{a}{c^2}\frac{\varphi \times \dot{\varphi}}{1 + \varphi^2}. \tag{23}$$

The quantity $\mathbf{S}$ is the density of spin angular momentum, which differs only by the gyromagnetic ratio γ from the magnetization

$$\mathbf{M} = \frac{a\gamma}{2c^2}\frac{\dot{\varphi} + \varphi \times \dot{\varphi}}{1 + \varphi^2}.$$

In the presence of an external magnetic field, an additional term $\mathbf{M}\cdot\mathbf{H}$ appears in the Lagrangian function (15). On calculating, as above, $\mathbf{S}$ and $\mathbf{M} = \gamma\mathbf{S}$ for the new Lagrangian function, we get

$$\mathbf{M} = \frac{a\gamma}{2c^2}\frac{\dot{\varphi} + \dot{\varphi} \times \varphi}{1 + \varphi^2} + \frac{a\gamma^2}{4c^2}\mathbf{H}.$$

Hence it follows that the equality $a\gamma^2/4c^2 = \chi$ must be satisfied, where χ is the magnetic susceptibility, which in a spin glass is of course isotropic.

It remains to write the additional terms in the Lagrangian function, that describe the relativistic anisotropy energy U_a. The value of U_a depends on the rotation angles themselves (and not on derivatives) and must be expressed in terms of the orthogonal matrix of spin rotations $U_{\alpha\beta}$, which according to (17) is

$$U_{\alpha\beta} = \delta_{\alpha\beta} + \frac{2}{1 + \varphi^2}(\varphi^\alpha\varphi^\beta - \varphi^2\delta_{\alpha\beta} + e_{\alpha\gamma\beta}\varphi^\gamma).$$

Since U_a is obtained from the Hamiltonian of spin-orbit and spin-spin interactions, which contains terms that are linear and quadratic in the spins, U_a may contain only combinations linear or quadratic in $U_{\alpha\beta}$ and, furthermore, invariant with respect to rotations. There are only two such combinations:

$$U_{\alpha\alpha} = 3 - \frac{4q^2}{1 + q^2}, \quad U_{\alpha\beta}U_{\beta\alpha} = 3 - \frac{16q^2}{(1 + q^2)^2}.$$

It is clear that the anisotropy energy of a spin glass may be written in the form

$$U_a = \frac{\alpha q^2 + \beta q^4}{(1 + q^2)^2}, \tag{24}$$

where α and β are positive anisotropy constants. It is important to notice that these constants are proportional to the fourth power of the ratio of the speed of the electrons to the speed of light; that is, they are considerably smaller than the anisotropy constants A and B of a disordered antiferromagnet that occur in formula (4). In fact, in order to calculate the anisotropy energy we must replace the spin operators $\sigma_1^\alpha, \sigma_2^\beta, \dots$ of the electrons in the Hamiltonian of the relativistic interactions, by $U_{\alpha\gamma}\sigma_1^\gamma, U_{\beta\delta}\sigma_2^\delta$, and carry out an averaging over the equilibrium state. If we neglect the relativistic interactions, then

$$\langle\sigma_1^\gamma\rangle = \langle\sigma_2^\delta\rangle = 0, \quad \langle\sigma_1^\gamma\sigma_2^\delta\rangle \sim \delta_{\gamma\delta},$$

and by virtue of the orthogonality of the rotation matrix, $U_{\alpha\gamma}U_{\beta\gamma} = \delta_{\alpha\beta}$. In order to obtain the dependence of the energy on the angle of spin rotation, it is necessary again to take account of the relativistic interactions. Thus the constants α and β are proportional to the square of the relativistic interactions.

We write the complete expression for the Lagrangian function of a spin glass, with allowance for the magnetic field and for the relativistic interactions[17]:

$$L = \frac{2\chi}{\gamma^2}\left[\frac{\dot{\varphi}^2 - c^2\varphi_i^2}{1 + \varphi^2} - \frac{(\varphi \cdot \dot{\varphi})^2 - c^2(\varphi \cdot \varphi_i)^2}{(1 + \varphi^2)^2}\right] + \frac{2\chi}{\gamma}\dot{\varphi} \cdot \frac{\mathbf{H} + \mathbf{H} \times \varphi}{1 + \varphi^2} - \frac{\alpha q^2 + \beta q^4}{(1 + q^2)^2}. \tag{25}$$

The linearized equations of motion in a constant field corresponding to this Lagrangian function have the form

$$\ddot{\varphi} - c^2\Delta^2\varphi + \gamma\mathbf{H} \times \dot{\varphi} + \frac{\alpha\gamma^2}{2\chi}\varphi = 0, \tag{26}$$

whence we easily find[17] the frequencies of spin waves polarized longitudinally (with respect to the magnetic field),

$$\omega_l^2 = \frac{\alpha\gamma^2}{2\chi} + c^2k^2 \tag{27}$$

and of spin waves polarized transversely,

$$\omega_t = \sqrt{\omega_l^2 + \left(\frac{\gamma H}{2}\right)^2} \pm \frac{\gamma H}{2}. \tag{28}$$

In the case $H = \alpha = 0$, these formulas reduce to the results of the paper of Halperin and Saslow.[16]

At sufficiently low frequencies, dissipative phenomena become important in spin glasses; they are analogous to the dissipation of elastic stresses in ordinary glasses or very viscous liquids. It is easy to carry out a calculation of these phenomena qualitatively, by analogy with the well-known Maxwell theory (see Ref. 26, §36). Equation (26) in the absence of relativistic interactions can be written as the law of conservation of the moment $\mathbf{M} = (2\chi/\gamma)\dot{\varphi} = (\chi/\gamma)\dot{\theta}$:

$$\frac{\partial\mathbf{M}}{\partial t} - \gamma\mathbf{M} \times \mathbf{H} = \frac{\partial\Pi_l}{\partial x_l}, \tag{29}$$

where Π_l is the moment flow, here playing the role of the elastic-stress tensor. In the high-frequency case considered above Π_l is expressed linearly in terms of the "strain" $\partial\theta/\partial x_l$:

$$\Pi_l = \frac{2\chi c^2}{\gamma}\frac{\partial\varphi}{\partial x_l} = \frac{\chi c^2}{\gamma}\frac{\partial\theta}{\partial x_l}.$$

The last relation is generalized in the usual way to the case of arbitrary frequencies, by introduction of a Maxwell time τ of dissipation of elastic stresses in the spin system:

$$\dot{\Pi}_l + \frac{1}{\tau}\Pi_l = \frac{\chi c^2}{\gamma^2}\frac{\partial\dot{\theta}}{\partial x_l} = c^2\frac{\partial\mathbf{M}}{\partial x_l}. \tag{30}$$

Formulas (29) and (30) give the desired generalization of the linear equations. In the limit of very low frequencies, $\omega \ll 1/\tau$, one gets the equation characteristic of paramagnets from (29) and (30):

$$\frac{\partial\mathbf{M}}{\partial t} - \gamma\mathbf{M} \times \mathbf{H} - D\Delta^2\mathbf{M} = 0.$$

The spin diffusion coefficient D is connected with the

time τ by the relation $D = c^2\tau$, which is analogous to the Maxwell expression for the viscosity of liquids. From this point of view, a spin glass is a paramagnet with very large diffusion.

6. NONCOLLINEAR AND DISORDERED ANTIFERROMAGNETS

The equations for noncollinear antiferromagnets, including disordered ones, are analogous to the equations for a spin glass. But the coefficients of the quadratic forms in (15), which determine the kinetic and potential energies, may contain, besides the metric tensor $g_{\alpha\beta}$, the components of the magnetic vectors $\mathbf{l}_a$. In an arbitrary state characterized by angles $\varphi^\alpha(x_i, t)$, the magnetic vectors are

$$\mathbf{l}_a(x_i, t) = \mathbf{l}_a^{(0)} + \frac{2}{1+\varphi^2}\{\varphi\ \varphi\cdot\mathbf{l}_a^{(0)} - \varphi^2\mathbf{l}_a^{(0)} + \varphi\times\mathbf{l}_a^{(0)}\}, \qquad (31)$$

where $\mathbf{l}_a^{(0)}$ are the constant values in the initial equilibrium state.

If we introduce the angular velocity $\boldsymbol{\Omega} = \delta\theta/dt$ of rotation in spin space and the analogous quantities $\boldsymbol{\Omega}_i = \delta\theta/dx_i$ that describe the variation of the angles in space, then the kinetic and potential energies of a spin glass are, respectively, $(\chi/2\gamma^2)\boldsymbol{\Omega}^2$ and $(\chi_c^2/2\gamma^2)\boldsymbol{\Omega}_i^2$.

For an arbitrary antiferromagnet, the kinetic energy is

$$T = \frac{1}{2\gamma^2}\chi_{\alpha\beta}(\varphi)\,\Omega_\alpha\Omega_\beta. \qquad (32)$$

By the method described above, it is easily shown that the quantities $\chi_{\alpha\beta}$ form the magnetic susceptibility tensor. We find simultaneously the terms that allow for the magnetic field and that must be added to T in order to obtain the Lagrangian function in the spatially homogeneous case, in the absence of relativistic interactions:

$$L = \frac{1}{2\gamma^2}\chi_{\alpha\beta}(\Omega_\alpha + \gamma H_\alpha)(\Omega_\beta + \gamma H_\beta). \qquad (33)$$

The form of the tensor $\chi_{\alpha\beta}$ is determined by the exchange symmetry of the antiferromagnet. For example, in UO_2 and in a disordered antiferromagnet the three mutually perpendicular magnetic vectors transform according to a single three-dimensional representation, which corresponds to cubic symmetry of the spin space. There is only one invariant $\boldsymbol{\Omega}^2$ of the form (32), so that $\chi_{\alpha\beta} = \chi\delta_{\alpha\beta}$. In $YMnO_3$ there are two invariants of the form (32), for which it is convenient to choose $\boldsymbol{\Omega}^2$ and $(\mathbf{n}\cdot\boldsymbol{\Omega})^2$, where $\mathbf{n} = \mathbf{l}_1 \times \mathbf{l}_2$. The susceptibility

$$\chi_{\alpha\beta} = \chi_\perp\delta_{\alpha\beta} + (\chi_\| - \chi_\perp)\,n_\alpha n_\beta$$

has two independent components $\chi_\|$ and $\chi_\perp$, in the direction $\mathbf{n}$ and in the plane perpendicular to it. In noncollinear antiferromagnets with magnetic vectors that transform only according to one-dimensional representations, the magnetic susceptibility has three independent components.

The potential energy in the general case has the form

$$U = \frac{1}{2}\Lambda_{\alpha\beta}^{ik}(\varphi)\,\Omega_{i\alpha}\Omega_{k\beta}, \qquad (34)$$

where the invariant tensor $\Lambda_{\alpha\beta}^{ik}$ plays the role of the elasticity-modulus tensor.

The complete Lagrangian function is the sum of the expressions (33) and (34) and of the relativistic anisotropy energy.

In $YMnO_3$ there are six independent invariants of the form (34) and three independent relativistic invariants, proportional to the square of the ratio of the speed of the electrons to the speed of light. The Lagrangian function of $YMnO_3$ is

$$L = \frac{\chi_\perp}{2\gamma^2}(\boldsymbol{\Omega} + \gamma\mathbf{H})^2$$
$$+ \frac{\chi_\| - \chi_\perp}{2\gamma^2}(\boldsymbol{\Omega} + \gamma\mathbf{H})\cdot\mathbf{n})^2 - \frac{1}{2}\{\Lambda_1\Omega_z^2 + \Lambda_2(\Omega_z\cdot\mathbf{n})^2 + \Lambda_3(\Omega_x^2 + \Omega_y^2)$$
$$+ \Lambda_4[(\Omega_x\cdot\mathbf{n})^2 + (\Omega_y\cdot\mathbf{n})^2] + \Lambda_5(\Omega_x\cdot\mathbf{l}_2 + \Omega_y\cdot\mathbf{l}_1)^2 + \Lambda_6(\Omega_x\cdot\mathbf{l}_1$$
$$- \Omega_y\cdot\mathbf{l}_2)^2\} - \frac{A}{2}\{(l_{1y} - l_{2x})^2 + (l_{1x} + l_{2y})^2\} - \frac{B}{2}(l_{1y} + l_{2x})^2$$
$$- \frac{C}{2}(l_{1x} - l_{2y})^2, \qquad (35)$$

where $\Lambda_1, \ldots, \Lambda_6$ are elastic constants and A, B, C are anisotropy constants. In the equilibrium state of $YMnO_3$, the nonvanishing components of the magnetic are $l_{1x}^{(0)} = -l_{2y}^{(0)} = 1$. The magnetic moment of $YMnO_3$, is determined by the formula

$$\mathbf{M} = \frac{\chi_\perp}{\gamma}(\boldsymbol{\Omega} + \gamma\mathbf{H}) + \frac{\chi_\| - \chi_\perp}{\gamma}\mathbf{n}\mathbf{n}\cdot(\boldsymbol{\Omega} + \gamma\mathbf{H}).$$

We give the spin-wave spectrum derived from the Lagrangian function (35) in the case when the magnetic field is directed along the z axis:

$$\omega_1^2 = \frac{\gamma^2}{\chi_\|}[4(B-C) + (\Lambda_1 + \Lambda_2)k_z^2 + (\Lambda_3 + \Lambda_4)(k_x^2 + k_y^2)], \qquad (36)$$

$$\omega_{2,3}^2 = \frac{1+\eta^2}{2}(\gamma H)^2 + \frac{\gamma^2}{\chi_\perp}\left[2|C| + \Lambda_1 k_z^2 + \left(\Lambda_3 + \frac{\Lambda_5 + \Lambda_6}{2}\right)(k_x^2 + k_y^2)\right]$$
$$\pm\left\{\frac{(1-\eta^2)^2}{4}(\gamma H)^4 + (\gamma H)^2(1-\eta)^2\frac{\gamma^2}{\chi_\perp}\left[2|C| + \Lambda_1 k_z^2\right.\right.$$
$$\left.\left. + \left(\Lambda_3 + \frac{\Lambda_5 + \Lambda_6}{2}\right)(k_x^2 + k_y^2)\right] + \frac{\gamma^4}{\chi_\perp^2}\frac{(\Lambda_5 - \Lambda_6)^2}{4}(k_x^2 + k_y^2)^2\right\}^{1/2},$$

where $\eta = (\chi_\| - \chi_\perp)/\chi_\perp$. There are three different branches, of which one, corresponding to oscillations of the angle of rotation about the z axis, is independent of the value of the magnetic field.

When $H = 0$, we have

$$\omega_2^2 = \frac{\gamma^2}{\chi_\perp}[2|C| + \Lambda_1 k_z^2 + (\Lambda_3 + \Lambda_5)(k_x^2 + k_y^2)],$$

$$\omega_3^2 = \frac{\gamma^2}{\chi_\perp}[2|C| + \Lambda_1 k_z^2 + (\Lambda_3 + \Lambda_6)(k_x^2 + k_y^2)].$$

When $\mathbf{k} = 0$, we get three different antiferromagnetic resonance frequencies:

$$\omega_1 = 2\gamma\sqrt{\frac{B-C}{\chi_\|}}, \qquad \omega_{2,3} = \sqrt{\frac{2\gamma^2}{\chi_\perp}|C| + \frac{(1+\eta)^2}{4}(\gamma H)^2} \pm \frac{1-\eta}{2}\gamma H. \qquad (37)$$

In the limit $\eta \to 0$, the formulas (37) reduce to the resonance frequencies of a spin glass obtained from (27) and (28). When $\eta = -1$, i.e., when $\chi_\| = 0$, there remain only two frequencies, which have the standard form for collinear antiferromagnets.

In UO_2, in addition to $\boldsymbol{\Omega}_i^2$, there is one other invariant of the form (34),

$$(\Omega_x\cdot\mathbf{l}_1)^2 + (\Omega_y\cdot\mathbf{l}_2)^2 + (\Omega_z\cdot\mathbf{l}_3)^2 = \frac{4}{(1+\varphi^2)^2}\left[\left(\frac{\partial\varphi_x}{\partial x} + \varphi_z\frac{\partial\varphi_y}{\partial x} - \varphi_y\frac{\partial\varphi_z}{\partial x}\right)^2\right.$$
$$\left. + \left(\frac{\partial\varphi_y}{\partial y} + \varphi_x\frac{\partial\varphi_z}{\partial y} - \varphi_z\frac{\partial\varphi_x}{\partial y}\right)^2 + \left(\frac{\partial\varphi_z}{\partial z} + \varphi_y\frac{\partial\varphi_x}{\partial z} - \varphi_x\frac{\partial\varphi_y}{\partial z}\right)^2\right],$$

where x, y, and z are Cartesian coordinates along the principal axes of the crystal, and there is one quadratic relativistic invariant,

$$l_{1x}^2 + l_{2y}^2 + l_{3z}^2 = 1 - \frac{8}{(1+\varphi^2)^2}(\varphi_x^2\varphi_y^2 + \varphi_x^2\varphi_z^2 + \varphi_y^2\varphi_z^2 + \varphi^2).$$

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The Lagrangian function of UO_2 is

$$L = \frac{2\chi}{\gamma^2}\left[\frac{\dot{\varphi}^2 - c_1^2\varphi_1^2}{1+\varphi^2} - \frac{(\varphi\cdot\dot{\varphi})^2 - c_1^2(\varphi\cdot\varphi_l)^2}{(1+\varphi^2)^2}\right] + 2\frac{\chi}{\gamma}\dot{\varphi}\cdot\frac{H + H\times\varphi}{1+\varphi^2}$$
$$- \frac{2\chi(c_2^2 - c_1^2)}{\gamma^2(1+\varphi^2)^3}\left[\left(\frac{\partial\varphi_x}{\partial x} + \varphi_z\frac{\partial\varphi_y}{\partial x} - \varphi_y\frac{\partial\varphi_z}{\partial x}\right)^2 + \left(\frac{\partial\varphi_y}{\partial y} + \varphi_x\frac{\partial\varphi_z}{\partial y} - \varphi_z\frac{\partial\varphi_x}{\partial y}\right)^2\right.$$
$$\left. + \left(\frac{\partial\varphi_z}{\partial z} + \varphi_y\frac{\partial\varphi_x}{\partial z} - \varphi_x\frac{\partial\varphi_y}{\partial z}\right)^2\right]$$
$$- \frac{A}{(1+\varphi^2)^2}(\varphi^2 + \varphi_x^2\varphi_y^2 + \varphi_x^2\varphi_z^2 + \varphi_y^2\varphi_z^2),$$

$$(38)$$

where A is an anistropy constant and where c_1 and c_2 are constants with the dimensions of velocity. The magnetic moment of UO_2 is

$$M = \frac{2\chi}{\gamma}\frac{\dot{\varphi} + \varphi\times\dot{\varphi}}{1+\varphi^2} + \chi H.$$

$$(39)$$

When the magnetic field is directed along one of the principal axes of the crystal, the frequencies of the three spin-wave branches are determined by the formulas

$$\omega_1^2 = c_1^2(k_x^2 + k_y^2) + c_2^2 k_z^2 + \frac{A\gamma^2}{2\chi},$$

$$\omega_{2,3}^2 = c_1^2 k^2 + \frac{1}{2}(c_2^2 - c_1^2)(k_x^2 + k_y^2) + \frac{A\gamma^2}{2\chi} + \frac{\gamma^2 H^2}{2}$$
$$\pm\left\{\frac{(c_2^2 - c_1^2)^2}{4}(k_x^2 - k_y^2)^2 + \gamma^2 H^2\left[c_1^2 k^2 + \frac{c_2^2 - c_1^2}{2}(k_x^2 + k_y^2) + \frac{A\gamma^2}{2\chi}\right] + \frac{\gamma^4 H^4}{4}\right\}^{1/2},$$

this agrees with the results of Dzyaloshinskiĭ and Kukharenko.[42] The magnetic resonance frequencies ($k=0$) for UO_2 are no different from the case of a spin glass.

In a disordered antiferromagnet (in the isotropic phase, in which $M_{i\alpha} = \delta_{i\alpha}$), there is, in addition to Ω_i^2, an invariant

$$(\Omega_i\cdot M_i)^2 = \left(\frac{\text{div}\varphi + \varphi\cdot\text{curl}\varphi}{1+\varphi^2}\right)^2$$

of the form (34). The anisotropy energy is determined by formula (4). The Lagrangian function has the form

$$L = \frac{2\chi}{\gamma^2}\left[\frac{\dot{\varphi}^2 - c_t^2\varphi_1^2}{1+\varphi^2} - \frac{(\varphi\cdot\dot{\varphi})^2 - c_l^2(\varphi\cdot\varphi_l)^2}{(1+\varphi^2)^2}\right] + \frac{2\chi}{\gamma}\dot{\varphi}\cdot\frac{H + H\cdot\varphi}{1+\varphi^2}$$
$$- \frac{2\chi}{\gamma^2}(c_l^2 - c_t^2)\left(\frac{\text{div}\varphi + \varphi\cdot\text{curl}\varphi}{1+\varphi^2}\right)^2 - \frac{A\varphi^2 + B\varphi^4}{(1+\varphi^2)^2},$$

where c_t and c_l are constants and are equal to the velocities of transverse and of longitudinal spin waves.

The magnetic moment of a disordered antiferromagnet differs from the expression (39) by the presence of an additional term $\mu\partial M_i/\partial x_i$ ($\mu =$ constant); this occurs, as in formula (8), because of the existence of an invariant $\Omega\cdot\partial M_i/\partial x_i$, which is a sum of total derivatives.

The linearized equations of motion of a disordered antiferromagnet in a constant field, corresponding to the Lagrangian function given above, have the form

$$\ddot{\varphi} - c_t^2\Delta^2\varphi - (c_l^2 - c_t^2)\nabla\,\text{div}\,\varphi + \gamma H\times\dot{\varphi} + \frac{A\gamma^2}{2\chi}\varphi = 0.$$

In general, one obtains a cubic equation for the spin-wave frequencies. We have

$$\omega^2 = \frac{A\gamma^2}{2\chi} + c_t^2 k^2 + \varkappa,$$

where $\varkappa$ satisfies the equation

$$\varkappa^3 + \varkappa^2[(c_l^2 - c_t^2)k^2 + \gamma^2 H^2] + \varkappa\gamma^2 H^2\left[(c_l^2 - c_t^2)k^2\cos^2\psi - \frac{A\gamma^2}{2\chi} - c_t^2 k^2\right]$$
$$+ \gamma^2 H^2\left(\frac{A\gamma^2}{2\chi} + c_t^2 k^2\right)(c_l^2 - c_t^2)k^2\cos^2\psi = 0;$$

here ψ is the angle between the directions of the mag-

netic field and of the wave vector.

When $H = 0$, the frequencies of the one longitudinal and two transverse spin waves are

$$\omega_l^2 = \frac{A\gamma^2}{2\chi} + c_l^2 k^2, \quad \omega_t^2 = \frac{A\gamma^2}{2\chi} + c_t^2 k^2.$$

When $k = 0$, there are three different magnetic resonance frequencies, the expressions for which coincide formally with formulas (27) and (28) for the resonance frequencies in a spin glass.

7. NONCOLLINEAR FERRIMAGNETS AND DISORDERED FERROMAGNETS

In the determination of the Lagrangian function of structures with nonzero spontaneous magnetization, it is necessary, in order to avoid exceeding the limitations of accuracy, to pay attention to the following facts.[17] In nonequilibrium states, the exact value of the magnetic moment of unit volume, which in the present chapter we shall denote by $\mathscr{M}$, in general differs from the spontaneous moment $M(x_i, t)$, expressed by formula (17) in terms of angles $\varphi(x_i, t)$ and of the equilibrium value M_0, by terms containing derivatives. Since M depends on the angles themselves, and not on derivatives, the ratio between $\mathscr{M}$ and M is noninvariant with respect to an arbitrary small redefinition of the angles, in which the new angles differ from the old by terms containing derivatives. It is clear that by using this arbitrariness, one can always achieve coincidence of the directions of the moments $\mathscr{M}$ and M. Thus, according to a definition of the rotation angles improved for the nonequilibrium case, we have $\mathscr{M} = \text{const } M$. In equilibrium, the constant is unity.

For ordinary ferromagnets or collinear (dielectric) ferrimagnets, the last condition is retained even in nonequilibrium states; this is due to the quantum-mechanical indistinguishability principle already mentioned in Chap. 4. The equality constant $= 1$ is an expression of this principle in our case, in analogy to the fact that, as in the theory of diatomic molecules with nonzero electronic moment (see Ref. 28, §82), equality is required of the projections on the molecule axis of the total and electronic orbital moments.

In nonequilibrium states of noncollinear ferrimagnets and of a disordered ferromagnet, the constant must be expressed as an expansion in the small quantities Ω and Ω_i. This expansion, if we take account of the symmetry with respect to time inversion, leads to the formula

$$\mathscr{M} = M + \frac{\chi_\parallel}{\gamma M^2}M\cdot\Omega M,$$

$$(40)$$

where $\chi_\parallel$ is a constant that has the meaning, as will be seen below, of susceptibility along the direction of M. Formula (40) is applicable to ferromagnets, despite the possibility that there may exist in them, along with $M\cdot\Omega$, invariants of the form $1\cdot\Omega$. The occurrence of such invariants in the expansion of the constant would break the symmetry of the derivatives $\partial M_\alpha/\partial\Omega_\beta = \partial^2 L/\partial\Omega_\alpha\partial\Omega_\beta$.

In order that the moment $\mathscr{M}$ may be determined by

formula (40), the Lagrangian function, in the spatially homogeneous case in the absence of relativistic interactions, must be

$$L = \frac{1}{\gamma}\,\mathbf{M}\cdot(\mathbf{\Omega}+\gamma\mathbf{H}) + \frac{\chi_\|}{2\gamma^2 M^2}\,[\mathbf{M}\cdot(\mathbf{\Omega}+\gamma\mathbf{H})]^2 , \qquad (41)$$

where the magnetic field has been taken into account in the usual manner. Allowance for it in the expression for $\mathcal{M}$ gives

$$\vec{\mathcal{M}} = \mathbf{M} + \frac{\chi_\|}{\gamma M^2}\,\mathbf{M}\mathbf{M}\cdot(\mathbf{\Omega}+\gamma\mathbf{H}) . \qquad (42)$$

The additional terms in formula (41) that result from spatial inhomogeneity and from relativistic interactions do not differ significantly from the similar terms in the cases, considered above, of a spin glass and of antiferromagnets. In noncollinear ferrimagnets, they have, as a rule, a complicated form. But the resulting equations and spectrum of oscillations in ferrimagnets and in a disordered ferromagnet are similar. Their specific properties are due to the presence in formula (41) of the first term, linear in the angular velocity. Below, we shall demonstrate these specific properties for the case of a disordered ferromagnet.

In this case there are only two invariants of the form (34) with spatial derivatives: $\mathbf{\Omega}_i^2$ and $(\mathbf{M}\cdot\mathbf{\Omega}_i)^2$. Since it is possible to construct from the components of the vector $\mathbf{M}$ a single invariant $\mathbf{M}^2$ that does not change in spin rotations, the anisotropy energy of a disordered ferromagnet, as of a spin glass, is proportional to the fourth power of the ratio of the speed of the electrons to the speed of light. Besides $U_{\alpha\alpha}$ and $U_{\alpha\beta}U_{\beta\alpha}$, it may contain the invariants

$$U_{\alpha\beta}M_\alpha M_\beta = \mathbf{M}^2 - \frac{2}{1+\varphi^2}[\mathbf{M}\varphi]^2, \quad U_{\alpha\gamma}U_{\gamma\beta}M_\alpha M_\beta = \mathbf{M}^2 - \frac{8\,[\mathbf{M}\varphi]^2}{(1+\varphi^2)^2},$$

and $U_{\alpha\beta}M_\alpha M_\beta U_{\gamma\gamma}$. The general expression for the anisotropy energy contains four constants $\alpha_1,\ldots,\alpha_4$:

$$U_a = \frac{\alpha_1\varphi^2 + \alpha_2\,[\nu\varphi]^2 + \alpha_3\varphi^4 + \alpha_4\varphi^2[\nu\varphi]^2}{(1+\varphi^2)^2},$$

where $\nu = \mathbf{M}/M$.

Thus the complete Lagrangian function of a disordered ferromagnet is

$$L = \frac{1}{\gamma}\,\mathbf{M}\cdot(\mathbf{\Omega}+\gamma\mathbf{H}) + \frac{\chi_\|}{2\gamma^2}[\nu\cdot(\mathbf{\Omega}+\gamma\mathbf{H})]^2 - \frac{aM}{2\gamma}[\nu\mathbf{\Omega}_i]^2 - \frac{\chi_\| c^2}{2\gamma^2}(\nu\mathbf{\Omega}_i)^2 - U_a . \qquad (43)$$

We write the linearized (with respect to the equilibrium state $\mathbf{M}_0 \| \mathbf{H}$) equations of motion in a constant field, derived from (43):

$$\gamma M\,[\nu\dot\varphi] - \chi_\|(\nu\ddot\varphi)\nu + a\gamma M\Delta^2\varphi + (\chi_\| c^2 - a\gamma M)(\nu\Delta^2\varphi)\nu$$
$$- \gamma^2 MH(\varphi - (\varphi\nu)\nu) - \frac{1}{2}(\alpha_1+\alpha_2)\varphi - \frac{1}{2}\alpha_2\nu(\nu\varphi), \qquad (44)$$

where we have neglected the term, small in comparison with the first term, $\chi_\|\gamma H\nu \times \dot\varphi$.

From (44) we obtain the spectrum

$$\omega_l^2 = c^2 k^2 + \frac{1}{2\chi_\|}\alpha_1\gamma^2$$

of longitudinal spin waves of antiferromagnetic type, and the spectrum

$$\omega_t = ak^2 + \gamma H + \frac{1}{2M}\gamma(\alpha_1+\alpha_2)$$

of transverse spin waves characteristic of ferromagnets. We emphasize in closing that this approach to

disordered ferromagnets (and antiferromagnets) is valid, as in the case of spin glasses, at not too low frequencies. Although in this case the system possesses long-range order, dissipative processes, of the elastic-stress dissipation type, are possible in it because disorder is also present.

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