Virial theorem for a system of charged particles

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Two formulations of the virial theorem are used in practice: one for separate particles and one for a continuous medium. The virial theorem for a system of charged particles which was given by Landau and Lifshitz in their book *The Classical Theory of Fields* should incorporate both these formulations. However, there is an error there in the derivation of this theorem, which is based on transformation from a discussion in terms of particles to a discussion in terms of a continuous medium. Specifically, the self-effect force of the charges is not eliminated. As a result, the infinite self-energy of the charges is not eliminated, and the corresponding final equation cannot be satisfied. In the present note, a refined formulation of the virial theorem for a system of charged particles is given. The renormalization of the total electromagnetic field energy is taken into account.

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In this note we examine the relationship among various formulations of the virial theorem. We find an error in the derivation and formulation of the virial theoren for a system of charged particles in the book *The Classical Theory of Fields* by L.D. Landau and E.M. Lifshitz. This error is repeated in other publications.

The virial theorem is one of the integral consequences of the equation of motion of a continuum or of a system of interacting particles. It determines the "global" conditions under which a system is confined to a finite volume (or the conditions that the motion of the particles is finite) without appealing to the particular structure of the system. For a system of particles with a Coulomb interaction, for example, this confinement condition is[1]

$$U=-2T. \tag{1}$$

In other words, the average potential energy must be negative (and this is possible only if particles with charges of different sign are present) and equal in magnitude to twice the kinetic energy.[1)] For a region in a continuous medium, e.g., a plasma in a volume V, the equilibrium condition is[2]

$$\int\limits_V \left(\rho v^2+3p+\frac{H^2+E^2}{8\pi}\right)dV = \oint\left\{\left(p+\frac{H^2+E^2}{8\pi}\right)\mathbf{r}\,d\mathbf{S}+\rho(\mathbf{vr})(\mathbf{v}\,d\mathbf{S})-\frac{(\mathbf{Hr})(\mathbf{H}\,d\mathbf{S})+(\mathbf{Er})(\mathbf{E}\,d\mathbf{S})}{4\pi}\right\}, \tag{2}$$

where ρ is the mass density, $\mathbf{v}$ is the local velocity of a fluid element of the volume, p is the pressure, and $\mathbf{H}$ and $\mathbf{E}$ are the magnetic and electric fields. The integral on the right is over the surface bounding the volume under consideration. For an isolated system this integral vanishes if the integration is extended to an infinite volume, and the equilibrium condition

$$\int\limits_V \left(\rho v^2+3p+\frac{H^2+E^2}{8\pi}\right)dV=0 \tag{3}$$

obviously does not hold. For large (astronomical) masses equilibrium can be maintained by gravitation; gravitation would provide a negative term $(-\nabla\Phi)^2/8\pi\gamma$ in the integrand, where Φ is the gravitational potential, and γ is the gravitational constant.[3] If gravitational forces are negligible, on the other hand, the confinement of a plasma to a bounded volume requires the use of external electromagnetic fields, and an external pressure[2] p_e is required to maintain the electromagnetic field in the finite volume. In neither of these cases does the surface integral in (2) vanish, and condition (2) in these cases tells us just what these external fields or the pressure p_e must be. The theorem expressed by Eq. (2) is also important for certain technological problems. For example, it immediately tells us that it is not possible to develop a completely force-free magnet coil.

A comparison of conditions (1) and (3) shows that these two forms of the virial theorem cannot be reconciled. For example, setting $p=0$ in (3) and using $\int\rho v^2dV=2T$, we find, instead of (1), the condition

$$\int\frac{\overline{E^2}+\overline{H^2}}{8\pi}\,dV=-2\overline{T}, \tag{4}$$

which differs from (1) in that its left side is clearly positive. The reason for this result is that Eq. (2) incorporates only an average over the volume element of the electromagnetic field, in which binary interactions are not taken into account, while condition (1), in contrast, takes only these binary interactions into account.

It is natural to suggest that a generalization of (1) and (2) should be a virial theorem for a system of charged particles which has the form, according to Ref. 4, of the vanishing of an integral over the sum of diagonal elements of the total stress tensor:

$$\int\overline{T}_{\alpha\alpha}\,dV=0. \tag{5}$$

From this condition a virial theorem is derived in the following form in Ref. 4:

$$\mathscr{E}=\sum_a \overline{m_a c^2\sqrt{1-\frac{v_a^2}{c^2}}}, \tag{6}$$

1)A condition of the type (1) is actually the basis for the classical-mechanics explanation of not only the confinement of planets or of electrons in an atom but also the confinement of atoms in the crystal lattice of a solid.

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where m_a and v_a are the mass and velocity of particle a, c is the speed of light, and $\mathscr{E}$ is the energy of the system. It follows from this derivation that

$$\mathscr{E}=\int \frac{\overline{E^2}+\overline{H^2}}{8\pi}\,dV+\sum \frac{m_a c^2}{\sqrt{1-(v_a^2/c^2)}}\,, \tag{7}$$

so that (6) is equivalent to the condition

$$\int \frac{\overline{E^2}+\overline{H^2}}{8\pi}\,dV+\sum \frac{\overline{m_a v_a^2}}{\sqrt{1-(v_a^2/c^2)}}=0, \tag{8}$$

which has the same form as the impracticable condition in (4). Contrary to expectation Eqs. (8) and (1) are mutually contradictory. The reason for this contradiction is an error in the derivation of condition (5) for the case of point charges. To clarify the question, let us go back to the derivation of Eqs. (1), (4), and (8).

Equation (1) is the result of multiplication of the equation of motion of particle a,

$$\frac{d\mathbf{p}_a}{dt}=\mathbf{F}(\mathbf{r}_a)\,, \tag{9}$$

by the radius vector of this particle $\mathbf{r}_a$, averaging over time, and a summation over the particles. Here it is assumed that $(\mathbf{F}(\mathbf{r}_a=-\partial U/\partial \mathbf{r}_a)$. In the derivation of Eqs. (2) and (4), on the other hand, the force $\mathbf{F}$ in the equation of motion of the fluid,

$$\rho\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\nabla\mathbf{v}=-\nabla p+\mathbf{F}, \tag{10}$$

$$\mathbf{F}=\rho_E\mathbf{E}+\frac{1}{c}[\mathbf{j}\mathbf{H}], \tag{11}$$

is transformed by means of Maxwell equations

$$4\pi\rho_E=\operatorname{div}\mathbf{E},\qquad \frac{4\pi}{c}\mathbf{j}=-\frac{\partial \mathbf{E}}{c\,\partial t}+\operatorname{rot}\mathbf{H} \tag{12}$$

into the divergence of the Maxwell stress tensor. As a result, when the continuity equation is taken into account, the time average of the momentum-transport equation becomes

$$\frac{\partial \overline{T}_{\alpha\beta}}{\partial x_\beta}=0, \tag{13}$$

where

$$T_{\alpha\beta}=\left(p+\frac{H^2+E^2}{8\pi}\right)\delta_{\alpha\beta}+\rho v_\alpha v_\beta-\frac{H_\alpha H_\beta+E_\alpha E_\beta}{4\pi}. \tag{14}$$

Multiplying by x_α and integrating over the volume, we find relations of the type of (5), (2), etc.

If ρ_E and $\mathbf{j}$ are understood to be the values of the electric charge density and current density, averaged over the volume element, we find a noncontradictory equilibrium condition (2) for a continuous medium. This condition does not necessarily have to be consistent with (1), since the equation of motion in this case contains an electromagnetic field averaged over the volume element which does not take binary interactions into account. If, on the other hand, we understand ρ_E and $\mathbf{j}$ to be the microscopic densities

$$\rho_E=\sum_a e_a\delta(\mathbf{r}-\mathbf{r}_a),\qquad \mathbf{j}=\sum_a e_a\mathbf{v}_a\delta(\mathbf{r}-\mathbf{r}_a), \tag{15}$$

then we should take into account the fact that at the points $\mathbf{r}=\mathbf{r}_a$ the electromagnetic field has a singularity, so that the force associated with the self-effect of the charges is incorporated in the expression for $\mathbf{F}$ in (11), which leads to (14) and thus to (8). Since nothing in these equations cancels the repulsive effect of the field of a charge on the charge itself, the result is the impracticable confinement condition in (8).

The self-effects should in actual fact be eliminated from the equations of motion (except for the radiation reaction force, but we will not deal with this effect here). Since Maxwell equations are linear, the electromagnetic field can be written as the sum of the fields produced by the various individual charges,

$$\mathbf{E}=\sum_a \mathbf{E}_a,\qquad \mathbf{H}=\sum_a \mathbf{H}_a, \tag{16}$$

so that, in continuous-medium terms, the self-effect force density is

$$\mathbf{F}_c=\sum e_a\left(\mathbf{E}_a+\frac{1}{c}[\mathbf{v}_a\mathbf{H}_a]\right)\delta(\mathbf{r}-\mathbf{r}_a). \tag{17}$$

The time average of this force can obviously be written as the divergence of $\sum_a(E_a^2+H_a^2)/8\pi$. Then instead of (8) we find a virial theorem in the form

$$\int \frac{\overline{E^2}+\overline{H^2}}{8\pi}\,dV-\int\sum_a \frac{E_a^2+H_a^2}{8\pi}\,dV+\sum \frac{\overline{m_a v_a^2}}{\sqrt{1-(v_a^2/c^2)}}=0. \tag{18}$$

The difference between the first two terms, which is equal to the total electromagnetic energy of the system of charges after the electromagnetic self-energy of the point charges has been subtracted, contains the interaction energy U in the nonrelativistic limit. This energy, even for a low-density plasma, is known to be negative[5]: $U=-TV/8\pi d^3$, where T is the plasma temperature, and D is the Debye length. A subtraction procedure is used directly in, for example, Ref. 6.

We thus see that the error in Ref. 4 is that in the transformation from the forces to the stress tensor the self-effect force is implicitly retained, and this force leads to an infinite field energy, which is not cancelled out in the confinement condition expressed by the virial theorem.

To find the correct result we should replace $\mathscr{E}$ in the equations in Ref. 4 by the renormalized total energy

$$\mathscr{E}'=\int \frac{\overline{E^2}+\overline{H^2}}{8\pi}\,dV-\int\sum_a \frac{E_a^2+H_a^2}{8\pi}\,dV+\sum_a \frac{\overline{m_a c^2}}{\sqrt{1-(v_a^2/c^2)}}\,. \tag{19}$$

The transformation to the scalar φ and vector $\mathbf{A}$ potentials of the electromagnetic field enables us to write

$$\mathscr{E}'=\sum_a e_a\left[\varphi(r_a)+\frac{\mathbf{v}_a}{c}\mathbf{A}(\mathbf{r}_a)\right]+\sum_a \frac{\overline{m_a c^2}}{\sqrt{1-(v_a^2/c^2)}}, \tag{20}$$

where $\varphi(\mathbf{r}_a)$ and $\mathbf{A}(\mathbf{r}_a)$ are the potentials generated by all charges other than charge a. With this change, the virial theorem in (6) takes a form in complete correspondence with (1). On the other hand, it obviously also applies to a continuous medium.

We might also note that this required renormalization of the electromagnetic field energy of a system of a point particles is also omitted by Rosenbluth and Stuart,[7] in their generalization of the virial theorem to the case of infinite motion (for which the time derivatives are conserved). This error went undetected because in that paper the theorem was applied directly to a light wave packet, i.e., to a region without any point particles.

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