L. I. Mandel'shtam and the modern theory of nonlinear oscillations and waves

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The principal models and phenomena of the modern theory of nonlinear oscillations and waves are briefly reviewed in observance of L. I. Mandel'shtam's centennial. Each part of the paper, broken down into sections on "Oscillators," "Self-oscillations," and "Modulation," begins with a discussion of the classical models and effects of the role they play in the modern theory. Attention is concentrated on an analysis of complex behavior of simple systems, collective phenomena and ensembles of nonlinear oscillators, the initiation of self-oscillations in space, stochastic self-oscillations and waves, and the excitation, transfer, and origins of modulation in nonlinear systems and media, etc. Oscillation-theory and wave-theory models are discussed in parallel.

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INTRODUCTION

Fate has been kind indeed to the scientific ideas of Leonid Isaakovich Mandel'shtam. This pertains in particular to his ideas in the field of nonlinear oscillations and one of its newly emerged subdivisions—the theory of nonlinear waves (or, when one wishes to stress the informality of the approach, the physics of nonlinear waves). Mandel'shtam obtained classical results in practically all areas of oscillation and wave theory, results that are still cited in original scientific papers and not only in textbooks and works on the history of science. Mandel'shtam nurtured a "nonlinear school of physicists,"[1] and his students and the students of his students have to their credit many fundamental results in nonlinear optics, nonlinear plasma theory, radiophysics, and other nonlinear sciences. But this was not the only role that Mandel'shtam had in the development of nonlinear-oscillation theory. Perhaps no less important than his personal scientific contribution and his preparation of students for the development of this theory and then, thirty years later, nonlinear-wave theory as well, was Mandel'shtam's original "oscillatory" line of thought, which he demonstrated in his papers and promoted all his life in lectures and informal visits. It is, of course, very difficult to present a brief discussion of modern oscillation and wave theory in such a way that Mandel'shtam's imprint on this most modern of sciences will be evident not only to the authors, but also to the reader: "On the one hand, the ability to encompass the complex variety of phenomena from a single perspective, to recognize in them with absolute clarity features of similarity and difference, and to reproduce everything that is significant in a simple and lucid model; on the other hand, acute interest in the concrete individuality of the physical phenomenon... ."[2] In this paper, we undertake the specific attempt to view the theory of nonlinear oscillations and waves in its most contemporary form through Mandel'shtam's eyes.

In the words of Mandel'shtam himself, the theory of oscillations and waves is a science with its own unique

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approach, which is based on the construction and study of basic elementary oscillatory or wave models, has its own "universal" language of oscillations, which it uses to describe the basic oscillatory phenomena (resonance, modulation, synchronization, scattering, etc.), and has at its disposal rather general analytic and qualitative methods that are adapted for analysis of these phenomena. Knowledge of the basic models and phenomena results in a special oscillatory intuition with whose aid for example, "dark areas in optics are illuminated, as by a searchlight, by study of oscillations in mechanics."[3] The effort to understand the mechanism of a phenomenon using the simplest possible model, reexamination of this model from all aspects until it is fully understood with the object of including it in the general arsenal of oscillatory conceptions, all are highly characteristic of Mandel'shtam's scientific and teaching activity. Only after an effect had been observed and was fully comprehended would he go from the basic elementary model to a concrete physical theory with all its inherent detail.

Our paper is therefore devoted to basic models and phenomena of the contemporary theory of nonlinear oscillations and waves. What are the present-day features of this theory?

The theory of nonlinear oscillations was capable of solving many problems and incorporated much knowledge even in Mandel'shtam's time. Nonlinear oscillations had been fully investigated, coupled oscillations of these oscillators had been analyzed, Andronov and van der Pol had done most of the work on self-oscillation theory, the phenomena of synchronization and competition had been discovered, and Vitt had even attempted construction of a theory of self-oscillations of distributed systems. In Mandel'shtam's time, however, the theory of nonlinear oscillations was, with a few exceptions, a theory of systems with a *small* number of degrees of freedom, systems that exhibited *simple* periodic or quasiperiodic behavior. The modern theory, on the other hand, is characterized by "lively interest" in other extreme cases—it deals for the most part with *strongly nonlinear* systems, investigates *complex* behavior (including randomization) in *simple* dynamic systems, and analyzes the response of a large number of nonlinear oscillators to an external field, i.e., investigates the *behavior of ensembles*. To make these present-day aspects more evident, we begin each part of the paper, which is subdivided into sections on "Oscillators," "Self-oscillations," and "Modulation," with a discussion of the classical models and effects. Where-ever possible, the models of oscillation and wave theory are set forth in parallel.

In speaking of the closeness of the nonlinear-oscillation and wave theories as seen historically, we should note that comparatively recently (during the 1960's), the theory of nonlinear waves basically was still relying heavily on the already accumulated experience of classical oscillation theory and developing in much the same way as nonlinear-oscillation theory did during the 1930's. Characteristically, many results of that time involve various methods of passing over to solutions described by differential equations in ordinary phase space. Here we refer, for example, to analysis of *stationary waves*—solitons, shock waves, etc., the interaction of a large number of waves, but in a narrow spectral interval—*modulation waves* and some others.

This was a time of very rapid development of nonlinear-wave theory—new effects were constantly being discovered and "synthesized" by broadening the range of activity, and exact and approximate methods were being evolved. It would perhaps not even be an exaggeration to say that this was a time in which results could be obtained with comparative ease. By the end of this period, a rather high level of understanding of the experimental results had already been attained, intuition had been developed, and it had become possible to explain most of the nonlinear phenomena. However, it must be stressed that nothing had as yet been constructed similar to the rigorous qualitative theory that had been produced by Poincaré and applied by Mandel'shtam and his students to oscillatory systems with a small number of degrees of freedom. It may prove to be totally impossible to derive such a theory because of insurmountable mathematical difficulties. So what can we count on? Progress in solution of individual classes of nonstationary problems with the aid of exact methods, extensive use of computer experiments and acquisition of rigorous results with the aid of computers, physical experiments, and the use of approximate methods give hope for construction of a sufficiently complete theory of nonlinear waves, also qualitative, but this time in a different sense—in the sense of combination of sufficiently lucid and simple concepts that enable us to select transparent models for a very broad class of phenomena and to determine the most adequate quantitative-analysis method.

When we speak of the analogies between oscillations and waves, we should note their great profundity and diversity. It is sufficient to mention the well-known analogy between spatial wave beats in stationary interaction in space and the time beats of oscillations. Just as far-reaching is the analogy with oscillations of waves interacting in time when their spatial structure is given. There are also less trivial analogies—those between nonstationary wave effects (for example, periodic modulation waves) and the interactions of oscillations in ensembles of coupled nonlinear oscillators (recovery, quasiperiodicity, etc.). In discussing these analogies, however, the question arises as to why and up to what point can a finite-dimensional (or, more precisely, one of a small number of dimensions) system be juxtaposed to a wave (distributed) system, i.e., when can the problem be reduced to an analysis of a phase space with a small number of dimensions?

The answer to this question, which is now almost obvious, became clear essentially by the beginning of the 1960's,[8,32] when nonlinear wave processes were analyzed and compared in two extreme "cases"—in media *with strong dispersion* and *small nonlinearity* and in nonlinear media with *weak dispersion*. For example, in propagation of a wave in a compressible gas or on the surface of shallow water (*no dispersion*), the crest of the wave will move faster than its base, the wave will be distorted continuously, and at a certain time it will break—its profile must become nonunique. This will happen with a wave of any finite amplitude (i.e., even with small nonlinearity). But such a process can no longer be described by a finite-dimensional model. It is convenient to explain this in the highly descriptive spectral language. In a medium without dispersion, small perturbations of all frequencies have the same phase velocity. And therefore all harmonics, even weak ones that appear as a result of nonlinearity, are in resonance with the main wave (*synchronism*) and, are effectively excited by it. Thus, if we wished to describe the process with the aid of a set of harmonics, we would find it necessary to consider an *infinite* number of them.

But if in the case of weak nonlinearity the dispersion is large (as, for example, in the case of media used in nonlinear optics), only a few waves may turn out to be in synchronism—*time and space resonance*—and direct analogies can be drawn with processes in oscillatory systems with a small number of degrees of freedom.

It is noteworthy that Mandel'shtam and his students and colleagues actually advanced these very modern conceptions as to the influence of dispersion (the *nonequidistant* nature of the spectrum for bounded systems) on the nature of the processes that take place in the nonlinear distributed system already in the mid-1930's. They were, it is true, concerned with bounded distributed systems with lumped nonlinearities, but this is not very fundamental here. This problem received quite a bit of attention in a paper that they presented to the Congress of the International Radio Union (1935) under the title "New Investigations of Nonlinear Systems." In particular, concerning systems *with strong dispersion*, in which the *distribution of "overtones is not harmonic"*[1)] this paper states that: "In this case the form of the stationary oscillations may be close to sinusoidal. With the aid of a theory analogous to the small parameter theory for systems with a finite number of degrees of freedom we can calculate the amplitude, resolve the stability question, etc.... ."[4] In the other limiting case, that of *no dispersion*, i.e., when there is a *"harmonic overtone distribution"* (a problem investigated by A. A. Vitt in connection with his analysis of the excitation of a violin string by a bow)—"stationary oscillations are always *sharply nonsinusoidal*."[4] It is also observed that in distributed electronic (diode) self-excited oscillators, "the exciting forces are small and inertia becomes significant, and this is why the oscillations are of nearly sinusoidal form."[4] There is no doubt that if powerful coherent-radiation sources had been available at the time and that a need had arisen to solve the corresponding nonlinear wave problems, there would have been someone available to take these problems in hand!

1. OSCILLATORS

a) The marble in the chute

Considering the nonlinear oscillatory circuit and the marble in the chute (Fig. 1) as his prime examples of nonlinear oscillatory systems, Mandel'shtam notes in his "Lectures on Oscillations" (1930–1932) that it is reasonable "to imagine the entire qualitative picture of the motions on the basis of the differential equation itself, without solving it."[2)] For the nonlinear oscillator (NO)—a conservative nonlinear system with one degree of freedom—this qualitative picture emerges complete from the form of its phase portrait (Fig. 2). The motion of the NO is fully determined by its initial energy. At low energies it describes small, harmonic oscillations. As the energy rises, the oscillations depart farther from harmonic—most of the time in the periodic motion is spent on the "slow" segments, on which the marble rolls up to the top of the hump (Fig. 1a), and, finally, at an initial energy equal to $E_0 = mgh$, the motion of the marble will no longer be periodic at all. On the phase plane (see Fig. 2), it is represented by the separatrix passing from one saddle point to the other. Thus, the motion of the NO is *nonisochronous*—the frequency of the oscillations depends on their amplitude (or energy). For motions not too close to the separatrix we can say that $\omega = \omega(A^2)$.

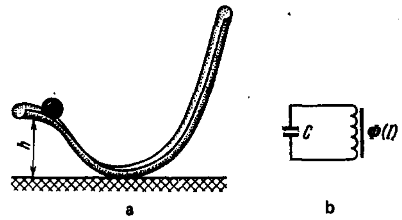

FIG. 1. Nonlinear oscillators.

To establish that a given dynamic system whose phase space is a plane belongs to the class of NO, i.e., to show that it is conservative, is by no means always as simple as, for example, in the case of the NO described by the equation

$$\ddot{u} - u\left(1 - \frac{1}{2}u\right) = 0, \tag{1.1}$$

whose phase plane appears in Fig. 2b. Actually, the integral of (1.1) is obvious: it is the energy integral $\dot{u}^2 - u^2 + u^3/3 = \text{const}$. Then the system

$$\dot{u}_1 = u_1(\nu_1 - \rho_1 u_2), \quad \dot{u}_2 = -u_2(\nu_2 - \rho_2 u_1), \tag{1.2}$$

which describes the ecological problem of the interaction between two biological forms—herbivores and carnivores—, appears at first glance to be nonconservative, and the integral found by Vitt,[5] $\rho_2 u_1 + \rho_1 u_2 - \nu_2 \ln u_1 - \nu_1 \ln u_2 = \text{const}$, looks nontrivial enough (see Fig. 2c for the phase portrait of this NO).

We draw attention to a singular solution of (1.1), which corresponds in the phase plane (see Fig. 2b) to the separatrix loop, a trajectory that is doubly asymptotic to point O. At the moment, interest in such solutions is very high in nonlinear-wave theory. For example, waves on the surface of "shallow water" can be

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[1)] I.e., the natural frequencies of the system are not multiples of one another.

[2)] In his next lecture on the same subject, Mandel'shtam notes: "The most difficult thing for the physicist is to obtain a measure of the required mathematical rigor. It would be more correct to say that he must know how to determine this measure" (Lectures on Oscillations, 1972, p. 73.

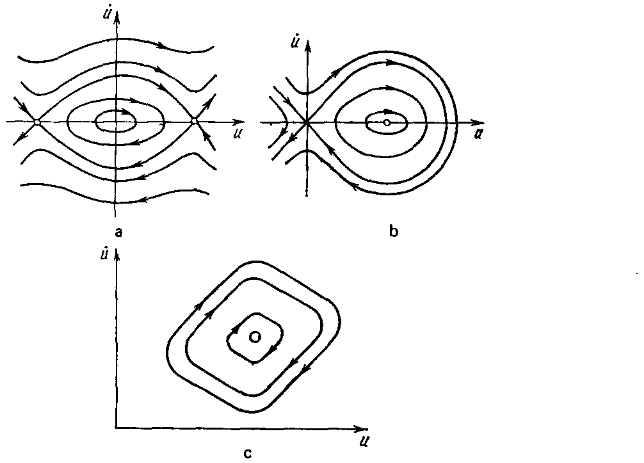

FIG. 2. Phase portraits of typical NO.

described approximately by the Korteweg–de Vries equation, which has become popular during the last 15 years (although it was discovered back in 1895):

$$u_t + v_0 u_x + u u_x + \beta u_{xxx} = 0. \tag{1.3}$$

If interest is limited to waves traveling at constant velocity and not undergoing profile changes, $u = u(x - Vt)$ (*stationary* waves), Eq. (1.3) with $V = V_0 + u$ yields the equation of the NO whose phase plane is shown in Fig. 2b. In this case, the doubly asymptotic path corresponds to a *soliton* or solitary wave, which drops off to zero at plus and minus infinity. Such essentially nonsinusoidal waves were quite familiar to mathematicians back at the beginning of the century, but they have attracted the attention of physicists only in recent decades.

b) The spring pendulum and nonlinear optics

In 1931, following the appearance of Fermi's paper[6] on the Raman spectra of the CO_2 molecule, which discussed the internal resonances of this molecule, Mandel'shtam suggested to A. A. Vitt and G. S. Gorelik that they investigate resonant-interaction effects of nonlinearly coupled oscillations in a surpassingly simple model—the spring pendulum (Fig. 3a),[7] whose equations, neglecting friction, have the form

$$\begin{aligned} &\ddot{u}_1 + \frac{k}{m} u_1 = l\left(\dot{u}_2^2 - \frac{g}{2l} u_2^2\right), \\ &\ddot{u}_2 + \frac{g}{l} u_2 = -\frac{1}{l}\left(\frac{g}{l} u_1 u_2 + 2u_1\ddot{u}_2\right). \end{aligned} \tag{1.4}$$

It was found on solution by the averaging method that when the parameter ratio $k/m \approx 4g/l$, i.e., when $\omega_{\text{vert}} \approx 2\omega_{\text{ang}}$, energy is pumped periodically from angular to vertical oscillations and vice versa, an effect that they also confirmed experimentally.[7]

Thirty years later, solving the problem of the stationary nonlinear operating regime of a parametric traveling-wave amplifier,[3)] R. V. Khokhlov, a second-generation student of Mandel'shtam, found that in propagation along the amplifier, the pump wave $2\omega_0$ parametrically amplifies the initial wave ω_0, transferring almost all its energy to it.[8] The reverse occurs in furthur propagation—the strong wave ω_0 generates a second harmonic and then everything is repeated from the beginning, i.e., we observe exactly the same phenomenon of periodic energy exchange between harmonics that was calculated and observed by Vitt and Gorelik (except in space rather than time) (see Fig. 3b). It is noteworthy that in the same year, 1961, second-harmonic generation was observed in propagation of a light wave from a ruby laser in optically transparent nonlinear crystals (Franken[10]). Together with Khokhlov's work, these experiments of Franken are regarded with complete justification as the beginnings of modern nonlinear optics.

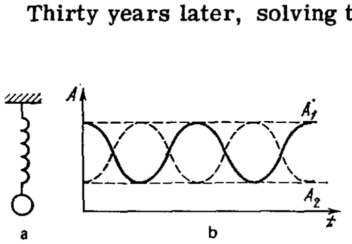

FIG. 3. Spring pendulum; periodic energy exchange between angular and vertical oscillations.

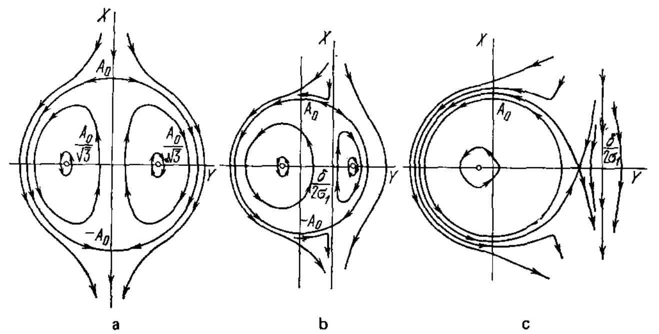

FIG. 4. Phase portraits of an NO describing energy exchange between harmonics in a system with quadratic nonlinearity. δ is the detuning. a) $\delta = 0$; b) $|\delta|/2\sigma_1 A_0 < 1$; c) $|\delta|/2\sigma_1 A_0 > 1$.

Assuming weak nonlinearity, the truncated (averaged) equations for the amplitudes and phases of the ω and 2ω oscillators interacting in time or in space are written in the form[8]

$$\left.\begin{aligned} &\dot{A}_1 = -\sigma_1 A_1 A_2 \sin\Phi, \\ &\dot{A}_2 = \sigma_2 A_1^2 \sin\Phi, \\ &\dot{\Phi} = -\left(2\sigma_1 A_2 - \sigma_2 \frac{A_1^2}{A_2}\right)\cos\Phi - \delta \end{aligned}\right\} \tag{1.5}$$

($\Phi = 2\varphi_1 - \varphi_2 - \delta t$, δ is the detuning from exact resonance). These equations are easily reduced to an NO equation by applying the energy integral $\sigma_2 A_1^2(t) + \sigma_1 A_2^2(t) = \text{const} = \sigma_1 A_0^2$ and introducing the new variables $X = A_2 \sin\Phi$, $Y = A_2 \cos\Phi$. Figure 4 shows the phase portraits of the resulting oscillator for various values of the detuning δ. It is seen that under the assumptions made as to the smallness of the nonlinearity (or, which is the same thing, smallness of the initial excitation energies), a system of two nonlinearly coupled oscillators demonstrates only very simple, quasiperiodic motions. From the physical point of view, the differences between different motions of this kind (see Fig. 4) consist merely in unequal depths of the energy beats between oscillators and the unequal periods of these beats. As we shall see, this simple behavior is also inherent

[3)] Such amplifiers were suggested in 1958 by P. Tien and H. Suhl.[9]

in many nonlinear systems that are at first glance very complex.

Generation of subharmonics is a degenerate case of the interaction of three resonantly coupled oscillators or waves:

$$\omega_3 = \omega_1 + \omega_2,\ \mathbf{k}(\omega_3) = \mathbf{k}(\omega_1) + \mathbf{k}(\omega_2), \tag{1.6}$$

where $\mathbf{k}(\omega)$ characterizes the dispersion law of the waves. With reference to the quantum-mechanical analogy, this process is often called *decay* [the conditions for frequency and wave-number resonance (1.6) can be regarded as energy and momentum conservation laws for the elementary event of merging of a pair of particles or decay of a single particle into a pair]. This process, which corresponds to the first zone of parametric instability, was first investigated for waves by Tien and Suhl, who proposed a distributed ferrite parametric amplifier. Another wave process, which corresponds to the second zone of parametric instability, is now also very well known; it is the decay of a pair of photons

$$2\omega_3 = \omega_1 + \omega_2,\quad 2\mathbf{k}(\omega_3) = \mathbf{k}_1(\omega_1) + \mathbf{k}_2(\omega_2), \tag{1.7}$$

in the same state. Processes of the type (1.7) generally become significant when simple decays (1.6) are forbidden (because the synchronism condition is not satisfied). It is precisely this situation that we have, for example, for waves on the surface of a deep liquid[11] and for waves in a plasma that have a nondecay dispersion law.[12]

It is possible to say a great deal concerning the properties of resonant oscillator interaction on the basis of the quantum analogy without really solving the problem. For example, quasiparticles can fuse only "if there is something to fuse with," i.e., if the number n_3 of quanta ω_3 [or $2n_3$ for the process (1.7)] produced in the merging process is exactly equal to the smaller of the quantum numbers n_1 or n_2 that existed at the initial time. The difference $n_1(0) - n_2(0)$ will, however, remain unused and, therefore, will be *preserved* for all t: $n_1(t) - n_2(t) = \text{const}$. It is also obvious that the sum of the number of quanta n_3 that have already been produced by time t and the number of quanta n_2 that have not been consumed by this time must also be constant, i.e., $n_3(t) + n_2(t) = \text{const}$. In wave theory, where $n_3 \sim |a_j|^2$,[76] these quantum-number conservation laws are usually referred to as the *Manley-Rowe relations*. Since quantum oscillators do not change quantum numbers on slow variation of the system parameters, the number of quanta is an *adiabatic invariant*.[74] The adiabatic invariant is violated if the oscillator transfers from one level to another, something that may result, for example, from resonant absorption of energy of an external field of frequency Ω by the oscillator. Under suitable conditions, this transfer, i.e., violation of adiabatic invariance, can occur even at high multiplicities of resonance: $\omega = m\Omega$, where $m \gg 1$, i.e., when the external field is varying very slowly. Applied to the classical oscillator, this result concerning the violation of the adiabatic invariant due to resonance was first obtained by Mandel'shtam and his students Andronov and Leontovich back in 1928.[86]

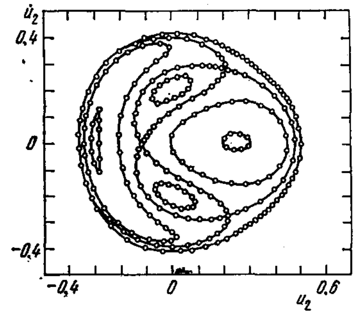

FIG. 5. Traces of paths on secant plane $u_1 = 0$ of phase space of system (1.8) at $\mu = 1$. The initial energy $E_0 < 1/12$.

In decays of the type (1.6) or (1.7), as in subharmonic generation, the response to the pump ω_3 becomes significant as the amplitudes of the amplified waves (oscillators) ω_1 and ω_2 increase, and the decay process gives way to a fusion process. All this is then repeated—in time for waves of specified spatial structure or in space for stationary harmonic waves. Thus, even a system of *three weakly nonlinear oscillators also exhibits only* simple periodic (or quasiperiodic) behavior.

c) Complex motions of a simple system

It might appear on the basis of the above examples that one could assert that a system of two (or even three) coupled oscillators is a very simple system in the sense that it demonstrates no "unforeseen" behavior. However, we shall not jump to conclusions, but instead consider how the system of two nonlinear coupled oscillators

$$\begin{aligned} &\ddot u_1 + u_1 = -\mu 2 u_1 u_2,\\ &\ddot u_2 + u_2(1 - \mu u_2) = -\mu u_1^2, \end{aligned} \tag{1.8}$$

which was investigated comparatively recently (in 1964[13]) not by averaging methods, but by detailed computer modeling, will behave.[4)] It is just as simple in form as (1.4). It is easily seen with the aid of the same averaging method that when $\mu \ll 1$ the oscillators exhibit simple, quasiperiodic behavior. This will also be the case when μ is not small ($\mu \sim 1$) but the initial excitation energies are [see Fig. 5, which shows the cross section of the trajectories cut by the plane $u_1 = 0$ in the three-dimensional $(u_1, u_2, \dot u_2)$ phase space of (1.8); this space becomes three-dimensional if the energy integral $-(\frac{1}{2} + u_2)u_1^2 + (\frac{1}{2} - u_2/3)u_2^2 + \frac{1}{2}(\dot u_1^2 + \dot u_2^2) = E$ is considered (at $\mu = 1$)]. We see that all the paths lie, as it were, on smooth surfaces (toruses), i.e., the motion of the systems is conditionally periodic for arbitrary initial conditions. System (1.8) is, after all, as simple as it appeared! But let us consider what will happen if we increase the oscillation energy of the oscillators. First, the motion of the second oscillator will become strongly nonlinear—motions near the separatrix of the single NO will appear (compare Fig. 2b), and because of the presence of the "external" force $u_1^2(t)$ we can no longer say whether they will remain quasiperiodic or whether

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4) This system is interesting for astrophysics: it models the behavior of a star in the field of a galaxy with the potential

$$U(u_1, u_2) = \frac{1}{2}u_1^2 + u_1^2 u_2 - \frac{1}{3}u_2^3 + \frac{1}{2}u_2^2.$$

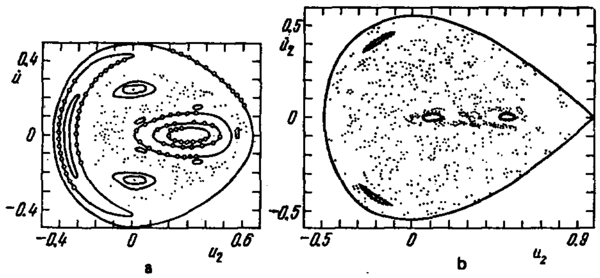

FIG. 6. Complex motions of a system of two NO (1.8). a) $E_0=0.125$; b) $E_0=0.167$.

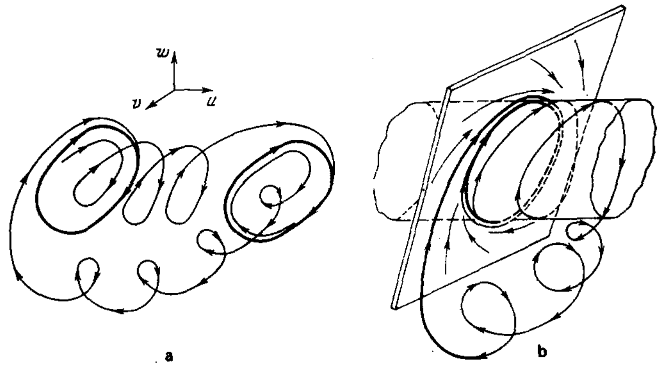

FIG. 7. Examples of saddle-point periodic paths: a) two saddle cycles; b) example of homoclinic path.

the type of motion will change from finite within the separatrix to nonfinite outside of it.

Figure 6 shows results of numerical experiments with two coupled NO (1.8) with initial energies $E_0>1/12$. We see that if the initial energy exceeds $E_0=1/12$, which still corresponds to simple motions, by only 0.004, the phase path no longer winds around any surface, but appears to wander at random in a bounded region of phase space! As E_0 increases further, the region occupied by the random motions becomes broader and that occupied by simple motions contracts (see Fig. 6b). Thus, the motion of two coupled NO in a simple model may be very complex.

What is the source of this complexity? It is this question that we shall now attempt to answer by examining a model simpler than (1.8)—an NO in a periodic field.

d) Oscillator in a pulsating potential well

In our coupled NO model we shall assume that the motion of one of the oscillators (u_1) is given and harmonic:

$$\ddot{u}-u+u^3=\mu\sin t. \qquad (1.9)$$

When $\mu=0$ we know everything about this oscillator (see Fig. 2b). Let us examine its behavior if $\mu\ll 1$. Physically, it appears obvious that a qualitative difference between nonautonomous and autonomous motions will appear if, under the action of an external force, the operator enters regions with different behaviors (inside or outside of the separatrix on the phase plane) at different times. This is easiest to see if the sinewave in (1.9) is replaced by a periodic sequence of square pulses—twice in every period, the phase portrait of Fig. 2b shifts to the left and then to the right by an amount of the order of μ. For low-amplitude oscillations (near the bottom of the well), these pulsations will go almost unnoticed—the motions will remain simple. On the other hand, motions near the separatrix may prove to be complex.[14] This complexity results from the existence, in the space of system (1.9), of the homoclinic structure discovered by Poincaré in connection with a study of the three-body problem back in 1889.[5)] A complete description of the paths within this structure was given comparatively recently.[49,69,90] It was found, in particular that this structure contains a denumerable set of unstable (saddle) periodic paths and that it is between these paths (with a broad range of initial conditions) that the "oscillator" wanders (Fig. 7).

e) Nonlinear Landau damping and Landau amplification

The problem of the behavior of a large number of oscillators, for example oscillators in the field of a periodic wave, is a very old one. A theory of the dispersion of light waves based on a model of oscillators embedded in an elastic ether had appeared even before Maxwell.[6)] Then there appeared the classical electron theory,[17] the theory of sound-wave dispersion in gases and the dispersion of electromagnetic waves in the ionosphere.[18] Mandel'shtam was also greatly interested in these problems; in particular, in 1941 he published a paper on the refractive indices of media with bound and free electrons. But all these are problems of the behavior of an ensemble of linear oscillators. What are the consequences of their nonlinearity? If even two coupled NO can behave in a highly complex fashion, how will an ensemble of these oscillators behave?

The first problems of this kind appeared about 20 years ago in electronics[19] and plasma physics, specifically in connection with problems in acceleration and heating of charged particles. Let us consider such a problem in the case of an electron flux whose electron velocity distribution function is represented in Fig. 8. In a coordinate system bound to the sinusoidal wave $E(x,t)=\varphi_0\cos(\omega t-kx)$ all particles can be classified as trapped or transiting. Those whose velocities lie in the range $\omega/k\pm\sqrt{e\varphi_0/m}$ do not have enough energy to overcome the potential barrier $e\varphi_0$, and they oscillate in the "well" of the wave, while those whose velocities are outside of this range take practically no notice of the wave (Fig. 9). Each i-th electron behaves like a pendulum in the field of the sinusoidal wave:

$$\ddot{u}_i+\omega_0^2\sin u_i=0,\quad i=1,\ 2,\ \ldots,\ N,\quad \omega_0^2=\frac{k^2e\varphi_0}{m}. \qquad (1.10)$$

Oscillations of the pendulum correspond to trapped electrons and rotations to transiting electrons (see Fig. 9). Thus, the particles in the field of the wave constitute an ensemble of identical nonlinear oscillators that differ only in the initial values of their energies. How

[5)] This structure appears in three-dimensional space in the neighborhood of a homoclinic path (see Fig. 7b).

[6)] The problem of light propagation in such a medium was solved by Rayleigh in 1869 as the answer to an examination question put to him by Maxwell (see Ref. 16).

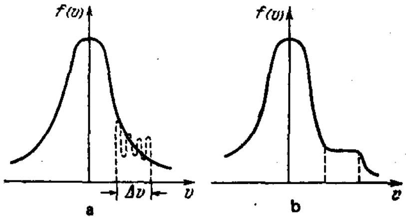

FIG. 8. Electron velocity distribution functions: a) appearance of oscillations in the field of a periodic longitudinal wave; b) formation of plateau.

will the ensemble behave in time? Since the interaction of the oscillators has not yet been taken into account, this question is answered quite simply on inspecting the motion of the oscillators on the phase plane. If $\partial f/\partial v|_{v=\omega/k}<0$, then at $t=0$ the greater portion of the trapped particles will be in the lower halves of the "cat's eyes" on the phase plane (Fig. 10). With time, the nonisochronism of the oscillators will change this region into a twisted spiral in which the number of turns will increase continuously. Therefore, the number of particles with different velocities will change continuously and the distribution function $f(V)$ will begin to pulsate in the interval Δv, becoming more and more dissected (see Fig. 8). If we wait long enough, all the oscillators should reassemble in the initial phase volume, since the motion of a conservative system (1.10) of N oscillators is reversible. Physically, however, it is obvious that no miracle will occur no matter how long we wait: the particles will mix as a result of their interaction, no matter how weak, with one another and with the waves, i.e., they will uniformly fill the entire region inside the separatrix and a plateau will form on the distribution curve. Since the average particle kinetic energy then increases, the sinusoidal wave in which the particles are oscillating loses part of its energy in accelerating them. This loss of energy by a monochromatic wave is often referred to as nonlinear Landau damping.[20]

If the particles have a nonequilibrium velocity distribution function, as, for example, in an electron beam-plasma system, the reverse process is also possible and a wave of finite amplitude may be amplified.[21] When the phase velocity of the wave "gets onto" the left slope of the nonequilibrium distribution curve (Fig. 11), the wave, building up as a result of nonlinear Landau amplification (there are fewer slow particles to take energy from the wave than there are fast ones to yield it), will increase in amplitude and will capture transit-

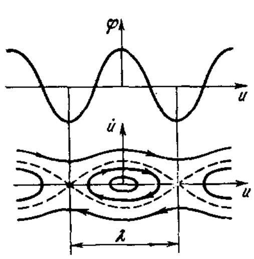

FIG. 9. Phase portrait of NO describing motions of trapped and transiting particles in the field of a wave.

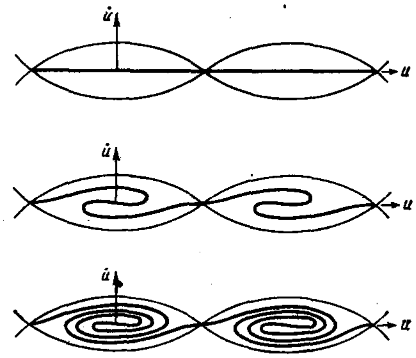

FIG. 10. Evolution of phase volume in an ensemble of noninteracting oscillator electrons.

ing particles. However, this process of amplification will obviously continue only until the numbers of fast and slow particles on the left slope of $f(V)$ are equal—until the plateau forms and the wave becomes nonlinear and stationary (*quasilinear relaxation*).

f) Chains of coupled nonlinear oscillators

These chains are an example of an ensemble of strongly interacting NO with ordered structure. Interest in the behavior of such ensembles appeared already at the beginning of the century in connection with the problem of the thermal conductivity and heat capacity of crystalline solids. It is usually assumed in analysis of thermal fluctuations in crystals that an energy kT in the classical theory or $h\nu/(e^{h\nu/kT}-1)$ in the quantum theory is associated with each normal oscillation (mode). But why is this universal distribution of energy over the degrees of freedom established for arbitrary initial conditions? How does thermalization occur? These were troublesome questions for everyone interested in the theory of heat capacity and, of course, for L. I. Mandel'shtam. The possibility of thermalization was naturally related to the nonlinearity of the oscillators. However, the first attempt to confirm the correctness of this general viewpoint by direct numerical computation (experiment) was not undertaken until 1952 by Fermi in collaboration with Pasta and Ulam. They used a computer to investigate the behavior of a chain of 64 nonlinearly coupled oscillators:

$$\ddot{u}_i+2u_i=(u_{i+1}+u_{i-1})+\alpha[(u_{i+1}-u_i)^n-(u_i-u_{i-1})^n],\quad i=1,\ 2,\ \ldots,\ 64, \tag{1.11}$$

where the exponent n of the nonlinearity was equal to two or three, and observed features in the behavior of the system "that surprised us from the very outset."[22] The system was not becoming thermalized! Instead, they first observed transfer of energy from the first strongly excited mode to higher modes, but then all the energy (to within 1%) was again collected in the

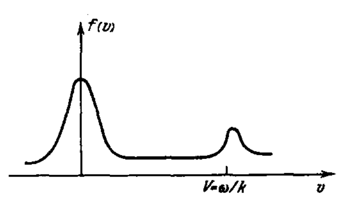

FIG. 11. Electron velocity distribution in a plasma-beam system.

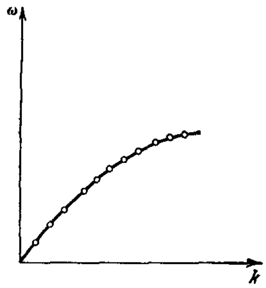

FIG. 12. Dispersion law for waves in a one-dimensional chain.

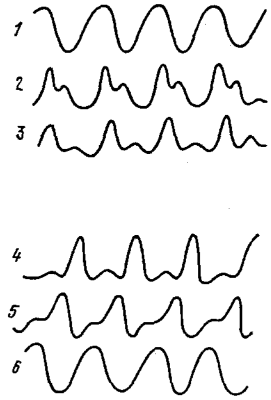

FIG. 13. Periodic evolution of nonlinear waves in LC networks.

first mode—the chain exhibited simple quasiperiodic behavior. Thus, it became clear that if thermalization is possible in chains of the type (1.11), its time is *anomalously large*. This "insubordination" of chain (1.11) to the prevailing conceptions was called the Fermi-Pasta-Ulam paradox. The key to this paradox, i.e., the answer to the question as to why such a complex system (and computer experiments were performed later with as many as 250 oscillators) exhibits only simple behavior, was found comparatively recently (in 1965 [23]). It developed that owing to a very unlikely combination of circumstances, chains of the type (1.11) were closely similar to fully integrable systems—minute "islands" in the space of all dynamic systems.[7] The fact that only simple behavior is possible in a fully integrable system follows clearly from its reducibility (using $N-2$ of the N integrals) to the phase plane of an NO, where all finite motions are periodic or lead to equilibrium. This is indeed an irony of fate: even two coupled NO can behave stochastically at a sufficiently high excitation energy, but here the entire chain suddenly becomes a nearly integrable system.

As we have seen, to make evident full integrability is not a simple matter even for a second-order system (when phase space is a plane), and it is the more difficult for systems of the nonlinear chain type. We now know, in particular, that a chain with an exponentially decreasing interaction potential is also fully integrable; this is the Tod chain:

$$\ddot{u}_n = \exp(u_{n+1} - u_n) - \exp(u_n - u_{n-1}). \tag{1.12}$$

Unfortunately, there are no *a priori* integrability criteria for such systems. But now we can answer the question as to why an increase in the number of oscillators in the chain has practically no influence on the nature of its behavior. Figure 12 shows the dispersion characteristic of a one-dimensional chain. The number of particles determines only the density of the points forming this characteristic (i.e., the number of normal oscillations of the chain), and has no effect at all on the shape of the curve—the type of dispersion remains unchanged. It is obvious that only the recovery time will increase on excitation of the first mode in a longer chain—a longer time is required for distribution of the initial energy over the larger number of normal oscillators, but the nature of the energy exchange between modes will not change. Nor will it change if we replace an infinite quadratic chain with a continuous medium having quadratic nonlinearity and a suitable dispersion law (see Fig. 12[8]):

$$u_{tt} - u_{xx} - (u^2)_{xx} - \beta u_{xxxx} = 0. \tag{1.13}$$

It was recently shown[24] that with periodic boundary conditions, this equation has an infinite set of independent integrals of motion, i.e., the *necessary* condition for full integrability is satisfied, although it has not yet been possible to prove full integrability. This has been possible (with arbitrary boundary conditions) for the single-wave analog of (1.13)—the Korteweg-de Vries (KDV) equation [see (1.3)]. The modified KDV equation corresponding to chain (1.11) with cubic nonlinearity is also found to be a fully integrable system:

$$u_t + u^2 u_x + \beta u_{xxx} = 0. \tag{1.14}$$

Even with the mixed nonlinearity $(\alpha_1 u + \alpha_2 u^2)u_x$, the KDV equation was found to be fully integrable.

Figure 13 shows the results of physical experiments with nonlinear *LC* networks that are described approximately by Eq. (1.3) or (1.14). Under sinewave excitation at the boundary, there was almost complete *restoration* along the network: the sinewave was transformed into a periodic sequence of solitons, i.e., a large number of harmonic oscillators was excited and the solitons when reverted to the sinusoid—all harmonics returned their energy to the first.

Apparently the integrable systems form a discrete set in the system space, and it is quite simple to "spoil" an integrable system by transforming it into a system with complex or stochastic behavior—it is only necessary to "jiggle" the dispersion law or the nonlinearity. If, for example, the dispersion law in (1.14) is made steeper, if u_{xxx} is replaced by u_{xxxxx}, the stochastic

[7] Unfortunately, we are unable to supply further details here, but despite the "low power" of the set of integrable systems, they play an exceptionally important role in the physics of nonlinear waves—both as specific examples from which certain general mechanisms of nonlinear phenomena can be surmised and as "standard" systems on the basis of whose known solutions it is possible to construct approximate solutions of "similar" nonintegrable systems (see Sec. 3).

[8] If we substitute a continuous "nonlinear string" without dispersion for the chain, we arrive at a curious variant of the "ultraviolet catastrophe": because of the unboundedness and equidistant property of the normal oscillator-mode spectrum of this string, the initial energy stored in a finite number of modes will move continuously upward through the spectrum, and there can be no periodic *exchange* of energy between modes.

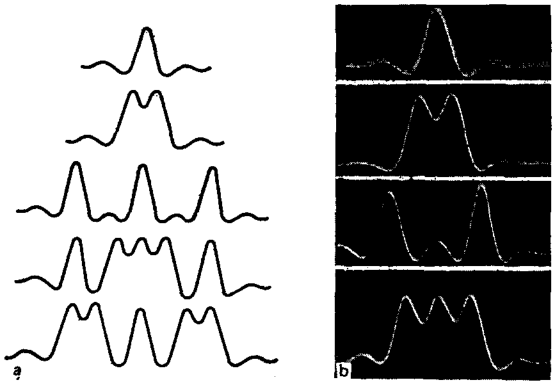

FIG. 14. Multisoliton solutions in nonlinear networks: a) numerical experiment; b) physical experiment.

property is observed in the resulting new one-dimensional medium even in the class of stationary waves $u = u(\xi = x - Vt)$. Such waves are described by the equation

$$\dddot{u} + \alpha\ddot{u} - Vu + u^2 = 0. \tag{1.15}$$

It has been shown[26] that there is a region in the phase space of this NO with complex behavior (homoclinic structure). Figure 14a shows numerical solutions of (1.15) and Fig. 14b oscillograms of similar waves that were observed in a nonlinear LC network with an equivalent circuit as shown in Fig. 15.

g) Solitons as particles

It would appear that solitons themselves, being rather complex formations, and soliton periodic lattices (*cnoidal* waves) should behave in very complex fashion on interacting with one another. However, to judge from numerous physical and numerical experiments, this impression is not always accurate. To the contrary, solitons often behave surprisingly simply when they interact: they are repelled, attracted, or oscillate relative to one another (Fig. 16) just like classical particles! It was recently established that this superficial analogy turns out to be quite a deep one with respect to weakly interacting solitons (or cnoidal waves). If the velocity difference (or, what is the same thing, the energy difference) between the solitons is small and the distance between their maxima remains large compared to their effective width throughout the entire process, their interaction is analogous in the literal sense to the interaction of particles and is described by Newton's equations. A soliton in the tail field of another soliton behaves like a marble in a chute. For example, for a pair of solitons we obtain the equation[27]

$$\frac{d^2u}{dt^2} - v'_E f(v, u) = 0, \tag{1.16}$$

where u is the distance between the solition maxima,

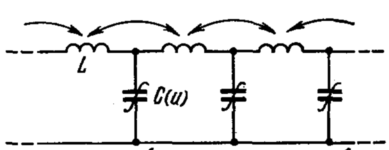

FIG. 15. Equivalent circuit of line in which the nonlinear waves shown in Fig. 14 were observed.

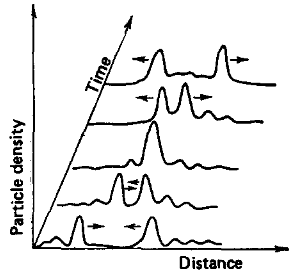

FIG. 16. Collision of ion-acoustic solitons.[25]

$f(u)$ describes the force field of the tail of one solition at the position of the other, and $v(E)$ is the soliton's velocity as a function of energy. For small interaction, equations similar to (1.16) can be derived from the initial equations for the waves by representing the field in the neighborhood of each soliton (its parameters are assumed to be slowly-varying) in the form of an asymptotic series and then applying boundedness requirements to the terms of this series.

After the "soliton-particle" analogy has been established [i.e., Eq. (1.16) has been derived], it is sufficient to know only the form of the force function $f(u)$, i.e., the nature of the soliton tails, to describe the soliton interaction. If $f(u)$ is monotonic, the solitions are repelled or attracted [Eq. (1.16) is, of course, no longer valid if their fields overlap strongly.][9] But if the solitons have oscillating tails, as in the cases of solitons of capillary-gravity waves on shallow water[29] or in a nonlinear artificial transmission line with inductive coupling between links,[27] then the function $f(u)$ is sign-variable and the solitons are alternatively repelled and attracted to form an oscillating pair *(a bound state;* Fig. 17).

The interactions of a large number of solitons of the same type can be analyzed in similar fashion because the nature of the tail does not depend on the number of solitons sitting on it.

To this we add that the analogy between nonlinear waves and oscillations is not so trivial as the mode analogs to which we have now become accustomed.

2. SELF-OSCILLATIONS

a) What are they?

As Mandel'shtam put it, when we are speaking of the *generation*, the *creation* of oscillations, we need an "arrangement that makes it possible for *stable* undamped oscillations to arise...;" "their oscillations are stable in the sense that if we start them off from some within broad limits arbitrary state, they oscillate with a definite period and a definite amplitude. They have a tendency, *irrespective of the initial conditions*, to settle into a defined regime."[30] Andronov, at the time one of Mandel'shtam's graduate students, called systems that possess this property *self-oscillatory* and was the first to give them a clear-cut mathematical definition when he related self-oscillations to

[9] Most of the exact solutions that have been found illustrate the repulsion of solitons.[28]

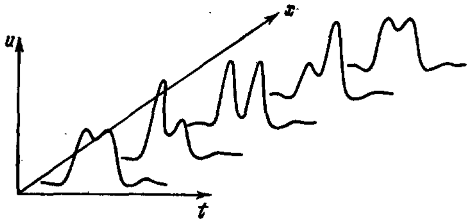

FIG. 17. Oscillating pair of solitons.

Poincaré limit cycles.[31]

The first self-excited oscillator "with a purpose" was invented and built in 1657 by Huygens, who adapted a pendulum to an old "pre-Galilean" clock and thus transformed it into a precision instrument with high rate stability (a theory of this clock was derived by N. N. Bautin, a student of Andronov's). Later, self-oscillations were investigated in a control system with dry friction,[33, 10)] generators that produced electromagnetic oscillations in the radio band appeared, and, finally, the "three-electrode cathode-ray-tube" or van der Pol generator[35] (Fig. 18). This circuit and the van der Pol equation that describes it

$$\ddot{u}-\mu(1-\alpha u^2)\dot{u}+\omega_0^2 u=0 \tag{2.1}$$

are still, half a century later, our basic model for self-oscillations in systems with one degree of freedom. Figure 19 presents phase portraits of (2.1) with various values of the nonlinearity parameter μ. When $\mu \ll 1$, the oscillations of the generator are nearly sinusoidal, and the nonlinear friction merely "selects" the amplitude of the stable limit cycle. An approximate (truncated) equation for the complex amplitude of the oscillations generated in this case has the form

$$\dot{a}=\mu\left(a-\frac{3}{4}\alpha|a|^2 a\right). \tag{2.2}$$

The stable equilibrium state $|a_0|^2=4/3\alpha$ corresponds to the limit cycle. Mandel'shtam and Papaleksi[36] showed that such an approximate solution of (2.1), i.e., $u(t)=|a_0|\cos(\omega_0 t+\varphi)$ is similar to the unknown exact solution not only on the limited time interval $T\sim 1/\mu$, but

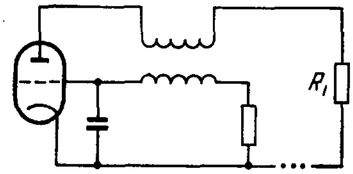

FIG. 18. Circuit of van der Pol generator.

also on an infinite interval, i.e., as $t\to\infty$.[11)]

But if the nonlinearity is not small, the oscillations in the generator will be essentially nonsinusoidal, and if $\mu \gg 1$ they will be of the relaxation type, consisting of segments of *fast* and *slow* motions. To find such discontinuous oscillations, Mandel'shtam and Papaleksi proposed the use of the "jump hypothesis," which takes account of the fact that the energy changes continuously in the jumps. This idea enabled Andronov and Vitt not only to understand the relaxation oscillations of the van der Pol generator more clearly, but also to solve a number of new problems, e.g., the problem of pulsed oscillations in a multivibrator.[38, 12)]

Since practically all the experience of the classical theory (at least for systems with nonsmall nonlinearity) was related to analysis of self-oscillations on the phase plane, the possibility of establishment of periodic motions that correspond to the limit cycle was associated exclusively with dissipative systems in which undamped oscillations occurred only at the expense of *nonperiodic* energy sources. Only a few years ago, no one would have thought of applying the term "self-oscillator" to a nonlinear oscillator with friction under the action of a periodic force:

$$\ddot{u}+\gamma\dot{u}-\alpha u(1-u^2)=f\sin\omega t. \tag{2.3}$$

But it is a self-oscillator: such a NO produces undamped oscillations whose parameters (intensity, frequency, and, in the more general case, *spectrum*, etc). *do not depend on a finite variation of the initial conditions* and depend weakly on changes in the external force. In particular, in the nonautonomous phase space $\dot{u}, u, t$ of (2.3) there are stable periodic motions to which, like to limit cycles of autonomous systems, correspond *stable stationary points* (in Poincaré's representation) if we view the system stroboscopically at the period of the external force.

Intensive studies of nonlinear dissipative systems with a three-dimensional phase space have, in recent years, made it possible to detect a completely new class of self-oscillatory systems. These are *noise* self-oscillators—dissipative systems that undergo *undamped random* oscillations, oscillations with a *continuous* spectrum, with energy drawn from nonnoise

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[10)] Both Mandel'shtam himself and his students are characterized by an exceptionally strict attitude to the facts of history of science and to the accuracy of acknowledgement of priority that this implies. In particular, Mandel'shtam and Andronov believed until 1931 that they had been the first to juxtapose generation with limit cycles, but when they found that this had been done intuitively nearly simultaneously with the discovery of the limit cycles themselves, they took every opportunity to point this out: "... The following preliminary remark is necessary if we are to avoid distortion of historical perspective. Ten years before the discovery of radio, in a study of self-oscillations in an automatic-control device, the French engineer Léauté (1885) investigated the phase space of this device and draw integral curves and limit cycles for it (without applying those names to them: he was apparently unfamiliar with the paper that Poincaré had published a bit earlier, in which limit cycles made their first appearance in mathematics). For reasons that we shall not discuss here, Léauté's remarkable studies had been almost completely forgotten."[34]

[11)] We note that the nontrivial physical approach to the proof of a purely mathematical problem that was used in this study was highly productive, in particular, in justifying similar approximate methods in the theory of nonlinear waves.[37]

[12)] Similar ideas were also used later in investigation of distributed nonlinear systems, in particular in deriving boundary conditions at a discontinuity in the theory of electromagnetic shock waves.[32]

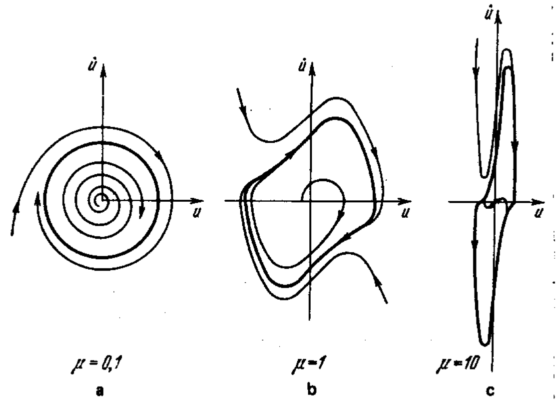

FIG. 19. Phase portraits of van der Pol generator at various values of nonlinearity: a) quasiharmonic oscillations; b) strongly nonsinusoidal oscillations; c) relaxation oscillations.

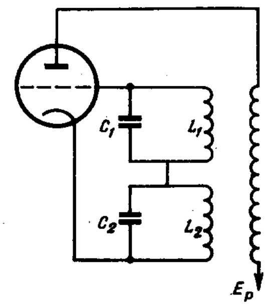

FIG. 20. Two-circuit self-excited oscillators.

sources.[13] It is noteworthy that even the familiar oscillator (2.3) is a noise self-oscillator over a broad range of parameters. The discovery of stochastic self-oscillations is perhaps the most brilliant achievement of modern theory. But why did it not appear until now?

b) Simple and complex attractors

This was because from Poincaré's time until recently, the limit cycle was the only example of a nontrivial attracting set—an *attractor*—in the phase space of nonlinear dissipative systems.[14] It is true that complex multiloop limit cycles corresponding to complex periodic self-oscillations were discovered quite a long time ago. They were, in particular, observed experimentally in an automatic temperature-control system by one of Andronov's graduate students[39] in the course of work on an assigned program entitled "Transition from a Plane to Three-Dimensional Space." Stable multiperiod motions were later observed in a study of synchronization of self-excited oscillators.[40] It would appear that the discovery of complex limit cycles and then also of bifurcations, which point the way to their further complication, might have served to broaden our view of self-oscillations. However, this did not actually take place until somewhat later, on publication of numerical experiments that demonstrated the existence of "nonperiodic phase fluxes" in dissipative nonequilibrium systems (E. Lorenz, 1963[41]). New mathematical objects, the complex attractors that Ruelle and Takens called "*strange attractors*," made their appearance at practically the same time in the abstract theory of dynamic systems.[42]

As an example of a strange attractor—an attracting set on which there are no stable paths and where they all behave in a complex and confused fashion—we might cite an attracting structure consisting of saddle-point cycles (when all paths that unwind from them tend to cycles of the same structure). Such a set of saddle-point cycles may be an "evolutionary vestige" of a homoclinic structure: the "homoclinic" itself is a "delicate thing" in a dissipative system—it vanishes as a result of a small change in parameters, but the stochasticism that is produced may remain. It is in precisely this way, in particular, that the strange attractor is produced in the phase space of the nonautonomous NO with friction [see (2.3)].

It is worthy of note that now, with this new view of stochastic self-oscillations (as the actual complex dynamics of a nonconservative system, and *not as a fluctuation amplifier*!), they are being observed in essentially very simple classical systems, such as coupled self-oscillators or the relaxation oscillator with one-and-a-half degrees of freedom.[43,44] They are being found because we now know precisely what to look for.[15] We shall return again to stochastic self-oscillations when we discuss concrete models. For the moment, however, we shall briefly discuss classical results.

c) New interest in old problems. Self-oscillations in space

Figure 20 shows the schematic of the two-circuit vacuum-tube oscillator that was investigated by van der Pol and Andronov and Vitt nearly half a century ago. Already then the most important effects characteristic of the interaction of "elementary self-oscillators," e.g., such as (2.2), had been observed. The averaged equations for the complex amplitudes of such self-oscillatory modes with independent frequencies have the form

$$\dot{a}_j = \mu h_j \left[1 - \alpha_j \left(|a_j|^2 + \sum_i \rho_{ji} |a_i|^2\right)\right] a_j, \quad j = 1, 2, \ldots, N. \qquad (2.4)$$

Figure 21 shows the phase portraits of this system for $N=2$ and various values of the parameters. They illustrate the classical effects of *competition* between modes, and of the *pulling* and *coexistence* of oscillations. Because of the specifics of the nonlinearity in the van der Pol generator, only the essentially trivial effect of simultaneous generation of two modes, which is possible when they are weakly coupled (see Fig. 21d) went unnoticed in Andronov and van der Pol's work (this case is typical, for example, for a gas laser with an inhomogeneously broadened active-medium line). The competition phenomenon that is observed with strong mode

[13] As in the case of definition of periodic self-oscillations (which is based on limit cycles, a more rigorous definition of stochastic self-oscillations requires an appropriate mathematical prototype. The strange attractor is such a prototype (see below).

[14] Only attractors correspond to long-lived oscillations in non-conservative systems.

[15] The same thing happened thirty years ago with limit cycles: after Andronov announced a "hunt" for them, they were discovered in chemistry, biology, ecology, and other often unexpected fields, and "there was no getting away from them."

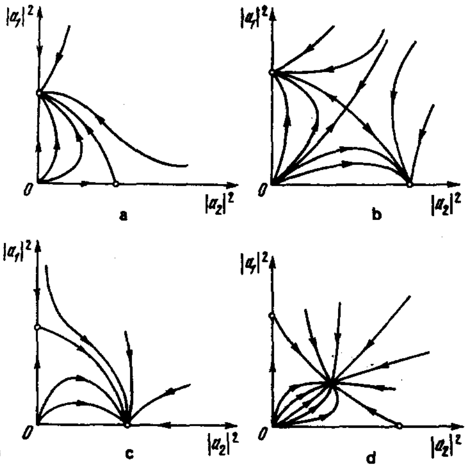

FIG. 21. Phase portraits of system (2.4), illustrating effects of competition, pulling, and coexistence of oscillations ($N=2$).

coupling is explained by the dependence of the nonlinear damping of one of the modes on the amplitude (energy) of the other. If the modes are equivalent and the coupling is mutual, the generation regime of the mode that predominated initially is established. A consequence of the dependence on the initial conditions is that in order to transfer the system from one regime to another is it necessary to change the frequency of one of the modes appreciabley, i.e., to change the tuning; here the values of the detuning are not the same for motion "out" and "back" (hysteresis). The range of detunings in which the generating frequency depends on prior history is known as the pulling range.

The last two decades have been a revival of interest in these classical effects, which, thanks to Mandel'shtam's school, have become almost commonplace. This interest is associated primarily with the appearance of active distributed systems (molecular and optical quantum generators [lasers], cyclotron-resonance masers, etc.) and the creation of systems with large numbers of active elements. In all cases when active devices are combined into organized spatial structures with the object of adding their powers or increasing their efficiency, the resulting systems become analogs of distributed systems. Only the type of dispersion of the resulting "medium" depends on the manner in which the active elements (Gunn diodes, LPD, etc.) are combined.

Classical oscillatory effects are often extended literally to waves because of the aforementioned space-time analogy between the interaction of normal oscillations (modes) in time and the stationary interaction of waves in space. As an example, Fig. 22 illustrates the spatial analog of the effect in which oscillations

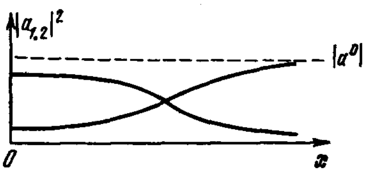

FIG. 22. Spatial competition of waves.

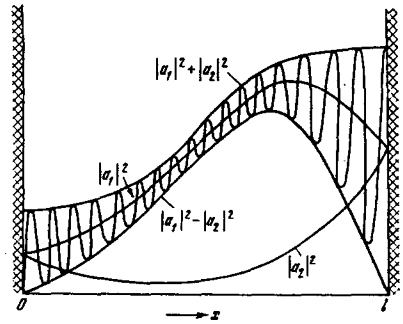

FIG. 23. Unsymmetric spatially inhomogeneous regime in a cavity with ideal reflection filled with a nonlinear medium.

compete in an active nonlinear medium with viscosity (high-frequency or low-frequency). This process is described by Eqs. (2.4) with x substituted for t. Working from the spatial-competition effect, we can, in particular, construct curious wave devices that take two or more *a priori* unknown quasiharmonic signals and separate the one with the highest (or lowest) frequency.[45]

It is this wave-competition effect that also explains the apparently altogether surprising establishment, in a *spatially symmetric* distributed self-excited oscillator (for example, with ideal reflection at the boundaries), of stationary field distributions that are *unsymmetric* along the x coordinate, with one of the colliding waves being dominant (Fig. 23). The equation for the amplitudes $a_{1,2}(x,t)$ of these waves using the simplest idealizations[46,47] can be written in the form

$$\frac{\partial a_{1,2}}{\partial t} \pm v\frac{\partial a_{1,2}}{\partial x} = \mu h[1-\alpha(|a_{1,2}|^2+2|a_{2,1}|^2)]a_{1,2} \qquad (2.5)$$

with the boundary conditions $|a_1(x,t)|=|_{x=0,l}|a_2(x,t)|$, where l is the length of the cavity. The distribution of intensities $|a_{1,2}(x)|^2$ in the stationary regime is easily reconstructed from the form of the paths on the phase plane of (2.5) when $\partial/\partial t\equiv 0$ (Fig. 24). In a short resonator, which does not allow time for competition to appear, only a banal standing-wave regime is possible, and a state of equilibrium on the line $|a_1|^2=|a_2|^2$ corresponds to it on the phase plane of Fig. 24. In a long resonator, on the other hand, colliding waves taking energy from a common source suppress each other over most of its length, evening out only near the reflecting walls. As a result, the standing-wave regime turns out to be unstable and one of the spatially inhomogenous regimes to which paths of the type *ebc* correspond in Fig. 24 is established.

d) Strong nonlinearity. More about nonlinear resonance

Another aspect of the interest in the classical problems of self-oscillation theory that has reappeared in

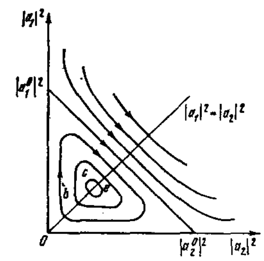

FIG. 24. Phase portrait of system (2.5).

recent years is associated with progress in the study of many essentially nonlinear, physically important systems with a *three-dimensional phase space*. They include both the aforementioned nonlinear oscillator with friction under the action of a periodic force and the Lorenz system now current in fluid dynamics, which describes thermal convection in a layer of fluid heated from below, and others.[43,48]

The groundwork for these recent advances was laid almost half a century ago by Mandel'shtam and his students, with Andronov leading the list. It must be said that when Mandel'shtam's attention was drawn to the generation of nonsinusoidal oscillations (about 1927), only isolated problems had been solved in this area and they appeared to be quite unique, if not casuistic. Thus, Papaleksi solved the problem of strongly nonlinear oscillations in a rectifier by a *storing-up method* [conversion of the system to a piecewise-linear system with subsequent conjugation of the integration constants (to obtain continuity of the solution and the derivatives with respect to t, or to satisfy "jump conditions")] (1911); then A. Sommerfeld investigated forced oscillations of an arc (1914), and Papaleksi considered periodic oscillations in a vacuum-tube oscillator with a piecewise-linear tube characteristic in 1922. But the question of greatest fundamental importance—that as to the *stability* of the periodic motions that were found—was not even posed. It appears that the stability problem was first solved by van der Pol for the particular case of the relaxation oscillator, using graphical constructions on the phase plane (1926). In 1927, Mandel'shtam suggested to Andronov that he derive a general method for investigation of the stability of periodic motions obtained by the storing-up method and "attempt to provide this method with a mathematical base." Out of this assignment arose Andronov's remarkable work "Poincaré Limit Cycles and Self-Oscillation Theory," which we have already mentioned and in appraising which Mandel'shtam noted that: "... here we have a mathematical formalism that is indeed adequate to our nonlinear problems and that does not have 'linear memories'.... Working from this formalism, it will be possible to create *new concepts* that are specific *for nonlinear systems*, to develop new insights that will enable us to *think nonlinearly*." Extending these studies Andronov and his colleagues with the aid of the Poincaré–Brouwer–Birkhoff method of point maps succeeded in solving several strongly nonlinear problems concerning self-oscillations in automatic control systems and, simultaneously, in answering Mandel'shtam's question as to the stability of periodic solutions found by the storing-up method. As Andronov himself put it, "this entire series of studies can be regarded from a certain point of view as an embodiment of Mandel'shtam's old idea of giving the storing-up method a mathematical education."

The general considerations formulated more than thirty years ago (1944) by Andronov concerning the application of point-transform theory to the study of concrete nonlinear systems, together with the idea of "historical" or "embryological" investigation of the dynamic system that he introduced into oscillation theory[50] (i.e., investigating the *evolution of phase-space structure* as the system parameters are varied plus use of Poincaré's theory of bifurcation points and stability reversal) have proven to be extremely productive. Recent progress in understanding three-dimensional dynamic systems, including the discovery of strange attractors, has grown out of precisely these ideas.

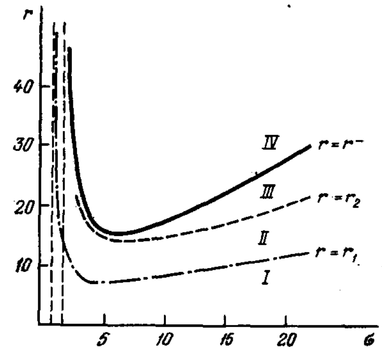

FIG. 25. Bifurcation diagram of Lorenz system.

The level of this understanding is, of course, still very far from the level on which we understand the dynamics of two-dimensional systems (and it is uncertain whether it will catch up), but it is even now high enough to permit bringing the entire power of modern mathematics to bear on the analysis of concrete systems. Also, and this is especially important, this level of understanding permits knowledgeable use of numerical and analog modeling not simply to compute cases, but for complete, including "embryological," investigation of system dynamics with the aid of computer curve generation and image analysis. This approach even enables us to produce fully rigorous results, or, in other words, to prove theorems with the aid of the machine.

Thus, for example, self-oscillations in the Lorenz system

$$\left.\begin{aligned} \dot{u} &= -\sigma(u-v), \\ \dot{v} &= -v+ru-uw, \\ \dot{w} &= -bw+uv. \end{aligned}\right\} \qquad (2.6)$$

have now been investigated in detail. Figure 25 shows the bifurcation diagram of the regime change of this system. Let us describe these bifurcations very briefly.[16] They were investigated by a computer analysis of the Poincaré transform of the points of the secant plane Σ, which passes transversally to the w axis through the nontrivial equilibrium states $C_{1,2}$ (Fig. 26). This two-dimensional map is found to be strongly compressed in one of the directions (θ on Σ; see Fig. 26) and stretched in the other. Repeated application of the transformation converts any cell on Σ into "lines" (they have Cantorian fine structure), and the analysis can therefore be confined to a *one-dimensional* mapping of the lines onto themselves and onto each other. Figure 27 shows the

[16] Much has been written recently on the Lorenz system and its applications (see, for example, Ref. 43); for this reason, we shall not even discuss the possible physical content of the variables. Our purpose is to illustrate Andronov's idea of an evolutionary approach to the study of dynamic systems as exemplified by this altogether nontrivial case.

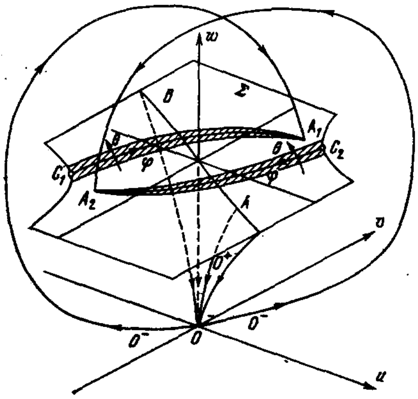

FIG. 26. Phase space of Lorenz system.

behavior of the unstable separatrices of the zero equilibrium state (saddle-node) that determines the properties of a one-dimensional map of this kind.

If $r < r_1$, the separatrices describe damped pulsations with the initial phase of the oscillations preserved; at $r = r_1$, the unstable separatrices are tangent to the stable two-dimensional separatrix AB (see Fig. 26), and at $r > r_1$ they have transferred from their stable focus C_1 or C_2 to an "alien" focus. Simultaneously, two symmetrically disposed limit cycles are produced from the loops of the separatrices, but they are unstable. At $r > r_2$, the separatrices tend to these newly formed cycles rather than to the equilibrium states C_1 and C_2, which, as before, remain stable: this is the time at which yet another (other than $C_{1,2}$) attractor—the *strange* attractor—makes its appearance.

Within this attractor, which is *bounded* by unstable *separatrices* and *cycles*, the paths behave in very complex fashion (the corresponding $v(t)$ oscillogram appears in Fig. 28). This complexity is associated, in particular, with the fact that a denumerable set of unstable cycles belongs to the attractor (which owe their origin to the homoclinic structure that existed in the past)—the path describes several revolutions around one cycle, is then thrown off onto another, spins around it, and so forth. Since, in addition to the strange attractor, there are two other "nonstrange" attractors in this parameter region, whether or not a static regime or a regime of stochastic pulsation is established in the system depends on the initial conditions. As r increases beyond r_2, the radii of the unstable cycles decrease and they "stick" to the equilibrium states C_1 and C_2 at $r = r^-$, passing their instability to them, i.e., only one attractor—the strange one—remains in the phase space of (2.6).

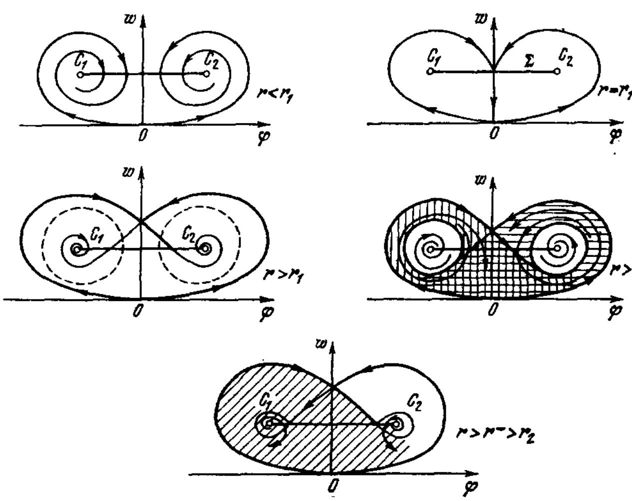

FIG. 27. Behavior of unstable separatrices in projection on plane.

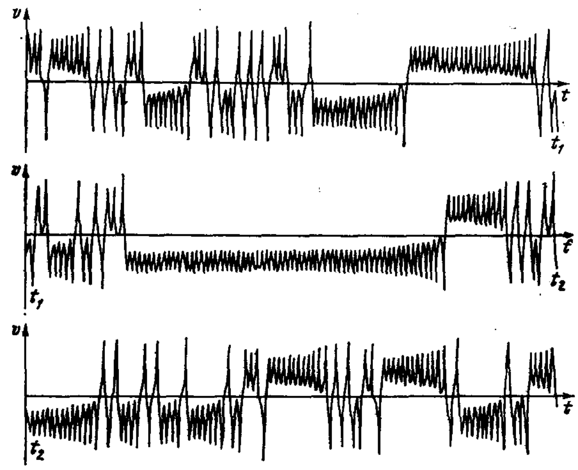

FIG. 28. Oscillogram of $v(t)$ oscillations in Lorenz system.

Use of results from the qualitative theory and computer analysis of point transforms has also made it possible to investigate, in just as much detail, the self-oscillations that arise in nonlinear resonance [see (2.3)]. Here, as in the nondissipative analog (1.9), it is possible to use Mel'nikov's method[15] if both friction and the external force are small for analytic determination of the value of the force f at which the separatrices of the saddle-point periodic motion touch and the homoclinic structure arises. As f increases further, the structure vanishes, but a denumerable number of periodic stable and unstable motions remains within the attractor $|\dot{u}| \leq f/\gamma$ (the region that all projectories only enter). Then, after a sequence of bifurcations—splittings of stable cycles—the motion becomes nonperiodic. Figure 29 illustrates the behavior of such an oscillator in time: that the motion is nonperiodic is clear.

e) Noise generators

The conceptions formed of stochastic self-oscillations made it possible to construct an actual radio-band noise generator.[51] It is obtained by making quite minor modifications to the classical circuit of the van der Pol generator (Fig. 30)—introduction of a nonlinear element with an S-shaped current-voltage characteristic into the grid circuit. This element might, for example, be a tunnel diode. Neglecting the nonlinearity of the tube

FIG. 29. Stochastic self-oscillations in system (2.3) (numerical experiment).[87]

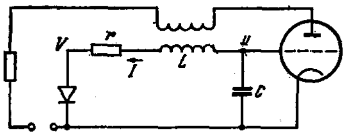

FIG. 30. Circuit of a simple noise generator.[51]

characteristic, we may write the equation of this system thus:

$$\begin{aligned} &\ddot{u}_1 - 2h\dot{u}_1 + u_1 = -au_2, \\ &\mu\dot{u}_2 = \dot{u}_1 - f(u_2), \end{aligned} \tag{2.7}$$

where $u_1 = I/I_m$, $u_2 = V/V_m$, $t = t_{\text{real}}/\sqrt{LC}$, $h = (MS - rC)/2\sqrt{LC}$ is the growth increment of the oscillations in the circuit, $a = V_m/\sqrt{L/C}I_m$ characterizes the influence of the nonlinear element on these oscillations, and μ, $\mu = (V_m/I_m\sqrt{LC})C_1 \ll 1$ is a small parameter that allows for the stray capacitance of the tunnel diode.

The operation of this generator can be described qualitatively as follows. As long as the current I and the voltage u_1 are small, the tunnel diode has no significant effect on the oscillations in the circuit, which build up because of the negative resistance introduced by the tube. Here the voltage across the tunnel diode, $V(I)$, is determined by the left-hand branch of the diode characteristic. When the current reaches the value I_m, the diode switches over and the voltage V_m is established. Then the current I decreases (here the voltage is determined by the right-hand branch of the characteristic), and the diode switches back (Fig. 31). In other words, when the amplitude of the oscillations in the circuit becomes high enough, the losses increase discontinuously and the amplitude of the oscillations drops. This means that the generated signal should take the form of a sequence of trains of growing oscillations, and this is the result obtained in experiments (Fig. 32). It cannot, of course, be proven that the oscillations are stochastic simply on the basis of these qualitative considerations. Here it is necessary to turn to the mathematical model (2.7) and to analyze the point maps.[51] Figure 33 shows the form of the map function for typical parameters of an actual working generator. The transformation is a stretching one in the region bounded by the light line (attractor). This means that all paths on the attractor are unstable[17)] and that the system forgets the initial conditions as $t \to \infty$—the probability density for obtaining a given value of u on repeated use of the transformation tends to an *invariant* distribution that does not depend on the probability density distribution of the initial fluctuations. To this we add that the statistical characteristics of the stationary-generation regime are stable not only to initial perturbations, but also to continuously active external fluctuations. This is indeed a noise generator!

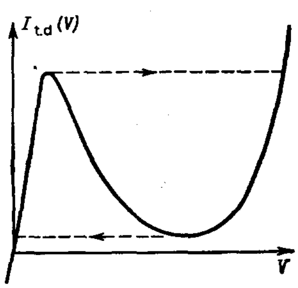

FIG. 31. Current-voltage characteristic of a tunnel diode.

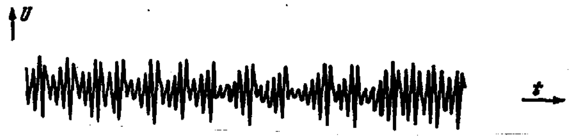

FIG. 32. Oscillogram of oscillations in circuit of Fig. 30.

f) Order from disorder. Synergetics

The disorganized behavior of a very simple nonlinear system (pendulum in a periodic field or van der Pol generator supplemented with a nonlinear element) is, of course, a perfectly astonishing phenomenon, but the opposite case—the regular, well-organized behavior of very complex disorganized systems with large and even infinite numbers of degrees of freedom—is no less surprising. Figure 34 shows spatial structures that arise in a plane-parallel horizontal layer of silicone oil when it is heated from below—an ordered structure of nontrivial form arises out of the *disordered initial disturbances* irrespective of the dimensions of the container or the geometry of its side walls. How can this macroscopic structure appear in a nonequilibrium medium that is homogeneous in the mean and in which disturbances of the most diverse and independent scales grow out of fluctuations as a result of instability? A similar question also arises in the attempt to explain, for example, the spiral galaxies in astrophysics or reverberators—spiral waves—in biology and chemistry.[52]

The problem of the formation of ordered temporal and spatial structures is a very general one; it is of interest to physicists, biologists, sociologists, and even physicians (epidemic waves). We now actually have a new field of science—synergetics (Greek "working together"),[18)] which is concerned in a general way with this and related problems of the existence, stability, and breakdown (generation of turbulence) of highly organized structures in nonequilibrium systems of various natures. But let us return to convection: how are we to explain the formation of structures (Benard cells) whose shape and scale do not depend, with finite limits, on the initial and boundary conditions? Other phenomena that were first investigated most completely before the war by Andronov are the *mutual synchronization* and *competition* of various *modes*. The simplest example for clarification is the case of convection in silicone oil (see Fig. 34), which has a strong dependence of viscosity on temperature, $\nu(T)$. Convective motions with a characteristic scale k_0 arise in the layer of oil just above the instability threshold (Fig. 35). The establishment of an elementary spatial structure in the

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[17)] More precisely, nearly all of them, since the fate of periodic paths that rest on the "crown" has not yet been clarified. However, even if they are stable, they have an attraction region so small that they are not realized because of the fluctuations present in the physical system.

[18)] The First International Symposium on Synergetics was held in 1972,[53] and two more have been convened since, in 1974 and 1977.

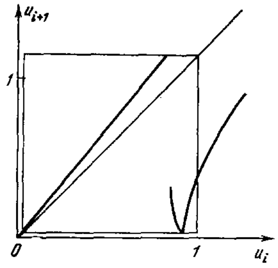

FIG. 33. Point map for system (2.7) with $\mu=0$.

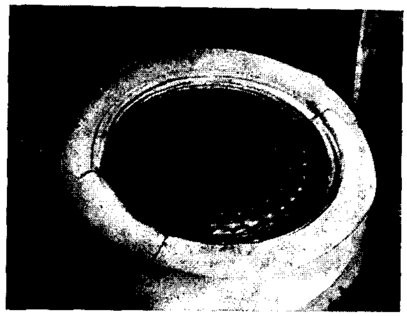

FIG. 34. Benard cells in thermal convection.

form of convection waves corresponds to growth of disturbances with a wave vector $\mathbf{k}_{01}$. But when viscosity depends on temperature,[19] this structure is found to be unstable to the generation of modes with other orientations of the vector $\mathbf{k}$, for example, waves disposed across the original waves (their coexistence results in the formation of rectangular structures). The $\nu(T)$ relation can usually be considered to be a quadratic one, and we then obtain resonant coupling among three modes of the same scale, $\mathbf{k}_{01}\pm\mathbf{k}_{02}=\pm\mathbf{k}_{03}$ (Fig. 36). It is the superposition of these modes with amplitudes that are equal and phases $-v_z(x,y)\sim\cos k_0/2x\cdot\cos\frac{1}{4}(k_0x+\sqrt{3}k_0y)\cdot\cos\frac{1}{4}(\ldots)$ that are synchronized in space that corresponds to nontrivial spatial structures in the form of hexahedral Benard cells (v_z is the vertical component of the velocity of the fluid)—fluid rises at the center of a cell and sinks near its faces (or vice versa if $\partial\nu/\partial T>0$). The orientation of the cells in space is arbitrary and depends on the initial conditions. On the other hand, competition between modes of unequal scales ensures stability of the particular structure against the appearance of others.

Let us cite one more example that indicates the role of the mode-synchronization effect in the appearance of ordered structures in nonequilibrium media. We refer to the establishment of solitons, in particular in a radio-band active medium, where one-dimensional waves are described by the equation

$$\frac{\partial u}{\partial t}+v_0\frac{\partial u}{\partial x}+\beta\frac{\partial^{2n+1}u}{\partial x^{2n+1}}=\alpha u^2-\nu_1u+\nu_2\frac{\partial^2u}{\partial x^2}; \tag{2.8}$$

here β characterizes the dispersion, $\nu_{1,2}$ the low-frequency and high-frequency dissipations, respectively, and α the active nonlinearity. If the dispersion is strong, the evolution of distrubances in this medium can be described with a small number of modes, for example, with frequencies ω and 2ω. The equations for their amplitudes and phases are found, accurate to the terms responsible for linear dissipation, to be similar to (1.5):

$$\begin{gathered}\frac{dA_1}{d\xi}=A_1A_2\cos\Phi-\delta_1A_1,\quad \frac{dA_2}{d\xi}=\frac{1}{2}A_1^2\cos\Phi-\delta_2A_2,\\ \frac{d\Phi}{d\xi}=-\left(2A_2+\frac{A_1^2}{2A_2}\right)\sin\Phi\qquad(\Phi=\varphi_2-2\varphi_1),\end{gathered} \tag{2.9}$$

but with the one fundamental difference that the signs in the right-hand sides of the equations for A_1 and A_2 are the same. Physically, this means that the harmonics are damped or grow simultaneously, i.e., the waves exchange energy *not with one another*, but with the nonequilibrium medium. Above the instability threshold, given a favorable phase difference ($\Phi=0,\pi$), the amplitudes of the harmonics increase without limit after a finite time (or on a finite distance) within the framework of this model: $A_{1,2}\sim1/(t^0-t)\,[t^0\sim1/A_{1,2}(0)]$ and we have *explosive instability*. It is very important that explosive instability is accompanied by rapid *mutual synchronization* of the phases of the interacting waves.[47] In the interaction of a large number of harmonics in a medium without dispersion ($\beta=0$),[20] this synchronization results in the establishment of nonlinear waves and solitons in particular: $u(x,t)=(3\nu_1/\alpha)\,\mathrm{ch}^{-2}\times[\sqrt{\nu_1/2\nu_2}\,(x-v_0t)]$, which have been observed experimentally (Fig. 37). Unlike "conservative" solitons (see Sec. I), these solitons propagate only at the linear-disturbance velocity v_0. The phase portrait of (2.8) for $\beta=0$ and stationary waves $u=u(x-v_0t)$ coincides with Fig. 2b.[21]

A much more complex structure appears as a result of mode synchronization in a nonequilibrium medium with more than one dimension: a downward flowing film of liquid. We can write an approximate equation for the deviation u of the film surface from the undisturbed level:

$$u_t+4uu_x+u_{xx}+\Delta_\perp^2u-nu_{yy}=0 \tag{2.10}$$

($n>0$, $\Delta_\perp^2=\partial^2/\partial x^2+\partial^2/\partial y^2$). Figure 38 shows the numerical solution of this equation with $u(x\to\pm\infty)=0$; it is a horseshoe-shaped soliton with an oscillating leading edge and a monotonically falling trailing edge. The most surprising thing about this solution is that it gives a rather good description of waves observed experimentally in a real downward flowing film (Fig. 39).

It is difficult to predict the future of synergetics, but it is already clear that this emerging science of self-organization, although it overlaps considerably with the theory of nonlinear oscillations and waves in nonequi-

[19] The temperature dependence of surface tension or other dissipative parameters also produces similar effects.

[20] In this case the number of interacting modes is limited by high-frequency damping.

[21] The fact that the phase portrait of a self-oscillatory system (for stationary waves) is the same as that of a conservative oscillator would appear at first glance to be paradoxical. However, this can be explained quite simply: owing to the absence of dispersion, the energy balance of dissipation and activity processes is satisfied simultaneously in this case for a continuous set of finite stationary waves traveling at velocity v_0. It is to these dissimilar waves that the continuum of paths on the "conservative" phase plane corresponds (see Fig. 2b).

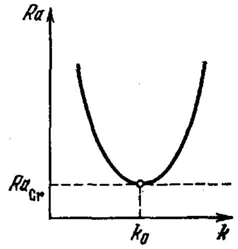

FIG. 35. Neutral curve for thermal convection in a layer.

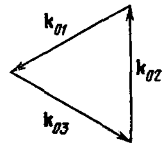

FIG. 36. Resonant mode triplet.

librium media, is still entirely independent and attracts specialists in a broad range of fields, just as the classical theory of nonlinear oscillations did in its day.

g) Generation of turbulence

The antipode of mutual synchronization of generators—*mutual randomization*—is new only as it applies to a system consisting of a small number of coupled self-oscillators.[43,44] But, as we have already noted, the fact of synchronization came as a surprise in ensembles consisting of large numbers of generators, while chaotic and turbulent behavior appeared natural and almost obvious. Therefore, the problem of the generation of turbulence—stochastic self-oscillations in a continuous medium[22)]—was associated almost exclusively with the excitation of a large number of "self-oscillatory" modes with noncommensurable frequencies and independent phases.[58] Now that we have more distinct conceptions of randomness in self-oscillatory systems, it is clear that, on the one hand, the very fact of excitation of a large number of degrees of freedom is no longer adequate to explain the appearance of turbulence and, on the other, that turbulence arises even when the number of modes excited in the medium is small. These are important and interesting problems, and many people are working on them, but we cannot discuss them in detail here and refer the reader to the recently published review articles (Refs. 42, 43, 56).

To this we add only that unlike the classical oscillation theory of Mandel'shtam's time, which considered only problems of the effects of fluctuations on nonlinear systems, i.e., the *transformation* of fluctuations, modern oscillation theory shows very great interest in the *production of "statistics"* in nonlinear dynamic systems.

3. MODULATION

a) Sinewave with variable amplitude and frequency

"Approximately simple" oscillations—oscillations that deviate slowly from the sinewave—were discussed by Rayleigh back in 1892, but the need for broad investigation of these oscillations appeared somewhat later in connection with problems in the reception and transmission of radio signals. "Without modulation there is no signal...that for which radio technology was created, transmission, does not exist."[51] [23)] Questions arose as soon as serious investigation of modulation started: What is a modulated oscillation—a "sinewave with variable amplitude and frequency" or a set of sinewaves with different frequencies and amplitudes? In other words, is there a difference between the temporal (spatial-temporal for waves) and spectral (mode) approaches? It was these questions that were actually discussed half a century ago when the problem of "narrowing" the spectral band of a radio transmitter's frequency-modulated signal arose; they reappeared a bit later in analysis of the beat spectrum that appeared on synchronization of a self-excited oscillator with a periodic external force and, finally, quite recently in connection with analysis of the dynamics of multimode (distributed) self-oscillatory systems with narrow generated spectra (lasers). Mandel'shtam was acutely aware of this problem; back in 1908, he presented an experiment at a Strasbourg University lecture in which he demonstrated that side frequencies actually do appear on modulation of an alternating current (a carrier). In his lectures, Mandel'shtam answered these questions as follows: "...It is necessary to know *why* we must speak of a single oscillation of variable amplitude and frequency, what is it that either we or nature intend to do with this oscillation." In other words, the correct result can be arrived at by using either approach, and which one is the more adequate depends on the problem.

The radioengineering term "modulation" was essentially introduced into physics by Mandel'shtam. He used the language of modulation when he discovered and described the scattering of light by acoustic lattice oscillations (Mandel'shtam–Brillouin scattering) or by atoms or molecules of the medium (combination [Raman] scattering). Mandel'shtam approached the problem of optical-image construction as though it were *spatial modulation*, and at the same time it was he who first posed the problem of frequency modulation in a self-excited oscillator.[24)]

Mandel'shtam regarded as modulation any process of *slow* changes in a *high-frequency* oscillatory system "in which it has time to describe many free oscillations before their amplitude, frequency, and phase change to any appreciable extent"; i.e., modulated oscillations in the classical theory are *quasiperiodic* oscillations with rather slowly varying parameters.

Does our understanding of modulation in the modern theory differ from the classical notion? For the most part, no; as before, we deal with oscillations or waves with slowly varying parameters. In the modern theory, however, the modulated oscillation or wave is by no means necessarily a "sinewave with slowly varying

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22) After the appearance of Landau's model,[58] G. S. Gorelik,[59] *a student of Mandel'shtam's, stated the relation of self-*oscillations to turbulence in the clearest terms.

23) Today we would say that without modulation there is no information.

24) This problem was solved by S. M. Rytov, a graduate student of Mandel'shtam's.[62]

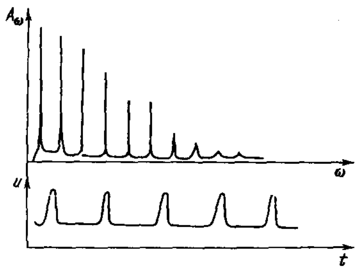

FIG. 37. Self-oscillations in the form of cnoidal waves (spectrum and oscillogram) as observed in an active line with imaginary (dissipative) dispersion.

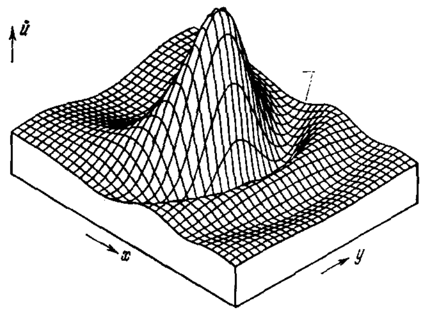

FIG. 38. Horseshoe soliton obtained in numerical solution of Eq. (2.10).[54]

amplitude and frequency"; the shape of the elementary oscillation or filling wave on which the modulation is superimposed can be arbitrary within broad limits (for example, a periodic cnoidal[25)] or sawtooth wave); we may speak with some justification of modulation as a slow change in certain parameters of motion even when the "filling" is not periodic. And while the analysis of modulated oscillations that are close to periodic nonsinusoidal oscillations has roots in classical oscillation theory, the investigation of slowly evolving nonperiodic oscillations or waves is characteristic of the modern theory alone. The first problem of this kind arose in study of the behavior of nonlinear waves in media with slowly varying parameters. Here we have a very clear-cut example—the development of sea waves as they approach the shore. This problem was solved by analyzing "quasisolitons"—waves similar to solitary stationary waves with amplitudes varying slowly due to the inhomogeneity of the medium (which arises due to the variability of depth near the shore). The most important distinguishing feature of the modern theory is essentially its broadening of the modulation concept to include not only processes in which modulation is transformed, but also processes in which it is produced—*self-modulation*.

Like oscillations, modulation may arise as a result of instability (self-modulation), may be forced (modulation is transferred to the carrier from an external source), or, finally, may be specified at the initial time (an analog of free oscillations). To this we add that there are now modulation analogs for nearly all nonlinear oscillatory or wave effects. This pertains to randomization and recovery effects, to nonlinear modulation waves, etc. It is even easy to conceive of certain effects in modulation on the basis of direct analogies with oscillations. This will perhaps seem less surprising when we recall that in many cases the transformations that occur in the modulation spectrum in a nonlinear system differ from the corresponding spectral changes of the oscillations (or waves) themselves only in that they occur at higher frequencies (are transplanted to the carrier frequency). Mandel'shtam attached great importance to these analogies and made masterful use of a general modulation approach: "Here he could and did produce much that was quite remarkable in the way of everything from fundamental physical discoveries to casual remarks."[57]

b) Mandel'shtam's traveling lattices. Modulation of waves by waves

Today, when we speak of modulation of waves by waves, the picture of a periodic traveling lattice on which the incident wave is diffracted (modulated) seems so natural that we do not concern ourselves with its origins. But Mandel'shtam was the first to see it. As early as 1913, analyzing the scattering of light at the interface between two media, he "materialized" the terms of the spatial Fourier series independently of Einstein and Debye, placing real periodic lattices in correspondence to them (just as, somewhat earlier, he had indicated the reality of spectral satellites in time modulation of alternating current). But they were still stationary lattices. Traveling lattices appeared five years later. By that time, Debye's theory of the heat capacity of solids, in which elastic (acoustic) waves were presented as "storages" for energy of thermal motion, was quite well known, and Mandel'shtam was the first to point out that light scattered by thermal fluctuations should be frequency-modulated by a traveling acoustic wave (lattice) and found the frequencies of the satellites $\nu_\pm : (\nu_\pm - \nu) = \pm 2\pi\nu (C_{s0}/C)\sin(\theta/2)$ (ν is the frequency of the incident light, C_{s0} and C are the velocities of sound and light, and θ is the scattering angle). This was a prediction of the scattering of electromagnetic waves by acoustic waves (Mandel'shtam–Brillouin

FIG. 39. Waves on downward flowing film.

[25)] A cnoidal wave, a periodic sequence of identical solitons, is represented on the phase plane (see Fig. 2) by a closed path near a separatrix.

scattering)[26]—the first example of the process scattering of waves by waves that is now under broad investigation in many areas.

Ten years later (in 1928) Mandel'shtam, working together with G. S. Landsberg on a study of the scattering of light in crystals, attempted observation of spectral satellites produced by modulation of light by sound. However, they observed much stronger splitting, which they explained in terms of modulation of the light by infrared vibrations of molecules. Thus was combination [Raman] scattering of light—scattering of waves by oscillators—discovered.[27] However, the scattering of light by sound was not observed experimentally until 1932 and then in France and the United States.[28] Mandel'shtam–Brillouin scattering (MBS) and Raman scattering (RS) constitute modulation as understood by Mandel'shtam: "... In much the same way as you inject your speech into the emission of a radio station by means of modulation, so do atoms oscillating in a molecule or crystal lattice tell us of their infrared vibrations, using the frequency of the emitted light as a carrier." This modulation of an incident wave by specified sources is usually called *spontaneous* scattering.

In the contemporary nonlinear theory, attention is concentrated on processes discovered in the early 1960's: *induced* scattering by oscillations (IRS, 1962) and by waves (IMBS, 1964).[63-65] In induced scattering, the incident wave itself amplifies the modulation sources—oscillations of atoms or molecules in RS or the sound wave in MBS. These processes constitute one manifestation of the parametric instability that we have already discussed—decay of the incident wave into a resonant pair of waves or into a wave and an oscillation (in RS).

When waves interact, modulation may either be produced or be transferred from one wave to another. The first effect of this kind was actually observed back in 1930, when Tellegen (Luxembourg) and Lbov (Gor'kiĭ) tuned receivers to the frequency of a local radio station and received the transmission (modulation) of a powerful station working on a totally different frequency—*cross-modulation*. The Luxembourg–Gor'kiĭ effect is explained quite simply (1934[60]): on passage of a strong modulated wave (pump) through a volume of ionospheric plasma, the coefficient of absorption for a weak wave also passing through the volume is changed in accordance with the modulation law specified by the plasma. Thus the pump modulation is transferred to a different carrier.

It is possible for modulation to cross not only from the strong (pump) wave, but also from the weak (signal) wave in the presence of an unmodulated pump. One of these possibilities is, as we know, embodied in the superheterodyne receiver—the modulation is mainly amplified at the intermediate frequency. The same kind of "superheterodyne" process can also be brought about for waves in a nonlinear medium with "intermediate"-frequency amplification.[61] The equations for the amplitudes of the parametrically coupled waves $\omega_3=\omega_1+\omega_0$ in such a medium at a given pump (heterodyne) field $a_0=\text{const}$ can be written in the form (a_3 is the amplitude of the signal)

$$\frac{\partial a_1}{\partial x}=i\sigma_1 a_0^* a_3+\gamma a_1, \quad \frac{\partial a_3}{\partial x}=i\sigma_3 a_0 a_1. \tag{3.1}$$

This mechanism is of interest, of course, only if the amplification of the intermediate wave is strong enough: $\gamma \gg \Gamma=|a_0|\sqrt{\sigma_1\sigma_2}$. Here the evolution of the signal and intermediate waves along the "receiver" is described by the following solution of (3.1):

$$\begin{aligned} a_1(x,\ t)&=a_3(0,\ t)\frac{i\sigma_1 a_0^*}{\gamma}(e^{\gamma x}-1),\\ a_3(x,\ t)&=a_3(0,\ t)(1-\delta e^{\gamma x}) \quad \left(\delta=\frac{\Gamma^2}{\gamma^2}\right); \end{aligned} \tag{3.2}$$

here $a_3(0, t)$ is the modulated signal wave at the entrance ($x=0$) into the nonlinear medium $[a_2(0, t)\equiv 0]$. The process of signal amplification in a wave superheterodyne receiver can be described as follows. First, there is slight amplification of the intermediate wave a_1, onto which the modulation that existed at the boundary is transferred from the signal wave, in the interval $0<x\lesssim 1/\gamma$; then the intermediate wave carrying the signal-wave modulation is strongly amplified in the range $1/\gamma\lesssim x\lesssim x_0=\ln(1/\delta)/\gamma$, and, finally, the amplified modulation is transferred to the signal wave: $x>x_0$. This process is obviously also possible with a low frequency pump.

The modulation of waves by waves is not always manifested in forms as familiar as slow variation of wave amplitudes or phases. Thus, when colliding waves interact even in an isotropic nonlinear medium, their *type of polarization* may also change—the plane of linearly polarized waves may rotate, linear polarization may be converted into elliptical, and so forth. Let us illustrate one of these effects with a specific example, considering the interaction, in time, of spatially homogeneous colliding waves of the same frequency in an optically active medium (laser). Let the angle between the field vectors of these linearly polarized waves be initially very small. What happens to them then? The polarization planes of the opposed waves will *rotate* in *opposite* directions. A possible mechanism of this is as follows. Each of the waves deexcites active particles whose dipole moments are aligned along its field; as a result, the component of the opposed wave with exactly the same polarization now propagates without amplification and only those of its components that have a somewhat different polarization are amplified. "Repulsion of polarizations" will result from this curious competition of colliding waves. The polarization rotation is totally different in the stationary case (Fig. 40). Here the polarization vectors of the colliding waves *are rotated in the same direction*.[66] This time the effect is fundamentally spatial. Indeed, a periodic lattice forms in the medium on passage of the colliding waves and re-

[26]By this time, Brillouin had already published some of his results on the scattering of light by sound.

[27]Combination scattering was discovered by Raman and Krishnan simultaneously with Mandel'shtam and Landsberg.

[28]The optical, Born branch of the dispersion curve had already been reported, but Mandel'shtam and his colleagues were unaware of these studies: exchange of information had not yet resumed after interruption by the war. They identified this branch independently (M. A. Leontovich[88]).

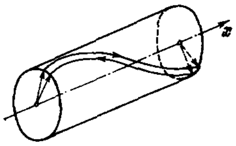

FIG. 40. Rotation of polarization in nonlinear interaction of colliding waves.

flects the components of each wave with mutually orthogonal polarization differently. It is because of this difference that the polarization vectors of the colliding waves are rotated.

The polarization-rotation effect of colliding waves was confirmed experimentally back in 1970 for a resonance isotropic active medium.[67] It has now been observed in a wide variety of isotropic media. Since the magnitude of the effect—the angle of rotation or the ellipticity of the polarization of the colliding waves—depends in a very subtle fashion on the properties of the nonlinear media, this effect has proven useful for their diagnosis giving rise to *nonlinear polarization spectroscopy*.[68]

c) Modulation recovery

It was observed in the very first experiments with IMBS and IRS in optics that the backscattered beam approximately repeats the evolution of the pump beam in the backward direction in time. It was then found that in many experimental situations the scattered wave exactly *reproduces* a complex-conjugate incident wave that is strongly modulated in the transverse direction.[70] Duplication of the backscattered (Stokes) wave in the backward direction of the optical path traversed by the pump signifies that a limited region in which scattering occurs behaves as a *mirror*. But this is not an ordinary mirror: the reflected wave duplicates the optical path of the incident wave in forward time only when its phase front is conjugate with the pump, i.e., $a_p(r) \sim a_0^*(r)$. Here the total phase of the wave $\exp[i\omega t - ikx + i\varphi]$ varies as it propagates in the $-x$ direction in the same way as that of the incident wave in the backward time direction. This is why effects in which the transverse modulation of the pump beam is reproduced in induced-scattering radiation have come to be known as *"wave-front inversions."*

The fact that the scattering volume acts as a nontrivial mirror is related to the selective manner of amplification of the Stokes wave (which grows out of noise) in the field of a pump broken up in r. If the pump phase front is unmodulated, Stokes waves with arbitrary transverse structure are amplified equally in its field; but if it is sufficiently cut up, a Stokes wave modulated in r in such a way that its maxima fall onto the minima of the pump and vice versa is not amplified as well as one that duplicates the pump profile. Formally, this can be explained as follows: the total power (averaged across the beam) of the backscattered wave is described by the equation[71] $dP/dx = -g(x)P(x)$, where the gain in the direction of propagation is

$$g(x) = \frac{G\int a_0(r)\,a_0^*(r)\,a_p(r)\,a_p^*(r)\,d^2r}{\int |a_p(r)|^2\,d^2r}. \qquad (3.3)$$

If, given the condition that $a_0(r)$ varies rapidly, the pump and initial-noise intensities are uncorrelated in r, the gain is $g = G\langle|a_0(r)|^2\rangle$ (quadruple correlations decay into pair correlations). But if $|a_p(r)|^2 \sim |a_0(r)|^2$, the increment will be twice as large. Since the two appear in the argument of the exponential and the total gain along x is quite large, it is certain that from the backscattered noise background there will be extracted the wave with the inverted wave front. Such effects are now being discussed widely in nonlinear optics in relation to the possibility of self-correcting transfer of powerful laser radiation over long distances—*adaptive nonlinear optics*.[71,72]

d) Self-modulation

Let us perform a simple experiment: at the boundary of an LC transmission line or chain of oscillators with a cubic nonlinearity [see (1.14)] we apply a sinusoidal oscillation whose frequency lies in the range of strong dispersion $\omega(k)$ (for example where the dispersion curve of Fig. 12 starts to bend), with the result that the harmonics arising due to the nonlinearity are out of synchronism with the main wave (and, consequently, do not build up). What kind of oscillation will we observe at the other end of the line? Figure 41 shows the answer in oscillogram form: the oscillations are found to be modulated! We would not have expected this because we intuitively link the appearance of modulation (in the narrow sense of the word) only to the transfer of information concerning a low-frequency signal to a high-frequency carrier. As we have already seen, the physical nature of this process may vary greatly, but there must be some source of modulation! But this is not evident in our experiment. This example illustrates the phenomenon of *self-modulation*—modulation occurs as a result of development of parametric instability along the line, in this case with the result that satellite waves appear with frequencies ω_1 and ω_2 near ω_0, where $\omega_1 + \omega_2 = 2\omega_0$ [compare the decay of a pair of quanta in the same state (1.7)]. This version of parametric instability is called *modulation instability* in the theory of nonlinear waves.[76,83]

To describe this and related phenomena in greater

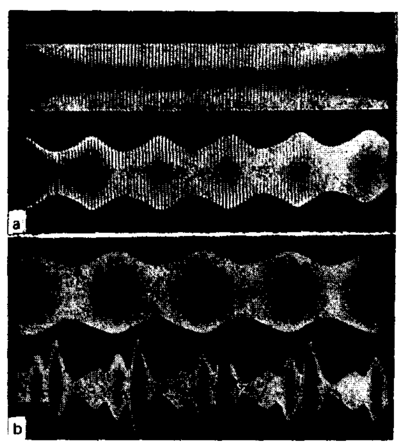

FIG. 41. Self-modulation of wave in nonlinear transmission line: a) appearance of modulation; b) evolution of sinusoidal modulation wave.

detail, we shall be obliged to return to the basic equation of the theory of modulated waves in nonlinear media—the nonlinear parabolic equation or nonlinear Schrödinger equation[73]:

$$\left(\frac{\partial a}{\partial t}+v\frac{\partial a}{\partial x}\right)-\frac{i}{2}\frac{d^2\omega}{dk^2}\frac{\partial^2 a}{\partial x^2}-\frac{i}{2k}\Delta_\perp a=i\varepsilon_n(|a|^2)a; \tag{3.4}$$

here a is the complex amplitude of the wave $\exp[-i(\omega t-\mathbf{kr})]$, $\mathbf{k}$ is its wave number, and ε_n characterizes the degree of nonlinearity of the medium; for light waves, for example, $\sqrt{\varepsilon_n}$ is a nonlinear increment to the refractive index. For the simpler case of plane waves we can write instead of (3.4)

$$\left(\frac{\partial a}{\partial t}+v\frac{\partial a}{\partial x}\right)-\frac{i}{2}\frac{d^2\omega}{dk^2}\frac{\partial^2 a}{\partial x^2}+i\alpha a|a|^2=0; \tag{3.5}$$

the terms in the parentheses describe modulation waves traveling at the group velocity in a linear medium without dispersion; the parabolic term $\sim d^2\omega/dk^2$ is responsible for dispersion spreading, and α for the magnitude and sign of the nonlinearity.[29)] As we shall presently see, modulation instability is possible only with a certain relation between the signs of the nonlinearity and the dispersion of group velocity: $\alpha d^2\omega/dk^2<0$.[75] The physical mechanism of this limitation (which is usually called Lighthill condition) is most easily understood by discussing self-modulation not in space-time language, i.e., not from an analysis of (3.5), but in spectral language, confining the analysis to the interaction of only three oscillator waves that form a wave with sinusoidal modulation.

Equations similar to (1.5) are derived from (3.5) for the complex amplitudes of the satellites $\omega_\pm$ and the carrier ω_0:

$$\dot{a}_0+i\frac{\alpha}{2}|a_0|^2a_0=0,\quad \dot{a}_\pm+i\left(\alpha|a_0|^2+\frac{1}{4}\frac{d^2\omega}{dk^2}k^2\right)a_\pm=-i\alpha a_0^2a_\mp^*; \tag{3.6}$$

here it has been recognized that because of the spectral proximity of the satellites, the detuning is given by $\delta=2\omega_0-\omega(k_0+k)-\omega(k_0-k)\approx(d^2\omega/dk^2)k^2$. The parametric increment γ with which the amplitude of the satellite increases in a given carrier field equals

$$\mathrm{Re}\,\gamma=\pm k\sqrt{-\frac{d^2\omega}{dk^2}\alpha|a_0|^2-\frac{k^2}{4}\left(\frac{d^2\omega}{dk^2}\right)^2}. \tag{3.7}$$

Since the spatial scale of the modulation may be arbitrary, a necessary (and sufficient as $k\to 0$) condition for modulation instability is $\alpha\omega_{kk}<0$. Its physical content is now also clear: for instability to appear, the nonlinear detuning $\sim\alpha|\alpha_0|^2$ must compensate the linear detuning $\sim(d^2\omega/dk^2)k^2$. The dependence of the increment on the modulation scale is shown in Fig. 42: for short-wave modulation, when $\Lambda^2\langle\pi^2/\alpha|a_0|^2)^{-1}d^2\omega/dk^2$, the nonlinear detuning is no longer capable of compensating the dispersion spreading and the modulation does not deepen (the increment becomes imaginary). The self-modulation effect was predicted in 1965[75] and was observed experimentally a year later for waves on the surface of a liquid.[11] Presumably,[76] this effect has a bearing on the explanation of the "seventh wave" phenomenon.

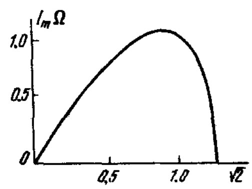

FIG. 42. Increment vs. modulation scale.

Let us turn to the analysis of the evolution of modulation waves within the framework of the linearized equation (3.4). For these waves we obtain the dispersion law

$$\Omega(k,k_\perp)=vk\pm\sqrt{\left(\frac{v}{2k_0}k_\perp^2+\frac{1}{2}\frac{d^2\omega}{dk^2}k^2\right)\left(2\alpha|a_0|^2+\frac{v_0}{2k_0}k_\perp^2+\frac{1}{2}\frac{d^2\omega}{dk^2}k^2\right)}, \tag{3.8}$$

which yields directly, specifically for one-dimensional waves $k_\perp\equiv 0$, the modulation-instability increment (3.7) that is already known to us. But what happens within the framework of our basic model (3.4) to small *multidimensional* disturbances? Assuming for simplicity in (3.8) that $k\equiv 0$, we find that for $k_\perp^2<4\alpha|a_0|^2k_0/v$ the quantity $\Omega(k_\perp)$ is purely imaginary—those multidimensional perturbations grow whose frequency is equal to the filling frequency! The physical manifestation of this is as follows. If a plane wave of frequency ω_0 is impressed on the boundary of a nonlinear medium whose dielectric permittivity increases with field strength, the wave is transformed to a periodic (in the transverse direction) system of beams as it propagates: it is *self-focused*.[77] This is a stationary spatial variant of parametric instability or decay of a pair of quanta in the same state $2\mathbf{k}_0\to\mathbf{k}_1+\mathbf{k}_2+\Delta\mathbf{k}(|a_0|^2)$ (Fig. 43).

e) Recovery

The nonlinear stage in the development of modulation instability depends on the asymptotic behavior of the initial disturbance as $|x|\to\infty$. If the perturbation is periodic in space, the sinusoidal modulation waves that build up as a result of modulation instability will undergo nonlinear distortion—one or more solitons will be formed in the period of the wave, but the solitons will then be smoothed and the wave will revert to its initial state, the whole process will then be repeated, and so forth. Modulation waves on the surface of a deep liquid behave in precisely this way (Fig. 44). A remarkable and astonishing phenomenon! And we would indeed be surprised if we had no "nonlinear experience" and had never observed something very similar: we recall Fig. 13, which illustrates the behavior of a periodic disturbance in a nonlinear network or a one-dimensional "medium." Exactly the same thing happens—the sinewave turns into a periodic sequence of solitons, a cnoidal wave, which then evolves back into a sinewave and so forth, i.e., a recovery effect is observed. The physical explanation of this similarity is easy. The nature of the nonlinear evolution is determined both for "waves without filling"—waves of the field itself—and

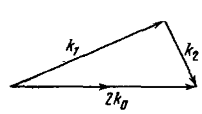

FIG. 43. Decay of pair of quanta in the same state.

[29)] The parabolic equation that describes beam diffraction was first derived by M. A. Leontovich in 1944.

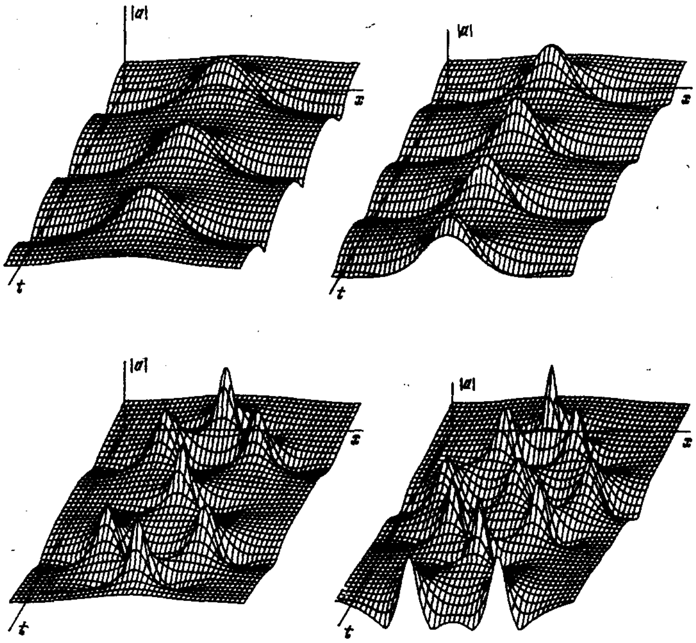

FIG. 44. Stationary waves and recovery for modulation waves on the surface of a deep liquid.[88]

for modulation waves by two competing effects—nonlinear contraction and dispersion spreading. The shape and other parameters of the periodic modulation wave must be specially adjusted if these effects are to offset each other exactly *throughout the entire space*. Such singular modulation waves do exist in the stationary modulation waves that were first investigated in 1966.[78] But this is an exception. For all other periodic disturbances, the "contraction" and "spreading" effects predominate by turns in the same way as kinetic energy becomes potential energy and vice versa in the oscillations of a pendulum. This is what determines the periodic evolution of a disturbance with periodic boundary conditions that develops as a result of modulation instability.

In its formal mathematical aspect, the effect of modulation-wave recovery in self-focusing media (or media with modulation instability) follows from the full integrability of the nonlinear Schrödinger equation with periodic boundary conditions.[79] In this case the nonlinear wave has a discrete spectrum (because of dispersion, higher-numbered harmonics can be regarded as nonresonant and hence the spectrum as limited), and a mode description can be used for more detailed understanding of the recovery mechanism. In the simplest case, modulation instability results in resonant interaction of only three modes—the carrier and satellites symmetric about it. The equations for their intensities A_0 and $A_1 = A_2$ bear a very close resemblance to the truncated equations of the spring pendulum [cf. (1.5)]:

$$\left.\begin{aligned} \dot{A}_0 &= 2A_0A_1 \sin\Phi, \\ \dot{A}_1 &= -A_0A_1 \sin\Phi, \\ \dot{\Phi} &= S + A_1 - A_0 + (2A_1 - A_0)\cos\Phi, \end{aligned}\right\} \tag{3.9}$$

$$\Phi = \left[\Delta\omega t + 2\arg\left(\frac{a_0}{a_1}\right)\right]\operatorname{sign}\alpha, \quad S = \operatorname{sign}\Delta\omega\cdot\alpha.$$

The phase portraits of the partially integrated system (3.9) in the variables $x = \sqrt{2A_0}\cos(\Phi/2)$, $Y = \sqrt{2A_0}\sin(\Phi/2)$ are shown in Fig. 45: almost all the motions are periodic, consistent with periodic energy exchange between the satellites and the carrier.

In the less trivial case in which many satellites grow simultaneously as a result of modulation instability, everything is essentially similar except that the shape of the nonlinear wave may be quite complex at the intermediate stage (see Fig. 46).

f) Radiosolitons

The analogy that we have established in the behavior of periodic field waves and modulation waves in nonlinear media can also be extended to nonperiodic waves, and to solitons in particular. As we shall soon see,

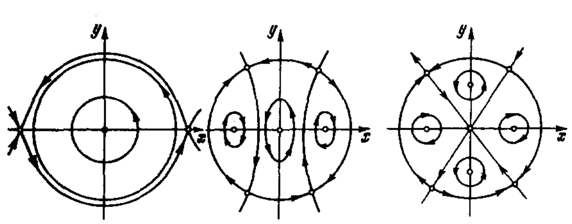

FIG. 45. Phase portraits of system (3.9).

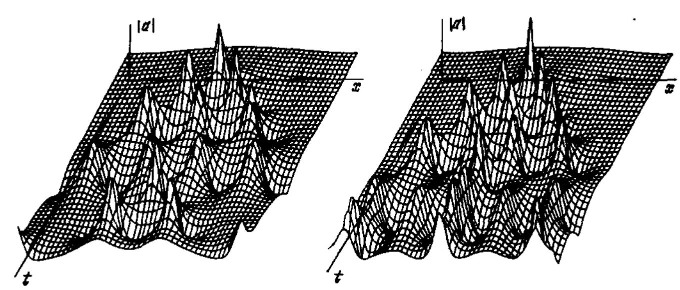

FIG. 46. Radiosolitons on deep water.

solitons of the nonlinear Schrödinger equation—*radiosolitons*—behave like videosolitons, including KDV soltions.[30)] Experiments with radiosolitons on deep water[80] indicate that their parameters do not change on colliding or overtaking, and there is only a jump in the phase of the filling. Radiosolitons are found to be stable formations within the framework of the one-dimensional theory.

However, most modulation solitions, like field solitons (see Ref. 81), are found to be unstable with respect to non-one-dimensional perturbations.

In particular, a waveguide channel of infinite length and a stationary packet with infinite front dimensions that has a finite wavelength in the direction of propagation are unstable. This can be made clear by quite graphic though not altogether rigorous energy considerations.[83] In (3.4) let $\varepsilon_n(|a|^2)a=\alpha|a|^2a$; then the field under consideration is characterized by an energy

$$H=\int dr\left[\frac{iv}{2}(a^*\nabla a-a\nabla a^*)+\frac{1}{2}\frac{d^2\omega}{dk^2}\left(\frac{\partial a}{\partial x}\right)^2+\frac{v}{2k}|\nabla_\perp a|^2+\alpha|a|^4\right]; \quad (3.10)$$

further, (3.4) has another integral $-N=\int|a|^2dr$, whose meaning is the number of quasiparticles (quanta) in the wave. Let the wave packet be characterized by the dimension l and the number of particles $N=\int|a|^2dr \approx (a)^2 l^m$, where m is the dimensionality of the packet. Then, recognizing that the number of particles in the packet is conserved, we have for its amplitude $a(t) \approx\sqrt{N}\,l(t)^{-m/2}$ and, for the energy, $H\approx(d^2\omega/dk^2)(N/l^2) -(\alpha N^2/l^m)$. Here the first term is responsible for diffraction *spreading* of the packet and the second term for its nonlinear *contraction*. This expression shows that a scale $l_0=(d^2\omega/dk^2)\,2/\alpha N$ for which the packet energy will be minimal $(\partial H/\partial l|_{l_0}=0)$ exists in the one-dimensional case, and we may hope that a soliton with these parameters will be stable. The behavior of the two-dimensional pulse $m=2$ will obviously depend on the initial conditions: if $\omega''>\alpha N\approx\alpha|a|^2/l^2$, then the energy minimum is reached as $l\to\infty$ and the pulse spreads; but if $\omega''<\alpha|a|^2/l^2$, the energy is minimal as $l\to 0$, and the soliton contracts to a point or *collapses*. The evolution of a three-dimensional soliton should also end in collapse: at $m=3$, nonlinear contraction predominates over diffraction spreading.

Bearing in mind the stability of radiosolitons in one-dimensional systems, it is natural to use such solitons as undisturbed (unmodulated) solutions for study of a broad range of models similar to the "standard" model—the nonlinear Schrödinger equation. Here we investigate the behavior of one of these models, namely:

$$a_t-\frac{i}{2}a_{xx}-|a|^2a=\gamma(a+a_{xx})-\rho|a|^2a, \quad (3.11)$$

which describes the nonlinear evolution of modulated waves in nonequilibrium media (the term $\sim\gamma$ describes the spectrally narrow increment of the waves and $\rho|a|^2a$ their nonlinear damping). In the approximation of small damping and a spectrally narrow increment, Eq. (3.11) has now been derived for Tollmien–Schlichting waves in a boundary layer, Langmuir waves excited by an electron beam, concentration waves of chemical reactions, etc. (see, for example, Ref. 84).

Like the standard conservative model, (3.11) has a solution in the form of an unmodulated harmonic wave with amplitude $a=\sqrt{\gamma/\rho}\exp(i\gamma t/\rho)$. With $\gamma\rho<1$, this wave is unstable to periodic disturbances with wave numbers $k<\sqrt{2\gamma(1-\gamma\rho)/\rho(1+\gamma^2)}$. At very small amplification and damping $(\gamma,\rho\ll 1)$, the development of this instability should result, as in the undisturbed model (for initial disturbances that decrease rapidly at infinity), in a steady-state solution in the form of a sequence of solitons

$$u(x,t)=\frac{A\exp\{(iA^2/2)t+iV^2(t/4)+V[(x-Vt)/2]\}}{\mathrm{ch}\,[A(x-Vt)/\sqrt{2}]}, \quad (3.12)$$

whose amplitude and velocity will now vary in time. Let us find equations for the parameters of such "modulated" radiosolitons. For this purpose, we determine the rate of change of the particle number $N=\int|a|^2dx$ and of the quasimomentum $P=\int i(a_x a^*-a_x^*a)dx$—the integrals of the conservative model—and then use (3.12) to find equations for $A(t)$ and $V(t)$:

$$\frac{dA}{dt}=2\gamma A\left(1-\frac{A^2}{6}-\frac{V^2}{4}\right)-4\rho\frac{A^3}{3},\qquad \frac{dV}{dt}=-2\gamma A^2\frac{V}{3}. \quad (3.13)$$

We see that as $t\to\infty$, all solitons stop and their amplitudes become equalized $A\to A_0=\sqrt{6/[1+(4\rho/\gamma)]}$. Thus, the initial periodic disturbance evolves into a lattice of modulation solitons.[31)]

If the amplification and damping are too small $(\gamma,\rho\sim 1)$, the initial perturbation evolves in a completely different manner: complex behavior arises as a result of the development of modulation instability.[85]

We have already encountered the fact that the appearance of simple or complex behavior in a dynamic system is related to the "distance" from the system to the nearest fully integrable system. For (3.11), this system is the nonlinear Schrödinger equation with periodic boundary conditions.[32)] In particular, at $\gamma\ll\rho\ll 1$, the distance between (3.5) and (3.11) is small and the behavior of (3.11) is simple, but as the distance increases $(\gamma,\rho\sim 1)$, numerical modeling[84] indicates that randomness arises in the system (3.11) with periodic boundary conditions.

This review of the modern theory of nonlinear oscillations and waves and its links to the creative output of L. I. Mandel'shtam does not, of course, pretend to completeness. Many aspects of this relation have not

30) Practically all the effects known in the theory of nonlinear "waves without filling" are also observed for modulation waves. In addition to those under discussion here, mention must also be made of simple and shock modulation waves.[82]

31) If the interaction of solitons via the exponentially falling tails is taken into account, the result of evolution will be ambiguous. In particular, it can be shown in the case $\gamma \ll\rho$ that Eq. (3.17) with periodic boundary conditions has a finite number of asymptotically stable periodic solutions. The periodic soliton lattice is the simplest.

32) The integrability of (3.5) with arbitrary boundary conditions has not yet been proven.

been touched upon or have been mentioned only in passing. However, we hope that the importance and productivity of Mandel'shtam's contribution to the science of oscillations and waves will be evident even from the material given above.

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