The renormalization group and ultraviolet asymptotics

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This article describes in detail the method of the renormalization group and outlines the possibilities of using it for the analysis of high-energy asymptotics in the framework of quantum field theory. The renormalization group formalism is constructed for an arbitrary scheme of renormalization. The exposition is based on the concepts of effective charge and effective mass. Principal attention is given to the problem of deriving reliable information about the ultraviolet properties of quantized-field models on the basis of perturbation theory calculations. A summary of the results of such calculations for a wide variety of models is given.

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CONTENTS

1. INTRODUCTION

The method of the renormalization group (RG) arose 25 years ago, after the development of the apparatus of renormalization in quantum field theory (QFT). It was pointed out[1] that the transformations of multiplicative renormalizations of quantized fields and coupling constants form a group, with observable quantities (matrix elements of the S matrix) as invariants under these transformations. It was also discovered[2-4] that this property of QFT (renormalization invariance) imposes definite restrictions on the form of the dependence of quantized-field Green's functions (GF) on their arguments, which in a number of cases make it possible to find some characteristics of this dependence (as a rule, the asymptotic form of the Green's functions for large momenta). Effectively, the application of the RG method corresponds to the summation of the leading asymptotic terms of infinite subclasses of Feynman diagrams.

The RG method is now widely used both for purely theoretical investigations in QFT (for example, for finding the ultraviolet and infrared asymptotics of the GF for various models, studying the problems of dynamic symmetry breaking, and so on) and also for studying the high-energy behavior of various physical processes, such as deep inelastic lepton-hadron interactions and electron-positron annihilation in hadrons, and so on. A modification of the RG method developed by Wilson has been used with success in statistical physics to describe critical phenomena (phase transitions).[5,6]

Apparently the best known formula associated with the term "renormalization group" is the expression for the effective charge in the leading-logarithm approximation,

$$\bar{g}\left(\frac{p^2}{\mu^2},\, g\right)=\frac{g}{1+ag\,\ln(p^2/\mu^2)}. \quad (1.1)$$

For $a<0$ (the situation of the so-called "zero of charge") $\bar{g}$ has an unphysical pole, and the corresponding quantum-field model is not internally consistent. If, on the

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other hand, $a>0$ ("asymptotic freedom"), then as the momentum increases the effective coupling constant g goes monotonically to zero, i.e., the interaction vanishes at small distances. This allows the use of asymptotically free models for the description of lepton-hadron processes with large momentum transfer.

The expression (1.1) for the effective charge is at present the basis of most of the applications of the RG in physics. Besides this, the RG method is a consistent scheme for the calculation of corrections to this formula. In the case of the zero of charge will the higher corrections be able somehow to straighten out the situation, and in the case of asymptotic freedom will they give further useful information about high-energy asymptotic behaviors, and improve the agreement between theory and experiment? To a considerable extent, our survey is devoted to a detailed analysis of these questions.

The basis of all quantized-field applications of the RG are the renormalization-group equations. In the literature there are somewhat different ways of writing the equations of the RG, largely due to the variety of possible schemes of renormalization in QFT. Owing to the property of renormalization invariance of quantum-field models, all renormalization schemes, like the different "versions" of the equations of the RG, are equivalent to each other and lead in the end to identical results. Therefore in the first part of this paper (chapters 2–6) we present what we believe is the most economical procedure for deriving the main relations of the RG, which enables us to treat all of the renormalization schemes from a unified point of view and includes all known versions of the RG as special cases. Special attention is also given to a comparison of the forms of the RG equations in different schemes of renormalization. We also examine in detail the passage to the ultraviolet asymptotic limit.

The second part (Chapters 7–9) is devoted to applications of the RG method in problems of high-energy physics. It contains a summary of the results of specific calculations of asymptotic properties of Green's functions made up to the present time. All of the results now in the literature on one-charge models are presented; among the many-charge theories those are considered which are most characteristic and interesting from the RG point of view. In Chapter 9 a brief review is given of applications of the RG to the analysis of a number of physical processes at large momenta.

As can be seen from the title of our article, we have not undertaken to describe the entire gamut of applications of the RG method to problems of particle physics. Therefore many extremely interesting examples of the use of the renormalization group which do not relate to the problem of high-energy asymptotics are not mentioned.

2. VARIOUS TYPES OF RENORMALIZATION PROCEDURES

In QFT giving the Lagrangian does not completely specify the quantitative properties of the corresponding system of fields. The set of parameters appearing in the Lagrangian (the masses m_i of the original fields and the coupling constants g_k) are not sufficient for the numerical description of transition probabilities, masses of physical particles, and so on.

The cause of this peculiar situation lies in the presence of ultraviolet divergences (we shall not consider at this point more delicate questions associated with spontaneous symmetry breaking or degeneracy of the vacuum). In the process of "reworking" and eliminating these divergences new parameters arise, some of which (such as cut-off momenta) disappear after the auxiliary regularization is removed, and some (for example, points of subtraction) remain and appear in the divergence-free final results of perturbation theory (PT) calculations. Therefore, in prescribing a Lagrangian for a QFT model we are essentially fixing only qualitative features of the theory (types of particles, selection rules, topology of diagrams, symmetry properties). To obtain quantitative results it is necessary to stipulate, in addition to the Lagrangian, a recipe for eliminating infinities.

Appropriate rules can be formulated unambiguously[4] and contain the above-mentioned arbitrary parameters $z_1, \ldots, z_N \equiv \{z\}$, which, at the end of the calculations, will appear in the expressions for the matrix elements of the S matrix

$$\mathscr{M} = \mathscr{M}(\{p\}, \{g\}, \{m\}, \{z\}); \tag{2.1}$$

here $\{p\}$ is the set of momenta of particles involved in the given process.

It is well known that in renormalizable theories (i.e., in cases when N, the number of arbitrary parameters z_i, is finite) the matrix elements $\mathscr{M}$ actually depend not on $\{g\}$, $\{m\}$, and $\{z\}$ separately, but on certain combinations of them $g_{\rm phys}(g, m, z)$ and $m_{\rm phys}(g, m, z)$

$$\mathscr{M} = \overline{\mathscr{M}}(\{p\}, \{g_{\rm phys}\}, \{m_{\rm phys}\}). \tag{2.2}$$

Therefore the arbitrariness in the parameters $\{z\}$ does not affect the values $\mathscr{M}$, since a change of these parameters

$$\{z\} \to \{z'\} \tag{2.3}$$

can always be compensated for by a change of the parameters of the Lagrangian (a renormalization)

$$\{g\} \to \{g'\}, \quad \{m\} \to \{m'\}, \tag{2.4}$$

so that in the final results $\{g_{\rm phys}\}$ and $\{m_{\rm phys}\}$ are unchanged. The numerical values of the "physical" constants $\{g_{\rm phys}\}$ and $\{m_{\rm phys}\}$ are determined from experiment. A knowledge of the explicit form of the function $\overline{\mathscr{M}}$ now gives an unambiguous recipe for calculating the matrix elements in the given (renormalized) model.

The transformations (2.3) of the parameters $\{z\}$ and the corresponding transformations (2.4) of the charges and masses form a group,[1] which is called the group of renormalizations or the renormalization group (RG). The basis of the construction of the entire RG apparatus is the fact that, as we have stated, $\mathscr{M}$ is actually independent of $\{z\}$.

As we shall show, the invariance of the theory under transformations of the RG can be used successfully in the study of the behavior of certain quantum-field quantities under a uniform scale transformation of all momenta. For these purposes it is sufficient (and convenient) to separate out from the entire set of parameters $\{z\}$ (whose number is determined by the number of different field structures in the original Lagrangian) a one-parameter family, thus making the renormalization scheme more specific. The corresponding parameter of the dimension of mass is retained as a supplementary argument (in actual fact a fictitious one) in all the expressions and is called the renormalizing parameter. Different schemes of renormalization, i.e., different recipes for selecting a one-parameter family from $\{z\}$, are equivalent in the sense of the calculation of physical quantities, since they can be reduced to each other by the transformation (2.3).

We shall give examples of the most widely used renormalization schemes.

a) Subtraction scheme

A renormalization procedure which is very widely used and convenient for practical calculation of matrix elements and probabilities is based on the subtraction of divergent integrals. One first introduces an auxiliary parameter, which regularizes (i.e., makes finite) the integrals over the internal momenta of Feynman diagrams. This parameter may be a cut-off momentum Λ or the mass M of a Pauli-Villars auxiliary field (or a dimensional-regularization parameter: see following section). From the regularized integrals one subtracts their values (in general, the first few terms of the Taylor's series) at certain fixed momenta $p_i=\tilde{p}_i$. Then one removes the auxiliary regularization (i.e., goes to the limit $\Lambda\to\infty$, $M\to\infty$, or $\varepsilon\to 0$). In the resulting final expressions there is still a dependence on the points of subtraction $\{\tilde{p}\}$.

A virtue of this method is that it is physically intuitive, since in a number of cases the momenta at which the subtractions are done can be chosen so that one can use for $\{g\}$ and $\{m\}$ the values of the charges and masses known from experiment. We note that in the usual formulation of the RG on the basis of the subtraction scheme[4] no mass renormalization at all is included among the transformations of the RG. This is achieved by imposing additional restrictions on the subtraction procedure and leads to some simplifications of the equations of the RG.

In order to have the possibility of working with a single renormalization parameter (a point of subtraction, commonly denoted by λ^2), we must set $\tilde{p}_i^2=\rho_i\lambda^2$. The coefficients ρ_i are fixed and characterize the concrete choice of renormalization procedure.

b) Dimensional renormalization scheme

In recent years a scheme for removing divergences based on dimensional regularization (see the review article, Ref. 8) has become very popular. This regularization involves changing in the formulas for integration over the virtual momenta from the natural physical dimensionality $n=4$ to $n=4-2\varepsilon$, where ε is a small parameter giving the deviation of the dimensionality n from its normal value. Then, for $\varepsilon\to 0$, all the integrals are finite, and the divergences appear as poles in the variable ε (of types $1/\varepsilon, 1/\varepsilon^2$, etc.).

Dimensional renormalization is a quite definite procedure of subtracting regularized integrals, consisting of subtracting only the poles in the variable ε (i.e., only the terms singular in ε in the expansion of the regularized integral in a Laurent series). This means that the counterterms (if we describe the subtraction by means of introduction of counterterms in the Lagrangian) in the scheme of dimensional renormalization are products of series in negative powers of ε and appropriate operators

$$\Big[\sum_{l=1}^{\infty}\frac{1}{\varepsilon^l}a_l(\{g\},\{m\})\Big]\prod_i\varphi_i(x).$$

The coefficients a_l are determined uniquely if we require that in the expressions after the subtraction the limit 0 can be taken, i.e., that the regularization can be removed.

The renormalization parameter in this scheme is a parameter of the dimensions of mass, denoted by μ, which assures the dimensionlessness of the regularization process and is introduced into the Lagrangian in multiplicative combinations with coupling constants of the type $g(\mu^2)^\varepsilon$.

Its economy is a virtue of this dimensional renormalization scheme (only one new parameter), as are also the preservation of symmetry properties and the simplicity of the technique of calculation. Moreover, this scheme belongs to the category of "massless" schemes, i.e., those in which the counterterms for renormalization of the wave functions and the dimensionless coupling constants do not depend on the masses,[9] i.e., $a_l=a_l(\{g\})$. The first example of a "massless" renormalization scheme was proposed by Weinberg.[10] It was shown in Ref. 11 that such renormalization schemes can be constructed on the basis of any regularization.

c) Cut-off scheme

This historically first and intuitively natural scheme of renormalization, based on an ultraviolet cut-off, has two main versions: The "Feynman cut-off," in which the the free propagator is modified in the following way,

$$\frac{1}{k^2}\to\frac{1}{k^2}-\frac{1}{k^2-\Lambda^2}=\frac{-\Lambda^2}{k^2(k^2-\Lambda^2)},$$

and a cutting off of the momentum integrals over the radial variable (after the "Wick rotation" of the axes of integration to get a Euclidean metric) at an upper limit Λ. After regularizing in this way one can, instead of eliminating the cut-off parameter Λ from the theory by subtractions, give to it the meaning of a renormalization parameter. To do this one must discard from the renormalized integrals all quadratic divergences (those proportional to Λ^2), and also all terms containing factors of the form $(\Lambda^2)^{-N}, N>0$ (which corresponds to using expressions that are asymptotic, in the sense $\Lambda^2\to\infty$). This procedure leads to yet another "massless" renormalization scheme which, despite some shortcomings, such as the absence of gauge invariance, is useful in scalar theories, since it makes the calculations remarkably simple.

If QFT still other renormalization schemes have found application (for example, so-called analytical renormalization), but we shall not need them in what follows and shall not discuss them.

3. MULTIPLICATIVE RENORMALIZATION OF GREEN'S FUNCTIONS

An important part is played in QFT by Green's functions (GF) (complete propagators) and by strong-coupling vertex functions. We shall be concerned here only with Green's functions that have been "made dimensionless" (i.e., scalar dimensionless factors with independent Lorentz and other structures, into which the original Green's functions are decomposed), and shall denote them by

$$\Gamma\left(\left\{-\frac{p^2}{\mu^2}\right\}, \left\{\frac{m^2}{\mu^2}\right\}, \{g\}\right)$$

(for propagators we shall sometimes use the special notation d), where μ^2 is the renormalization parameter. We agree to normalize GF so that the contribution to them from the corresponding Born diagrams will be equal to unity (in the cases we shall need this can always be done). Then

$$\Gamma\left(\left\{-\frac{p^2}{\mu^2}\right\}, \left\{\frac{m^2}{\mu^2}\right\}, \{0\}\right)=1. \tag{3.1}$$

It must be emphasized here that by the coupling constants $\{g\}$ we mean either the coefficients of products of field operators in the interaction Lagrangian, or the squares of these coefficients, if they are the actual parameters of the perturbation theory expansion. For example, in quantum electrodynamics the coupling constant g is the square of the electronic charge, e^2.

In renormalizable theories the dimensions of coupling constants in mass units can be larger than or equal to zero. From the point of view of the RG, dimensionless constants are decidedly different from the dimensional ones, and for the latter the renormalization is very similar in structure to that for masses. Owing to this we do not put dimensional constants in a special category, but consider them together with the masses; then all of the $\{g\}$ are assumed to be dimensionless, which allows a simplification of the exposition with practically no loss in generality.

A transformation of the RG, i.e., replacement of μ with μ', accompanied by a suitable change (2.4) of the charges and masses, leads[4] to multiplication of the Green's function Γ by a real factor Z_Γ independent of the momenta

$$\Gamma\left(\left\{\frac{-p^2}{\mu'^2}\right\}, \left\{\frac{m'^2}{\mu'^2}\right\}, \{g'\}\right)=Z_\Gamma\left(\frac{\mu'^2}{\mu^2}, \left\{\frac{m^2}{\mu^2}\right\}, \{g\}\right) \times\Gamma\left(\left\{\frac{-p^2}{\mu^2}\right\}, \left\{\frac{m^2}{\mu^2}\right\}, \{g\}\right). \tag{3.2}$$

The quantities $\{g'\}$, $\{m'\}$, and Z_Γ are uniquely determined, depend on the set of arguments $(\mu'^2,\mu^2,\{m^2\},\{g\})$ and can be calculated with perturbation theory (PT). We note that in the scheme of dimensional renormalization (and in general in "massless" schemes) Z_Γ and $\{g'\}$ do not depend on the masses and can be expressed in terms of only μ'^2/μ^2 and $\{g\}$.

The relation (3.2), which expresses the multiplicative nature of the renormalization of the GF, is the basis of the construction of the entire formalism of RG equations. Usually this is done with the use of auxiliary objects ξ_g, called invariant charges and defined as the product of the charge g and the corresponding vertex function Γ_g, multiplied also by the square root of the complete propagator for each external line at this vertex,

$$\xi_g=g\Gamma_g\prod_i \sqrt{d_i}. \tag{3.3}$$

It can be shown[4] that this quantity does not change under transformations of the RG,

$$\xi_g\left(\left\{\frac{-p^2}{\mu'^2}\right\}, \left\{\frac{m'^2}{\mu'^2}\right\}, \{g'\}\right)=\xi_g\left(\left\{\frac{-p^2}{\mu^2}\right\}, \left\{\frac{m^2}{\mu^2}\right\}, \{g\}\right). \tag{3.4}$$

Therefore the RG equations for the invariant charges are simpler than those for the GF; they are solved in short order and used for the further analysis of the model. Here, however, we shall present another possible approach to the construction of the RG formalism. The auxiliary objects used are not the invariant charges, but other quantities; these are the effective charges and effective masses, which characterize the renormalization of charges and masses under transformations of the RG.

Since at present several different renormalization schemes are widely used in QFT, it is of considerable interest to carry out a comparative analysis of them from the point of view of the RG. For this purpose we need a formula connecting the GF of different schemes

$$\Gamma\left(\left\{\frac{-p^2}{\mu^2}\right\}, \left\{\frac{m^2}{\mu^2}\right\}, \{g\}\right)=Z\left(\frac{\tilde{\mu}^2}{\mu^2}, \left\{\frac{m^2}{\mu^2}\right\}, \{g\}\right) \times\tilde{\Gamma}\left(\left\{\frac{-p^2}{\tilde{\mu}^2}\right\}, \left\{\frac{\tilde{m}^2}{\tilde{\mu}^2}\right\}, \{\tilde{g}\}\right). \tag{3.5}$$

It is important to emphasize that the transition from one scheme to another, which is a special case of the general transformation (2.3), (2.4), involves not only the change $\mu\to\tilde{\mu}$, $\{g\}\to\{\tilde{g}\}$, $\{m\}\to\{\tilde{m}\}$ in the arguments of the GF, but also a change in the form of the function itself, since a major difference between Eqs. (3.2) and (3.5) is that in the latter the quantities $\tilde{\Gamma}$ and Γ on the right and left sides are different functions of their respective arguments.

We shall illustrate the renormalization properties of the Green's function with the example of a scalar field with the Lagrangian

$$\mathcal{L}=\frac{1}{2}\partial_\mu\varphi\,\partial_\mu\varphi-\frac{m^2}{2}\varphi^2-\frac{16\pi^2}{4!}g\varphi^4. \tag{3.6}$$

Let us consider the four-terminal GF Γ_4. The first two terms of its expansion in g correspond to the diagrams of Fig. 1 and are of the form

$$\Gamma_4=1+\frac{g}{2}[f(s)+f(t)+f(u)], \tag{3.7}$$

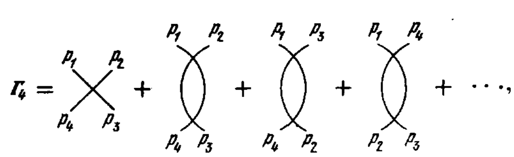

FIG. 1.

where

$$s=(p_1+p_2)^2,\quad t=(p_1+p_3)^2,\quad u=(p_1+p_4)^2,$$

and the function f is represented by the logarithmically diverging integral

$$f(s)=\frac{i}{\pi^2}\int\frac{d^4k}{(k^2-m^2)[(p_1+p_2-k)^2-m^2]}. \tag{3.8}$$

We shall carry out the calculation of the integral (3.8), and so of the function Γ_4, in three different renormalization schemes.

a) Cut-off scheme

Making a Wick rotation of the axes of integration and changing to Euclidean momenta, $k=(k_0,\mathbf{k})\to k_e=(\mathbf{k},ik_0)$, we cut off the integral over k_e at some upper limit Λ:

$$f^{(\Lambda)}(s)=-\frac{1}{\pi^2}\int\limits_{|k_e|<\Lambda}\frac{d^4k_e}{(k_e^2+m^2)[(p_1+p_2-k)_e^2+m^2]}$$

and, applying formulas of integration from Ref. 13, we get

$$f^{(\Lambda)}(s)=J\left(\frac{s}{m^2}\right)+\ln\frac{m^2}{\Lambda^2}+1, \tag{3.9}$$

where the function

$$J(z)=\int\limits_0^1 dx\ln[1-zx(1-x)]$$

is real for $z\leqslant 4$. Substitution of Eq. (3.9) in Eq. (3.7) gives an expression for $\Gamma_4^{(\Lambda)}$,

$$\Gamma_4^{(\Lambda)}=1+\frac{g}{2}\left[J\left(\frac{s}{m^2}\right)+J\left(\frac{u}{m^2}\right)+J\left(\frac{t}{m^2}\right)+3\ln\frac{m^2e}{\Lambda^2}\right] \tag{3.10}$$

(e is the base of natural logarithms), from which it is seen that a change $\Lambda\to\Lambda'$ of the renormalization parameter is equivalent to this order in g to multiplying $\Gamma_4^{(\Lambda)}$ by a factor $Z_4^{(\Lambda)}$

$$\Gamma_4^{(\Lambda')}=Z_4^{(\Lambda)}\Gamma_4^{(\Lambda)},\qquad Z_4^{(\Lambda)}=1-\frac{3g}{2}\ln\frac{\Lambda'^2}{\Lambda^2}+O(g^2). \tag{3.11}$$

As for the renormalization of the mass and the charge, because to first order in g there is no correction to the propagator, the mass is not renormalized

$$m'=m+O(g^2), \tag{3.12}$$

and to determine g' we can use the property (3.4) of invariance of ξ_g, which in our approximation is

$$g=g'Z_4+O(g^3), \tag{3.13}$$

from which we have

$$g'=g+\frac{3g^2}{2}\ln\frac{\Lambda'^2}{\Lambda^2}+O(g^3). \tag{3.14}$$

b) Subtraction scheme

In this scheme the integrals f, regularized in any way (for example, by means of a cut-off) are subtracted at points $s=\rho_s\lambda^2, t=\rho_t\lambda^2, u=\rho_u\lambda^2$

$$f^{(\lambda)}(s)=J\left(\frac{s}{m^2}\right)-J\left(\rho_s\frac{\lambda^2}{m^2}\right).$$

(The choice of the subtraction points is restricted by the requirement that the counterterms must be real, so that $\rho\lambda^2\leqslant 4m^2$.) Substituting this expression in Eq. (3.7) we get

$$\Gamma_4^{(\lambda)}=1+\frac{g}{2}\left[J\left(\frac{s}{m^2}\right)+J\left(\frac{u}{m^2}\right)+J\left(\frac{t}{m^2}\right)-J\left(\rho_s\frac{\lambda^2}{m^2}\right)-J\left(\rho_u\frac{\lambda^2}{m^2}\right)-J\left(\rho_t\frac{\lambda^2}{m^2}\right)\right]. \tag{3.15}$$

From this we get the expression for the renormalization constant $Z_4^{(\lambda)}$ corresponding to the change $\lambda\to\lambda'$,

$$Z_4^{(\lambda)}=1-\frac{g}{2}\left\{\left[J\left(\rho_s\frac{\lambda'^2}{m^2}\right)-J\left(\rho_s\frac{\lambda^2}{m^2}\right)\right]+[t]+[u]\right\}. \tag{3.16}$$

The form of g' is clear from Eq. (3.13). A characteristic feature of this scheme is that $Z_4^{(\lambda)}$ (and consequently also g') depends on the coefficients $\{\rho\}$, which fix the points of subtraction.

c) Dimensional renormalization scheme

For the dimensionally regularized integral

$$f_\varepsilon^{(\mu)}(s)=\frac{i(\mu^2)^\varepsilon}{\pi^2}\int\frac{d^{4-2\varepsilon}k}{(k^2-m^2)[(p_1+p_2-k)^2-m^2]}$$

we use formulas given in Ref. 14 to get

$$f_\varepsilon^{(\mu)}(\varepsilon)=-\frac{1}{\varepsilon}+\ln\frac{m^2}{\mu^2}+J\left(\frac{s}{m^2}\right)+O(\varepsilon).$$

After subtracting the term $1/\varepsilon$ which is singular in ε and taking the limit $\varepsilon\to 0$, we get the renormalized expression for $f^{(\mu)}$:

$$f^{(\mu)}(s)=\ln\frac{m^2}{\mu^2}+J\left(\frac{s}{m^2}\right),$$

which gives

$$\Gamma_4^{(\mu)}=1+\frac{g}{2}\left[J\left(\frac{s}{m^2}\right)+J\left(\frac{u}{m^2}\right)+J\left(\frac{t}{m^2}\right)+3\ln\frac{m^2}{\mu^2}\right]. \tag{3.17}$$

From this, making the change $\mu\to\mu'$ in $f^{(\mu)}$, we find

$$Z_4^{(\mu)}=1-\frac{3g}{2}\ln\frac{\mu'^2}{\mu^2}, \tag{3.18}$$

$$g'=g+\frac{3g^2}{2}\ln\frac{\mu'^2}{\mu^2}. \tag{3.19}$$

In the approximation considered the functions Γ_4 in all three schemes have the same dependence on the momenta, given by the function $J(z)$, and differ from each other only by additive constants. In this approximation the connection between g and the physical coupling constant g_{phys} is also simple in form. For example, if we define g_{phys} as the value of ξ_g that corresponds to the symmetric point $s=u=t=4m^2/3$, i.e., set by definition

$$g_{phys}=g^{(i)}\Gamma_4^{(i)}\left(s=u=t=\frac{4}{3}m^2\right), \tag{3.20}$$

then inserting the expressions (3.10), (3.15), and (3.17) in Eq. (3.20), and solving for $g^{(i)}$ we get

$$\left.\begin{aligned}
g^{(\Lambda)}&=g_{phys}-\frac{3}{2}g_{phys}^2\left[J\left(\frac{4}{3}\right)+\ln\frac{m^2e}{\Lambda^2}\right],\\
g^{(\lambda)}&=g_{phys}-\frac{1}{2}g_{phys}^2\left[3J\left(\frac{4}{3}\right)-J\left(\rho_s\frac{\lambda^2}{m^2}\right)-J\left(\rho_u\frac{\lambda^2}{m^2}\right)-J\left(\rho_t\frac{\lambda^2}{m^2}\right)\right],\\
g^{(\mu)}&=g_{phys}-\frac{3}{2}g_{phys}^2\left[J\left(\frac{4}{3}\right)+\ln\frac{m^2}{\mu^2}\right].
\end{aligned}\right\} \tag{3.21}$$

In higher orders in g the Green's functions of various schemes will show differences also in the momentum dependence. In fact, in diagrams that have a number of closed loops and therefore require an equal number of additional integrations over internal momenta, the differences by a constant that arise in the first integration will as a result of the other integrations inevitably be converted into differences by a function of the momenta. However, according to the relation (3.5) this discrepancy between the GF of the various renormalization schemes can be completely compensated by redefinition of the charges and masses and multiplication by a common momentum-independent factor Z.

4. EFFECTIVE CHARGE AND EFFECTIVE MASS

A direct consequence of the multiplicative nature of the renormalization of the GF and the coupling constants is the functional equations of the RG, which can be derived from the relation (3.2). In a number of cases these equations can be analyzed in general form, and their solution can also be written in functional form.[4,15] This formulation, however, is equivalent to the solutions of the differential equations of the RG, which can be derived and analyzed more simply. For this reason we shall not consider the functional equations of the RG and will adopt a scheme of exposition based on the differential equations.

To start with, we confine ourselves to the case of a model with a single dimensionless coupling constant g and a single mass m. The relation (3.2) takes the form

$$\Gamma\left(\left\{\frac{-p^2}{\mu'^2}\right\}, \frac{m'^2}{\mu'^2}, g'\right) = Z_\Gamma\left(\frac{\mu'^2}{\mu^2}, \frac{m^2}{\mu^2}, g\right)\Gamma\left(\left\{\frac{-p^2}{\mu^2}\right\}, \frac{m^2}{\mu^2}, g\right). \tag{4.1}$$

Differentiating this with respect to μ'^2 and then setting $\mu'=\mu$, we get an equation for Γ:

$$\left[\mu^2\frac{\partial}{\partial\mu^2} + \beta\left(\frac{m^2}{\mu^2}, g\right)\frac{\partial}{\partial g} + \gamma_m\left(\frac{m^2}{\mu^2}, g\right) m^2\frac{\partial}{\partial m^2} - \gamma_\Gamma\left(\frac{m^2}{\mu^2}, g\right)\right] \times\Gamma\left(\left\{\frac{-p^2}{\mu^2}\right\}, \frac{m^2}{\mu^2}, g\right) = 0, \tag{4.2}$$

where the so-called renormalization-group functions β, γ_m, and γ_Γ are the logarithmic derivatives with respect to μ'^2, taken at the point $\mu'=\mu$, of g', $\ln m'^2$, and Z_Γ, respectively, and

$$\beta\left(\frac{m^2}{\mu^2}, g\right) = \mu'^2\frac{\partial}{\partial\mu'^2} g'\left(\frac{\mu'^2}{\mu^2}, \frac{m^2}{\mu^2}, g\right)\Big|_{\mu'=\mu}, \tag{4.3a}$$

$$\gamma_m\left(\frac{m^2}{\mu^2}, g\right) = \frac{\mu'^2}{m'^2}\frac{\partial}{\partial\mu'^2} m'^2(\mu'^2, \mu^2, m^2, g)\Big|_{\mu'=\mu}, \tag{4.3b}$$

$$\gamma_\Gamma\left(\frac{m^2}{\mu^2}, g\right) = \mu'^2\frac{\partial}{\partial\mu'^2}\ln Z_\Gamma\left(\frac{\mu'^2}{\mu^2}, \frac{m^2}{\mu^2}, g\right)\Big|_{\mu'=\mu}. \tag{4.3c}$$

In the subtraction scheme Eq. (4.2) is known as Ovsyannikov's equation.[15] In the literature Eq. (4.2) is also often called the Callan-Symanzik equation (on this point see the end of Chapter 6).

The functions β, γ_m and γ_r play an exceptionally important part in the apparatus of the RG. Their explicit calculation is the fundamental problem that arises in the renormalization group approach to the investigation of QFT models. Effective methods for finding the first few terms of the PT series for the RG functions are described in the literature.[16-19]

In the example of the φ^4 model which we have considered the RG functions can be calculated very simply in the lowest (one-loop) approximation. Carrying out according to Eq. (4.3) the differentiation with respect to the primed renormalization parameter in the explicit expressions for Z_4 and g', we get in the cut-off and dimensional renormalization schemes

$$\gamma_\Gamma = -\frac{3}{2}g, \quad \beta = \frac{3}{2}g^2, \tag{4.4}$$

and in the subtraction scheme

$$\gamma_\Gamma = -\frac{g}{2}\left[3 - \left(\int_0^1 \frac{dx}{1-\rho_s(\lambda^2/m^2)x(1-x)}\right) - (t) - (u)\right], \quad \beta = -g\gamma_\Gamma. \tag{4.5}$$

We note the explicit dependence of the RG functions of this scheme on the coefficients $\{\rho\}$. In the approximation considered $\gamma_m=0$ for all schemes, this being a consequence of Eq. (3.12). In the subtraction scheme $\gamma_m=0$ in all orders in g, since here, as already stated, the mass is not included among the transformations of the RG.[4]

As can be seen from Eq. (4.4), the RG functions of the cut-off and dimensional renormalization schemes turned out to be independent of the mass to the order considered. This property follows from the "masslessness" of these schemes and can be proved for an arbitrary order of PT.[9,11]

Now, regarding the RG functions as known (in practice this means that they have been calculated to some particular approximation), let us find the expression for the GF, which satisfies Eq. (4.1), or equivalently is the solution of the RG differential equation (4.2). For this we first derive in explicit form the dependence of g', m', and Z_Γ on μ' and the other arguments (μ, m, and g). We introduce the following notations:

$$\mu'^2/\mu^2 = t, \quad m^2/\mu^2 = y,$$
$$g' = \bar g(t, y, g), \quad \frac{m'^2}{\mu'^2} = \bar y(t, y, g). \tag{4.6}$$

The functions $\bar g$ and $\bar y$ are called the effective charge and effective mass and describe the renormalization of g and m under the renormalization group change $\mu\to\mu'$. From Eq. (4.6) we have the normalization relations for $\bar g$ and $\bar y$:

$$\bar g(1, y, g) = g, \quad \bar y(1, y, g) = y. \tag{4.7}$$

Let us now derive the differential equation which $g(t,y,g)$ satisfies. Represent the RG transformation $\mu\to\mu''$ as a succession of transformations $\mu\to\mu'\to\mu''$. This makes $g\to g'\to g'' = g(t', y', g')$, where $t'=\mu''^2/\mu'^2$. Consequently

$$g'' = \bar g(t', \bar y(t, y, g), \bar g(t, y, g)) = \bar g(tt', y, g). \tag{4.8}$$

We rewrite the relation (4.3a) in the notation (4.6)

$$\beta(y, g) = t\frac{\partial}{\partial t}\bar g(t, y, g)\Big|_{t=1}. \tag{4.9}$$

Differentiating Eq. (4.8) with respect to $\ln t'$ at the point $t'=1$ and using Eq. (4.9), we get

$$t\frac{\partial}{\partial t}\bar g(t, y, g) = \beta(\bar y(t, y, g), \bar g(t, y, g)). \tag{4.10}$$

In a similar way we derive the equation for $\bar y$,

$$t\frac{\partial}{\partial t}\ln\bar y(t, y, g) = \gamma_m(\bar y(t, y, g), \bar g(t, y, g)) - 1, \tag{4.11}$$

and that for Z_Γ,

$$t\frac{\partial}{\partial t}\ln Z_\Gamma(t, y, g)=\gamma_\Gamma(\bar{y}(t, y, g), \bar{g}(t, y, g)), \tag{4.12}$$

with the normalization condition

$$Z_\Gamma(1, y, g)=1. \tag{4.13}$$

From these relations we find

$$Z_\Gamma(t, y, g)=\exp\Big[\int_1^t \frac{du}{u}\gamma_\Gamma(\bar{y}(u, y, g), \bar{g}(u, y, g))\Big]. \tag{4.14}$$

Substituting Eqs. (4.6) and (4.14) in (4.1) and denoting $-p_i^2/\mu'^2$ by x_i, we get the solution of the RG equation (4.2) in the form

$$\Gamma(\{xt\}, y, g)$$
$$=\Gamma(\{x\}, \bar{y}(t, y, g), \bar{g}(t, y, g))\exp\Big[-\int_1^t \frac{du}{u}\gamma_\Gamma(\bar{y}(u, y, g), \bar{g}(u, y, g))\Big]. \tag{4.15}$$

This relation for the GF Γ, in which its RG properties are fully taken into account, describes the effect of a simultaneous scale transformation of all momentum arguments of $\Gamma, \{x\}\to\{xt\}$. The explicit form of $\bar{y}$ and $\bar{g}$ must be determined from the system (4.10), (4.11) on the the basis of available information about β and γ_m. Some qualitative characteristics of such systems have been studied in the literature,[20] but their exact solution requires a knowledge of the RG functions in all orders of PT. In actual situations we know only the first few terms of the relevant expansions. Whether (and how) it is possible to get reliable information about any properties of the full RG on the basis of these few terms—this is the main question in which we shall be interested throughout this paper.

Let us now return to the φ^4 model. In the one-loop approximation the system of equations for $\bar{g}$ and $\bar{y}$ is [with the expression (4.4) used for β]

$$t\frac{\partial}{\partial t}\bar{g}=\frac{3}{2}\bar{g}^2, \quad t\frac{\partial}{\partial t}\ln\bar{y}=-1. \tag{4.16}$$

Solving Eq. (4.16) and considering the normalization (4.7), we get

$$\bar{g}=\frac{g}{1-(3/2)g\ln t}, \quad \bar{y}=\frac{y}{t}. \tag{4.17}$$

Equation (4.17) for $\bar{g}$ denonstrates a typical and very important characteristic of the RG approach; the expressions that are obtained contain contributions of all orders in the coupling constant. This is due to the fact that in any finite order in g the GF does not have the required RG properties (multiplicative renormalization). Therefore the solution of the RG equations on the basis of the functions β, γ_m, and γ_Γ, as calculated to a given order of PT, is equivalent to summing an infinite series of contributions from diagrams with an arbitrary number of closed loops. In other words, we have gone beyond the ordinary PT ordered in powers of g, and have in effect regrouped and partially summed its terms, to a new PT in the effective charge $\bar{g}$.

This analysis of the RG structure of GF can be generalized to the case of several charges and masses, and can also be extended to the situation in which there are gauge fields in the theory. There then appear additional effective quantities corresponding to the new coupling constants, masses, and gauge parameters, but the structure of the fundamental relations (4.10), (4.11), and (4.15) is completely preserved. We shall write out the many-charge relations in explicit form in connection with our treatment of the ultraviolet asymptotics of GF.

It is interesting to note that in the subtraction scheme the effective charge $\bar{g}(t,y,g)$ for $t>0$ is identical with the invariant charge ξ_g defined by Eq. (3.3). For $t\leq 0$ the charge $\bar{g}$, unlike ξ_g, is a as rule not defined, since the requirement that the counterterms be Hermitean makes the ratio $\mu'^2/\mu^2\equiv t$ positive. In the other renormalization schemes the charges $\bar{g}$ and ξ_g are unequal.

5. ULTRAVIOLET ASYMPTOTICS OF GREEN'S FUNCTIONS

The final equation of the preceding section, Eq. (4.15), can be used successfully for the analysis of the ultraviolet asymptotics of a number of QFT models. Since the relation (4.15) describes the effect of a homogeneous transformation of all the momentum arguments of the GF (the possibility of using the RG method to study momentum dependences of another kind, for example the asymptotics of the amplitudes for processes on the mass shell, will be discussed in Chapter 9), we can from the very beginning regard the GF as a function of a single momentum argument p^2, and to which all the p_i^2 are proportional, $p_i^2=\rho_i p^2$. The coefficients $\{\rho\}$ specify the asymptotic conditions and are involved (implicitly) in the determination of the corresponding GF. In particular, in the subtraction scheme the renormalization functions depend on the choice of these coefficients.

It is natural to expect that in the ultraviolet limit, i.e., for $|p^2|/m^2\to\infty$, the dependence of the GF on the masses will be unimportant. Let us consider first the case in which the renormalization parameter is large in comparison with the mass, $\mu^2\gg m^2$. Since in each order of PT the passage to the limit $m^2\to 0$ is regular,[21] in the region $|p^2|$, $\mu^2\gg m^2$ the mass arguments $y=m^2/\mu^2$ and $\bar{y}$ in (4.15) can be taken equal to zero. The relation (4.15) then becomes

$$\Gamma(xt, g)=\Gamma(x, \bar{g}(t, g))\exp\Big[-\int_1^t \frac{du}{u}\gamma_\Gamma(\bar{g}(u, g))\Big], \tag{5.1}$$

where we regard x as fixed, $t\to+\infty$, and the two-argument function $\Gamma(x,g)=\Gamma(x,0,g)$ is the GF of the massless theory.

In what follows the argument x will as a rule be set equal to 1, which corresponds to the case of spacelike momenta. To study asymptotics in the timelike region one must use the formula (5.1) with $x<0$.

Returning to the general case of the asymptotics of $\Gamma(t,y,g)$ for $t\to\infty$ and arbitrary fixed y, we use a "massless" renormalization scheme. Then, because the RG functions (4.3) are independent of the mass, the effective charge, determined from

$$t\frac{\partial}{\partial t}\bar{g}(t, g)=\beta(\bar{g}(t, g)), \quad \bar{g}(1, g)=g, \tag{5.2}$$

does not contain the argument y at all, and for $t \to \infty$ Eq. (4.15) goes over into (5.1) if

$$\bar{y}(t, y, g) \xrightarrow[t\to\infty]{} 0. \tag{5.3}$$

The relation (5.2) is satisfied under the condition that the right side of Eq. (4.11) is less than zero for $t \to \infty$, i.e., if

$$\lim_{t\to\infty} \gamma_m (\bar{y}(t, y, g), \bar{g}(t, y, g)) < 1. \tag{5.4}$$

This is certainly true in the subtraction scheme, where $\gamma_m \equiv 0$, and in asymptotically free theories (see below), in which the effective charge $\bar{g}$, which is the parameter on the left side of the inequality (5.4), goes to zero for large t. In other cases PT calculations are not sufficient to test the inequality (5.3). Nevertheless, we shall assume that Eq. (5.4) is correct, possibly thereby limiting the applicability of the formulas now to be derived.

In other schemes (not "massless") the renormalization functions depend on y, and the result of this is that the asymptotic of $\Gamma(t,y,g)$ is not the same as $\Gamma(t,0,g)$. We can, however, find the connection between these functions in explicit form, since, transferring the RG transformation μ from the region $\mu^2 \sim m^2$ into the region $\mu^2 \gg m^2$, we thus make y go to zero. This procedure and its consequences are described in detail in Chapter 6. The conclusion we reach is as follows: The asymptotic expressions for the Green's function are identical to those obtained in the massless theory up to corrections $\sim\bar{y}$ (so-called mass corrections). If the inequality (5.4) is strictly satisfied, these corrections fall off according to a power law as t increases. Calculation of the mass corrections is a separate problem, with which we shall not deal in this article.

Thus we are to take the relations (5.1) and (5.2) as the basis for our further arguments. The RG functions $\beta(g)$ and $\gamma_\Gamma(g)$ that appear in them are often called the Gel-Mann–Low function and the anomalous dimension, respectively. As can be seen from Eq. (5.1), the analysis of the asymptotics of the GE $\Gamma(t,g)$ requires information about the behavior of the effective charge $\bar{g}(t,g)$ for $t \to \infty$, which is completely determined by the function $\beta(g)$.

Let us consider some characteristic possibilities for the ultraviolet behavior of $\bar{g}(t,g)$ and $\Gamma(t,g)$ (a more detailed exposition is given in Ref. 4). It is convenient to write Eq. (5.2) in integral form

$$\ln t = \int_g^{\bar{g}(t,g)} \frac{du}{\beta(u)}. \tag{5.5}$$

In renormalizable theories the expansion of $\beta(u)$ in power series in u begins with the second power of u. We first assume that the first coefficient of this series is positive, i.e.,

$$\beta(u) = au^2 + O(u^3), \quad a > 0. \tag{5.6}$$

This means that on some interval $(0, \delta)$ the function $\beta(u)$ is positive. Let g be in this interval, and suppose that the integral (5.5) diverges at its upper limit for $\bar{g} = g_0 < \infty$. For this to be so, the function $\beta(u)$ must have a zero of at least the first order at the point $u = g_0$. This situation is shown qualitatively in Fig. 2, a. From Eq. (5.5) we can conclude that $\bar{g}(t,g) \xrightarrow[t\to\infty]{} g_0$. This sort of behavior of the effective charge $\bar{g}$ corresponds to a finite renormalization of the charge, and the point g_0 is called an ultraviolet-stable point. As can be seen from Eq. (5.1), the asymptotics of the GF has a power-law form

$$\Gamma(t, g) \xrightarrow[t\to\infty]{} \Gamma(1, g_0)\, t^{-\gamma_\Gamma(g_0)}.$$

If the integral (5.5) diverges only at an infinite value of the upper limit, then $\bar{g}(t,g) \xrightarrow[t\to\infty]{} \infty$, i.e., an infinite renormalization of the charge occurs, while if the integral (5.5) converges for $\bar{g} \to \infty$,

$$\int_g^\infty \frac{du}{\beta(u)} = \ln t_0 < \infty,$$

a pole of the effective charge appears at the point $t_0, \bar{g}(t_0, g) = \infty$ (a situation well known under the name "zero of charge", see Ref. 22), which indicates that the model in question is internally inconsistent.

In the calculation of the function $\beta(u)$ [Eq. (5.6)] with the first few terms of the PT series, two of these possibilities may be realized: A "zero of charge" [for example, if we use only the first term, cf. Eq. (4.7) for the φ^4 model], and a finite renormalization, if $\beta(u)$, in this case a polynomial of finite degree, has a zero at $u = g_0 \neq 0$. However, in either of these situations we cannot draw any reliable conclusions from PT about the asymptotics of $\Gamma(t,g)$, since, first, to find the quantity g_0 it is necessary to include all orders in the coupling constant, and second, in the case of (5.6) the effective charge $\bar{g}(t,g)$ itself increases as $t \to \infty$ and therefore cannot serve as a parameter of the expansion in Eq. (5.1). In other words, for $t \to \infty$ there is a departure from the weak-coupling region and PT in terms of the effective charge cannot be applied.

The picture is entirely different when

$$\beta(u) = -bu^2 + O(u^3), \quad b > 0 \tag{5.7}$$

(Fig. 2, b). A range of values of u near zero, $0 < u < \delta$, can always be found such that the first term on the right side of Eq. (5.6) dominates. Setting $\beta(u) = -bu^2$ in Eq. (5.5), we get the effective charge in the form

$$\bar{g}(t, g) = \frac{g}{1 + bg \ln t},$$

i.e., $\bar{g}(t,g) \sim 1/(b \ln t) \to 0$ for $t \to \infty$. This sort of asymp-

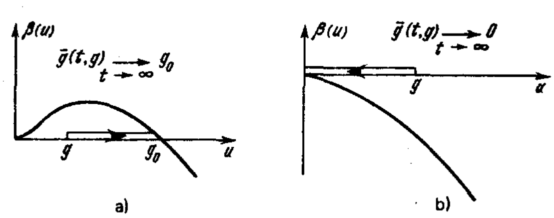

FIG. 2.

totic behavior of $\bar{g}(t,g)$ is known as asymptotic freedom (AF).[23] As we shall show, AF is typical for nonabelian gauge theories.

An analysis of the neglected terms in the right member of Eq. (5.7) shows, on the assumption $g < \delta$, that

$$\bar{g}(t,\ g)=\frac{1}{b\ln t}+O\left(\frac{\ln\ln t}{\ln^2 t}\right). \tag{5.8}$$

In renormalizable theories the expansion of $\gamma_\Gamma(g)$ in series in g is of the form $\gamma_\Gamma(g)=cg+O(g^2)$, and it follows from Eq. (5.1) that

$$\Gamma(t,\ g)=\mathrm{const}\cdot(\ln t)^{-c/b}\left[1+O\left(\frac{\ln\ln t}{\ln t}\right)\right]. \tag{5.9}$$

It can be seen from Eqs. (5.8) and (5.9) that in models with AF (and only in them, i.e., only if the first term of the expansion of $\beta(g)$ in g is negative) one can use PT in the high-energy limit, since there the effective charge is a small parameter for an "improved" perturbation theory and the asymptotics of the GF are governed by the first terms of the PT for the RG functions $\beta(g)$ and $\gamma_\Gamma(g)$.

As is well known, in a massless theory the GF $\Gamma(t,g)$ can be expanded in a double series in g and $\ln t$

$$\Gamma(t,\ g)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}c_{nm}g^n\ln^m t.$$

It can be shown[4] that by expanding the right side of Eq. (5.1) in an analogous series with functions β, γ_Γ, and $\Gamma(1,g)$ calculated in the one-loop approximation we get the correct coefficients of $g^n\ln^n t$ for all n. Inclusion of two-loop contributions allows determination also of the coefficients of $g^n\ln^{n-1}t$, and so on. This means that starting from the N-loop expressions for the RG functions and for $\Gamma(1,g)$ we have the possibility of using the RG method to sum all the leading logarithmic terms of PT (of the type of $g^n\ln^n t$) and also the $N-1$ lower degrees in the logarithms, down to $g^n(\ln t)^{n-N+1}$. However, only in the case of AF, Eq. (5.7), is such a summation really useful, since in that case it leads to an improved PT with a new small parameter, the effective charge $\bar{g}$.

In studying the ultraviolet asymptotics of specific models of QFT we have to deal repeatedly with many-charge theories. Therefore we give here the main RG formulas for that case. All of the changes in the RG equations that are required in going from one to many charges can be seen already from the example of a two-charge model. We shall confine ourselves to a discussion of such a theory, with coupling constants g and h.

In the two-charge case Eq. (5.2) is replaced by a system of equations for two effective charges $\bar{g}$ and $\bar{h}$:

$$t\frac{\partial}{\partial t}\bar{g}(t,\ g,\ h)=\beta_g(\bar{g},\ \bar{h}),\quad \bar{g}(1,\ g,\ h)=g,$$
$$t\frac{\partial}{\partial t}\bar{h}(t,\ g,\ h)=\beta_h(\bar{g},\ \bar{h}),\quad \bar{h}(1,\ g,\ h)=h. \tag{5.10}$$

By using the fact that Γ, the Green's function, is dimensionless, we can write the differential equation for it in the form

$$\left[x\frac{\partial}{\partial x}-\beta_g(g,\ h)\frac{\partial}{\partial g}-\beta_h(g,\ h)\frac{\partial}{\partial h}+\gamma_\Gamma(g,\ h)\right]\Gamma(x,\ g,\ h)=0$$

(here $x=-p^2/\mu^2$), and its solution is

$$\Gamma(xt,\ g,\ h)=\Gamma(x,\ \bar{g},\ \bar{h})\exp\left[-\int_1^t\frac{du}{u}\gamma_\Gamma(\bar{g}(u,\ g,\ h),\ \ \bar{h}(u,\ g,\ h))\right]. \tag{5.11}$$

The investigation of the asymptotics of the GF requires primarily an analysis of the system (5.10). Since in the right sides of these equations the argument t does not appear explicitly, the solution can be represented in the phase plane of the effective charges $\bar{g}$ and $\bar{h}$ (for two charges; for more than two, in a phase space) by curves which show the motion of the point $(\bar{g},\bar{h})$ as t varies. In particular, a finite renormalization corresponds to the existence on the phase plane of an ultraviolet-stable point (g_0, h_0) to which the phase curves converge,

$$\bar{g}(t,\ g,\ h)\xrightarrow[t\to\infty]{}g_0<\infty,\quad \bar{h}(t,\ g,\ h)\xrightarrow[t\to\infty]{}h_0<\infty. \tag{5.12}$$

In the case of AF the origin of coordinates is a stable point,

$$\bar{g}\xrightarrow[t\to\infty]{}0,\quad \bar{h}\xrightarrow[t\to\infty]{}0.$$

6. CONVERSION FORMULAS

Let us now explain how important the dependence of the ultraviolet asymptotics of the GF on the renormalization scheme used is. To do this we first find the explicit connection between the RG functions of the different schemes. The change $\Gamma\to\tilde{\Gamma}$ of the GF on going to a different renormalization scheme is described by Eq. (3.5), which in the high-energy limit of a one-charge theory takes the form

$$\Gamma\left(\frac{-p^2}{\mu^2},\ g\right)=Z\left(\frac{\tilde{\mu}^2}{\mu^2},\ g\right)\tilde{\Gamma}\left(\frac{-p^2}{\tilde{\mu}^2},\ \tilde{\bar{g}}\left(\frac{\tilde{\mu}^2}{\mu^2},\ g\right)\right). \tag{6.1}$$

Setting $\tilde{\mu}=\mu$ and using the notations

$$-\frac{p^2}{\mu^2}=x,\quad \tilde{\bar{g}}(1,\ g)=q(g),\quad Z(1,\ g)=p(g),$$

we get a relation connecting the GF's of different schemes:

$$\Gamma(x,\ g)=p(g)\,\tilde{\Gamma}(x,\ q(g)). \tag{6.2}$$

Differentiating Eq. (6.2) with respect to x and using the RG equation

$$\left[x\frac{\partial}{\partial x}-\beta(g)\frac{\partial}{\partial g}+\gamma_\Gamma(g)\right]\Gamma(x,\ g)=0 \tag{6.3}$$

and the analogous equation for $\tilde{\Gamma}$ (with the changes $\beta\to\tilde{\beta}, \gamma_\Gamma\to\tilde{\gamma}_\Gamma$), we find the required connection between the RG functions[14,24]

$$\tilde{\beta}(q(g))=\beta(g)\frac{dq(g)}{dg}, \tag{6.4}$$

$$\tilde{\gamma}_\Gamma(q(g))=\gamma_\Gamma(g)-\beta(g)\frac{d\ln p(g)}{dg}. \tag{6.5}$$

We call Eqs. (6.4 and 6.5) conversion formulas.

Because of the fact that there is no dependence on the renormalization scheme in the lowest (Born) approximation, the expansions of $q(g)$ and $p(g)$ in power series in g are of the form

$$q(g) = g + O(g^2), \quad p(g) = 1 + O(g).$$

From this and Eqs. (6.4) and (6.5) it follows that in a given order in g the conversion from $\beta(g)$ to $\tilde{\beta}(g)$ and from $\gamma_\Gamma(g)$ to $\tilde{\gamma}_\Gamma(g)$ requires knowledge of the conversion functions $q(g)$ and $p(g)$ only in lower orders. Therefore in the one-loop approximation the RG of all schemes are identical.

It is curious that the equality $\beta(g) = \tilde{\beta}(g)$ holds (in a one-charge theory) also in the two-loop approximation.[25] In fact, substituting the first terms of the series

$$q(g) = g + Ag^2 + Bg^3 + \dots,$$
$$\beta(g) = ag^2 + bg^3 + cg^4 + \dots,$$
$$\tilde{\beta}(g) = \tilde{a}g^2 + \tilde{b}g^3 + \tilde{c}g^4 + \dots$$

in Eq. (6.4) and equating the coefficients of g^2, g^3, and g^4, we get

$$\tilde{a} = a, \quad \tilde{b} = b, \quad \tilde{c} = c - aA^2 + aB - bA. \tag{6.6}$$

The difference between the functions β and $\tilde{\beta}$ begins only in the three-loop approximation. This gives a reflection of the fact that for given $\beta(g)$ and $\tilde{\beta}(g)$ the function $q(g)$ is determined only up to an arbitrary constant, since from Eq. (6.4) we have the relation

$$\tilde{\psi}(q(g)) = \psi(g) + \text{const}, \tag{6.7}$$

where ψ and $\tilde{\psi}$ are indefinite integrals of the form

$$\psi(g) = \int^g \frac{du}{\beta(u)}, \quad \tilde{\psi}(g) = \int^g \frac{du}{\tilde{\beta}(u)}.$$

An analysis of the conversion formulas from the point of view of PT series, such as we have made for Eq. (6.4), shows that generally speaking the difference between the RG functions of different schemes begins at the level of two loops. This is the case also for the functions β in many-charge theories, and for the functions γ_Γ, for an arbitrary number of coupling constants. Many-charge conversion formulas are given explicitly in Ref. 24.

We now discuss some consequences of these formulas. As was remarked earlier, the values of observable quantities (matrix elements of the S matrix), as calculated in QFT, are independent of which renormalization scheme is used. It can be seen from Eq. (6.2) that the same statement is also true for the GF up to a common factor $p(g)$. In fact, if we drop this factor, we find that $\tilde{\Gamma}$ differs from Γ only by the replacement of the argument g by $q(g)$. But this is precisely the sort of renormalization of the coupling constant that accompanies a change to a different renormalization scheme. Taking this redefinition of the charge into account, we see that the dependence of the GF on x (i.e., on the momentum) is just the same in both cases.

To illustrate this general proposition let us examine in more detail an expansion of the type of Eq. (5.9) in asymptotically free theories. We put the relation (5.1) in the form

$$\Gamma(xt, g) = \Gamma(x, \bar{g}(t, g)) \exp\left(-\int_g^{\bar{g}(t, g)} \frac{du\gamma_\Gamma(u)}{\beta(u)}\right), \tag{6.8}$$

and Eq. (5.5) in the form $\ln t = \psi(\bar{g}(t,g)) - \psi(g)$. We take as a new expansion parameter L the quantity $L \equiv \ln t + \psi + \psi(g)$. In the literature the parameter L is often written in the form $\ln(p^2/\Lambda^2)$ to emphasize the logarithmic nature of the dependence of L on the momentum. In this form the quantity $\Lambda^2(\mu^2, g)$, having taken into itself the entire dependence on μ and g, is the single parameter of the theory. In view of the fact that according to Eq. (4.8) the effective charge does not change under RG transformations $\mu \to \mu', g \to g'(\mu'^2/\mu^2, g)$, the parameter L is also an invariant of the RG. Solving the equation $L = \psi(\bar{g})$ for $\bar{g}$, we get

$$\bar{g}(L) = \frac{a}{L} + \frac{b \ln L}{L^2} + \frac{d}{L^2} + O\left(\frac{\ln^2 L}{L^3}\right). \tag{6.9}$$

Substituting Eq. (6.9) in (6.8), we find

$$\Gamma(t, g) = f(g) L^{-A}\left[1 + \frac{B \ln L}{L} + \frac{C}{L} + O\left(\frac{\ln^2 L}{L^2}\right)\right], \tag{6.10}$$

where f is some function; the numbers a and A are calculated from PT in the one-loop approximation, and b, d, B, and C, in the two-loop approximation, and so on.

Do the coefficients of the expansion (6.10) change in the transition to a different renormalization scheme? Let us write Eq. (6.10) in the form $\Gamma(t,g) = f(g)\Phi(L)$. Then, according to Eqs. (6.2) and (6.7),

$$\Phi(L) = \Phi(\ln t + \psi(g)) = \tilde{\Phi}(\ln t + \tilde{\psi}(q(g))) = \tilde{\Phi}(L + \text{const}).$$

Consequently, the only effect of a change of the renormalization scheme [not counting a change of the common factor $f(g)$] is a shift of the parameter by a constant. [We note that the definition of L from the very beginning allowed this amount of arbitrariness, since $\psi(g)$ is an indefinite integral.] The numerical value of this constant (we denote it by Δ) can be found by comparing C and $\tilde{C}$:

$$\tilde{C} = C + A\Delta.$$

In order to fix the determination of L unambiguously [and thus fix all of the coefficients in the expansion (6.10)] we can, for example, require that $C = \tilde{C} = 0$. By imposing this kind of condition, we assure that all of the coefficients in Eq. (6.10) are independent of the renormalization scheme.

The index $\gamma_\Gamma(g_0)$ of the power law asymptotic is also independent of the renormalization scheme in theories with a finite renormalized charge. In fact, suppose $\beta(g_0) = 0$. Then according to Eq. (6.4) in a different scheme $\tilde{\beta}(\tilde{g}_0) = 0$, where $\tilde{g}_0 = q(g_0)$ (provided only that the quantity $dq(g)/dg_{g=g_0}$ is finite), and it follows from Eq. (6.5) that the stated relation $\tilde{\gamma}_\Gamma(\tilde{g}_0) = \gamma_\Gamma(g_0)$ holds. It is true, however, only in the complete theory, and is violated in any finite order of PT, since to find the quantity $\gamma_\Gamma(g_0)$ one must take into account all orders in g. By breaking off the PT series, we get only an approximation for $\gamma_\Gamma(g_0)$, for which the approximate expressions for $\gamma_\Gamma(g_0)$ in different renormalization schemes are not necessarily equal.

In concluding this chapter we apply the conversion formulas to study the ultraviolet asymptotics of the GF $\Gamma(x,y,g)$ in the case $y\neq 0$. As was shown in Chapter 5, to do this we must convert from the value $\mu^2\sim m^2$ of the renormalization parameter to a value $\mu'^2\gg m^2$. By Eq. (3.2)

$$\Gamma(x,y,g)\equiv\Gamma\left(\frac{-p^2}{\mu^2},\frac{m^2}{\mu^2},g\right)=Z_\Gamma\left(\frac{\mu'^2}{\mu^2},\frac{m^2}{\mu^2},g\right)\Gamma\left(\frac{-p^2}{\mu'^2},\frac{m'^2}{\mu'^2},g'\left(\frac{\mu'^2}{\mu^2},\frac{m^2}{\mu^2},g\right)\right).$$

This equation, being true for arbitrary p^2 and μ'^2, holds also in the asymptotic region $|p^2|,\mu'^2\gg m^2$. Taking account of Eq. (5.3), we have

$$\Gamma_{ac}(x,y,g)\equiv\Gamma_{ac}\left(\frac{-p^2}{\mu^2},\frac{m^2}{\mu^2},g\right)=Z_{ac}\left(\frac{\mu'^2}{\mu^2},\frac{m^2}{\mu^2},g\right)\Gamma\left(\frac{-p^2}{\mu'^2},0,g'_{ac}\left(\frac{\mu'^2}{\mu^2},\frac{m^2}{\mu^2},g\right)\right), \quad (6.11)$$

where we take the ratio $m^2/\mu^2\equiv y$ as fixed, and the asymptotic functions Γ_{as}, Z_{as}, and g'_{as} are obtained from Γ, Z_Γ, and g' by discarding all terms that go to zero in the limit $|p^2|/m^2\to\infty$ or, correspondingly, $\mu'^2/m^2\to\infty$. There then remains in the function $\Gamma_{as}(x,y,g)$ only a logarithmic dependence on the momentum. Comparing Eqs. (6.11) and (6.1), we conclude that the conversion from $\Gamma_{as}(x,y,g)$, $y=\text{const}\neq 0$ to $\Gamma(x,0,g)$ is analogous to that from one renormalization scheme to another. Consequently, the ultraviolet asymptotics of $\Gamma(x,y,g)$ is given, as in the case of $\Gamma(x,0,g)$, by Eq. (5.1), but now with different RG functions $\beta(y,g)$ and $\gamma_\Gamma(y,g)$, which are connected with $\beta(g)\equiv\beta(0,g)$ and $\gamma_\Gamma(g)\equiv\gamma_\Gamma(0,g)$ by the conversion formulas of the type of Eqs. (6.4), (6.5).

Written in differential form, the RG equation for Γ_{as}

$$\left[x\frac{\partial}{\partial x}-\beta(y,g)\frac{\partial}{\partial g}+\gamma_\Gamma(y,g)\right]\Gamma_{ac}(x,y,g)=0 \quad (6.12)$$

in the subtraction scheme with $y=1$ is exactly the same as the asymptotic form of the well known Callan-Symanzik equation.[26] The correspondence between the RG equation (4.2) and the Callan-Symanzik equation has been studied in papers by Lowenstein[7] and Collins[27]. The extension of the arguments of these papers[27] to the case of an arbitrary renormalization scheme leads to the conclusion that the difference between these two types of RG equations is entirely due to the dependence of the RG functions on masses. In "massless" schemes these equations are identical. For precisely this reason the logarithmic asymptotics of $\Gamma(x,y,g)$ and $\Gamma(x,0,g)$ coincide, as was shown in Chapter 5.

7. ONE-CHARGE MODELS

In this chapter we present a list of the results so far obtained on the ultraviolet asymptotics of one-charge quantum field models, i.e., the results of calculations of the corresponding Gell-Mann–Low functions, and discuss their consequences.

a) Spinor electrodynamics (usually called quantum electrodynamics):

$$\mathcal{L}=i\bar\psi\hat\partial\psi-\frac{1}{4}f_{\mu\nu}f_{\mu\nu}+e\bar\psi\hat A\psi,\quad \frac{e^2}{4\pi}\equiv\alpha.$$

Calculations of $\beta(\alpha)$ in the subtraction scheme have been carried to the three-loop approximation[28]

$$\beta_{subt}(\alpha)=\frac{\alpha^2}{3\pi}+\frac{\alpha^3}{4\pi^2}+\frac{\alpha^4}{8\pi^3}\left(\frac{8}{3}\zeta(3)-\frac{101}{36}\right), \quad (7.1)$$

where ζ is the Riemann zeta function. Substituting this expression in Eq. (5.5) we see that the integral on the right side of (5.5) converges when the upper limit is infinite, i.e., the situation is qualitatively not different from that of the one-loop approximation; as before a "zero of the charge" occurs.

However, the reliability of this conclusion may be put in question, if only by a very simple argument.[4] Starting from Eq. (7.1), let us expand the function $1/\beta(\alpha)$ in a series in α, and take into account only the first three terms,

$$\frac{1}{\beta(\alpha)}=\frac{3\pi}{\alpha^2}\left[1-\frac{3\alpha}{4\pi}+\frac{\alpha^2}{\pi^2}\left(\frac{155}{96}-\zeta(3)\right)+\cdots\right]. \quad (7.2)$$

Substitution of this result into Eq. (5.5) does not lead to a "zero of the charge", but to an infinite renormalization. This illustration gives another indication of the inadequacy of PT alone for the analysis of the high-energy picture in QFT models that do not have the property of asymptotic freedom.

It is curious to compare the result (7.1) with the expression for the function $\beta(1,\alpha)$ from the Callan-Symanzik equation and the function $\beta_{dim}(\alpha)$ calculated in the dimensional renormalization scheme. In the first case the conversion formula is

$$\beta(1,\alpha)\frac{dq(\alpha)}{d\alpha}=\beta_{subt}(q(\alpha)), \quad (7.3)$$

and the conversion function $q(\alpha)$ can be expressed in terms of the complete photon propagator,

$$q(\alpha)=\alpha d_{ac}(1,1,\alpha),$$

and in the three-loop approximation is given by[30]

$$q(\alpha)=\alpha-\frac{5}{9}\frac{\alpha^2}{\pi}+\left[\zeta(3)+\frac{65}{648}\right]\frac{\alpha^3}{\pi^2},$$

from which, using Eq. (7.3), we get $\beta(1,\alpha)$:

$$\beta(1,\alpha)=\frac{\alpha^2}{3\pi}+\frac{\alpha^3}{4\pi^2}-\frac{121}{288}\frac{\alpha^4}{\pi^3}. \quad (7.4)$$

For $\beta_{dim}(\alpha)$ we get the following expression:

$$\beta_{dim}(\alpha)=\frac{\alpha^2}{3\pi}+\frac{\alpha^3}{4\pi^2}-\frac{31}{288}\frac{\alpha^4}{\pi^3}. \quad (7.5)$$

We note that $\zeta(3)$ has disappeared from Eqs. (7.4) and (7.5) and also that the sign of the α^4 term is opposite to that in (7.1).

b) The scalar model with quartic interaction

$$\mathcal{L}=\frac{1}{2}\partial_\mu\varphi\partial_\mu\varphi-\frac{(4\pi)^2}{4!}g\varphi^4. \quad (7.6)$$

has been the most completely studied case because of the relative simplicity of the calculations. The calculations with PT have been carried out to the four-loop approximation.[17-19,31] In the subtraction scheme the function $\beta(g)$ is[19]

$$\beta_{subt}(g)=\frac{3}{2}g^2-\frac{17}{6}g^3+19.3g^4-146g^5, \quad (7.7a)$$

where the last two coefficients are calculated approximately. In the cut-off scheme the calculations can be done analytically,[18]

$$\beta_{\text{cut}}(g)=\frac{3}{2}g^2-\frac{17}{6}g^3+\left[\frac{109}{8}+6\zeta(3)\right]g^4 -\left[60\zeta(5)+18\zeta(4)+\frac{69}{2}\zeta(3)+\frac{1115}{12}\right]g^5 \approx 1.5g^2-2.83g^3+20.84g^4-216.09g^5. \tag{7.7b}$$

The correspondence between the coefficients of g^4 and g^5 in Eqs. (7.7a) and (7.7b) has been checked with the conversion formula (6.4).

We also give the result for $\beta(g)$ found (in collaboration with D. I. Kazakov) in the dimensional renormalization scheme:

$$\beta_{\text{dim}}(g)=\frac{3}{2}g^2-\frac{17}{6}g^3+\left[\frac{145}{16}+6\zeta(3)\right]g^4 -\left[60\zeta(5)-9\zeta(4)+39\zeta(3)+\frac{3499}{96}\right]g^5 \approx 1.5g^2-2.83g^3+16.27g^4-135.80g^5. \tag{7.7c}$$

Unlike Eq. (7.1), these results from PT have an alternating sign of the coefficients; also they increase rapidly. [Because of this, breaking off the series for $\beta(g)$ at different powers of g leads alternately to the "zero of charge" situation and to finite renormalization.] Here there is a reflection of the fact that asymptotically the coefficients β_n of the expansion

$$\beta(g)=\sum_{n=2}^{\infty}(-g)^n\beta_n$$

contain a factorial for large n. For example, in the subtraction scheme

$$\beta_n \xrightarrow[n\to\infty]{} \tilde{\beta}_n = 1.096n^{7/2}n! \tag{7.8}$$

Equation (7.8) was derived by the method of steepest descent from the functional integral.[32] It shows that the power series for $\beta(g)$ has no region of convergence, being an asymptotic series. A detailed analysis[33] shows that partial sums of this series, such as Eqs. (7.7), can provide a basis for numerical calculations of the function $\beta(g)$ in the range of values $0<g<0.1$, in which $\beta(g)$ rises monotonically like a quadratic parabola and differs only slightly from the one-loop approximation $(3/2)g^2$. At the same time the exact coefficients $\beta_n(n=2,3,4,5)$ differ considerably from the values of $\tilde{\beta}_n$ given by the large-n approximation, so that a synthesis of the PT data with the asymptotic information (7.8) is not a simple problem. Despite various efforts in this direction,[33,34] there are *no convincing results* so far. It is not excluded that to obtain reliable information about the behavior of $\beta(g)$ for $g \gtrsim 1$ will require methods not based on PT, i.e., strong-coupling methods.

The model (7.6) becomes asymptotically free[35] if we change the sign of the coupling constant, $g \to -g$,

$$\mathcal{L}_- = \frac{1}{2}\partial_\mu\varphi\partial_\mu\varphi + \frac{(4\pi)^2}{4!}g\varphi^4, \quad g>0, \tag{7.9}$$

since then

$$\beta_-(g) = -\beta(-g),$$

where the function $\beta(g)$ is given by Eqs. (7.7). Accordingly, the coefficients in the series for $\beta_-(g)$ keep the same sign, and $\beta_-(g)$ itself is negative, which indeed corresponds to the case of AF.

However, the classical Hamiltonian of the model (7.9) has no lower bound, and a doubt arises as to the existence of a lowest state in the quantum case. An analysis based on the concept of the continuous integral and a functional steepest-descent method[36] shows that the GF of the model (7.9) has an imaginary part in the region $g>0$, which is proportional to $\exp(-1/g)$ and corresponds to a tunnel-effect transition from the state of the free-field vacuum, which ceases to be the lowest state when there is an interaction $g>0$. Accordingly, the model (7.9) is internally inconsistent.

c) The model with a cubic nonlinearity in six-dimensional space-time,

$$\mathcal{L}_{\text{int}} = \frac{g}{3!}\varphi^3_{(6)}, \tag{7.10}$$

is another simple renormalizable model with the dimensionless coupling constant g^2. Calculations have been done in the two-loop approximation,[37,38]

$$\beta(g^2) = -\frac{3}{4}\frac{g^4}{(4\pi)^3} - \frac{125}{144}\frac{g^6}{(4\pi)^6}. \tag{7.11}$$

The model (7.10) possesses asymptotic freedom. The questions of its *freedom from contradictions and of the existence* of a vacuum state are, however, as yet unsettled.

d) A scalar model in three-dimensional space-time

$$\mathcal{L}_{\text{int}} = -\frac{g}{6!}\varphi^6_{(3)}. \tag{7.12}$$

Calculations in the two-loop approximation give[37]

$$\beta(g) = \frac{10}{3}\frac{g^2}{(8\pi)^2} - 74.4\frac{g^3}{(8\pi)^4}. \tag{7.13}$$

The change of sign $g \to -g$, as in the φ^4 model, changes the zero-of charge situation to asymptotic freedom, but again the problem of the absence of a lowest state in the theory arises.

e) Yang-Mills field interacting with fermions. We pass to a consideration of nonabelian gauge theories.[39] In these, as a rule, AF is realized. It is this that explains the present interest in nonabelian models and the widespread belief that they will be the basis of a future theory of strong interactions.

Let us consider the model of a nonabelian gauge field (Yang-Mills field) interacting with fermions. (If the gauge group G is identified with the "color" group and the fermions transform according to its fundamental representation, this model is known as quantum chromodynamics,[40] or the model of interaction between "colored" quarks and gluons.) The Lagrangian is of the form

$$\mathcal{L} = \mathcal{L}_{\text{YM}} + i\bar{\psi}_i^A\hat{\mathcal{D}}\psi_i^A, \tag{7.14}$$

where

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} - \frac{1}{2\alpha}(\partial_\mu B^a_\mu)^2 - \partial_\mu\bar{\eta}^a\partial_\mu\eta^a + gf^{abc}\bar{\eta}^a B^b_\mu\partial_\mu\eta^c,$$
$$F^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + gf^{abc}B^b_\mu B^c_\nu,$$
$$\mathcal{D}_\mu\psi_i^A = \partial_\mu\psi_i^A - igB^a_\mu(R^a)_{ij}\psi_j^A,$$

the η are fictitious scalar fields (Faddeev-Popov "ghosts"), α is a gauge parameter, and f^{abc} are the structure constants of the gauge group. The index A

$=1,2,\dots N_f$ distinguishes types of fermions, and the index i or j, the "colors." The gauge group G is characterized by the value of the Casimir operator $C_2(G)$,

$$f^{abc}f^{dbc}=C_2(G)\,\delta^{ad},$$

and the matrices R^a of the fermion representation R, which satisfy the commutation relations $[R^a,R^b]=if^{abc}R^c$, by the numbers $C(R)$ and $T(R)$:

$$R^aR^a=C(R)\,I,\quad \mathrm{Sp}\,(R^aR^b)=T(R)\,\delta^{ab},$$

where I is the identity matrix. In these notations, the Gell-Mann–Low function of the model (7.14) is, in the two-loop approximation,[41,42]

$$\begin{aligned}&\beta(g^2)=A\frac{g^4}{(4\pi)^2}+B\frac{g^6}{(4\pi)^4},\\&A=-\frac{11}{3}C_2(G)+\frac{4}{3}N_fT(R),\\&B=-\frac{34}{3}C_2^2(G)+\frac{20}{3}C_2(G)N_fT(R)+4C(R)N_fT(R).\end{aligned}\qquad(7.15)$$

If the fermions transform according to the fundamental representation of the group $SU(N_c)$ (quantum chromodynamics with N_c colors), then, since in this case

$$C_2(G)=N_c,\quad T(R)=\frac{1}{2},\quad C(R)=\frac{N_c^2-1}{2N_c},$$

the expressions for A and B are:

$$\begin{aligned}&A=-\frac{11}{3}N_c+\frac{2}{3}N_f,\\&B=-\frac{34}{3}N_c^2+N_f\left(\frac{13}{3}N_c-\frac{1}{N_c}\right).\end{aligned}\qquad(7.16)$$

From Eqs. (7.15) and (7.16) it can be seen that this model is asymptotically free both in the case of a pure Yang-Mills field ($N_f=0$) and also if the number of fermion multiplets is not too large $[N_fT(R)<(11/4)C_2(G)]$. This fact is very important from the point of view of physics, since it assures the possibility of constructing models with fermions possessing AF. We note also that the coefficients A and B increase linearly with increasing N_f, with B passing through zero and becoming positive earlier than A. Therefore for any gauge group there are values of N_f for which A is small and negative and B is positive. Under these conditions the curve of the function $\beta(g^2)$ is of the form shown in Fig. 3. From this figure and Eq. (5.5) we see that the point $g^2=g_0^2$ is infrared-stable, i.e., the effective charge $\bar{g}^2(t,g^2)$ goes to g_0^2 for $t\to 0$. In calculating the values of the anomalous dimensionalities $\gamma_\Gamma(g_0^2)$ at this point one can use the small quantity A as expansion parameter.[41] A virtue of this method is that the coefficients of powers of this parameter found from PT are independent of the renormalization scheme used.

f) Supersymmetric gauge models. The simplest model of this class has the Langrangian[43]

$$\begin{aligned}\mathcal{L}=\mathcal{L}_{\mathrm{YM}}+\frac{i}{2}\bar{\lambda}^a\hat{\mathcal{D}}\lambda^a+\sum_n\Big\{&\frac{i}{2}\bar{\psi}_i^n\hat{\mathcal{D}}\psi_i^n+\frac{1}{2}(\mathcal{D}_\mu A_i^n)^2\\&+\frac{1}{2}(\mathcal{D}_\mu B_i^n)^2-ig\bar{\lambda}^a(R^a)_{ij}[A_i^n+\gamma_5B_i^n]\,\psi_j^n-\frac{g^2}{2}[(R^a)_{ij}A_i^nB_j^n]^2\Big\}.\end{aligned}\qquad(7.17)$$

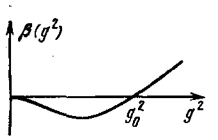

FIG. 3.

Owing to the high degree of symmetry (gauge + supersymmetry) in this model, it has been possible to connect by means of one constant g a large number of fields of different types: The Yang-Mills field, a Majorana (i.e., charge-conjugation invariant) spinor λ^a, and N multiplets made up of a scalar A_i^n, a pseudoscalar B_i^n, and a Majorana spinor ψ_i^n. The fields λ^a transform according to the adjoint representation of the gauge group G, which gives the expression for the covariant derivative

$$\mathcal{D}_\mu\lambda^a=\partial_\mu\lambda^a+gf^{abc}B_\mu^b\lambda^c,\qquad(7.18)$$

and the fields A_i^n, B_i^n, and ψ_i^n, according to a representation R.

$$\mathcal{D}_\mu A_i^n=\partial_\mu A_i^n-ig(R^a)_{ij}B_\mu^aA_j^n,$$

and analogous formulas for B_i^n and ψ_i^n.

A calculation of $\beta(g^2)$ for the model (7.17), carried out in the two-loop approximation,[44] gave the following result:

$$\begin{aligned}&\beta(g^2)=A\frac{g^4}{(4\pi)^2}+B\frac{g^6}{(4\pi)^4},\\&A=NT(R)-3C_2(G),\\&B=4NT(R)\,C(R)+2NT(R)\,C_2(G)-6C_2^2(G).\end{aligned}\qquad(7.19)$$

The expression (7.19) differs from the result (7.15) for the previous model by an interesting feature. Namely, it is easy to find a representation R for which the coefficient A is zero. For example, let R be the adjoint representation of the group $SU(N_c)$. Then $C_2(G)=C(R)=T(R)=N_c$ and $A=N_c(N-3)$, $B=6N_c^2(N-1)$.

Setting $N=3$, we get $A=0$, $B=12N_c^2$; i.e., in the lowest approximation there is no charge renormalization.

An analysis of the two-loop diagrams for the model (7.17) shows[44] that for the coefficient B to go to zero along with A, the contribution of the Yukawa interaction in the expressions (7.19) must be doubled. It turned out[45] that this is just what happens in a different supersymmetric gauge model which was proposed in Ref. 46

$$\begin{aligned}\mathcal{L}=\mathcal{L}_{\mathrm{YM}}&+\frac{i}{2}\bar{\lambda}_k^a\hat{\mathcal{D}}\lambda_k^a+\frac{1}{2}(\mathcal{D}_\mu A_i^a)^2+\frac{1}{2}(\mathcal{D}_\mu B_i^a)^2\\&-\frac{g}{2}f^{abc}\bar{\lambda}_k^a[(\alpha^i)_{kl}A_i^b+(\beta^i)_{kl}\gamma_5B_i^b]\lambda_l^c+\frac{g^2}{4}[(f^{abc}A_i^bA_j^c)^2\\&+(f^{abc}B_i^bB_j^c)^2+(f^{abc}A_i^bB_j^c)^2].\end{aligned}\qquad(7.20)$$

The gauge group of this model is the group $SU(N_c)$, with the fields (λ, A, and B) transforming according to its adjoint representation, so that the covariant derivatives of these fields are all constructed according to Eq. (7.18). Besides this, the model (7.20) is $SU(4)$ supersymmetric and contains three multiplets of fields A and B [the fields ψ and λ shown in Eq. (7.17) are combined in λ_k^a]. The indices run through the following values:

$$a,\ b,\ c=1,\ \dots,\ N_c^2-1,\qquad k,\ l=1,\ \dots,\ 4,\qquad i,\ j=1,\ 2,\ 3.$$

The six four-rowed square matrices α^i,α^i satisfy the relations

$$[\alpha^i,\ \alpha^j]_+=[\beta^i,\ \beta^j]_+=-2\delta^{ij},\quad [\alpha^i,\ \beta^j]_-=0.$$

The calculations[45] showed that in this model the two-loop contribution of the Yukawa interaction to the function $\beta(g^2)$ twice as large as in the case of Eq. (7.17), which makes both the first two coefficients vanish, $A = B = 0$. Since in the two loop calculations practically all of the features of the model (7.20) already appear [the four-scalar interaction does not contribute directly to $\beta(g^2)$ in this approximation, but it must be included to maintain the RG structure of the GF], the vanishing of the coefficients A and B very probably is not just a play on numbers, but indicates that there is no charge renormalization in this theory. However, no additional arguments have so far been proposed in favor of this possibility.

8. MANY-CHARGE MODELS

We begin our consideration of the many-charge case with two models for which the Gell-Mann–Low functions have been calculated in the two-loop approximation. The first of these is scalar electrodynamics, i.e., the theory of the interaction of the electromagnetic field (A_μ) and a charged scalar field φ:

$$\mathcal{L} = (\partial_\mu + ieA_\mu)\varphi^* (\partial_\mu - ieA_\mu)\varphi - \frac{h}{3!}(\varphi^*\varphi)^2. \tag{8.1}$$

This model was studied in Ref. 47 (in the one-loop approximation); two-loop calculations were carried out in Ref. 48. We give the results (the dimensional renormalization scheme was used)

$$\begin{aligned} \beta_e(e^2, h) &= \frac{1}{3}\frac{e^4}{(4\pi)^2} + 4\frac{e^6}{(4\pi)^4}, \\ \beta_h(e^2, h) &= \frac{1}{(4\pi)^2}\left(\frac{5}{3}h^2 - 6he^2 + 18e^4\right) \\ &\quad + \frac{1}{(4\pi)^4}\left(-\frac{10}{3}h^3 + \frac{28}{3}h^2e^2 + \frac{158}{3}he^4 - 208e^6\right). \end{aligned} \tag{8.2}$$

Since in this order no dependence of β_e on h has yet appeared, the second charge $\bar{h}$ is not involved in the equation for the effective charge $\overline{e^2}$. Consequently, in scalar electrodynamics the effective electromagnetic coupling constant shows zero-charge behavior at large momenta in both the one-loop and the two-loop approximations. The associated departure from the weak-coupling region makes it impossible to get reliable information about the asymptotics of the model (8.1) on the basis of the approximation (8.2).

Two-loop calculations have also been done for several models of the Yukawa type, i.e., for a trilinear interaction of fermions with (pseudo) scalars (the one-loop approximation for the Yukawa model was investigated in Ref. 49). Let us consider the simplest variant of this theory,

$$\mathcal{L}_{\text{int}} = g\bar{\psi}\gamma_5\varphi\psi - \frac{h}{4!}\varphi^4. \tag{8.3}$$

In the dimensional renormalization scheme the result of the calculation of β_g and β_h is[24].

$$\begin{aligned} \beta_g(g^2, h) &= 5\frac{g^4}{(4\pi)^2} + \frac{1}{(4\pi)^4}\left(\frac{h^2g^2}{12} - 2g^4h - \frac{57}{4}g^6\right), \\ \beta_h(g^2, h) &= \frac{1}{(4\pi)^2}\left(\frac{3}{2}h^2 + 4g^2h - 24g^4\right) \\ &\quad + \frac{1}{(4\pi)^4}\left(-\frac{17}{6}h^3 - 6g^2h^2 + 14g^4h + 192g^6\right). \end{aligned} \tag{8.4}$$

Substitution of Eq. (8.4) in the system of equations (5.10) leads to a picture in the phase plane of the effective charges $\overline{g^2}$ and $\bar{h}$ which is shown in Fig. 4, a (the arrows show the direction in which the momentum

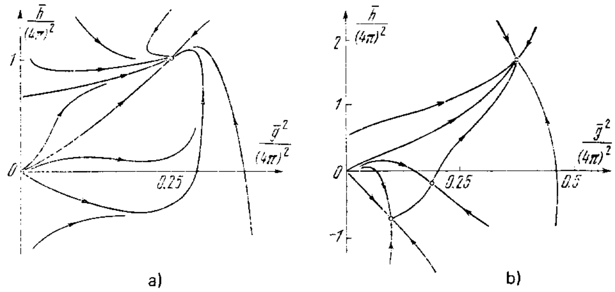

FIG. 4.

variable t increases).

It has turned out that in this model there exists (in the two-loop approximation) an ultraviolet-stable point (g_0^2, h_0) at which the functions β_g and β_h go to zero,

$$\begin{aligned} &\beta_g(g_0^2, h_0) = \beta_h(g_0^2, h_0) = 0, \\ &\frac{g_0^2}{(4\pi)^2} \approx \frac{1}{4}, \quad \frac{h_0}{(4\pi)^2} \approx 1. \end{aligned} \tag{8.5}$$

According to Eq. (5.12) the values g_0^2 and h_0 are the high-energy limits of the effective charges $\overline{g^2}$ and $\bar{h}$. Using the two-loop expressions found in Ref. 24 for the anomalous dimensionalities of the fermion and boson propagators

$$\begin{aligned} \gamma_\psi(g^2, h) &= -\frac{1}{2}\frac{g^2}{(4\pi)^2} + \frac{13}{8}\frac{g^4}{(4\pi)^4}, \\ \gamma_\varphi(g^2, h) &= -2\frac{g^2}{(4\pi)^2} + \frac{1}{(4\pi)^4}\left(5g^4 - \frac{h^2}{12}\right), \end{aligned} \tag{8.6}$$

and Eqs. (5.11) and (8.5), we get power-law asymptotics for these propagators in the form

$$d_\psi(t, g^2, h) \underset{t\to\infty}{\sim} t^{0.02}, \quad d_\varphi(t, g^2, h) \underset{t\to\infty}{\sim} t^{0.27}. \tag{8.7}$$

However, as the previously considered φ^4 model shows, inclusion of higher orders could decidedly alter the entire picture. Therefore neither the numerical values of the exponents in Eq. (8.7) nor even the existence of an ultraviolet-stable point can be regarded as reliably established.

The model (8.3) has also been studied in the two-loop approximation in Ref. 25, by the use of the method of invariant charges, or, what is the same thing, of effective charges, in the subtraction scheme. An essential feature of this scheme is that the RG functions depend on the ratios of the external momenta, and this was manifested not only in a change of the numerical values of the exponents in the power-law asymptotics of the GF, but also in a change of the number of ultraviolet stable points in the phase plane on conversion from set $\{\rho\}$ of fixed ratios of the momenta to another set. (Compare, for example, Fig. 4, a with Fig. 4, b which is taken from Ref. 25.)

In Refs. 50, 51 the model (8.3) has been generalized to the case of $SU(2)$ and $SU(3)$ symmetries of the Yukawa interaction, and also the possibility of the simultaneous presence in the theory of both scalar and pseudoscalar fields, and also Majorana spinors, has been taken into account. The results of the calculations in the two-loop approximation do not lead to important differences from Eq. (8.4), and the qualitative appearance of the phase plane is essentially the same.

The foregoing analysis of a large number of models has confirmed the previously stated general conclusion that PT is inadequate [even in the case of comparatively small numerical values of the anomalous dimensionalities, as in Eq. (8.7), and *a fortiori* in the case of a "zero of the charge"] for the determination of the ultraviolet properties of a theory which does not possess AF. Proceeding to the consideration of many-charge asymptotocally free models, we note that the construction of models in which all of the effective charges go to zero for $t \to \infty$ and a lowest energy state exists is not possible in four-dimensional space-time without the introduction of nonabelian gauge fields.[52] Moreover, as is readily shown from analysis of one-loop corrections to the photon propagator, any model that includes an Abelian gauge field (photon) cannot be asymptotically free.

From these considerations, and also from the requirement that we construct models that are realistic in the set of fields and their properties (for example, in order to make nonabelian vector fields massive by means of the Higgs mechanism, one must introduce scalar particles into the theory), it is natural to conduct an examination of a very general Lagrangian, containing, along with nonabelian gauge fields also spinor and scalar fields transforming according to arbitrary representations. This Lagrangian must have several coupling constants (a gauge constant and, generally speaking, some independent constants for Yukawa and four-scalar interactions), and the main question is whether one can assure that all effective charges will simultaneously go to zero, i.e., whether one can find an asymptotically free model of this type. This problem has been investigated in a number of papers,[17,53] considering a broad class of gauge groups and their various representations. The corresponding analysis of many-charge RG equations is almost impossible to survey, since for the various groups and representations there are different numbers of independent types of interactions of scalar fields. To illustrate the general relationships (they turn out to be inherent in most of the models) we shall consider a very simple case of a theory which includes nonabelian gauge fields, spinor fields, and pseudoscalar fields and has been studied in Ref. 17. The formulas of that paper can be applied in all cases in which there is only one type of four-scalar interaction, i.e., in particular, for the group $SU(2)$. Therefore we shall consider here the case in which the gauge group in $SU(2)$, the scalar fields transform according to its adjoing representation, and the spinors transform according to some representation R,

$$\mathcal{L} = \mathcal{L}_{\mathrm{YM}} + \bar{\psi}_i (i\hat{\partial} + g\hat{B}^a (R^a)_{ij}) \psi_j + \frac{1}{2} (\partial_\mu \varphi^a + g f^{abc} B^b_\mu \varphi^c)^2 + \varkappa \bar{\psi}_i \gamma_5 (R^a)_{ij} \psi_j \varphi^a - \frac{h}{8} (\varphi^a \varphi^a)^2. \tag{8.8}$$

In the one-loop approximation the system of equations for the effective charges $\overline{g^2}$, $\overline{\varkappa^2}$, and $\bar{h}$ is of the form

$$t \frac{\partial \bar{g}^2}{\partial t} = -\frac{C}{(4\pi)^2} \bar{g}^4, \tag{8.9a}$$

$$t \frac{\partial \bar{\varkappa}^2}{\partial t} = \frac{1}{(4\pi)^2} (A\bar{\varkappa}^4 - B\bar{g}^2\bar{\varkappa}^2), \tag{8.9b}$$

$$t \frac{\partial \bar{h}}{\partial t} = \frac{1}{(4\pi)^2} (\alpha\bar{h}^2 - \gamma\bar{h}\bar{g}^2 + \sigma\bar{h}\bar{\varkappa}^2 + \delta\bar{g}^4 - \rho\bar{\varkappa}^4), \tag{8.9c}$$

where

$$C = 7 - \frac{4}{3} T(R), \quad A = 3C(R) + 2T(R) - 2,$$

$$B = 6C(R), \quad \alpha = \frac{11}{2}, \quad \gamma = 12, \quad \sigma = 4T(R),$$

$$\delta = 12, \quad \rho = \frac{8}{5} T(R) [3C(R) - 1].$$

The constants $T(R)$ and $C(R)$ can be expressed in terms of the isospin j which characterizes the representation R of the group $SU(2)$:

$$T(R) = \frac{1}{3} j(j+1)(2j+1), \quad C(R) = j(j+1).$$

Let us first consider the first two equations of the system (8.9), since they are independent of the third equation. The behavior of $\overline{g^2}$ and $\overline{\varkappa^2}$ as functions of t can be seen from the diagram of the phase plane (Fig. 5). The region in which the effective charges approach zero (i.e., the region under the separatrix $\overline{\varkappa^2} = k\overline{g^2}$, exists in the first quadrant only for $k = (B - C)/A > 0$. This condition is satisfied for isospin $j \geq 0$. We note two features of the phase plane of Fig. 5:

a) The singular solution $\overline{\varkappa^2} = k\overline{g^2}$ is asymptotically free, but unstable, since in order to "keep on it" one must rigorously establish that the constants are proportional to each other, $\varkappa^2 = kg^2$.

b) In the region of AF the effective Yukawa constant $\overline{\varkappa^2}$ goes to zero more rapidly than g^2 and for $t \to \infty$ becomes negligibly small compared with $\overline{g^2}$.

Therefore further analysis of the existence of asymptotic freedom of the solutions of Eqs. (8.1), including the third order, is worth doing for two cases: 1) When $\overline{\varkappa^2} \ll g^2$, and 2) when $\overline{\varkappa^2} = kg^2$. Let us consider the first case.

Equation (8.9c) takes the form

$$t \frac{\partial \bar{h}}{\partial t} = \frac{1}{(4\pi)^2} (\alpha\bar{h}^2 - \gamma\bar{h}\bar{g}^2 + \delta\bar{g}^4) = \frac{1}{(4\pi)^2} \left(\frac{11}{2} \bar{h}^2 - 12\bar{h}\bar{g}^2 + 12\bar{g}^4 \right). \tag{8.10}$$

The right side of this equation is always larger than zero; the charge $\bar{h}$ increases and AF is impossible.

It is also impossible in the case of the opposite sign of the quartic interaction, $h \to -h$. As we noted before, for the purely scalar theory there would be AF, but in the present case the originally negative charge $\bar{h}$ increases with increasing t, passes through zero and increases to infinity. The general pattern in the phase plane is shown in Fig. 6.

Analysis of a large number of groups and representations[53] has shown that the properties found here ("dying away" of the Yukawa effective constant in comparison with that of the gauge field and the absence of asymptotically free solutions) are characteristic of the majority

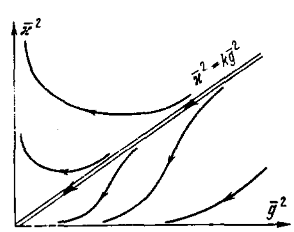

FIG. 5.

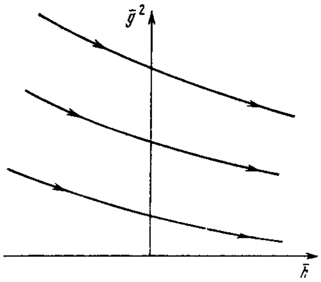

FIG. 6.

of the models, except for a few, based on representations of rather high dimensionalities, for which AF is possible. In particular, there has been no success in using the Higgs effect to give mass to all the vector mesons and at the same time preserve AF, if we don't stay exactly on the separatrix $\bar{\varkappa}^2 = k\bar{g}^2$ in the phase plane of Eqs. (8.9a) and (8.9b).

Let us turn to the examination of this singular solution

$$\bar{\varkappa}^2 = k\bar{g}^2, \qquad k = \frac{B+C}{A} = \frac{6C(R)+(4/3)T(R)-7}{3C(R)+2T(R)-2}. \tag{8.11}$$

The instability of the solution (8.11) will be no obstacle if we postulate that the required relation $\varkappa^2 = kg^2$ is satisfied exactly. Then for the effective charges the relation (8.11) holds for all t, and the regime of AF is realized for this particular solution. However, we must still assure that the charge $\bar{h}$ will go to zero. To do this we take $\bar{h} = mg^2$, i.e., we look for an asymptotically free singular solution of the system (8.9) for which all three charges are proportional to each other. The coefficient m is determined from a quadratic equation, which is obtained by setting $\bar{h} = mg^2$ in Eq. (8.9c) and using (8.9a) and (8.11). The requirement that m be positive puts no more new restrictions than the condition $k > 0$ on the choice of the representation R, since for any isospin value $j \geq 1$ a solution with $m > 0$ exists [and, like the solution (8.11), is unstable[11]].

Thus, a regime of AF can be achieved with unstable singular solutions of the many-charge RG equations.[54] Inclusion of higher order corrections must, in general, distort the straight-line singular solutions.[11] Then the connection between the constants g^2, $\varkappa^2$, h will be given by some functions expandable in PT series; i.e., the conditions $\varkappa^2 = kg^2$, $h = mg^2$ will be replaced with $\varkappa^2 = f_1(g^2)$, $h = f_2(g^2)$, where

$$f_1(g^2) = kg^2 + O(g^4), \quad f_2(g^2) = mg^2 + O(g^4).$$

In such an asymptotically free model the coefficients of the field structures in the Lagrangian are rather complicated functions (known only in the form of PT expansions) of a single constant g^2; i.e., we are actually dealing here with a single-charge theory.

Relations between the coupling constants of the type of Eq. (8.11) can arise in a natural way as consequences of some symmetry of the Lagrangian.[55] In fact, if symmetry considerations connect the coefficients of various structures in the Lagrangian in a definite way, then these coefficients will be renormalized in a self-consistent way, so that an analogous connection will hold between the corresponding effective charges, i.e., there will be corresponding special solutions of the RG equations. As a rule, connections imposed by symmetry are very simple, of the type of $\varkappa = g$, $h = g^2$. There is such a connection, for example, in the supersymmetric model (7.17). Inclusion of higher orders of PT naturally does not change these relations governed by symmetry. Therefore one way to seek out new symmetries in QFT may be to look for special solutions of the EG equations that do not change when two-loop corrections are included.[51]

The question of the existence of singular solutions has been investigated for a broad class of gauge models, both for the purpose of constructing asymptotically free models capable of aspiring to describe hadron interactions,[11,56] and also to look for new symmetries.[51] An interesting model in this connection, based on the group $SU(2)$, has been considered in Ref. 11.

$$\mathcal{L} = \mathcal{L}_{\mathbf{YM}} + i\sum_{k=1}^{10} \bar{\psi}_k \left(\hat{\partial} - ig\frac{\tau^a}{2}\hat{B}^a\right)\psi_k + i\bar{\xi}\hat{\partial}\xi + \left|\left(\partial_\mu - ig\frac{\tau^a}{2}B_\mu^a\right)\varphi\right|^2 - \varkappa(\bar{\psi}_1\xi\varphi + \varphi^+\bar{\xi}\psi_1) - \frac{h}{8}(\varphi^+\varphi)^2, \tag{8.12}$$

where τ^a are the isotopic Pauli matrices, ψ and φ are isodoublets, and ξ is an isosinglet. The boundary of the region of AF in this model is the singular solution $h = g^2$ $2\varkappa^2$. This simple relation between the constants suggests the presence of a hidden symmetry of the Lagrangian (8.12). Two-loop calculations, which have so far not been undertaken with this model, might considerably advance the solution of this question.

9. HIGH ENERGY ASYMPTOTIC PROPERTIES OF PHYSICAL PROCESSES

Besides the applications to the ultraviolet properties of the GF of quantized fields which we have considered, the RG method has many applications to the analysis of the high-energy asymptotics of physical processes. The direct use of the formula (4.15) is difficult here, however, since in the amplitudes for actual processes, unlike GF, all of their quadratic momentum arguments, p_i^2 and $p_i p_j$, cannot be made to approach infinity simultaneously, since some of them must stay on the mass shell. Furthermore some of the ratios of the type of $p_i p_j / p_i^2$, which were regarded as fixed in Eq. (4.15), now become large and contribute to the logarithmic asymptotic terms. And the RG methods alone are insufficient for the summation of such contributions, because ratios $p_i p_j / p_i^2$ are not affected by the scale transformations of the RG.

The relation (4.15) is modified for such purposes as follows:

$$F(\{xt\}, \{z\}, y, g) = t^{D/2} F\left(\{x\}, \left\{\frac{z}{t}\right\}, \bar{y}, \bar{g}\right) \exp\left[-\int_1^t \frac{du}{u}\gamma_F(\bar{y}, \bar{g})\right], \tag{9.1}$$

where D is the dimension of the amplitude F in mass units, and $\{z\}$ denotes those arguments of F that lie on the mass shell. The essential difference between Eqs. (9.1) and (4.15) is that on the right side of (9.1) the momentum arguments $\{z/t\}$ go to zero for $t \to \infty$. This can lead to the appearance of new singularities of F as a function of t, which manifest themselves in the PT series as logarithms. These "low-energy" logarithms, unlike the "ultraviolet" ones that get into F via the effective charge $\bar{g}$, are accordingly not summable by the RG method, which means that the method we have described cannot be applied to the study of such amplitudes. If, however, the "low-energy" logarithms are suppressed and the passage to the limit $z/t \to 0$ in the function $F(\{x\}, \{z/t\}, \bar{y}, \bar{g})$ is smooth, then all of the RG apparatus developed in the foregoing can be fully applied to the investigation of the ultraviolet asymptotics of such processes.

Analysis of the Feynman diagrams in an arbitrary order of PT has shown that suppression of the "low-energy" logarithms actually occurs in some processes for a number of QFT models. Namely, it has been proved that the RG can be applied to find the asymptotics of form-factors[57] and amplitudes for elastic scattering at fixed angles[58] in renormalizable theories without vector fields. In vector theories, in particular in quantum chromodynamics, the contribution of "low-energy" logarithms is not suppressed for most processes, and here, as a rule, *the RG method cannot be applied.*

In one case, in the study of e^+e^- annihilation into hadrons, application of the RG method does not encounter the difficulties we have just now noted. In fact, the cross section for this process (Fig. 7), is proportional to the spectral density of the photon propagator,[59]

$$\sigma^{\text{tot}}_{e^+e^-\to h} = \frac{8\alpha^2\pi^2}{p^2}\Pi(p^2),$$

$$d^{-1}(p^2) = 1 + 2\alpha p^2 \int\limits_{4M^2}^{\infty} \frac{ds\Pi(s)}{s(s-p^2-i0)},$$

i.e., it depends on only one momentum argument p^2. In particular, in nonabelian theories the asymptotic form of the cross section for e^+e^- annihilation is[59,60]

$$\sigma^{\text{tot}}_{e^+e^-\to h} = \frac{4\pi\alpha^2}{3p^2}\left(\sum_i Q_i^2\right)\left(1 + \frac{c}{\ln(p^2/\mu^2)} + \cdots\right), \quad (9.2)$$

where the constant c is determined by the gauge group and the number of quarks, and the Q_i are the electric charges of the quarks in units of the charge e of the electron.

Here also, however, the application of the RG formulas that lead to Eq. (9.2) is not entirely justified, since in using them we are ignoring threshold singularities and cuts, which are characteristic of the photon propagator for $p^2 > 0$. The permissibility of transferring the results established by the RG method for the asymptotics in the Euclidean domain $p^2 < 0$, where the GF is real, into the domain of timelike momenta requires special analysis. In Ref. 61 a method is proposed for combining RG and dispersion methods, which makes it possible to do the analysis of e^+e^- annihilation into hadrons without extrapolating the RG results into the region $p^2 > 0$, since the RG equations are here written not for the observable quantities themselves but for dispersion integrals containing them.

The application of the RG together with Wilson's technique of operator expansions[62] has had distinguished success in describing deep inelastic lepton-hadron processes on the basis of nonabelian gauge theories. There is an extensive literature on this topic (see, for example, Ref. 23). Here we do not go into the details, and only point out the main features of the approach and discuss the results.

The process of deep inelastic lepton-hadron scattering (Fig. 8) is characterized by the kinematic variables $Q^2 = -q^2$ and $x = Q^2/2pq$, which lie in the region $Q^2 \gg M^2$, $0 \leqslant x \leqslant 1$, where M^2 is the square of the hadron mass. The cross section for the process is expressed in terms of the structure functions $F_i(x, Q^2)$, which are connected by a Fourier transformation with the matrix element of the commutator of the electromagnetic currents between hadron states, $\langle p|[J(x), J(0)]_-|p\rangle$. The moments $M_n(Q^2) \equiv \int_0^1 dx x^{n-2} F(x, Q^2)$ satisfy the relation

$$M_n(Q^2) = \sum_\alpha A_n^\alpha C_n^\alpha(Q^2), \quad (9.3)$$

where in the terms of the Wilson operator expansions,[62]

$$J(x)J(0) \sim \sum_{n=0}^{\infty}\sum_\alpha \tilde{C}_n^\alpha(x^2)x_{\nu_1}\ldots x_{\nu_n}O^\alpha_{\nu_1\ldots\nu_n} \quad (9.4)$$

the constant A_n^α is connected with the matrix element of the operator $\hat{O}^\alpha$, $A_n^\alpha p_{\nu_1}\cdots p_{\nu_n} \sim \langle p|0^\alpha_{\nu_1}\cdots{\nu_n}|p\rangle$, and $C_n(Q^2)$ is the Fourier transform of the coefficient function $\tilde{C}_n^\alpha(x^2)$.

It was found that the behavior of the moments $M_n(Q^2)$ at large Q^2 could be studied by means of RG methods.[63] With attention to the dependence of the function $C_n(Q^2)$ on both the coupling constant g and the renormalization parameter μ, one can derive for it the RG differential equation

$$\left(\mu^2\frac{\partial}{\partial\mu^2} + \beta(g)\frac{\partial}{\partial g} - \gamma_n(g)\right)C_n\left(\frac{Q^2}{\mu^2}, g\right) = 0,$$

whose solution can be expressed in the form, analogous to Eq. (6.8),

$$C_n(t, g) = C_n(1, \bar{g}(t, g))\exp\left(-\int\limits_g^{\bar{g}(t,g)} du\frac{\gamma_n(u)}{\beta(u)}\right), \quad (9.5)$$

where $t = Q^2/\mu^2$. In asymptotically free models the leading asymptotics of the $C_n(t, g)$ are determined by the lowest order of PT and according to Eq. (5.9) have the following form[64]

$$C_n(t, g) \sim (\ln t)^{c_n/b}, \quad (9.6)$$

where $b < 0$ and $c_n > 0$ are the one-loop coefficients of the expansion of $\beta(g)$ and $\gamma_n(g)$ in power series in g^2. Substituting this result in Eq. (9.3), we get a logarithmic decrease of the moments $M_n(Q^2)$, i.e., a logarithmic

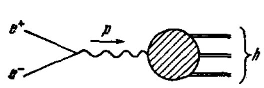

FIG. 7.

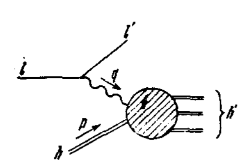

FIG. 8.

breaking of Bjorken scaling [for exact scaling $M_n(Q^2)$ is independent of Q^2 for large Q^2]. This behavior of $M_n(Q^2)$ is in good agreement with experiment.[65]

In the literature there are results of calculations of corrections $\sim\bar{g}^2$ [i.e., of the type of Eq. (6.10)] to the formula (9.6), determined both by expansion of $C_n(1,\bar{g})$ in series in $\bar{g}^2$,[66,67] and also by taking into account the two-loop coefficient in $\gamma_n(g)$.[68,69] All of these calculations have been corrected and brought together in Refs. 67 and 69. The corrections $\sim\bar{g}^2$ turn out to be rather small [a few percent of the leading contribution (9.6)], which again is evidence in favor of the applicability of PT in asymptotically free theories.

We note that a logarithmic deviation from scaling can also occur in theories with a finite renormalization of charge,[70] for example, in the model (8.8), under the condition that the scalar effective constant $\bar{h}$ approaches a finite value for $t\to\infty$.

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