# From the equivalence principle to the equations of gravitation

V. P. Vizgin and Ya. A. Smorodinskiĭ

*Institute of the History of Natural Science and Technology, Academy of Sciences of the USSR and I. V. Kurchatov Institute of Atomic Energy*
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The main stages in the discovery by Einstein of the equations of the general theory of relativity in November 1915 are recounted. The evolution of his ideas from the first formulation of the equivalence principle in 1907 is followed. The part played by Hilbert in the finding of the final equations of the theory is discussed.

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*"During my long life, I have recognized one truth–our entire science appears primitive and undeveloped when compared with reality, but it is still the greatest treasure we possess . . ."*

*A. Einstein*

## PROLOGUE

To the ancient philosophers it was obvious that the celestial bodies cannot satisfy terrestrial laws. They executed circular motions in accordance with a higher harmony. Thus taught Aristotle, and thus was constructed the Ptolemaic system.

Newton was the first to decide otherwise. Discovering that the laws of fall of bodies on the Earth are the same as the ones that control the motion of planets, he opened up to scientists the limitless expanse of the Universe. But many years were still to pass before they could follow Newton. It was only in the 19th century that serious discussion about the nature of gravitational forces began, and even at the dawn of the present century little was known about these forces. Of course, it must not be thought that everything said by Einstein was completely new. Even Einstein's most radical idea of identifying a physical interaction (gravitation) with the geometry of space-time, which is the basis of the general theory of relativity, had its predecessors. We shall give here some striking remarks, which now appear prophetic. However, such prophetic anticipations were rare, and few heard them. In this sense, the path to the general theory of relativity was not a continuous logical thread; there were no tributaries merging at a

*Translator's note*. Because I do not have ready access to many of the original quotations from Einstein and other authors given in Russian in this paper, I have had to translate the already translated Russian in many cases.

  

confluence into a broad river. Einstein's work is to be seen rather as a waterfall encountered in a comparatively peaceful river.

But anticipations there were. Among the prophetic voices, Nikolai Lobachevskiĭ's in 1835 can be most clearly heard:

"In nature we actually recognize only motion, without which sense impressions are impossible. Thus, all other concepts, for example, geometrical, are created by our intellect artificially, being taken from the properties of motion; therefore, space, by itself, does not exist separately for us. There cannot therefore by any contradiction in our thinking if we suppose that some forces in nature follow one particular Geometry and others a different Geometry. And even if this is a pure assumption for which other convincing arguments must be found, there is no doubting the fact that forces produce everything: motion, velocity, time, mass, even distance and angles".[1] On June 10, 1854, Bernhard Riemann read his habilitation address at Göttingen University (it was published only in 1868 by Dedekind). The lecture was entitled "On the hypotheses which lie at the base of geometry". In this lecture, Riemann said that space must have some material base, something "real" and that if space is identified with a continuous manifold"... it is necessary to explain the occurrence of metrical relationships by something outside it, in binding forces which act upon it".[2] However, he ended his lecture with the words: "This carries us over into the sphere of another science, that of physics, into which the character and purpose of the present discussion will not allow us to enter."[2] In 1870, the Englishman William Clifford wrote: "I hold in fact (1) That small portions of space *are* in fact of a nature analogous to little hills on a surface which is on the average flat; namely, that the ordinary laws of geometry are not valid in them. (2) That this property of being curved or distorted is continually being passed on from one portion of space to another after the manner of a wave. (3) That this variation of the curvature of space is what really happens in that phenomenon which we call the *motion of matter*, whether ponderable or etherial. (4) That in the physical world nothing else takes place but this variation, subject (possibly) to the law of continuity."[3]

But none of these comments, in which we now find ancestral wisdom, bore fruit at their time. It was not possible to construct a theory of gravitation from them. It was necessary to add to them physical principles, the fundamental nature of which became clear to Einstein alone. Above all, it was necessary to have the requirements of the special theory of relativity and the equality of inertial and gravitational mass. The importance of this last was already recognized by Newton. Using a pendulum with weights of different materials, he proved this equality with an error of less than 0.1%. In 1828, Bessel increased the accuracy of Newton's experiments by 60 times. It was only after more than half a century that Eötvös reawakened interest in the problem of the equality of the two masses, the inertial and the gravitational. Following Newton and Bessel, he saw in this fact one of the most important laws of nature and reduced the error to $10^{-7}-10^{-8}$ (1889–1909).[1)] And although many understood the importance of the equality of the masses and did sometimes attempt to explain it, no one before Einstein attempted to take it as the basis of the theory of gravitation.

Another approach to the theory of gravitation also appeared possible. It is remarkable that as early as 1801 Soldner calculated the amount by which a ray of light passing close by the Sun must be deflected.[2)] Assuming that light is a material body,[3)] he concluded that the path of light must be a hyperbola (as in the case of a comet), and obtained the magnitude of the deflection from elementary theory. Inexplicably, as a result of a slip Soldner obtained the result $2\times 0.84''$. A calculation in accordance with the formulas of classical mechanics should have given $0.84''$, a value close to that obtained by Einstein in his first attempts to treat the influence of a gravitational field on the propagation of light and half the correct result obtained on the basis of the general theory of relativity and confirmed by experiment.

Laplace was the first to pose the problem of the velocity of propagation of gravitation. Within the framework of Newtonian theory, Laplace attempted to take into account retardation. As a result, he established a lower limit for the velocity of propagation of gravitation, but made a fatal mistake in that he introduced retardation and calculated the effect in the first order in the ratio of the velocity of a planet to the velocity of light (in fact, this is an effect of second order!). If there is to be no contradiction with observations, the velocity of propagation of gravitation must (according to Laplace) exceed the velocity of light by $10^8$ times!

On another occasion, the formulas were more favorably disposed to Laplace. Using an incorrect formula for the kinetic energy of light, he correctly calculated the gravitational radius and predicted the existence of black holes. "... the attractive force of a heavenly body could be so large, that light could not flow out of it", a quotation that has now become classical.[4] As this remark was made during the last years of the 18th century, Laplace's percipience is remarkable.[4)]

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[1)]In our time, Dicke's group has reduced the error to $10^{-11}$. The highest accuracy has been achieved by V. B. Braginskiĭ and V. I. Panov, who have reduced the error even further to $10^{-12}$. At this accuracy, one can already assert that even the weak forces do not violate the great principle. This principle is also satisfied by systems in which there is an appreciable gravitational energy (the Moon in the field of the Sun), gravitational energy making equal contributions to both masses in this case too.

[2)]The calculation was published in the astronomical yearly Berliner Astronomisches Jahrbuch; an English translation can be found in the paper of S. L. Jaki, Found. of Phys., **8**, 927 (1978). (cf. the even earlier investigations of Refs. 90 and 91.)

Soldner's work was used by Lenard in his attacks on Einstein. This story is discussed by Jaki.

[3)]"... It must not be thought that the objects which exist and act on our senses do not have the properties of matter".

[4)]Laplace's arguments disappeared in later publications (beginning with the third edition of his "Exposition of the System of the World").

This remark of Laplace was recalled only recently when the fantastic "black holes" came to be regarded as real (at least in modern theories).

The Newtonian theory of gravitation did not explain the mechanism of gravitation. In the 18th and 19th centuries not a few ether mechanisms were proposed for gravitation; these were based either on analogy with the mechanics of continuous media and explained the attraction by the pressure of a medium or by vibrational wave effects in a medium, or used Le Sage's model (ether-kinetic hypotheses) and explained the attraction by the screening of gravitating bodies "bombarded" by ether particles. None of these attempts, still less the attempts at an electrodynamic derivation or generalization of Newton's law, met with success or has left any visible trace in the modern theory of gravitation.[5]

While theory advanced erratically, contradictions began to accumulate in the observational material. They were not so drastic as, for example, in the case of the ultraviolet catastrophe, but the pedantic astronomers still spoke of persistent deviations from the predictions of Newtonian mechanics. In 1859, Laplace's student Leverrier published a communication stating that the motion of the planets does not agree with the calculations. The calculated motion of the perihelia of Mercury and Mars due to the influence of the other planets turns out to be smaller than the observed value by 38″ per century for Mercury and by 25″ (corrected value 7″) for Mars. Leverrier's own attempt to explain this anomalous advance by a new hypothetical planet Vulcan (whose orbit was to lie within Mercury's) did not lead to success. Just as unsuccessful were attempts to explain the anomalies of Mercury by a ring of asteroids between it and the Sun (Tisserand, 1891), a satellite of Mercury (Haerdtl, 1894), and other similar hypotheses based on hidden masses in the solar system. Various modifications of Newton's law based on the analogy with action-at-a-distance electrodynamics, or power-law and exponential modfications (see below) also failed to solve the problem.[5] Without going into the details of the history of celestial mechanics, let us merely say that by the end of the 19th century the need for more radical changes in our ideas about gravitational forces was obvious. The analysis of the motion of the planets made by Newcomb was also very important for Einstein. In 1926, Einstein wrote a letter to Newcomb's daughter (it was read at the unveiling of the memorial to her father in 1935): "Your father was the last of the great scientists who, considering this problem [the perturbations in the motions of the planets] calculated with great care the motions in the solar system. This problem is so grandiose that only a few could work independently and sufficiently critically on its solution".[6] The motion of the perihelion of Mercury was to be become one of the touchstones of the general theory of relativity.

But not only discrepancies between theory and observation in figures long after the decimal point in the formulas of celestial mechanics warned of difficulties; real major paradoxes were ripening. They lay in wait for anyone who attempted to turn from the solar system to an analysis of events on the scale of the Universe.

The first such danger was noted in 1874 by Neumann: "If one assumes that the stars extend infinitely in all directions, and that the mean density of this matter is constant, and if Newton's law is valid, then the force with which these stars act on our terrestrial globe is quite undetermined, and can have any direction and magnitude. Newton's law leads in such a case to an absurd result and is therefore invalid in this case."[7] This paradox was analyzed in detail by Seeliger (and for this reason it is usually named after him).

A resolution of the paradox was sought by Neumann and Seeliger in exponential modifications of Newton's law.

It should be said that Seeliger's paradox does not appear today so serious. For a nonstatic universe, one can obtain correct formulas using Newton's theory of gravitation,[5)] but at that time the universe was regarded as a static system of fixed stars, and it did not seem possible to eliminate the paradox.

A similar paradox was discovered even earlier by de Chéseaux (1744) and Olbers (1826). They showed that under the conditions of an infinite and static universe filled uniformly with stars that have existed for ever the energy flux must be infinite. In such a universe, the night sky cannot be dark, and the universe must be filled with radiation in thermal equilibrium with the stars, i.e., it must have a very high temperature. In fact, the universe does not have a hot background but only the cooled cosmic microwave background radiation at a temperature of 2.7 °K. To avoid the danger posed by Olbers's paradox, it is sufficient that the stars be of finite age and that the universe be subject to the laws that follow from Einstein's equations. Only these laws have led to a "cooling" of the sky that we observe. The darkness of the night is striking evidence for the power of these equations.

The successes of the Maxwellian theory of the electromagnetic field also suggested the possibility of a field approach to the theory of gravitation. In this way, one could hope to link gravitation to electromagnetism, understand the propagation of gravitation in space, and eliminate some of the difficulties mentioned above. Maxwell himself already thought of a theory of the gravitational field but concluded that the energy density of the field would be negative and he therefore abandoned the further development of this approach.

At the turn of the present century, Volterra attempted to develop a scalar field theory, and in 1900 Lorentz constructed an electromagnetic theory of the gravitational field, interpreted in the general case as a vector field; these theories also failed to solve the principal difficulties.

Thus, at the start of the present century the situation with regard to the theory of gravitation was clearly unsatisfactory, although not so critical as in the case of the electrodynamics of moving bodies or the theory of

[5)]This was pointed out by Ya. B. Zel'dovich.

radiation. It was felt that the empirical anomalies in celestial mechanics could be eliminated by a small correction to the inverse-square law. For this, it would be sufficient to replace the square by the power $2+1.6\cdot 10^{-7}$ (Hall and Newcomb). The discovery of the special theory of relativity not only posed the problem of reconciling it with Newton's theory of gravitation but also generated new hopes for the solution of the problem of gravitation. However, the development of Lorentz-covariant generalizations of Newton's law by Poincaré (1906) and Minkowski (1908), which did lead to the conclusion that gravitation propagates with the velocity of light, still left the main difficulties unsolved.

In 1907, when Einstein's published his paper in which he formulated for the first time the principle of equivalence and discussed on its basis the influence of the gravitational field on the propagation of light, he was still occupying the post of a technical expert of the third class at the patent office in Bern. Einstein's pioneering approach to the problem of gravitation associated with the idea of extending the restricted principle of relativity was not supported by other physicists, and Einstein himself encountered great difficulties in extending it to inhomogeneous gravitational fields. Moreover, in the period 1908–1910 he was intensively occupied with problems of quantum theory. It was only in 1911 that Einstein returned to a theory of gravitation based on the equivalence principle and predicted that the velocity of light should depend on the gravitational potential.

On the basis of this dependence, the German theoretician Abraham put forward in 1911–1912 a scalar theory of the gravitational field, which, however, is not consistent with the principle of relativity. In 1912–1913, the Finnish physicist Nordström and the German physicist Mie engaged actively in the development of gravitational theories. They developed Lorentz-covariant theories of the gravitational field that are completely satisfactory from the logical point of view but do not explain the anomalous advance of Mercury's perihelion and do not predict deflection of light in a gravitational field. Discussions with these physicists and consideration of their theories was of great importance for Einstein.

Leading German theoreticians such as Planck and von Laue, who enthusiatically supported the special theory of relativity, did not recognize the depth of Einstein's ideas leading to the extension of relativity theory, but they did not enter into an open polemic with him during these years (1911–1915). Planck indirectly made an important contribution to the development of the relativistic theory of gravitation. He was one of the first to pose the problem of the equality of inertial and gravitational mass in the light of the special theory of relativity. To him and Minkowski we owe the four-dimensional variational formulation of the principle of inertia, which, as we shall see, played a key part in forming the geometrical concept of gravitation that is the basis of the general theory of relativity. As early as 1899 Planck had also drawn attention to the fact that the constant of gravitation in conjunction with the velocity of light and the constant that bears his name gives rise to the so-called "natural system of units" (in which Boltzmann's constant is also included). The fundamental length $l_p=\sqrt{\hbar\gamma/c^3}$ introduced by him in this system[6] plays a fundamental part in the quantum theory of gravitation currently emerging. Nevertheless, Planck reacted to the ideas of Einstein that led to the general theory of relativity with great reserve. One cannot but see in this similarities with the position of Einstein himself with regard to quantum mechanics, one of the founders of which was the very same Einstein. The creations of the human mind not infrequently outgrow their creators!

Although Einstein acutely felt the lack of understanding and support of the majority of his colleagues, he was not completely alone. We must mention here especially three of his friends. There was the astronomer Freundlich, who actively prepared for astronomical observations with a view to obtaining experimental verification of the principle of equivalence and the general theory of relativity; then there was the mathematician and student friend of Einstein Marcel Grossmann, who helped him find and master the necessary mathematical formalism; and finally there was the remarkable physicist Ehrenfest, whose subtle and critical intellect made him an inestimable interlocutor for many physicists in the first third of the twentieth century.

In the spring of 1913, Einstein and Grossmann completed a paper in which the new theory takes an almost convincing form. This sketch of a tensor geometrical and generally covariant theory of gravitation did not yet contain the correct equations of the gravitational field. Erroneous arguments hindered the authors from correctly evaluating the possibilities of the Riemann-Christoffel and Ricci tensors for establishing the field equations, and they strayed from the true path in giving up the key requirement of general covariance of these equations. After his transfer from Zurich to Berlin, Einstein continued the work alone and was actually engaged in an argument with himself, thinking up new arguments only to refute them in a following paper. It was only in November 1915 that he returned to the point of departure—the requirement of general covariance of the field equations.

A few months before this, in the summer of 1915, Einstein had met at Göttingen the patriarch of mathematicians—Hilbert. Hilbert was not only well acquainted with the mathematical formalism of the theory of relativity but for several years had already been deeply interested in fundamental problems of physics. It is probable that in discussions with Hilbert Einstein went through his arguments and doubts, which could have helped him reject his own objections that were hindering him from the final triumph. For his part, Hilbert found in the theory of gravitation presented by Einstein (in July 1915, the physical part of the theory was almost completely clear) the desired field for his

[6]Of course, in 1899 Planck did not yet have the constant $h$ at his disposal and to define the system of units he used the constant from Wien's law, which in modern theory is equal to the ratio $h/k$.

ideas concerning the axiomatization of physics and, grasping the greatness of Einstein's ideas, joined in the search for the most rational way of obtaining the equations.

In November 1915, at a session of the Acadeny of Sciences at Berlin, Einstein presented his new final variant of the theory. During the same November at Göttingen Hilbert presented equations that he had obtained in a completely different way on the basis of Einstein's earlier papers and, possibly, on the basis of what Einstein had told him.

These communications in November 1915 of Einstein and Hilbert strikingly reveal two different styles of scientific investigation. Einstein believed in the harmony of the world, and he doggedly sought and found global physical laws in nature. Hilbert, the rigorous logician, just as determinedly created an axiomatic science.

As a result, the general theory of relativity is at once a deeply physical and also rigorous mathematical theory, redolent of beauty and greatness.

The establishment of the general covariant equations of gravitation ended the exhausting search made by Einstein for more than eight years (with some interruptions). The relativistic theory of gravitation, or the general theory of relativity, was basically his creation. Grossmann helped in the finding and development of the mathematical formalism of the theory. Hilbert convincingly demonstrated the great heuristic possibilities of the variational principle and endowed the theory with regal splendor and the perfection of mathematics. For many years, the general theory of relativity was to become the almost unattainable ideal for all future physical theories.

In the following sections, we shall follow in greater detail the process of construction of the general theory of relativity. Our main attention will be concentrated on the formation of the geometrical concept of gravitation (1911–1913) and the setting up of the generally covariant equations of the gravitational field (1913–1915).

## I. TOWARD A GEOMETRICAL CONCEPTION OF GRAVITATION

The history of the creation of the relativistic theory of gravitation can be divided into the following four stages:

1) Discovery of the principle of equivalence and the prediction on its basis of two gravitational-optical effects, 1907–1911.

2) Scalar theories, 1911–1912.

3) Development of the tensor geometrical concept of gravitation, 1912–1913.

4) Search for the equations of the gravitational field, 1913–1915.

In the present paper, we shall be concerned only in passing with the first two stages to the extent that they are important for understanding the subsequent development.

### 1. Equivalence principle and scalar theories

The principle of equivalence was discovered by Einstein when he attempted to introduce gravitation in the framework of the special theory of relativity and was then confronted with the remarkable fact of the equality of inertial and gravitational masses. Einstein interpreted this equality from the relativistic point of view as "... the complete physical identity of a [homogeneous][7)] gravitational field and a corresponding acceleration of the frame of reference (1907)".[8] On the basis of this principle, he immediately concluded that the rate of physical processes must be changed in a gravitational field; in particular, he predicted a shift in the frequency of the radiation of an atom on the surface of the Sun and, finally, concluded that the velocity of light should depend on the gravitational potential in accordance with the formula

$$c = c_0\left(1+\frac{\Phi}{c_0^2}\right),$$

where $\Phi$ is the Newtonian scalar potential. From this one can deduce directly a deflection of light in the gravitational field of the Sun equal to 0.85", i.e., a deflection equal to half of the correct amount calculated in 1915 on the basis of the general theory of relativity (with allowance for the curvature of space).

The difficulties in the way of extending the equivalence principle to inhomogeneous gravitational fields, which are associated with a need to go beyond Lorentz covariance and mean that the coordinates lose a direct metrical meaning, and also intensive investigations in quantum theory caused a delay of several years in the further development of the theory of gravitation. It was only in 1911 that Einstein returned to the problem of gravitation, which to no small degree may have been due to his recognition at that time of the possibility of an experimental verification of light deflection in the field of the Sun at the time of an eclipse.[9]

The equivalence principle actually provided the theory of only homogeneous fields. For this, Einstein used the concept of the Newtonian scalar potential. The first scalar theory of arbitrary fields was created by Abraham, who worked within the framework of the electromagnetic program and did not accept the special theory of relativity. He saw the downfall of the special theory of relativity in the dependence of the velocity of light on the potential that follows from the equivalence principle, and used this dependence to construct a scalar theory based on a generalization of Poisson's equation. He proposed a four-dimensional wave equation of the second order in which the density of rest mass, which is not a scalar, appears as source of the field on the right-hand side. Thus, Abraham's theory was not consistent with the principle of relativity and encountered difficulties when interpreted from the point of view of the equivalence principle.

Much more satisfactory was the Lorentz-covariant scalar theory of Nordström, the first variant of which

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[7)] *Translator's Note.* Explanatory interpolations of the Russian authors are given throughout in square brackets.

was developed by the Finnish theoretician in the fall of 1912. In the second variant of this theory, which was formulated by him in the summer of 1913 and perfected somewhat later by Einstein himself in collaboration with the Dutch physicist Fokker (see below), the source of the field is the trace of the energy-momentum tensor (which is a scalar!). In this theory, the four-dimensional Minkowski interval is multiplied by an arbitrary function of the space-time coordinates and is transformed into

$$ds^2 = \Phi(x, y, z, t)(dx^2 + dy^2 + dz^2 - c^2dt^2).$$

The equation of the light cone

$$ds^2 = 0$$

is unchanged, and in this theory there is of course no influence of the gravitational field on the propagation of light.[8)] Ultimately, it was necessary to reject for this reason both Nordström's theory, and also Mie's, which is essentially equivalent to the first variant of Nordström's theory.

However, in 1913–1915, when Einstein had already developed the tensor geometrical concept of gravitation, he regarded the scalar theory as a perfectly valid competitor, since it satisfied the four main requirements that in Einstein's opinion must be met by any rea-reasonable theory of the gravitational field. He formulated these requirements in the fall of 1913: "1. Fulfillment of the laws of conservation of momentum and energy. 2. Equality of the inertial and gravitational masses of closed systems. 3. Validity of the theory of relativity in a restricted sense, i.e., the system of equations must be covariant under linear orthogonal substitutions (generalized Lorentz transformations). 4. The observed laws of nature must not depend on the absolute values of the gravitational potential (or the gravitational potentials)".[10]

Einstein, stimulated partly by Abraham's work, made the next step along the path from the equivalence principle to the theory of arbitrary fields in the direction of a scalar theory. However, in contrast to Abraham, he allowed the scalar approach only for treating static fields. At the start of 1912, he formulated two variants of a scalar theory of the static field, which, although they did not satisfy Einstein (they were not completely consistent with the equivalence principle and concerned only static fields), nevertheless contained very important ideas that played an important role in the development of the tensor-geometrical theory of gravitation (which in what follows we shall call the geometrical theory).[11,12] Einstein's scalar theory included the idea of nonlinearity of the field equations and the idea of representing the equation of free motion of a material point in a gravitational field in the form of a variational principle for the four-dimensional interval (the geodesic principle).

[8)]There are two sources (each giving 0.85") of the deflection of light: the "mass" of light, which is an effect due to the special theory of relativity, and the curvature of space. In *the scalar theory, the two effects compensate each other*; in the tensor theory, they are added.

## 2. Preliminary remarks on the geometrical theory of gravitation

There are grounds for believing (see below) that already in the summer of 1912 Einstein recognized the inadequacy of the scalar approach and the need for using a tensor approach. The use of tensors, or rather symmetric tensors of the second rank, was suggested by two circumstances. First, adoption of the four-dimensional Minkowski approach for describing static fields led to a tensor $g_{ik}$ that determines the metric of space-time, a concept that arises from a generalization of the metric of the special theory of relativity. Second, in a relativisitic theory it would be more natural to regard as the source of the gravitational field the energy-momentum tensor $T_{ik}$, in contrast to scalar theories in which the tract $T$ of this tensor is the field source. In conjunction with the first consideration, this ultimately led to a Riemannian structure of space-time and to the Ricci tensor $G_{ik}$ as its main tensor characteristic of second rank. It was the tensor approach that could be more naturally reconciled with the requirements of relativity theory and the equivalence principle.

It is curious that the tensor theory was criticized by Abraham, who felt that the introduction of 10 "gravitational forces" unjustifiably complicates the theory. In fact, as we now know, the tensor approach is necessary in a consistent four-dimensional metric conception, and only such a theory includes the equivalence principle in a natural manner.

On his transfer from Prague to Zurich in October 1912, Einstein and the mathematician Marcel Grossmann, who was a Professor at the Zurich Polytechnic and had been a student friend of Einstein, intensively developed the tensor geometrical theory of gravitation. In a letter to Sommerfeld on October 29, 1912, Einstein wrote:

"I am now working exclusively on the problem of gravitation and hope, with the assistance of a companion here, a mathematician, to eliminate all difficulties. But I have never worked so hard in my life and I now gain great respect for mathematics, the subtleties of which, in my limitation, I previously regarded as a luxury. Compared with this problem, the original theory of relativity was child's play".[13]

The new theory was published in the joint paper of Einstein and Grossmann: "Outline of a generalized theory of relativity and theory of gravitation". The date on which the paper was received by the editor of the journal is not indicated,[14] but there are grounds for believing that the paper was finished in April or May of 1913 and submitted at that time.[9)]

[9)]This is confirmed by letters of Einstein to Laub on July 22, 1913 (Ref. 15) and to Mach on June 25, 1913 (Ref. 16). In the first he writes that "two months ago I came to terms" with the problem of gravitation. In the second he expresses the hope that Mach has already obtained his "new paper on relativity and gravitation", which "I have at last finished after unending labor and painful doubts". In addition, in a paper submitted to the journal on July 24, 1913, Nordström refers to the "Outline".[17]

## 3. Einstein and the genesis of the geometrical theory of gravitation

Einstein himself spoke approximately 20 years after the event about the development of the ideas forming the basis of the "Outline" and, thus, the geometrical concept of gravitation. In the Gibson lecture delivered at the University of Glasgow (1933), having described the essence of the problem associated with the coordinates losing a direct physical (metrical) meaning on the transition to accelerated systems, Einstein described the further development of his thought as follows: "... For a long time I could not understand what coordinates in physics could mean. The solution of this dilemma was found only in 1912, through the following argument. It was still necessary to find a new formulation of the law of inertia, which, in the absence of a true "gravitational field in an inertial coordinate system," would go over into the Galilean formulation of the principle of inertia. According to this last, a material point on which no forces act is represented in four-dimensional space by a straight line, i.e., by a shortest or, more precisely, extremal curve. This concept presupposes the existence of the length of a line element, i.e., a metric. In the special theory of relativity, as was shown by Minkowski, this metric is quasi-Euclidean, i.e., the square of the "length" $ds$ of the line element is a definite quadratic function of the coordinate differentials.

If one now introduces other coordinates by means of a nonlinear transformation, then $ds^2$ remains a homogeneous function of the coordinate differentials, but the coefficients of this function ($g_{\mu\nu}$) will not be constants but certain functions of the coordinates. Mathematically, this means that the physical (four-dimensional) space has a Riemannian metric. The time-like external curves of this metric determine the motion of a material point subject to no other forces than gravitational. The coefficients of this metric ($g_{\mu\nu}$) simultaneously describe the gravitational field in the chosen coordinate system. In this way, there was found a natural formulation of the equivalence principle, the extension of which to arbitrary gravitational fields appeared very natural.[10)]

Thus, the above dilemma was resolved as follows: it is not the coordinate differentials that have a real physical meaning but only the Riemannian metric corresponding to them. This laid the foundations of the general theory of relativity. However, two problems still remained unsolved.

1. If the field equations are expressed in terms of the special theory of relativity, how are they extended to the case of a Riemannian metric?

2. What are the differential equations that determine the Riemannian metric itself (i.e., $g_{\mu\nu}$)?

I worked on these problems from 1912 to 1914 in collaboration with my friend Marcel Grossmann. We found that mathematical methods for the solution of the first problem already existed in a ready form in the absolute differential calculus of Ricci and Levi-Civita. With regard to the second problem, for its solution we required differential equations of second order in the $g_{\mu\nu}$. We soon saw that these expressions were already provided by Riemann (the curvature tensor). Already two years before the publication of the general theory of relativity we studied the correct equations of the gravitational field, but we were not persuaded of their physical applicability. In contrast, I even supposed that they could not be confirmed experimentally. Moreover, it seemed to me, on the basis of very general considerations, that one could show that a law of gravitation invariant under arbitrary coordinate transformations is incompatible with the principle of causality. This error cost me two years of extremely hard work until I saw the error at the end of 1915 and found a connection between the theory and data of astronomical observations, after which I returned with repentance to the Riemannian curvature".[18]

The difficulties standing in the way of extending the equivalence principle to arbitrary gravitational fields (the loss by the coordinates of a direct physical meaning and the absence of indications how the Lorentz group should be extended) arose already in 1907 and, essentially, interrupted further advance. The way out of these difficulties was found by Einstein shortly before his transfer from Prague to Zurich (probably, in the summer of 1912). Of decisive importance was the analysis of "reducible" gravitational fields, i.e., fields that can be eliminated by a coordinate transformation, in the framework of four-dimensional Minkowskian geometry. The principal role then passed from the coordinates to the metric, and the principle of inertia obtained a simple geometrical formulation. The transition to uniformly accelerated frames of reference transformed the pseudo-Euclidean metric into a Riemannian ("reducible") metric: $ds^2=g_{ik}dx_idx_k$. In accordance with the equivalence principle, the coordinates of the metric tensor characterized not only space-time but also the gravitational field. The inertial motion of a material point was described as motion along a geodesic in a Riemannian space. This geometrization of "reducible" fields opened up the way to the construction of a theory of arbitrary gravitational fields: it was only necessary to give up the condition that the general Riemannian metric be reducible to a pseudo-Euclidean metric by a simple coordinate transformation. This led to a completely new view of gravitation and, with it, of the geometry of space-time, the new view being associated with the transition to Riemannian geometry and interpretation of its metric tensor as a gravitational potential. The special principle of relativity was generalized in such a way that the Lorentz group was replaced by the group of arbitrary con-

[10)]It is clear that not all $g_{\mu\nu}$ correspond to a true gravitational field. Here, Einstein is speaking of a field relative to a *chosen* coordinate system, which may be eliminated in a different system. (Footnote appended by the Russian authors.)

tinuous coordinate transformations.[11)]

According to Einstein's recollections, the ideas which provided the foundation of the general theory of relativity were developed before he moved to Zurich. The solution of the further problems associated with correct allowance for the influence of gravitation on other physical processes, the search for the differential equations for the $g_{\mu\nu}$ (i.e., the equations of the gravitational field) and so forth required the use of a completely new field of mathematics. With this stage there begins the joint work of Einstein and Grossmann, who was the author of a number of papers on differential and nonEuclidean geometry. Grossmann found the appropriate mathematical formalism: the absolute differential calculus of Ricci and Levi-Civita (tensor analysis in an $n$-dimensional Riemannian space). It was evidently this mathematics that "inspired great respect" in Einstein and, compared with this first outline of the general theory of relativity the original theory of relativity did indeed appear as "child's play".

## 4. Premises of the geometrical theory (Einstein's papers in 1912)

The history described in Einstein's recollections can also be followed in his papers. Essentially, these are the four papers written by Einstein between February and July 1912 during his stay in Prague: the first two papers (February[11] and March[12]) were devoted to scalar theories of the static field (published May 23); the third paper, in which he discussed Machian effects in these theories, was probably written in May and published in July[19]; the fourth, which is devoted to the polemic with Abraham and entitled "Answer to Abraham's comment", was received by the editor of the Annalen der Physik on July 4 and published on August 13 (Ref. 20).

The fundamental difficulties associated with the coordinates losing their direct physical meaning and the absence of a clear indication of the form of the extended transformation group are not mentioned explicitly in either the review of 1907 (Ref. 8) nor in the paper of 1911 (Ref. 9). The difficulties are indicated very clearly in the paper of Ref. 20: "...The equivalence principle opens up an interesting avenue—the equations of the theory of relativity including gravitation must also be invariant under transformations of acceleration (and rotation). However, the route to this aim appears very difficult to us. It can already be seen from the very special case of gravitation of masses at rest so far considered that the space-time coordinates lose their simple physical meaning and one cannot predict what form the general equations of space-time transformations should have".[21]

Not long before this Einstein's attitude to Minkowski's four-dimensional concept, which for several years he had regarded as mere formalism, changed. In 1910, he considered briefly for the first time the possibility of a four-dimensional representation of Lorentz transformations, emphasizing its "formal" nature.[22] In 1911, in his talk "The theory of relativity" at the meeting of the Society of Natural Scientists at Zurich he said: "Finally, a few words on the extremely interesting mathematical direction that the theory has been given principally by the mathematician Minkowski, who has unfortunately died so prematurely... . The further use of this formal equality [of the spatial coordinates and the time coordinate] has led to an extremely perspicuous exposition of the theory of relativity and greatly facilitated its applications. Physical events are represented in a four-dimensional world, and the space-time relationships between them are represented in this four-dimensional world by geometrical theorems".[23] It was the use of the four-dimensional approach to the theory of static fields, added by Einstein in the corrections of the proofs of the March paper, that was, after the equivalence principle, the *second* important step toward the geometrical interpretation of gravitation. In contrast to Abraham, Einstein did not use the four-dimensional technique in his papers on the scalar theories. A certain justification for this was the fact that he restricted himself to considering static fields and the conviction that the special theory of relativity loses its validity in the presence of gravitational fields. In addition, Einstein also allowed the possibility of deviation of the geometry of space from Euclidean geometry.[12)] The changed attitude to the four-dimensional formulation of special relativity and the realization that the four-dimensional approach is applicable in the presence of gravitation (stimulated by Abraham's papers) led to the discovery of the possibility of extending the four-dimensional formulation of the principle of inertia in relativistic mechanics to the case of motion of a material point in a static gravitational field. Important here were, of course, considerations associated with the correspondence principle that Einstein used effectively in the construction of new theories. He noted that the four-dimensional variational formulation of the equations of motion of a free particle in special relativity, found as early as 1906 by Planck,[25]

$$\delta \int ds = 0, \tag{1}$$

(where $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ is the four-dimensional interval in special relativity) also remains valid

[11)]The path that led to Einstein's equations is very different from the path taken in the development of quantum field theories, which led physics to generalizations of Maxwell's equations. If the searches for equations of gravitation by Lorentz, Abraham and possible successors had led them to wave equations with spin 2, they would have led to correct results. However, this was recognized only in the work of Thirring (1961), Feynman (1963), and Ogievetskiĭ and Polubarinov (1963), which was based on the quantum theory of relativistic fields. But these theories, which have no geometrical content, cannot lead to cosmological consequences (at least, not naturally). Thus, Einstein's path is still unique.

[12)]He pointed out just such a possibility in the February paper: "Thus, for example, it is very probable that they [i.e., the Euclidean relations] do not hold in a uniformly rotating frame, in which, because of the Lorentz contraction, the ratio of the circumference to the diameter must differ from $\pi$ when our definition of length is used".[24]

in the theory of static fields if the velocity of light is regarded as a function of the coordinates: $c=c(x, y, z)$. Thus, Einstein actually arrived at the conclusion that the four-dimensional interval, or the metric of space-time, in a static gravitational field has the form[13)]

$$ds^2 = c^2 (x, y, z)\, dt^2 - dx^2 - dy^2 - dz^2. \tag{2}$$

The fundamental nature of the concept of the interval in the special theory of relativity was demonstrated by Minkowski, Planck, von Laue, and others.[27] It now became essential in the relativistic theory of gravitation as well. This also indicated the way to the resolution of the difficulty associated with the coordinates losing their direct physical meaning: it is the metric and not the coordinates themselves that must acquire a real physical meaning.

In principle, the metric (2) had already implied a curved space with nonzero curvature and Ricci tensors[27]:

$$R^i_{4j4} = -c\frac{\partial^2 c}{\partial x_i\, \partial x_j} \qquad (i, j = 1, 2, 3),$$

$$R_{ij} = \frac{1}{c}\frac{\partial^2 c}{\partial x_i\, \partial x_j}, \quad R_{i4} = 0, \quad R_{44} = -c\nabla^2 c.$$

However, in the presence of only a linear dependence of the velocity of light on the coordinates, i.e., in the case of "reducible" (homogeneous) fields, the curvature tensor vanishes and space-time remains flat.

However, at this stage, Einstein was not yet using the formalism of the theory of curvature. He restricted himself to the remark: "The Hamiltonian equation written down at the end [i.e., Eq. (1) with the corresponding interval (2)] gives an indication of how the equations of motion of a material point in a dynamical gravitational field must be constructed".[28] It followed that the equations expressing the principle of inertia in the presence of gravitation can be interpreted by means of the equations of geodesics in a four-dimensional space with generalized (differing from pseudo-Euclidean) metric (2). As the next step, one could consider the Riemannian metric

$$ds^2 = g_{ik}\, dx_i\, dx_k \tag{3}$$

and the equations of geodesics in Riemannian space. However, in the spring of 1912 it was not yet clear to Einstein how this could be reconciled with the scalar nature of the field potential. But already in the July polemical note there is a striking passage that reveals Einstein's awareness that neither the scalar nor the vector approach to the problem of gravitation could succeed: "If the gravitational field can be interpreted in the sense of our present relativity theory, this can be done in only two ways. The vector of the gravitational field can be represented either as a four-vector [scalar potential] or a six-vector [vector potential]. In each of these cases, one obtains transformation formulas for the transition to a uniformly and rectilinearly moving frame of reference. Using these formulas and the formulas for transforming the ponderomotive forces one can find the forces that in both cases act on a material point moving in a static gravitational field. However, one then obtains results that contradict the consequences drawn above on the basis of the gravitational mass of energy [i.e., from the equivalence principle]. Thus, a gravitational field vector cannot apparently be introduced without contradictions into the scheme of the present theory of relativity".[29]

This conclusion could have led Einstein to the idea of using a tensor potential had he remained within the framework of the special theory of relativity. But Einstein saw a way out in extending the relativity principle on the basis of the equivalence principle, leaving open the question of the nature of the potential. Still, the conviction that the scalar and vector approaches and no prospects in conjunction with the understanding of the part played by non-Euclidean Riemannian geometry led right up to the tensor geometrical concept of gravitation.

Although Einstein's papers published in 1912 and directly preceding the "Outline" did not contain explicit references to the Riemannian structure of space-time, the identification of the metric tensor with the gravitational potential, and so forth, we see clearly the roots from which the composite parts of the future theory were to develop.

A further argument in favor of the correctness of the chosen path associated with the equivalence principle and extension of the relativity principle was seen by Einstein in the fact that in this manner (in the framework of the scalar theory of static fields) the Machian idea that the inertial mass of a material point should depend on the masses surrounding it finds confirmation. This idea was the subject of the paper of Ref. 19, which was completed in May and appeared in July. The result obtained in this paper is that the "presence of the shell $K$ of inertial mass ($M$) increases the inertial mass ($m$) of the material point $P$ within it" in accordance with the formula

$$m' = m + \frac{kmM}{Rc_0^2},$$

where $R$ is the radius of the shell, $k$ is the gravitational constant, and $c_0$ is the velocity of light in vacuum. Einstein comments further that "This suggests that the inertia of a material point is *entirely* due to the influence of all other masses through an interaction of some kind."[30] For the first time, he here refers to Mach: "This completely coincides with the point of view put forward by Mach in his ingenious investigations of this problem". He hoped to realize this idea fully in a theory of arbitrary fields. However, such an idea now appears incomprehensible. In the expression written down above, $m'$ depends explicitly on the potential, which should not lead (on account of the equivalence principle) to observable effects.[14)]

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[13)]By a transformation of the time $c_0 dt' = c(x, y, z)dt$ ($c_0$=const) that is different at different points of space the interval is reduced to the Minkowski interval with different clock rates at different points.

[14)]Mach's idea that there is a connection between the inertia of bodies and the action of the masses of distant stars ("Mach's principle") appeared promising to Einstein. He discussed this idea in connection with closed models of the universe. It has now become clear that more realistic models do not satisfy Mach's principle, which Einstein himself was forced to abandon.

## 5. Einstein in Prague—scientific encounters

Einstein's Prague publications and his recollections about the Prague period of his life convincingly demonstrate the key role of this period in the development of the tensor-geometrical theory of gravitation. Comparison of this material with the biographical literature, the recollections about Einstein, and the correspondence belonging to this period indicates that Einstein's various scientific contacts with other scientists were extremely fruitful. We must mention here above all the mathematician G. Pick, the physicist Ehrenfest, and the astronomer Freundlich. Finally, it was precisely in this period that his discussion with Abraham developed most intensively. Einstein even earlier had felt a great influence of Mach,[15)] but in the German University in Prague, of which the first rector had been Mach and at which his students were working (A. Lampa, G. Pick, and others), this influence of Mach's ideas obtained a new momentum. At the beginning of his stay in Prague, Einstein became acquainted with Ehrenfest. The acquaintance was through correspondence,[16)] a correspondence beginning between them in April (Einstein's first letter to Ehrenfest was dated April 12, 1911).[32] One of the question that they seem to have discussed in the correspondence was the so-called "Ehrenfest paradox". This paradox, which played a large part in the development of the relativistic theory of rigid bodies, was described by Ehrenfest in 1909 in a short note, which, on the basis of the special theory of relativity, proved the impossibility of setting an absolutely rigid disk into uniform rotation about its central axis.[33] Assuming the possibility of such rotation, Ehrenfest arrived at his paradox, which takes the form that, through the Lorentz contraction, the length of the circumference decreases whereas the radius remains unchanged. As it happens, in May 1911 Einstein submitted to the Physikalische Zeitschrift a short paper in which, referring to Ehrenfest's thought experiment, he proves the reality of the Lorentz contraction.[34] In accordance with the equivalence principle, uniformly accelerated rectilinear motion of a frame of reference is equivalent to a homogeneous gravitational field; one could attempt to associate with a more complicated field uniform rotation, which, in accordance with Ehrenfest's thought experiment, would lead to violation of the Euclidean relationships. Thus arose the idea of curvature of space in the presence of gravitation. Indeed, in the February paper on the theory of a static field there is a reference to "Ehrenfest's disk (or cylinder)". It is also pointed out there that the use of absolutely rigid rods in accelerated frames of reference could lead to non-Euclidean relationships.[24] Subsequently, Einstein frequently used Ehrenfest's thought experiment with the rotating disk to demonstrate how non-Euclidean relationships can be generated in noninertial frames of reference.

There is no doubt that Einstein did not fortuitously turn his attention to Ehrenfest's paradox, which he related organically to the difficulties encountered in transforming the equivalence principle into a consistent theory of arbitrary gravitational fields.

In the above papers on the scalar theory of static fields, Einstein used so-called "pocket" measuring instruments (i.e., instruments that could be carried along with an observer). This term was also due to Ehrenfest. The introduction of this concept indicates that Einstein recognized the need for the development of "infinitesimal thinking" but initially attempted to develop it on a physical, operational-measuring basis, whereas the problem required solution of the equations of four-dimensional Riemannian differential geometry.

We should comment briefly on the contacts between Einstein and the astronomer Freundlich and the mathematician Pick. Freundlich had heard of Einstein's predictions of deflection of light and of Einstein's desire to make contact with astronomers. The contact soon took place. The correspondence of Einstein with Freundlich reveals the great importance that Einstein attached to experimental verification of consequences of the equivalence principle.[36] Einstein's conviction of the possibility of such verification of the theory by astronomical observations, which was energetically supported by Freundlich, was undoubtedly an important stimulus for the further development of the theory.

It is known from the biographical literature that one of Einstein's closest friends in Prague was the mathematician G. Pick. According to the evidence of P. Frank, who succeeded Einstein in the department of theoretical physics after his departure for Zurich and who knew Pick well, it was Pick who drew Einstein's attention to "absolute differential calculus" and Riemannian geometry as the most appropriate mathematical formalism for constructing a relativistic theory of gravitation.[37] However, Einstein never mentioned Pick in connection with the mathematical formalism of the general theory of relativity.

Among Einstein's correspondents during the Prague period, we can mention the eminent Polish physicist M. Smoluchowski, in a letter to whom dated March 24, 1912 he wrote, in particular: "However, I have not yet

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[15)] In a letter to Mach in August 1909 he wrote: "Of course, I know well your main works, among which I particularly value your book on mechanics".[31] In this book, Einstein was greatly impressed by the criticism of the basic concepts of Newtonian mechanics, i.e., "absolute space" and "absolute time". The understanding of the relative nature of acceleration and the very possibility of a different viewpoint with regard to established concepts must have been close to Einstein. However, Mach's philosophical ideas and his interpretation of natural science as "analysis of sensations" could not be reconciled with the development of the theory of quanta and the special theory of relativity. For Einstein, they rapidly lost all value.

[16)] Einstein and Ehrenfest became personally acquainted in January 1912. Shortly after meeting Einstein, Ehrenfest wrote to A. F. Ioffe: "...I was with Einstein—Einstein is absolutely unique. I was simply overwhelmed by the inexhaustibility of his ideas, on the one hand, and the absolute accuracy and asceticism (!!) of his thinking, on the other! In addition, he is extremely simple, full of the joy of living, healthily natural, and very acute, —he is exceptionally sincere and musically gifted—".[35]

succeeded in finding the dynamical laws of the gravitational field. A simple scheme of four dimensions on an equal footing in the form used by Minkowski is here invalid".[38] This letter confirms once more that from the spring of 1912 Einstein was persistently seeking a way of modifying Minkowski's four-dimensional concept with a view to applying it to the theory of arbitrary gravitational fields. In Prague, Einstein entered a world in which his interests met an active response, and his mathematical accoutrement was probably significantly extended, and the part played by mathematics in his work became more significant.[39] By the fall of 1912, the problem of gravitation appeared to him in a different aspect.

## 6. Einstein's third letter to Mach

From October 1912 to April–May 1913, after the transfer to Zurich, Einstein, in contact with Grossmann, who helped him master the new field of mathematics, worked persistently on the foundations of the geometrical theory, the first published exposition of which was the "Outline".[14] However, there are good grounds for assuming that there is a much earlier albeit very brief exposition of the main ideas of the theory given by Einstein and contained in the third of the four letters which he is known to have written to Mach. Although the letter is not dated, it was evidently written on the eve of the New Year (1913), since it ends with New Year greetings, and the contents of the letter rule out other possibilities with a high degree of probability. If this dating is accepted, the letter contains the first known outline of the geometrical theory of gravitation: "I am very grateful for the friendly interest you show in the new theory [Mach's letters to Einstein have not survived, and one can only assume that Einstein sent him reprints of his papers on the theory of a static field]. The mathematical difficulties that one encounters in following up these ideas are unfortunately very difficult for me too. I am greatly pleased that the development of the theory reveals clearly the depth and importance of your investigations into the foundations of classical mechanics. I still cannot understand how Planck, whom I otherwise have learnt to value more than almost anyone else, could have shown so little understanding of your aims.[17)] I may say he also does not accept my new theory.

I cannot reproach him for that; for so far all that I can bring forward in favor of my new theory is the following epistemological argument. For me, it is absurd to ascribe physical properties to "space". The totality of masses generates a $G_{\mu\nu}$ field (the gravitational field), which, in its turn, controls the unfolding of all processes, including the propagation of light rays and the behavior of rods and clocks. Initially, events are referred to four absolutely arbitrary space-time variables. If the conservation laws for momentum and energy are to be satisfied, these must then be specialized in such a manner that only (and all) linear transformations lead from one allowed frame of reference to another. The frame of reference is, as it were, adapted to the existing world by means of the energy theorem and loses its nebulous *a priori* existence.

I shall shortly send you some accounts of the matter in which the formal aspect is kept as short as possible and the factual aspect is developed as far as possible. But I do not succeed fully in these abstract matters in making a complete separation of the essence of the matter from the form.

With best wishes for the New Year

Very respectfully, yours

A. Einstein".[18)]

Thus, if the proposal about the dating of this letter to December 1912 or January 1913 is correct, Einstein and Grossmann had by the beginning of 1913 already advanced far in the creation of a geometrical theory of gravitation based on the principle of general covariance. However, they had not apparently succeeded in giving a generally covariant formulation to the law of conservation of energy and momentum, and they had come to the conclusion that the linearly covariant form of this law restricts the class of allowed transformations to linear transformations. Judging from the letter, this solution did not satisfy Einstein, since it led to difficulties in the search for the equations of the gravitational field. It may be assumed that difficulties of both physical and mathematical nature delayed the publication of the paper.

## 7. "How did Einstein do this?"[19)]

But let us return to the "Outline". The physical part of the "Outline", which was written by Einstein, begins as follows: "The theory presented here arose from the conviction that the proportionality between inertial and gravitational mass is an exact law of nature that must be reflected already in the very foundations of theoretical physics".[14] Einstein passes from the equality of the masses to the equivalence principle. The next link in the chain of arguments, which Einstein had already found in the spring of 1912, is the four-dimensional formulation of the equations of motion of a material point in a static field, this having the same form as the equations of motion of a free material point in the special theory of relativity but with a coordinate-dependent velocity of light:

$$\delta\int ds=\delta\int\sqrt{-dx^2-dy^2-dz^2+c^2\,dt^2}=0, \tag{4}$$

where $c=c(x,y,z)$. The generalization of the relativity principle that follows from this entails the existence of

[17)] In 1908–1910, Planck violently attacked Mach's philosophical views but gave him his due in drawing the attention of natural scientists to the operational—measurement aspects of scientific theory. However, Planck did not appreciate Mach's ideas about the relativity of motion as developed in Mach's Mechanics, in contrast to Einstein.

[18)] This letter was published for the first time in 1963.[40] A Russian translation, which contains some inaccuracies, is given in the book of Ref. 41.

[19)] This is the title of a section of a paper by R. Dicke which briefly describes the evolution of Einstein's thought on the path to the general theory of relativity.

a class of allowed coordinate transformations broader than the Lorentz group, that leave Eq. (4) invariant in the presence of a static field. The interval $ds^2$ takes the form

$$ds^2 = \sum g_{ik}\, dx_i\, dx_k,$$

where $g_{ik} = g_{ik}(x, y, z, t)$.

In the special theory of relativity, the interval has the form

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2,$$

and in the case of a static field the velocity of light $c = c(x, y, z)$ is a measure of the gravitational potential.

Thus, "on the basis of the part played by $ds$ in the law of motion of a material point," Einstein summarizes, "the invariant $ds$ must be an absolute invariant".[44] This approach is consistent with Einstein's aim to achieve a generalization of the relativity principle that would in no way restrict the class of admissible frames of reference. We may mention that in this he saw a realization of Mach's ideas aimed at eliminating space-time absolutes. This kind of thought led directly to the principle of general covariance: "In the ordinary theory of relativity, only linear orthogonal transformations are allowed. We shall show that to describe the influence of a gravitational field on material processes it is necessary to write down equations covariant under arbitrary transformations".[44] It followed from the invariance of $ds$ that the gravitational potential $g_{ik}$ is a covariant tensor of second rank. This justified Einstein's assertion from the summer of 1912 that scalar and vector potentials are unsatisfactory. Simultaneously, one of the main problems—the loss by the coordinates of their direct physical meaning—was solved. Since the metric $ds$ now took over the main physical role, and was understood as "an invariant measure for the distance between two neighboring space-time points", the distance corresponding to given differentials "could be measured only if the $g_{\mu\nu}$ that determine the gravitational field are known". This meant that the "the gravitational field influences measuring rods and clocks in a completely definite manner". In other words, the geometry of space-time, like the gravitational field, is determined by the tensor $g_{\mu\nu}$ and acquired a Riemannian structure. The idea of a non-Euclidean space, which Einstein had been mulling over at least since the spring of 1912 also acquired a precise expression. The non-Euclidean nature of the geometry was of an infinitesimal nature. Einstein especially emphasized the local validity of the special theory of relativity and the possibility of using "pocket" rigid rods and clocks.

Thus, the part played by the interval $ds$ in the emerging theory and in its mathematical formalism was established; the properties of $ds$ are described in the "Mathematical Part" of the "Outline". "The mathematical formalism for constructing the vector analysis of a gravitational field characterized by the invariant element of length

$$ds^2 = \sum g_{\mu\nu}\, dx_\mu\, dx_\nu,$$

—which is how Grossmann begins his exposition—is essentially contained in the fundamental paper of Christoffel on the transformation of quadratic differential forms. Using Christoffel's results, Ricci and Levi-Civita developed their method of absolute, i.e., independent of a coordinate system, differential calculus, which makes it possible to cast the differential equations of mathematical physics in an invariant form".[43]

This formalism was then used by Einstein in the "Physical Part" to derive the energy—momentum conservation law (in differential form) of a material system characterized by an energy—momentum tensor in the presence of a gravitational field. The study of the conservation law begins with the case of "continuously distributed uncoupled masses", and it is then asserted that the obtained result

$$\sum_{\mu\nu} \frac{\partial}{\partial x_\nu} (\sqrt{-g}\, g_{\sigma\mu}\theta_{\mu\nu}) - \frac{1}{2} \sum_{\mu\nu} \sqrt{-g}\, \frac{\partial g_{\mu\nu}}{\partial x_\sigma}\, \theta_{\mu\nu} = 0 \tag{5}$$

is also valid for arbitrary material systems with energy—momentum tensor $\theta_{\mu\nu}$. The proof of the general covariance of (5) is based on a representation of the left-hand side of (5) by means of the covariant divergence of the tensor $\theta_{\mu\nu}$. Einstein comments that the second term on the left-hand side "expresses the influence of the gravitational field on the material process". In Sec. 6, he shows further that the equations of all physical processes that take place in the gravitational field can be obtained by the generally covariant reformulation of the corresponding Lorentz-covariant equations. In this manner, one can take into account the influence of the gravitational field on these physical processes. Simultaneously, one obtains the possibility of representing them in generally covariant form. Einstein realizes this procedure by taking the example of Maxwell's equations, in connection with which he also notes the work of the Viennese theoretician Kottler, who is also referred to by Grossmann in the "Mathematical Part" as one of those who used the calculus of Ricci and Levi-Civita in physics before Einstein.[20)]

## II. GENERALLY COVARIANT EQUATIONS OF GRAVITATION. EINSTEIN'S PATH

The problem of the equations of the gravitational field now became the central problem. An appreciable fraction of both parts of the "Outline" is devoted to precisely this problem. It was only after two and a half

[20)] Kottler's dissertation "Über die Raumzeitlinien der minkowskischen Welt" was presented at the session of the Viennese Academy of Sciences on July 4, 1912 and published in October of that year.[45] The main idea of the paper is to apply to the problems of electron theory the methods of the theory of integral forms of Goursat and the theory of invariants of differential quadratic forms of Ricci and Levi-Civita. The possibility of using these methods was based on the four-dimensional formalism of Minkowski. In particular, Kottler gave a generally covariant formulation of Maxwell's equations. He did not consider the problem of gravitation and did not relate this formulation to the geometry of space-time. Kottler also referred to a paper of the English theoretician Bateman, who already in 1910 attempted to use the theory of integral forms and the calculus of Ricci and Levi-Civita.[46] In this paper, Bateman actually established the conformal invariance of Maxwell's equations.

years that Einstein after "unending labor and painful doubt" was able to find the correct solution to the problem of the field equations.

## 1. "Two years before the publication of the general theory of relativity we had already studied the correct equations of the gravitational field. . ."

Thus wrote Einstein in his recollections in 1938.[18] Indeed, the modern reader, studying the "Outline", cannot but be surprised by seeing how close Einstein and Grossmann were to the correct solution of the problem of the field equations.

On the basis of the correspondence principle, Einstein posed the problem of a generally covariant and tensor generalization of the Poisson equation for the scalar potential:

$$\Delta\varphi = 4\pi\varkappa\rho,$$

where $\rho$ is the "matter" density, and $\varkappa$ is the gravitational constant. In the most general form, the first, as yet devoid of concrete content equations are written down:

$$\Gamma_{\mu\nu} = \varkappa\theta_{\mu\nu}, \tag{6}$$

where $\theta_{\mu\nu}$ is the energy—momentum tensor of the "matter", which in a relativistic approach must replace the noninvariant scalar "matter" density $\rho$, and $\Gamma_{\mu\nu}$ is some as yet unknown generally covariant tensor of the second rank that generalizes the Laplacian and is therefore composed of derivatives of the potential $g_{\mu\nu}$ up to second order. The rank of the tensor $\Gamma_{\mu\nu}$ is determined by the rank of the energy—momentum tensor $\theta_{\mu\nu}$. The requirement of general covariance of Eq. (6) is determined by the general principle of relativity, while the actual form of these equations is determined by the correspondence principle (in the limit of weak fields and low velocities, Eqs. (6) must go over into Poisson's equation). In a space free of "matter", the required equations take the form

$$\Gamma_{\mu\nu} = 0,$$

i.e., they are completely analogous to the Laplace equation. The problem of the tensor $\Gamma_{\mu\nu}$ reduces to finding a nontrivial generally covariant tensor of second rank composed of the derivatives of the $g_{\mu\nu}$ with respect to the coordinates up to the second order that is simultaneously a sufficiently general characteristic of curved space.

The "absolute differential calculus" had in its arsenal a tensor of the required form, namely, the Ricci tensor $G_{\mu\nu}$, which is the contraction of the Riemann-Christoffel tensor.[21)] In the "Mathematical Part" of the "Outline", Grossmann wrote[47]: "These generalized differential tensors may also be helpful for forming the differential equations of the gravitational field. Indeed, one can immediately point out the covariant tensor $G_{im}$ of second rank and second order that could occur in these equations, namely

$$G_{im} = \sum_{kl} \gamma_{kl}\,(ik,\,lm) = \sum_{k} \{ik,\,km\}."$$

Here, $\gamma_{\mu\nu}$ is the contravariant metric tensor (which is now denoted by $g^{\mu\nu}$), and $(ik, lm) \equiv G_{iklm}$ is the Riemann-Christoffel tensor. Thus, on the basis of arguments deduced from the requirement of covariance and the correspondence principle, one would then have expected Einstein and Grossmann to write down the equations

$$G_{im} = -\varkappa T_{im}, \tag{7}$$

$$G_{im} = 0, \tag{8}$$

(the latter for matter-free space-time), which, as is well known, are the corresponding field equations characteristic of the general theory of relativity. Equations (7) differ from the correct equations by the absence of the term $+(\varkappa/2)g_{im}T$ on the right-hand side (or the term $-(1/2)g_{im}G$ on the left-hand side). In empty space, both scalars ($T$ and $G$) vanish. Therefore, Eqs. (8) are correct. The term containing one of these scalars could be found on the basis of arguments associated with the energy-momentum conservation law (as was done by Einstein in November 1915).

## 2. Abandonment of general covariance of the field equations

However, the authors of the "Outline" did not adopt the generally covariant equations (7)–(8) as the equations of the gravitational field. Why did they reject such a natural choice? The prime reason for this fatal rejection was the circumstance that, as Grossmann noted, "in the special case of an infinitely weak static gravitational field, this tensor does not reduce to $\Delta\varphi$".[47] In other words, the generally covariant field equations could not be reconciled (as they thought) with the correspondence principle.

The reason for this can be seen by substituting in the expression for the Ricci tensor $G_{ik}$ the value for the metric tensor $g_{ik}$ corresponding to a weak field:

$$g_{ik} = \delta_{ik} + \varepsilon h_{ik}, \tag{9}$$

where $\delta_{ik}$ is the metric tensor of flat space-time, $h_{ik}$ is an arbitrary symmetric tensor of second rank, and $\varepsilon$ is an infinitesimally small parameter such that terms containing $\varepsilon^2$ can be ignored. The result of this substitution is

$$G_{ik} = \frac{\varepsilon}{2}\left[\frac{\partial^2 h_{ik}}{\partial x_j\,\partial x_l}\,\delta^{jl} - \frac{\partial}{\partial x_k}\left(\frac{\partial h_{ij}}{\partial x_j} - \frac{1}{2}\,\frac{\partial h_{jj}}{\partial x_i}\right) - \frac{\partial}{\partial x_i}\left(\frac{\partial h_{kj}}{\partial x_j} - \frac{1}{2}\,\frac{\partial h_{jj}}{\partial x_k}\right)\right]. \tag{10}$$

The first term in square brackets has the form of the Laplacian, but it is not easy to understand the significance of the remaining terms. It appeared to Einstein and Grossmann that precisely these terms prevent $G_{ik}$ reducing in the Newtonian limit to $\Delta\varphi$, since they could not give for them a reasonable physical interpretation.[22)]

In the proof corrections to the "Physical Part" of the "Outline", Einstein gave a second (also erroneous)

[21)]The Ricci tensor is now denoted by $R_{\mu\nu}$, and $G_{\mu\nu}$ is used to denote the expression $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$; however, we shall not change Einstein's notation.

[22)]As will be seen from what follows, Einstein at that time did not understand the part played by coordinate conditions. We now know (as V. A. Fock has shown) that, for example, in a "harmonic" coordinate system the last two "obstreperous" brackets vanish.

argument against generally covariant field equations; this is associated with a different methodological principle of physics—the principle of causality. He describes a procedure that shows that generally covariant equations of gravitation would apparently lead to a violation of the principle of causality or, more precisely, to an ambiguous correspondence between the distribution of the energy—momentum of matter and the gravitational potential. He considers a region $L$ of space-time in which the energy-momentum tensor of matter vanishes, $T_{\mu\nu}=0$. Then the gravitational potentials, both within $L$ and outside it must be determined by the tensor $T_{\mu\nu}$ outside $L$. If it is assumed that the field equations for the gravitational potential $g_{\mu\nu}$ are generally covariant, then one can, in particular, make a coordinate transformation of the form

$$x'_\mu = x_\mu \quad \text{outside} \quad L,$$
$$x'_\mu \neq x_\mu \quad \text{within} \quad L$$

(at least at one point of $L$ and at least for one index $\mu$). As a result of this transformation, $g'_{\mu\nu}$ will differ from $g_{\mu\nu}$ at least at one point of $L$:

$$g'_{\mu\nu} \neq g_{\mu\nu}.$$

But $T'_{\mu\nu}=T_{\mu\nu}$ everywhere, both outside $L$, where $x'_\mu = x_\mu$, and within $L$, where $T_{\mu\nu}=0=T'_{\mu\nu}$. It would therefore appear that the same distribution of the energy-momentum of matter, expressed by the tensor $T_{\mu\nu}$, could generate at least two different systems of gravitational potentials $g_{\mu\nu}$ and $g'_{\mu\nu}$. "Therefore," concluded Einstein, "if ... we adhere to the requirement that $\theta_{\mu\nu}$ [i.e., $T_{\mu\nu}$] be completely determined by the value of $\gamma_{\mu\nu}$ [i.e., $g_{\mu\nu}$], we are forced to restrict the choice of the coordinate system".[48] In other words, the classical understanding of causality associated with the requirement of a unique correspondence between the distribution of the matter energy-momentum $T_{\mu\nu}$ and the gravitational field $g_{\mu\nu}$ seemed to Einstein to be in conflict with the principle of general covariance for the field equations. The authors of the "Outline" considered that this argument explained the unsuccessful attempt to use the Ricci tensor $G_{ik}$ to construct the field equations. The failure of the generally covariant equations (7) and (8) to satisfy the correspondence principle had received an original theoretical justification.

It is now even hard to understand the origin of this nonrigorous argument. Why a change in the components of the tensor $g_{\mu\nu}$ resulting from a change of the coordinates (within $L$) should be a flaw in the theory remains incomprehensible. For this property simply reflects the freedom in the choice of the coordinates in the general theory of relativity. In fact, this argument was thought up by Einstein post factum to justify a sentence already passed on the equation. "Sentence first—verdict afterwards"—this was the logic in the trial of Alice.[23)]

In the comments on the "Physical Part", Einstein actually gave a third argument associated with the principle of energy—momentum conservation. This argument also appeared to be directed against the requirement of general covariance and actually seemed to indicate a quite definite covariance group for the field equations, namely the group of linear transformations. If, following Einstein, one introduces the concept of the energy-momentum tensor $t_{\mu\nu}$ of the gravitational field, then the law of conservation of energy-momentum of matter and of the gravitational field can be written in the form

$$\sum_{\mu\nu} \frac{\partial}{\partial x_\nu}[\sqrt{-g}\, g_{\sigma\mu}(T_{\mu\nu}+t_{\mu\nu})]=0. \tag{11}$$

However, Einstein then noted that these conservation equations for the energy-momentum tensor with allowance for gravitation are "covariant only under linear transformations, so that in the theory developed above only linear transformations can be regarded as admissible".[48] As we know, this argument too was spurious.

The same argument against a systematic generally covariant approach to the field equations is contained in the well-known Christmas letter of Einstein to Mach written on the eve of 1913: "Initially, events are referred to four absolutely arbitrary space-time variables. If the conservation laws for momentum and energy are to be satisfied, these must then be specialized in such a manner that only (and all) linear transformations lead from one allowed frame of reference to another".[50]

### 3. "Groping in the dark" (attempts at noncovariant solution of the problem of the field equations)

Thus arose a working hypothesis. The equations of the gravitational field cannot be generally covariant despite the generally covariant idea underlying the new theory. Arguments associated with the law of conservation of energy and momentum suggested that the field equations should still be linearly covariant. Therefore, the authors of the "Outline", using this requirement, and also the correspondence principle, the conservation of energy-momentum, and the principal idea of the new theory—the geometrical form of gravitation—constructed linearly covariant equations of the second order for the tensor potential $g_{\mu\nu}$ (or $\gamma_{\mu\nu}$, i.e., $g^{\mu\nu}$).

A natural linearly covariant generalization of the Laplacian is the operator

$$\sum_{\alpha\beta} \frac{\partial}{\partial x_\alpha}\left(\gamma_{\alpha\beta}\frac{\partial}{\partial x_\beta}\right). \tag{12}$$

Therefore, the required tensor $\Gamma_{\mu\nu}$ was sought in such a form that it contained the expression (12), this reducing in the weak-field limit to the wave operator:

$$-\left(\frac{\partial^2\gamma_{\mu\nu}}{\partial x_1^2}+\frac{\partial^2\gamma_{\mu\nu}}{\partial x_2^2}+\frac{\partial^2\gamma_{\mu\nu}}{\partial x_3^2}-\frac{1}{c^2}\frac{\partial^2\gamma_{\mu\nu}}{\partial x_4^2}\right). \tag{13}$$

In the static case, $\gamma_{\mu\nu}$ reduces to the single component

23) In his remarkable encyclopedia article on the theory of relativity, Pauli, emphasizing that the general solution of the generally covariant field equations must contain four arbitrary functions and that there must be four identities among the ten field equations, continues: "The contradiction with the principle of causality is merely apparent, since all possible solutions of the field equations differ from one another only formally, all being physically equally valid".[49]

24) This is Einstein's expression referring to the period preceding the discovery of the general theory of relativity.[18]

$\gamma_{44}$, and the expression (13) to the left-hand side of Poisson's equation. But the tensor $\Gamma_{\mu\nu}$ can also include linearly covariant tensor expressions that vanish in the weak-field limit. They can be found on the basis of arguments derived from the energy-momentum conservation law. This led to the establishment of the following linearly covariant differential equations:

$$\Delta_{\mu\nu} = \varkappa\,(\theta_{\mu\nu} + \vartheta_{\mu\nu}), \tag{14}$$

where $\theta_{\mu\nu}$ is the contravariant energy-momentum tensor of the matter, $\vartheta_{\mu\nu}$ is a tensor expression that depends on $g_{\mu\nu}$ and its first derivatives and is interpreted as the energy-momentum tensor of the gravitational field, and $\Delta_{\mu\nu}$ is a tensor expression that depends on $g_{\mu\nu}$ and its first and second derivatives. The energy-momentum conservation law for the system as a whole is then written in the divergence form

$$\sum_{\mu,\nu} \frac{\partial}{\partial x_\nu} [\sqrt{-g}\, g_{\sigma\mu} (\theta_{\mu\nu} + \vartheta_{\mu\nu})] = 0.$$

The gravitational field equations (14) can also be written down in the simple and perspicuous form

$$\sum_{\alpha,\beta,\mu} \frac{\partial}{\partial x_\alpha} \left( \sqrt{-g}\, \gamma_{\alpha\beta} g_{\sigma\mu} \frac{\partial \gamma_{\mu\nu}}{\partial x_\beta} \right) = \varkappa\,(T_{\sigma\nu} + t_{\sigma\nu}), \tag{15}$$

where

$$T_{\sigma\nu} = \sum_\mu \sqrt{-g}\, g_{\sigma\mu} \theta_{\mu\nu}, \quad t_{\sigma\nu} = \sum_\mu \sqrt{-g}\, g_{\sigma\mu} \vartheta_{\mu\nu} \tag{16}$$

are the mixed energy-momentum tensors. Einstein saw a great advantage of these equations in the circumstance that "besides the components of the energy-stress tensor $T_{\sigma\nu}$ of matter, the components of the tensor of the gravitational field (namely $t_{\sigma\nu}$) occur with equal right as field sources; this is obviously a necessary requirement, since the gravitational influence of a system cannot depend on the physical nature of the energy serving as a source of the field".[51] Even more important was the fact that these equations satisfied the correspondence principle. For sufficiently small deviations of the $g_{\mu\nu}$ from the pseudo-Euclidean values,

$$g_{\mu\nu} = \delta_{\mu\nu} + g^*_{\mu\nu},$$

Eqs. (16) reduce to the "wave" form

$$\Box g^*_{\mu\nu} = \varkappa T_{\mu\nu}. \tag{17}$$

Poisson's equations are obtained from them under the following additional conditions: "1) among the field sources, only uncoupled masses are taken into account; 2)...the field is assumed to be static; 3) ...the velocities and accelerations (of a material point) are regarded as small quantities and only quantities of the lowest order are retained".[52] The component $g_{44}$ is identified with the Newtonian potential.

For all that, the authors of the "Outline" felt from the start the imperfection of their theory; the linearly covariant nature of the field equations runs counter to the generally covariant scheme of the geometrical theory: "...Initially, the most natural thing is to require covariance of the system of equations under arbitrary transformations. However, this requirement is contradicted by the fact the equations of the gravitational field we have constructed do not have this property. We have been able to show that the equations of the gravitational field are covariant only under arbitrary linear transformations, but we do not know whether there exists a general group of transformations under which these equations are covariant. The question of the existence of such a group of transformations for the system of equations... is of great importance for the problem considered here. Whatever the case, at the present state of the theory we cannot require covariance of the equations under arbitrary transformations".[53]

At the beginning of 1914, there was a definite shift in the development of the problem of the field equations. Einstein made an attempt to show that the previously obtained equations (15) of the gravitational field admit not only linear transformations but also a larger class of nonlinear transformations, these including accelerated motions and rotation. The requirement of only linear covariance essentially deprived the theory of its physical basis—the interpretation of the equivalence principle in the spirit of equal validity of uniformly accelerated frames of reference. Critics of the Einstein approach, above all Abraham and Mie, regarded this achievement of the Einstein-Grossmann theory as its main weakness. And indeed, the authors of the "Outline" themselves, as we have seen, were not satisfied by the linearly covariant solution of the problem of the equations of the gravitational field.

Under these conditions, Einstein's attention was again drawn to a Lorentz covariant scalar field theory (second theory of Nordström). Together with the young Dutch theoretician Fokker he wrote in February 1914 a paper, submitted on February 18, in which Nordström's theory was formulated in a simple and natural manner by means of a new mathematical formalism, "the absolute differential calculus".[54] Einstein and Fokker under-pinned their formalism with a generalized relativistic foundation: "Since in nature there do not exist frames of reference to which one can refer objects, we shall initially refer the four-dimensional manifold to completely arbitrary coordinates... and we shall restrict the choice of the frames of reference only when the problem we consider suggests such a restriction".[54] In this way there was established a certain correspondence between the Nordström theory formulated in this manner and the geometrical Einstein-Grossmann theory: the condition that the potential be a scalar and the condition of Lorentz covariance imposed on the generally covariant tensor scheme yielded Nordström's theory. A fundamental part in the derivation of the field equations was played here by the Riemann-Christoffel tensor and its contraction with respect to all four indices. As a result, the field equations were initially written in the generally covariant form

$$R = \varkappa T,$$

and only then $g^{\mu\nu}$ (or $\gamma_{\mu\nu}$) were chosen in such a way as to satisfy the principle of the constancy of the velocity of light, this yielding the equations of Nordström's theory:

$$\Phi \Box \Phi = \varkappa T.$$

A striking feature of this theory was its lack of parallelism with the line of argument adopted by Einstein and Grossmann; this had led to the equations (14) or (15) of the gravitational field, which do not use the

Riemann-Christoffel tensor although the approach was persistently linked to a generally covariant scheme of argument and proved so effective in the derivation of the equations of Nordström's theory. It is not surprising that, returning to the tensor geometrical theory, Einstein (with Fokker) acutely felt this flaw in the theory. And again he thought of using the Ricci tensor to obtain the field equations: "Finally, the part played in the present investigation by the differential Riemann-Christoffel tensor suggests that one could find a way of deriving the Einstein-Grossmann gravitational equations that is independent of physical assumptions".[55] As follows from the foregoing, this last expression is obviously a reference to the geometrical argumentation based on general covariance. Here, there is also a very interesting comment which indicates that the pendulum of Einstein's doubts was again swinging in the direction of the generally covariant approach to the problem of the field equations. "The proof of the existence or absence of a connection of such kind [i.e., a connection of the Ricci tensor and the field equations]," Einstein and Fokker conclude their paper, "would amount to important theoretical progress". There then follows the comment: "The argument against such a connection given in §4 of the "Outline" disappears after a more careful analysis".[55] Nevertheless, in a paper from January 24, 1914, answering a criticism of Mie, Einstein continues to argue for the validity of the choice of a special coordinate system (i.e., limited covariance of the equations), again invoking the principle of causality—the ambiguous determination of the potentials $g_{ik}$ from given distribution of the matter energy-momentum tensor and fulfillment of the energy-momentum conservation laws as expressed in the usual form, i.e., vanishing of the divergence of the corresponding tensor.

In February 1914, as we have seen, Einstein was inclined to return to the generally covariant approach and, in particular, the use of the curvature tensor to construct the equations of the gravitational field. But in March of the same year there was a new turn; Einstein again left the correct path. In a letter to Besso, dated March 1914, he speaks of finding a larger class of transformations admitted by the field equations, these including not only linear but also nonlinear transformations corresponding to accelerated frames of reference: "With regard to the theory of gravitation, the following is new. From the gravitational equation we have

$$\sum_{\alpha\beta\mu} \frac{\partial}{\partial x_\alpha}\left(\sqrt{-g}\,\gamma_{\alpha\beta} g_{\sigma\mu} \frac{\partial \gamma_{\mu\nu}}{\partial x_\beta}\right) = k\,(T_{\sigma\nu} + t_{\sigma\nu}),$$

and from the conservation law

$$\sum_{\alpha\beta\mu\nu} \frac{\partial^2}{\partial x_\nu}\frac{\partial}{\partial x_\alpha}\left(\sqrt{-g}\,\gamma_{\alpha\beta} g_{\sigma\mu} \frac{\partial g_{\mu\nu}}{\partial x_\beta}\right) = 0. \qquad (18)$$

These are four equations of third order for the $g_{\mu\nu}$ (respectively, $\gamma_{\mu\nu}$), which can be regarded as conditions for a special choice of the frame of reference. For brevity, let us call them

$$B_\sigma = 0.$$

I have succeeded in proving by a simple calculation that the gravitational equations are valid for any frame of reference satisfying these conditions. And it follows from this that there hold acceleration transformations of very different forms that carry the equations into themselves (for example, rotations as well), so that the hypothesis of equivalence is confirmed, and moreover to an unexpectedly large extent... I am now completely satisfied and no longer doubt the correctness of the complete system irrespective of whether or not the solar eclipse observation succeeds. The common sense of this matter is obvious...at present, I have no burning desire to work, since I have driven myself terribly hard until finding the solution described above. The general theory of invariance was essentially only a hindrance. The direct path proved to be the only one feasible. What I cannot understand is how I could grope in the dark for so long before finding what was so close".[56]

It now seemed that it had been possible to overcome the difficulty in the original theory with linearly covariant field equations: the allowed frames of reference, which were subsequently called "adapted" ("adapted to the gravitational field"), included accelerated frames of reference. True, the geometrical (or kinematic) meaning of the condition of being "adapted" remained obscure. The conclusion that "acceleration transformations" are allowed—it was subsequently seen to be erroneous—not only accorded naturally with the equivalence principle but also meant that the solution of the problem of gravitation entailed an extension of the theory described mathematically by the conditions of being "adapted". It only remained to describe the corresponding class of frames of reference. The "absolute differential calculus" with its basically generally covariant mathematical formalism was unexpectedly reduced to only ancillary significance. As Einstein wrote: "The general theory of invariance was essentially only a hindrance".

The concept of "being adapted" was published initially in a small joint paper by Einstein and Grossmann, apparently written in the spring of 1914 (Ref. 57), and then in a comprehensive summary by Einstein completed in November 1914 (Ref. 58). Although the development of this concept was a move away from the requirement of general covariance, it did contain a germ of success. These papers brought recognition of the need for extending the group of linear transformations since the "acceleration transformations" required by the principle of equivalence were nonlinear: "This hypothesis [i.e., the principle of equivalence] becomes particularly convincing if it turns out that the "fictitious" gravitational field existing in an accelerated coordinate system can be regarded as a "true" gravitational field, i.e., if the theory admits acceleration transformations (in other words, nonlinear transformations)".[59] The equations of gravitation remained the same as in the "Outline" (i.e., of the form (15)), but it was now shown that the transformation of the energy-momentum conservation equation with allowance for the field equations (15) gives the conditions (18), which already occurred in the March letter to Besso, and defines the class of "allowed" transformations, this being in Einstein's opinion sufficiently large to include acceleration transformations, which, however, was not proved.

It is true that this paper has an important comment indicating that the objection to the generally covariant approach associated with the energy-momentum conservation law in the form in which it had appeared earlier had lost its force: "... the assertion about the restriction on the choice of the coordinate system is incorrect; it follows from the relation (III) [i.e., the divergence equation of energy-momentum conservation] only if one allows only linear transformations for which the quantities $t_{\mu\nu}/\sqrt{-g}$ are ascribed a tensor character, for which, as has been found, there is no justification".[60] In other words, in order to go beyond the linearly covariant approach, Einstein gave up the requirement that the energy-momentum components of the gravitational field should be ascribed a tensor nature. And although the restriction of the class of allowed transformations to "permitted" or "adapted" transformations was based essentially on the divergence formulation of the energy-momentum conservation law of the system, the conclusion that a nontensor nature of the energy-momentum complex of gravitation could be allowed ultimately helped to overcome the objections to the generally covariant approach to the solution of the problem of the field equations. This was the last joint paper of Einstein and Grossmann and it was completed not later than April 1914, when Einstein transferred to Berlin.

In the long paper[58] completed at the end of November 1914, Einstein continued to develop the concept of "adapted" coordinate systems. Against the general covariance of the field equations there is here adduced a single argument associated with the breakdown of the causality principle. In this paper, the concept of adapted coordinate systems is formulated more clearly; these are specially constructed in such a way that the coordinate transformations used in the argument leading to violation of the causality principle are eliminated. The conditions distinguishing such systems proved to be rather complicated (in general, they differ from the conditions (18)):

$$B_\mu = \sum_{\alpha\sigma\nu} \frac{\partial^2}{\partial x_\sigma \, \partial x_\alpha} \left( g^{\nu\alpha} \frac{\partial H \sqrt{-g}}{\partial g_\sigma^{\mu\nu}} \right) = 0,$$

and their geometrical meaning remained obscure. In any case, Einstein assumed that the class of "adapted" coordinate systems was sufficiently large to include accelerated frames of reference. The corresponding equations of the gravitational field actually had the same structure as established already in the "Outline":

$$\sum_{\alpha\beta} \frac{\partial}{\partial x_\alpha} \left( \sqrt{-g} \, g^{\alpha\beta} \frac{1}{2} \sum_\tau g^{\nu\tau} \frac{\partial g_{\sigma\tau}}{\partial x_\beta} \right) = -\varkappa \left( T_\sigma^\nu + t_\sigma^\nu \right). \tag{19}$$

As Einstein noted, these equations "despite their complexity, admit a simple physical interpretation". Under the condition that the three-index quantities $(1/2)\sum_\tau g^{\nu\tau}\, \partial g_{\sigma\tau}/\partial x_\beta$ are interpreted as gravitational field intensities (as became clear subsequently, this interpretation was unfortunate), the left-hand side is the divergence of the field intensity, which is determined by the total energy-momentum tensor (the right-hand side). Einstein saw an important confirmation of the validity of Eqs. (19) in the fact that the "energy tensor of the gravitational field, like the energy tensor of matter, itself generates the field". In addition, Eqs. (19) were derived from a variational principle with a Lagrangian proportional to the square of the gravitational field intensity ($\Gamma^\tau_{\nu\sigma} = (1/2)\sum_\mu g^{\tau\mu}\partial g_{\mu\nu}/\partial x_\sigma$):

$$L = -\sum g^{\tau\tau'} \Gamma^\rho_{\mu\tau} \Gamma^\mu_{\rho\tau'}, \tag{20}$$

and was analogous to the Lagrangian of the electromagnetic field quadratic in the intensities.

But in this last variant of the concept of "being adapted" the Einstein-Grossmann theory retained its main defects: the incomplete covariance of the theory, the absence of a clear physical (or geometrical) meaning of the "adapted" coordinate systems, the insufficient physical justification for the choice of the Lagrangian of the theory in the form (20), and the absence of a rigorous proof that acceleration transformations are included among the permitted ("adapted") transformations. Einstein's persistent opponent Abraham, in a long review of the "newest theories of gravitation" written in December 1914, made a number of deep critical comments about the "adapted" variant of the Einstein-Grossmann theory. In particular, he comments completely correctly: "It would be interesting and important to establish what transformations besides the linear transformations are contained in this class of transformations [i.e., class of "adapted" transformations]? And what physical meaning (uniformly accelerated motion, rotation, etc.) can be ascribed to them? Only then could one speak of a certain "generalized" theory of relativity if the equal validity of the frames of reference postulated by the relativity principle of 1905 for uniformly and rectilinearly moving systems were now extended to such systems that are in a state of accelerated motion or rotation relative to one another. For the moment, this extension of relativity has not succeeded".[61] He also drew attention to the insufficiently justified choice of the Lagrangian (20) of the theory: "In the recently published general presentation of the "general theory of relativity" [i.e., in the paper Ref. 58], Einstein derives the differential equations of his theory from a certain variational principle on the basis of certain restrictions whose physical meaning is not explained".[62]

Noting the vulnerability of the Machian interpretation of inertia—Einstein regarded the possibility of such an interpretation in his theory as a great advantage compared with scalar theories—and also the greater complexity of the tensor-geometrical theory, Abraham expressed his preference for the scalar approach.[25)]

## 4. "Breakthrough to clarity"[26)]

In November 1915, Einstein finally returned to the requirement of general covariance of the field equa-

[25)]"If one bears in mind the extreme complexity resulting from increasing the number of gravitational potentials to ten and from the curvature of the four-dimensional world, then from the point of view of Mach's "economy of thought" one should surely give preference to scalar theories until experimental confirmation has been found that there are ten potentials of gravitation rather than one".[63]

[26)]This is Einstein's expression from Ref. 18.

tions, which led him almost immediately to the correct equations of the gravitational field. This was "...one of the most exciting and tense periods of my life", wrote Einstein in a letter to Sommerfeld on November 28, 1915 (Ref. 13) three days after the publication of these equations.

For almost a complete year Einstein had published nothing on the theory of gravitation. Despite Abraham's criticism, Einstein had evidently for a certain time regarded the creation of the fundamentals of the theory as completed. Under these conditions, questions relating to the physical interpretation of the theory, in particular its experimental confirmation, became more important. Interest in experiments brought realization that Nordström's second theory was a competitor. In addition, Freundlich's expedition, which should have decided the question relating to the deflection of light rays in the field of the Sun and, thus, decide in favor of one of these two theories, was abandoned because of the outbreak of the First World War in August 1914. Finally, Einstein's recent papers had been excessively burdened with complex mathematical calculations and he was, perhaps by contrast, now particularly attracted by questions involving a simpler physical interpretation and especially experiment. In particular, this found reflection in the experiments he made in the period January–May 1915 with the Dutch physicist de Haas.[64] On February 12, 1915, he wrote to Besso, referring to the experiments on the Einstein-de Haas effect: "The experiments will soon be completed... a wonderful experiment, and a pity that you cannot see it. But how crafty is nature when you wish to approach her in an experiment! In my old age I begin to sicken for experiments".[65] In the same letter he writes about gravitation. He mentions only one thing—verification of the "red shift" effect by studying the spectra of binary stars. On the basis of Freundlich's investigation, who had used the spectral measurements of binaries by Campbell and Ludendorf, Einstein concluded that there had been obtained an "approximate quantitative verification of the theory giving satisfactory agreement". It should however be noted that Freundlich's paper had been justifiably criticized by Seeliger, and, ultimately, hopes of confirming the "red shift" effect through the observations of the spectra of binary stars were not justified.[66] Interest in the experimental side of the theory of gravitation was thus stimulated by the contact with Freundlich, who at this time intensively studied the problem of the anomalous precession of Mercury's perihelion, in particular Seeliger's hypothesis that the zodiacal light could have a perturbing effect; this hypothesis had already been advanced at the end of the 19th century to explain the anomaly. It so happened that in February 1915 Freundlich completed his critical analysis of this hypothesis and concluded that none of the hypotheses (based on Newtonian theory) assuming hidden masses in the solar system could explain the Mercury anomaly. Subsequently, when Einstein again spoke after a long interrruption about the anomalous precession of Mercury's perihelion, he referred to Freundlich: "Freundlich has recently written about the impossibility of finding a satisfactory explanation for the anomaly in Mercury's motion on the basis of Newton's theory".[67] But these words date from November 1915, after his return to the requirement of general covariance of the equations of the gravitational field.

It can be assumed that work on the theory of gravitation recommenced in the summer of 1915, when Einstein was in Göttingen and Zurich. Well known is his letter to Sommerfeld on July 15, which gives an indirect indication of this. First, in answer to Sommerfeld's suggestion that a new edition of the collection of classic papers "The Principle of Relativity" should contain an exposition of the general theory of relativity, Einstein noted that "the volume should appear without changes and the inclusion of the general theory of relativity because none of the existing expositions of the latter is complete."[13] This important admission meant that the long review on the theory of gravitation published at the end of 1914 (Ref. 58) was no longer regarded by Einstein as sufficiently complete or fully correct. Second, he writes about his visit in Göttingen, where he had discussions with Hilbert, which could, as we believe (see below) have had a significant influence on Einstein's thinking.

Finally, he mentions a paper of Freundlich, in all probability the one devoted to Mercury's anomalous precession, calling it "undoubtedly fundamental". This suggests that at that time Einstein already thought of explaining the anomaly of Mercury in the framework of the geometrical theory.

November 1915 became the month of the final and headlong assault. In the four November communications[68-70,67] presented at the sessions of the Prussian Academy of Sciences on 4, 11, 18, and 25 November, respectively, Einstein solved the problem of the equations of the gravitational field and as a result achieved general covariance of the theory and also, on the basis of these equations, explained the anomalous precession of Mercury's perihelion and for the first time gave the correct value for the deflection of light by the Sun.

In the first communication[68] the decisive step is taken: Einstein returned "...to the requirement of a more general covariance of the field equations, which he had abandoned with heavy heart when working with .. ...Grossman".[68] However, he still imposed a certain restriction on the arbitrary continuous transformations —the determinant of these transformations must be equal to unity, i.e., he imposed the condition of unimodularity, which significantly simplifies the calculations and makes the basic formulas more perspicuous.[27)]

The logic for deriving the field equations that had already been used in the "Outline" led directly to the

[27)] "Just as the special theory of relativity," wrote Einstein, "is based on the postulate that its relations must be covariant under linear orthogonal transformations, the theory presented here is based on the postulate of covariance of all systems of equations under transformations with determinant 1".[68]

field equations

$$R_{\mu\nu} = -\varkappa T_{\mu\nu}, \tag{21}$$

where $R_{\mu\nu}$ is the Ricci tensor subject to the unimodularity condition. We may mention that Einstein did not regard this condition as a serious restriction on the admissible transformations. At the end of the paper, he specially returns to this question and shows that rotation and a motion of one frame of reference with respect to another in which the coordinate origin of the new system moves arbitrarily with respect to the old system are included among the allowed transformations.

He also notes that, normalizing the coordinate system in some natural manner, for example, by means of the condition

$$\sum_{\beta} \frac{\partial g^{\alpha\beta}}{\partial x_\beta} = 0,$$

one can readily obtain the Newtonian approximation from Eqs. (21).[28)] This suggests that Einstein at that time already clearly understood the error in his argument demonstrating the conflict between general covariance of the field equations and the causality principle (based on uniqueness).

However, the equations in the form (21) were internally inconsistent, as was shown by their comparison with the energy-momentum conservation law. First, it was found that the unimodularity condition was satisfied everywhere only if the trace of the energy-momentum tensor of the "matter" vanishes. Second, on the introduction of $t_{\mu\nu}$, the energy-momentum pseudotensor of the gravitational field, the following formula (which, it is true, was not written down by Einstein) was obtained for the traces $T$ and $t$ of the energy-momentum tensors):

$$\frac{\partial^2 g^{ik}}{\partial x_i\,\partial x_k} + \varkappa\,(T - t) = 0, \tag{22}$$

in which $T$ and $t$ enter with opposite signs. This asymmetry between the contributions to the energy from matter and from the gravitational field did not have a physical justification. Initially, Einstein was actually disturbed by only the first inconsistency.

Its elimination was the subject of the second communication,[69] in which there is actually advanced the assumption of an "electromagnetic-like" structure of "matter", this being expressed by the vanishing of the trace of the "matter" energy-momentum tensor: $T = 0$.

Because of the covariance of this condition, Einstein assumed it to be possible to postulate completely generally covariant field equations in the form

$$G_{ik} = -\varkappa T_{ik} \tag{23}$$

and it was only to facilitate the calculations that he proposed using the unimodularity condition, which now, i.e., in the case when the hypothesis $T = 0$ is adopted, does not lead to an inconsistency.

The third communication[67] of November 18 contained an explanation of the anomalous precession of Mercury's perihelion on the basis of the field equations (23) for empty space:

$$G_{ik} = 0 \tag{24}$$

and the equations of motion of a material point in the field:

$$\frac{d^2 x_\nu}{ds^2} = \sum_{\sigma\tau} \Gamma^{\nu}_{\sigma\tau} \frac{dx_\sigma}{ds} \frac{dx_\tau}{ds},$$

i.e., the geodesic equation. His result—an advance of Mercury's perihelion by 43″ per century—agreed well with the data of the astronomers (45″ ± 5″) and, as another important result, Einstein obtained a value for light deflection in the field of the Sun equal to 1.7″ (instead of 0.85″), which obviously did not depend on the adoption of the hypothesis $T = 0$. This communication is also remarkable in that it seems to anticipate the correct generally covariant field equations, since Einstein says that the hypothesis $T = 0$ is unnecessary, which would be possible only if the equations are augmented by the term with the scalar $T$ or $G$: "In a paper to be published shortly, it will be shown that this hypothesis too [i.e., the assumption $T = 0$] is superfluous".[67] Evidently, he regarded publication of the calculation demonstrating the remarkable empirical confirmation of the geometrical theory and the generally covariant field equations as a more important matter than the derivation of the general form of the field equations and their justification and publication.

These equations were the content of the last November publication (November 25).[70] Adding to the right-hand side of the equations the term with the scalar $T$, Einstein finally obtained completely generally covariant equations of the gravitational field that do not require an additional assumption about the structure of the energy-momentum tensor $T_{ik}$ of "matter":

$$G_{im} = -\varkappa\left(T_{im} - \frac{1}{2} g_{im} T\right). \tag{25}$$

"Thus, finally," wrote Einstein at the end of the paper, "the construction of the general theory of relativity as a logical scheme has been completed".[71]

How did he justify the equations that to this day constitute the core of the general theory of relativity?

Einstein showed that multiplication of both sides of Eq. (25) by $g^{im}$ and subsequent contraction over the indices $i$ and $m$ yields the equation

$$\frac{\partial^2 g^{ik}}{\partial x_i\,\partial x_k} - \varkappa\,(T + t) = 0, \tag{26}$$

which is analogous to Eq. (22) but includes the traces of the energy-momentum tensors of "matter" and of the gravitational field "in the same manner", i.e., with the same sign. This becomes clear if we replace the tensor $T_{ik}$ in Eq. (21) by $T_{ik} - (1/2)g_{ik}T$. Then in Eq. (22) it is necessary to replace $T$ by $-T$, and both the scalars $T$ and $t$ occur in the equation with the same signs. A different argument, which is used in the majority of textbooks and monographs written subsequently, takes the form that the absence of the term $-(1/2)g_{im}T$ on the

[28)]In the absence of the term with the scalar curvature, which appeared only in the last November communication,[70] the Newtonian approximation was guaranteed only if the unimodularity condition is adopted.

left-hand side or, equivalently, of the term $-(1/2)g_{im}G$ on the right-hand side leads to nonvanishing of the covariant divergence of the energy-momentum tensor of "matter", as follows directly from the contracted Bianchi identities. However, Einstein did not use the well-known properties of the curvature tensor, and this considerably complicated his path to the correct field equations.

As we shall see, Hilbert proceeded from a variational principle and immediately obtained on the left-hand side, not $G_{ik}$, but the necessary combination $G_{ik} - (1/2)g_{ik}G$.

Let us now return to the analysis of the reasons and circumstances that led Einstein to abandon the noncovariant attempts at solution of the problem of the equations of the gravitational field and revert to the path of general covariance, which led him to his triumphant finale. In the first November communication, he wrote: "...a renewed analysis has shown that, following the proposed path, absolutely nothing can be proved; what had nevertheless appeared to be achieved was based on confusion. The postulate of relativity in the extent that I required is always satisfied when Hamilton's principle is taken as basis, but actually it does not give one the possibility of determining the Hamilton function $H$ of the gravitational field. In fact, the relation (77) in Ref. 58 restricting the choice of $H$ simply reflects the fact that $H$ must be invariant under linear transformations, and such a requirement has nothing in common with the relativity of acceleration .... For these reasons, I completely lost faith in the field equations I had obtained and began to seek a path that would limit the possibilities in a natural manner. I thus returned to the requirement of general covariance of the field equations, which I had abandoned with a heavy heart when I was working with my friend Grossmann. At that time, we actually approached very close to the solution of the problem proposed here".[68]

Three days after the famous communication to the Prussian Academy of Sciences at Berlin on November 25, 1915, in which he presented the correct field equations (25), Einstein wrote a letter to Sommerfeld (on November 28), in which he gives one further important reason for rejecting the concept of "adapted transformations": "Namely, I recognized that my previous equations of gravitation were entirely without meaning. This is indicated by the following considerations:

1) I showed that the gravitational field in a uniformly rotating system does not satisfy the field equations.

2) For the motion of Mercury's perihelion one obtains 18″ per century instead of 45″.

3) During the last several years I had not succeeded in obtaining the Hamilton $H$ function by the covariant treatment. After a suitable generalization, it is an arbitrary function. It follows that covariance with respect to "adapted" coordinate systems was devoid of content".[72]

Einstein rejected his own ideas just as vigorously as he had defended them when he was persuaded of their correctness. The first and third arguments actually repeat what was said in the first November communication (though in a somewhat different form it is true). However, the second argument associated with the impossibility of explaining Mercury's anomaly in the framework of the "adapted theory" is mentioned here for the first time. It can be assumed that Einstein's interest in the anomalous displacement of Mercury's perihelion was again awakened by Freundlich's paper devoted to this problem and completed already in February 1915; this was probably the paper that Einstein in the July letter to Sommerfeld called "undoubtedly fundamental".[73]

The definite lack of satisfaction with the "double covariance" of the 1913–1914 theory (general covariance of the equations of motion of "matter" and the equations describing the interaction of "matter" with gravitation, but only linear or "adapted" covariance for the equations of the gravitational field itself), which Einstein had frequently felt earlier, grew through the summer and fall of 1915 into the conviction that such a theory was wrong. Moreover, two of the three main arguments against it given in the letter to Sommerfeld have a clear physical origin and are directly related to experiments (disagreement with the equivalence principle and incorrect value for the perihelion advance).

It is here appropriate to mention one further circumstance with a deep physical meaning that helped Einstein return to general covariance of the field equations. It is mentioned in both the first November paper and the letter to Sommerfeld on November 28 and is concerned with the question of what quantities in the geometrical theory should be identified with the gravitational field intensities. Writing the equation for the energy-momentum conservation law for "matter" in the form

$$\sum_{\nu} \frac{\partial T_{\sigma}^{\nu}}{\partial x_{\nu}} = \frac{1}{2} \sum_{\mu} \frac{\partial g_{\mu\nu}}{\partial x_{\nu}} T_{\sigma}^{\mu},$$

Einstein noted in his first paper: "This conservation equation has persuaded me to regard the quantities $1/2\sum_{\mu} g^{\tau\mu}\partial g_{\mu\nu}/\partial x_{\sigma}$ as the natural expression for the components of the gravitational field, although, bearing in mind the formulas of the absolute differential calculus, it would have been better to introduce the Christoffel symbols $\left\{ {\nu\sigma \atop \tau} \right\}$ instead of these quantities. This was a fatal prejudice".[74]

In the letter to Sommerfeld, he emphasizes that the correct identification of the field intensities with the Christoffel symbols played a decisive part in the establishment of the connection between the generally covariant equations (25) and their Newtonian approximation. Writing the correct equations in a coordinate system in which $\sqrt{-g} = 1$:

$$\sum_{l} \frac{\partial \left\{ {im \atop l} \right\}}{\partial x_l} + \sum_{\alpha\beta} \left\{ {i\alpha \atop \beta} \right\} \left\{ {m\beta \atop \alpha} \right\} = -\varkappa \left( T_{im} - \frac{1}{2} g_{im} T \right),$$

Einstein writes: "Already three years ago I discussed with Grossmann these equations (without the second term on the right-hand side), but we then decided that they do not contain the Newtonian approximation, which was erroneous. The key to this solution was given by recognition of the fact that it is not $\sum_\alpha g^{i\alpha}\partial g_{\alpha i}/\partial x_m$ that is the natural expression for the "components" of the gravitational field but rather the related Christoffel symbol $\{{}^{im}_{l}\}$. If this is understood, the equation given above can be readily represented, since there does not arise a temptation, for the sake of a more general interpretation, to transform them by calculation of the symbols".[72]

The use of the Christoffel symbols as field components was one of the last main steps to the final aim. When they were used, all the relations took on a simpler form amenable to a comparatively clear physical interpretation.

As we have seen, in the period 1913–1915 Einstein frequently returned to the idea of general covariance of the field equations, and two of the three arguments against adopting equations of this kind had in fact already been rejected by him. In all probability, the argument based on the correspondence principle lost its force not later than January–February 1914. Soon after this, probably in the spring of 1914, Einstein recognized the falsity of the argument against general covariance of the field equations based on the requirement that the energy-momentum components have a tensor nature. The only objection that apparently remained was that based on the false causality paradox. In the first two November communications, this question is not elucidated. But in the third communication, devoted to the calculation of the motion of Mercury's perihelion, there is a remark which indicates that in this question too Einstein had now achieved clarity. Noting the covariance of the equations of the gravitational field under arbitrary continuous transformations, he continues: "Nevertheless, it would appear correct to assume that such transformations carry all solutions into each other, and, therefore (for given boundary conditions) they differ from each other only formally, and not physically".[75] This argument would imply that the ambiguity in the solution of the generally covariant field equations that had previously appeared to Einstein as a serious objection to their adoption was in fact only formal and not physical in nature and therefore not a paradox.[29)] To this it is now worthwhile to add that the solution in matter-free space is nevertheless not completely determined by the energy-momentum tensor of matter. The existence of gravitational waves renders the problem of the uniqueness of solutions more complicated. The derivation of Einstein's equations is a clear example of how general physical principles made it possible to write down equations containing a great deal more than was known at the time of their derivation. At the end of November 1915, the general equations were finally written down. What had appeared an insuperable difficulty for several years became a simple matter in one month. Old prejudices were laid aside, and the palm was yielded to the general principles that should have been followed without hesitation.

The theory was completed with the long classical paper "The foundations of the general theory of relativity" submitted to the Annalen der Physik on March 20, 1916 (Ref. 76). This paper reveals no trace of the laborious work that had preceded it.

## III. GENERALLY COVARIANT EQUATIONS OF GRAVITATION. HILBERT'S PATH

On November 20, 1915 the eminent German mathematician Hilbert gave a lecture entitled "The foundations of physics" to the Göttingen Mathematical Society.[77] In it, he derived generally covariant equations of the gravitational field equivalent to Einstein's equations (25). In the third November publication, Einstein pointed out that the hypothesis $T=0$ is now superfluous, and this could be regarded as a kind of anticipation of the generally covariant field equations. He presented the correct equations a week later, on November 25. The coincidence is of course remarkable. Analysis of the corresponding publications of Einstein and Hilbert and the evidence of F. Klein, M. Born, H. Weyl, and W. Pauli, who knew both men very well, justify the conclusion that the problem of the general covariance of the equations of gravitation was solved independently in Berlin and Gottingen [but see the end of this section].

Before we turn to an analysis of Hilbert's lecture, let us briefly consider the route he took to his principle achievement in the field of physics.

### 1. Hilbert's sixth problem

One of the best known pupils of Hilbert, H. Weyl, distinguished six main periods in the work of his teacher. The basic theme of the fifth period, which lasted from 1910 to 1922, was physics. But before this too Hilbert had been interested in fundamental problems of physics.

Among the 23 famous mathematical problems put forward by Hilbert at the Third International Congress of Mathematicians in Paris (August, 1900) there was one, the sixth, which referred directly to physics. It consisted of the "axiomatic construction in accordance with the same scheme [i.e., the scheme of axiomatic investigations in geometry] of the physical disciplines in which mathematics already plays a leading part". Hilbert also associated the axiomatization of physics with the introduction into it of the powerful method of group theory: "If we are to treat the physical axioms after the manner of geometry, we should attempt initially to describe a class of physical processes which is as general as possible by means of a small number of axioms and then, adding successively special axioms, arrive at the more specialized theories—and here it is possible that a classification principle could be taken over from Lie's deep theory of infinite transformation groups."[78] Einstein's relativistic ideas could be naturally related to geometrical and group theories. F. Klein's "Erlangen program" here found fruitful soil.[79]

In 1905, Hilbert and his friend H. Minkowski organized a seminar on the electrodynamics of moving bodies, at which there was a discussion of the Michelson-Morley experiments and the papers of Lorentz and Poincaré. Two years later, Minkowski created his famous four-dimensional invariant-theoretic conception of the special theory of relativity, which was com-

[29)]See also Pauli's comment in the earlier footnote 23.

pletely in accord with the axiomatic group-theoretic program of Hilbert and Klein. Investigations into integral equations opened up for Hilbert the possibility of participating directly in the development of problems in physics, first in the kinetic theory of gases and then in the theory of radiation. It was in these fields of physical theory that he hoped by means of the formalism of integral equations to implement his project of the axiomatization of physics. At this time, Einstein was already working intensively on the relativistic theory of gravitation, and Mie had advanced his apparently very promising unified electromagnetic field theory. Hilbert became more and more attracted to the problems of the electron theory of the structure of the atom. In the summer of 1914, P. Debye, at Hilbert's request, organized a seminar at Göttingen on the structure of matter. Hilbert's interests were shifted toward fundamental problems of physics associated with the construction of a unified field theory of matter.

In the middle of the summer of 1915, Einstein traveled to Göttingen and sometime later wrote an enthusiastic letter to Sommerfeld about Hilbert and the mutual understanding that had developed between them: "I had great joy at Göttingen and was understood to the last detail. Hilbert completely charmed me. An outstanding man!".[13]

## 2. "Foundations of physics" and the equations of gravitation

Thus, on November 20, Hilbert gave his lecture "The foundations of physics", in which, in a completely different manner, he obtained generally covariant equations of gravitation equivalent to the Einstein equations (25) (Ref. 77).

In this investigation, Hilbert combined his axiomatic aims with the idea of constructing a unified physical field theory on the basis of a powerful mathematical formalism containing Riemannian geometry, tensor analysis, the theory of Lie groups, and variational calculus. This time, it was not some particular physical theory that was to be axiomatized but physics as a whole, and the creation of the axioms would simultaneously amount to the creation of a unified physical theory. Hilbert was inspired by the work of Mie on nonlinear electrodynamics and Einstein's idea of general covariance. "The grandiose problems posed by Einstein," write Hilbert "and also the methods ingeniously developed for their solution, his far reaching ideas and the formation of concepts by means of which Mie constructed his electrodynamics have opened up new paths for investigations into the foundations of physics. In what follows, using the axiomatic method and proceeding essentially on the basis of two axioms, I wish to derive a new system of basic equations of physics. These equations, which have perfect elegance, contain simultaneously the solution to the problems of Einstein and Mie".[77]

Mie's theory was forgotten, but the method used by Hilbert has become part of the arsenal of methods of modern theoretical physics. Hilbert proceeded from the variational principle and immediately introduced a "world function", the Lagrangian, choosing it for the gravitational field in the form of the scalar curvature $G$ and for the electromagnetic field in the usual form (albeit with allowance for the generalization characteristic of Mie's theory). Further, proceeding in the now standard manner, Hilbert immediately obtained Einstein's equations, on the left-hand side of which the Ricci tensor $G_{in}$ was now replaced by the combination $G_{\mu\nu}-(1/2)g_{\mu\nu}G$:

$$G_{\mu\nu}-\frac{1}{2}g_{\mu\nu}G=-\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}L}{\partial g^{\mu\nu}} \tag{26}$$

(it is this combination that is now denoted by $G_{ik}$). The right-hand side of the equation contains the energy-momentum tensor of matter expressed in terms of the derivatives of the corresponding part of the Lagrangian.[30)]

Here, success awaited Hilbert. The trace of the energy-momentum tensor of the electromagnetic field is zero, and it therefore follows from Hilbert's equation that the scalar curvature $G$ is also zero. But Hilbert's equations (26) have a more general nature and are also true when a tensor whose trace is nonvanishing occurs on the right-hand side.

Einstein long searched for a way to change the right-hand side of the equation, and ultimately replaced $T_{\mu\nu}$ by $T_{\mu\nu}-(1/2)g_{\mu\nu}T$.

Hilbert immediately obtained the left-hand side of a different but equivalent equation, deriving from the variational principle the correct expression for the left-hand side without thinking about the properties of the right-hand side. The two equations are equivalent to each other by virtue of the obvious equation $G=-\varkappa T$, which follows from either of them. (Already in 1916, Einstein referred to Hilbert's derivation.)[80]

"The differential equations of gravitation obtained in this manner—as we read in the published variant of Hilbert's lecture—are, it seems to me, in harmony with the grandiose general theory of relativity advanced by Einstein in his recent papers".[81]

The greater part of Hilbert's lecture was devoted to analyzing the problem of energy-momentum conservation in this theory, which differed from the general theory of relativity only by the specialization in the "material" part of the Lagrangian, which was assumed to correspond to Mie's electrodynamics. Hilbert showed, in particular, that the conservation laws for energy and momentum in generally covariant theories have an identical nature and thus differ fundamentally from the conservation laws in theories based on flat space-time.[31)] Emmy Noether soon elucidated the origin of this difference on the basis of her theorems on invariant variational problems.[82]

Thus, the two great theoreticians proceeded towards the same goal by different routes.

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[30)] In Hilbert's paper, $L$ is the Lagrangian of Mie's nonlinear electrodynamics.

[31)] Hilbert's contribution to the investigation of this problem has been considered in detail in a book of one of the present authors.[83]

Hilbert did not set himself the task of constructing a theory of gravitation. His aim, stated already in 1900, was to axiomatize physics on the basis of fundamental mathematical structures. Hilbert regarded his work as a development and generalization of Einstein's theory that would be capable subsequently of solving basic problems of fundamental physics. This is eloquently attested by the final paragraph of the published lecture: "As we see, if properly interpreted, a few simple assumptions, expressed in the axioms I and II, turn out to be sufficient for the construction of a theory that will not only radically transform our ideas about space, time, and motion in the direction indicated by Einstein but also, as I am convinced, through the equations derived here, explain the secret and hitherto hidden phenomena within the atom, and on their basis it must be possible generally to reduce all physical constants to mathematical constants. In this manner we approach the possibility in principle of transforming physics into a science like geometry, which is undoubtedly an excellent example of the axiomatic method, using in the given case the services of powerful instruments of mathematical analysis, namely the variational calculus and the theory of invariants".[84]

Let us now compare the approaches of Einstein and Hilbert to the problem of the equations of gravitation as evaluated by authorities such as Klein, Weyl, Pauli and Born, who knew both men extremely well and probably heard the history at first hand. In 1920, Klein wrote: "In this matter, there can be no talk of priority, since the two authors followed completely different lines of thought (and, moreover, such that initially compatibility of their results did not seem to be guaranteed). Einstein proceeded intuitively and had in mind arbitrary material systems. Hilbert proceeded deductively, introducing the aforementioned ...restriction to electrodynamics, from a higher variational principle. He proceeded, in particular, from Mie's theory".[85] Pauli's opinion was as follows (1921): "Simultaneously with Einstein and independently of him, Hilbert established the generally covariant field equations. Hilbert's exposition however, appealed little to physicists, since Hilbert, first, introduced the variational principle axiomatically, and, second, which is more important, his equations were derived, not for an arbitrary material system, but specially on the basis of Mie's theory of matter".[86] Thirteen years after the described events, Weyl recalled: "In his investigations into the general theory of relativity, Hilbert combined Einstein's theory of gravitation with Mie's program of a unified field theory. The more sober approach of Einstein, which was unrelated to Mie's very speculative program, proved to be more helpful. Hilbert's paper can be regarded as a precursor of a unified theory of gravitation and electromagnetism".[87] The approach and thinking of Hilbert enjoyed popularity: "At that time, there was a very happy atmosphere in Hilbert's circle; the dream of a universal law that controls both the cosmos as a whole and all atomic nuclei seemed almost to be realized".[87] But these hopes were not destined to be fulfilled. Only the deep physical thinking of Einstein created the theory that was to become the living basis of all physics of our time, although Einstein himself from the beginning of the twenties embarked on the path to a unified field theory opened up by Hilbert, but without success.[32)]

After our paper had been submitted, it was reported that the Einstein archive in Princeton has letters of Einstein and Hilbert dated November 1915 (Ref. 89). These letters fill an important gap in our story.

It turns out that in November 1915 the two theoreticians worked in close contact with each other: they exchanged letters and the text of their papers and each of them knew what the other was doing. This correspondence beautifully complements the lectures of Einstein in Berlin and Hilbert in Göttingen.

## EPILOGUE

From the vague but essentially brilliant anticipations of Lobachevskiĭ, Riemann, and Clifford on the connection between space and matter to the systematic, mathematically developed theory founded on experiment the distance was very great. For the prophetic utterances of the great geometers to acquire real physical content there were needed decades in the development of physics that led to the field concept and the extension of the classical relativity principle.

The "germ" of the new theory arose in 1907 in the attempt to understand the influence of the gravitational field on the propagation of light and to include the gravitational field in the general scheme of the special theory of relativity. The idea of the "falling lift" changed the direction of the development of the theory from a comparatively simple question to a fundamental principle.

If one wishes to find a historical parallel for Einstein, the image that comes to mind is that of the great dreamer and natural scientist Keppler. Working frenziedly on the theory of Mars, he pictured his scientific investigation as a battle in which nature springs ambushes. In his works, he described his achievements as a triumph on the field of battle. He regarded this war as one of the episodes in the mastering of nature, in the recognition of what he called the harmony of the world. The path of a bold dreamer has led to triumph in physics more than once.

Einstein "conquered" the equations of gravitation in battles no less heavy. For him too the dream of the "harmony of the world" was a no less inexhaustible source of energy. The only difference was that he described his battles in the restrained style adopted in our century.

Einstein's equations formed the basis of a great science. Only two years later Einstein asserted the all encompassing nature of the new law of universal gravi-

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[32)]Hilbert's contribution to the general theory of relativity is also discussed in the interesting Ref. 88.

tation. In his paper "Cosmological considerations on the general theory of relativity" he boldly described the entire universe by a single equation. But this is already another theme.

[33)]*Translator's Note*. This Russian collection of Einstein's scientific works will henceforth be referred to in the bibliography as Sobranie.

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[1]N. I. Lobachevskiĭ, "Novye nachala geometrii s polnoĭ teorieĭ parallel'nykh [New foundations of geometry with complete theory of parallels (Introduction)], in: Ob Osnovaniyakh Geometrii (On the Foundations of Geometry), Gostekhizdat, Moscow, 1956, p. 64.

[2]B. Riemann, "Über die Hypothesen, welche der Geometrie zu Grunde liegen," Abh. Königl. Gesell. der Wissen. zu Göttingen, Band 13 (Russian translation published in the book in Ref. 1, p. 324).

[3]W. K. Clifford, "On the space-theory of matter," Proc. Cambr. Phil. Soc. **21** (1870); in: W. K. Clifford, Mathematical Papers, Chelsea, New York (1882), p. 21; quoted in Russian translation by I. B. Pogrebysskiĭ in his book: Ot Lagranzha k Eĭnshteĭnu (From Lagrange to Einstein), Nauka, Moscow, 1966, p. 260; see also Ref. 92.

[4]P. S. Laplace, Allgemeine geographische Ephemeriden, verfasst von einer Gesellschaft Gelehrter, Band IV, Weimar (1799); cited in: S. W. Hawking and G. F. R. Ellis, The Large Scale Structure of Space-Time, CUP, 1973 (Russian translation published by Mir, Moscow, 1977).

[5]J. Zenneck, Gravitation, in: Encyclopädie der mathematischen Wissenchaften, Band 5, Tl. 1, Leipzig (1903).

[6]A. Einstein, "Appreciation of Simon Newcomb," Science **69**, 249 (1929) [Russian translation published in the Russian collection of Einstein's Scientific Works (Sobranie Nauchnykh Trudov) Vol. 1, Nauka, Moscow (1965), p. 106].[33)]

[7]C. Neumann, "Ueber die Kräften elektrodynamischer Ursprungs zuzuschreibenden Elementargesetze," Abhandlungen Math. Kl. Königl. Sächsischen Gesellschaft Wissenschaft (Leipzig) **10**, pp. 417–524 (1973); quoted in the Candidate's Dissertation: M. M. Kuchment, Osnovnye Étapy Formirovaniya N'yutonovskoĭ Kosmologii (Main Stages in the Formation of the Newtonian Cosmology), Institute of the History of Natural Science and Technology, USSR Academy of Sciences, Moscow (1972).

[8]A. Einstein, "Über das Relativitätsprinzip und die aus demselben gezogenen Folgerungen," Jahrb. d. Radioaktivität u. Elektronik **4**, 411–462 (1907), Sec. 18 (Sobranie, Vol. I, p. 106).

[9]A. Einstein, "Über den Einfluss der Schwerkraft auf die Ausbreitung des Lichtes," Ann. Phys. **35**, 898–908 (1911) (Sobranie, Vol. I, p. 165).

[10]A. Einstein, "Zum gegenwärtigen Stande des Gravitationsproblems," Phys. Z. **14**, 1249–1262 (1913) (Sobranie, Vol. I, p. 273).

[11]A. Einstein, "Lichtgeschwindigkeit und Statik des Gravitationsfeldes," Ann. Phys. **38**, 355–369 (1912) (Sobranie, Vol. I, p. 189).

[12]A. Einstein, "Zur Theorie des statischen Gravitationsfeldes," Ann. Phys. **38**, 443–458 (1912) (Sobranie, Vol. I, p. 202).

[13]From the Correspondence between Sommerfeld and Einstein, in: A. Sommerfeld, Puti noznaniya v fizike (Paths to knowedge in physics), Russian translation published by Nauka, Moscow (1973).

[14]A. Einstein and M. Grossmann, "Entwurf einer verallgemeinerten Relativitätstheorie und Theorie der Gravitation," Z. Math. und Phys. **62**, 225–261 (1913) (Sobranie, Vol. I, p. 227).

[15]Quoted in: K. Zelig, Al'bert Eĭnshteĭn (Albert Einstein), Atomizdat, Moscow (1964), p. 120.

[16]Quoted in the paper: H. Hönl, "K Istorii Printsipa Makha (On the History of Mach's Principle) in: Eĭnshteĭnovskiĭ Sbornik, 1966 (Einstein Collection. 1966), Nauka, Moscow (1968), p. 262.

[17]G. Nordström, "Zur Theorie der Gravitation vom Standpunkt des Relativitätsprinzip," Ann. Phys. **42**, 533 (1913).

[18]A. Einstein, "Einiges über die Entstehung der allgemeinen Relativitätstheorie," George A. Gibson Foundation Lecture, Glasgow (20th June, 1933) (Sobranie, Vol. II, pp. 405–406).

[19]A. Einstein, "Gibt es eine Gravitationswirkung, die der elektrodynamischen Induktionswirkung analog ist?" Vierteljahrschr. Gerichtl., Med., Ser. 3, **44**, 37–40 (1912) (Sobranie, Vol. I, p. 223).

[20]A. Einstein, "Relativität und Gravitation. Erwiderung auf eine Bemerkung von M. Abraham," Ann. Phys. **38**, 1059–1064 (1912) (Sobranie, Vol. I, p. 217).

[21]A. Einstein, *ibid*., p. 221.

[22]A. Einstein, Principe de relativité et ses conséquences dans la physique moderne," Arch. Sci. Phys. Natur., Ser. 4, **29**, 5–28, 125–144 (1910), Sec. 9 (Sobranie, Vol. I, p. 161).

[23]A. Einstein, "Die Relativitätstheorie," Naturforsch. Gesellschaft, Vierteljahresschrift, Zürich, **56**, 1–14 (1911) (Sobranie, Vol. I, p. 186).

[24]A. Einstein, see Ref. 11, p. 190 in the Sobranie.

[25]M. Planck, "Printsip Otnositel'nosti i Osnovnye Uravneniya Mekhaniki," (The relativity principle and the basic equations of mechanics), in: Izbrannye Trudy (Selected Works), Nauka, Moscow (1975), p. 445.

[26]W. Pauli, Relativitätstheorie, in: Enzyklopädie der mathematischen Wissenschaften, II, Teubner, Leipzig [Russian translation published by Gostekhizdat, Moscow-Leningrad (1947)].

[27]B. Hoffman, "Einstein and tensors," Tensor **26**, 157 (1972).

[28]A. Einstein, see Ref. 12, p. 216 in the Sobranie.

[29]A. Einstein, see Ref. 20, p. 220 in the Sobranie.

[30]A. Einstein, see Ref. 19, p. 225 in the Sobranie.

[31]Quoted in the paper: G. Holton, "Eĭnshteĭn o Fizicheskoĭ Real'nosti," (Einstein on physical reality), in: Eĭnshteĭnovskiĭ Sbornik. 1969–1970 [Einstein Collection. 1969–1970, Nauka, Moscow (1970), p. 212].

[32]V. Ya. Frenkel', Paul' Érenfest (Paul Ehrenfest), Atomizdat, Moscow (1977).

[33]P. Ehrenfest, "Ravnomernoe Vrashchatel'noe Dvizhenie Tverdykh Tel i Teoriya Otnositel'nosti," ("Uniform rotational motion of rigid bodies and the theory of relativity,") in: P. Ehrenfest, Otnositel'nost', Kvanty, Statistika (Relativity, Quanta, Statistics), Nauka, Moscow (1972), p. 37.

[34]A. Einstein, "Zum Ehrenfastschen Paradoxon," Phys. Z. **12**, 509, 510 (1911) (Sobranie, Vol. I, p. 187).

[35]Ehrenfest-Ioffe, Nauchnaya Perepiska, 1907–1933 (Scientific Correspondence, 1907–1933), Nauka, Leningrad (1973), p. 81.

[36]L. R. Pyenson, "Einstein's early scientific collaboration," Hist. Studies Phys. Sci. **7**, 83 (1976).

[37]P. Frank, Einstein, his Life and Times, Knopf, New York (1947).

[38]H. Z. Teske, "Einsteinovy Prazske korrespondence," Acta Historiae Rerum Naturalium Necnon Technicarum (Praha) **7**, 228 (1962).

[39]J. Illy, Albert Einstein in Prague (unpublished paper).

[40]F. Herneck, "Die Beziehungen Zwischen Einstein und Mach," Wiss. Z. Friedrich-Schiller-Univ. Jena, Math.-Naturwiss. Reihe **15**, 1 (1966).

[41]P. S. Kudryavtsev, Istoriya Fiziki. Tom 3 (The History of Physics, Vol. 3), Prosveshchenie, Moscow, 1971, p. 82–83.

[42]R. Dicke, Teoriya Gravitatsii i Nablyudeniya (Theory of Gravitation and Observation), Quoted in Ref. 31, p. 115.

[43]A. Einstein, see Ref. 14, p. 248 in the Sobranie.

[44]A. Einstein, see Ref. 14, p. 232 in the Sobranie.

[45]F. Kottler, "Ueber die Raumzeitlinien der Minkowskischen

Welt," Wien. Ber. Abt. IIA **121**, 1659 (1912).
[46]H. Bateman, "The transformations of the electrodynamica equations," Proc. London Math. Soc. Ser. 2, **8**, 223 (1910).
[47]A. Einstein and M. Grossmann, see Ref. 14, p. 262 in the Sobranie.
[48]A. Einstein and M. Grossmann, see Ref. 14, p. 265 in the Sobranie.
[49]W. Pauli, see Ref. 26, p. 233 of the Russian translation.
[50]F. Herneck, see Ref. 40, p. 8 of the Russian translation.
[51]A. Einstein, "Physikalische Grundlagen einer Gravitationstheorie," Naturforsch. Gessellschaft, Zürich, Vierteljahrschr. **58**, 284–290 (1913) (Sobranie, Vol. I, p. 270).
[52]A. Einstein, See Ref. 10, p. 293 in the Sobranie.
[53]A. Einstein and M. Grossmann, see Ref. 14, p. 243 of the Sobranie.
[54]A. Einstein and A. D. Fokker, "Die Nordströmsche Gravitationstheorie vom Standpunkt des absoluten Differentialkalküls," Ann. Phys. **44**, 321–328 (1914) (Sobranie, Vol. I, p. 305).
[55]A. Einstein and A. D. Fokker, *ibid.*, p. 312.
[56]Correspondence between A. Einstein and M. Besso, in: Eĭnshteĭnovskiĭ Sbornik. 1974 (Einstein Collection. 1974), Nauka, Moscow, 1976, p. 40.
[57]A. Einstein and M. Grossmann, "Kovarianzeigenschaften der Feldgleichungen der auf die verallgemeinerte Relativitätstheorie gegründeten Gravitationstheorie," Z. Math. und Phys. **63**, 215–225 (1914) (Sobranie, Vol. I, p. 399).
[58]A. Einstein, "Die formale Grundlage der allgemeinen Relativitätstheorie," Sitzungsber. preuss. Akad. Wiss. **2**, 1030–1085 (1914) (Sobranie, Vol. I, p. 326).
[59]A. Einstein and M. Grossmann, see Ref. 57, p. 400 in the Sobranie.
[60]A. Einstein and M. Grossmann, see Ref. 57, p. 402 in the Sobranie.
[61]M. Abraham, "Die neuere Gravitationstheorien," Jahrb. Radioakt. Elektron **11**, 514 (1914).
[62]M. Abraham, *ibid.*, p. 517.
[63]M. Abraham, *ibid.*, p. 520.
[64]A. Einstein and W. J. de Haas, "Experimenteller Nachweis der Ampèreschen Molekularströme," Verhandl. Dtsch. Phys. Ges. **17**, 152–170 (1915) (Sobranie, Vol. III, p. 363).
[65]Correspondence between A. Einstein and M. Besso, see Ref. 56, p. 44.
[66]W. Pauli, see Ref. 26, p. 224 of the Russian translation.
[67]A. Einstein, "Erklärung der Perihelbewegung der Merkur aus der allgemeinen Relativitätstheorie," Sitzungsber. preuss. Akad. Wiss. **47**, 2, 831–839 (1915) (Sobranie, Vol. I, p. 439).
[68]A. Einstein, "Zur allgemeinen Relativitätstheorie," Sitzungsber. preuss. Akad. Wiss. **44**, 2, 778–786 (1915) (Sobranie, Vol. I, p. 425).
[69]A. Einstein, "Zur allgemeinen Relativitätstheorie. (Nachtrag)," Sitzungsber. preuss. Akad. Wiss. **46**, 2, 799–801 (1915) (Sobranie, Vol. I, p. 435).
[70]A. Einstein, "Die Feldgleichungen der Gravitation," Sitzungsber. preuss. Akad. Wiss. **48**, 2, 844–847 (1915) (Sobranie, Vol. I, p. 448).
[71]A. Einstein, *ibid.*, p. 450.
[72]Correspondence between Sommerfeld and Einstein, see Ref. 13, p. 193 of the Russian translation.
[73]E. Freundlich, "Ueber die Erklärung der Anomalien im Planetensystem durch die Gravitationswirkung interplanetarer Massen," Astron. Nachr. **201**, 49 (1915).
[74]A. Einstein, see Ref. 68, p. 430 of the Sobranie.
[75]A. Einstein, see Ref. 67, p. 440 of the Sobranie.
[76]A. Einstein, "Die Grundlage der allgemeinen Relativitätstheorie," Ann. Phys. **49**, 769–822 (1916) (Sobranie, Vol. I, p. 452).
[77]D. Hilbert, "Die Grundlagen der Physik," in: Gesammelte Abhandlungen, Vol. 3, Springer, Berlin (1935), p. 258 [Russian translation published in: Variatsionnye Printsipy Mekhaniki (Variational Principles of Mechanics), Fizmatgiz, Moscow, 1959, p. 589].
[78]D. Hilbert, "Mathematische Probleme," in: Gesammelte Abhandlungen, Vol. 3, Springer, Berlin (1935), p. 290 [Russian translation published in: Problemy Gil'berta (Hilbert's Problems), Nauka, Moscow, 1969, pp. 34–35].
[79]F. Kein, "Comparative review of recent researches in geometry (Erlangen Program)," Am. Math. Soc. Bull. **2**, 35 (1893) (Russian translation published in: Ob Osnovaniyakh Geometrii (On the Foundations of Geometry), Gostekhizdat, Moscow, 1956, p. 399.
[80]A. Einstein, "Das hamiltonisches Prinzip und allgemeine Relativitätstheorie," Sitzungsber. preuss. Akad. Wiss. **2**, 1111–1116 (1916) (Sobranie, Vol. I, p. 524).
[81]D. Hilbert, see Ref. 77, p. 596 of the Russian translation.
[82]E. Noether, "Invariante Variationsprobleme," Kgl. Ges. d. Wiss. Nachrichten. Math.-Phys. Klasse **2**, 17 (1918) (Russian translation published in the same collection as the translation of Ref. 77, p. 611).
[83]V. P. Vizgin, Razvitie Vzaimosvyazi Printsipov Invariantnosti s Zakonami Sokhraneniya v Klassicheskoĭ Fizike (Development of the Interconnection between Invariance Principles and Conservation Laws in Classical Physics), Nauka, Moscow, 1972.
[84]D. Hilbert, see Ref. 77, p. 597 of the Russian translation.
[85]F. Klein, Gesammelte mathematische Abhandlungen, Vol. 1, Springer, Berlin, 1921, p. 568.
[86]W. Pauli, see Ref. 26, p. 211 of the Russian translation.
[87]H. Weyl, "David Hilbert and his mathematical work," in: C. Reid, Hilbert, Berlin, 1970 (Russian translation by Nauka, Moscow, 1977, p. 359).
[88]J. Mehra, Einstein, Hilbert and the Theory of Gravitation, Dordrecht-Boston, 1974.
[89]J. Earman and C. Glymour, "Einstein and Hilbert: two months in the history of general relativity," Arch. Hist. Exact Sci. **19**, 291 (1978).
[90]J. Michell, Philos. Trans. R. Soc. London **34**, Pt. I, 35 (1784).
[91]H. Cavendish, Scientific Papers, Vol. II, Cambridge, 1921, p. 437.
[92]Al'bert Éĭnshteĭn i Teoriya Gravitatsii (Albert Einstein and the Theory of Gravitation), Mir, Moscow, 1979.

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