How light conquers darkness (W.R. Hamilton and the concept of group velocity)

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During the past decades, references have begun to appear in the literature on science and the history of science to the fact that the concept of group velocity was known to Hamilton even before Stokes and Rayleigh. What was written about this a quarter of a century ago was quite vauge: "In 1839 Hamilton presented a paper to the Irish Academy and promised to publish further details, but no report appeared."[1] What is written nowadays is as follows: "The concept of group velocity of a wave was first introduced by Hamilton[2,3] and subsequently developed in the works of Rayleigh."[4] Such references exhaust the information about Hamilton's works, and it remains unclear what part of his work has been published and to what extent his results anticipated the work of Rayleigh.[5]

During his lifetime, Hamilton published only two short communications[2,3] in which he gave a very brief and incomplete account of his investigations on the velocity of propagation of waves. He provided a very clear exposition of his opinions and results in a letter to John Herschel (February 1839), which was published 100 years later in the second volume of Hamilton's collected works.[6] This same volume contains the first publication of two scientific notebooks (125 pages of large size), perusal of which shows that Hamilton was ahead of his time by about this period of 100 years. This can also be seen from publications during his lifetime[2,3] and from his letters to Herschel, which have the character of preprints by today's standards.

Like the majority of investigators of his time, Hamilton adopted a model in which a medium is regarded as a discrete lattice described by a system of differential-difference equations. Many physicists had studied waves in such lattices, but before Hamilton only steady oscillations of the entire lattice (proper waves) were considered. Attention was devoted mainly to the problem of choosing a law of interaction between the centers of the lattice which would provide at least a qualitatively correct description of the experimentally observed dispersion of light. But nobody even posed the problem of the penetration of perturbations into an initially stationary region of a medium. In Hamilton's words, "Much had been done, perhaps, in the dynamics of light; little, I thought, in the dynamics of darkness." This problem was formulated and solved for the first time by Hamilton.

The problem was posed as follows. Suppose that at the initial instant of time a perturbation having the structure of a proper wave of the lattice occupies a limited region of the medium and that the rest of the medium is at rest. What will happen after the lapse of a sufficiently long interval of time? This problem was considered by Hamilton for lattices of various types (one-dimensional, two-dimensional, three-dimensional, and even n-dimensional—we must not forget the Hamilton was also an outstanding mathematician!) and for various types of coupling between the centers. The most detailed analysis was naturally made for the simplest one-dimensional lattice (a Lagrange chain) whose transverse oscillations are described by equations of the form

$$\ddot{z}_{x,t} = a^2 (z_{x+1,t} - 2z_{x,t} + z_{x-1,t}), \tag{1}$$

where x is an integral coordinate (the unit of length is chosen to be equal to the period of the lattice), so that the constant a has the dimensions of velocity. The proper waves in such a lattice have the form

$$z = A\cos(2x\nu - 2at\sin\nu + \psi),$$

i.e., their phase velocity has the value

$$V = a\frac{\sin\nu}{\nu}, \tag{2}$$

where ν is a constant determined by the frequency or wavelength.

Suppose that at the initial instant of time $t=0$ the left-hand side of the lattice ($x<0$) is perturbed according to the wave law

$$\begin{aligned} z_{x,0} &= 1 - \cos 2x\nu, \\ \dot{z}_{x,0} &= -2a\sin\nu\cdot\sin 2x\nu, \end{aligned} \qquad (x<0) \tag{3}$$

and that the right-hand side ($x>0$) is at rest. Then for $t>0$ the solution of Eq. (1) subject to the initial conditions (3), in Hamilton's own notation, has the form

$$z_{x,t} = \sin^2(x\nu - at\sin\nu) - \frac{\sin^2\nu}{2\pi}\int_0^\pi \frac{\sin(2x\theta - 2at\sin\theta)}{\sin\theta(\cos\theta - \cos\nu)}\,d\theta. \tag{4}$$

Analogous solutions were constructed by Hamilton for more complicated lattices (including multi-dimensional ones), both with initial conditions of the form (3) (a semi-infinite train) and with an initial perturbation that occupies a finite region.

Hamilton then considered the behavior of the solutions of the form (4) for large values of at, i.e., in the language of modern physics, he studied the asymptotic behavior of these solutions. The final conclusions of this investigation, which occupies most of Hamilton's scientific notebooks and is summarized in Refs. 2 and 3, are as follows. In a lattice of the general type that admits proper waves of the form

$$\cos(kx - st + \psi),$$

the "velocity of propagation of vibratory motion" has the value ds/dk and is not equal to the "velocity of transmission of phase," which has the value s/k. Moreover, there is still a third velocity, which characterizes the arrival of the first perturbation in the initially stationary medium. For the Lagrange chain, this is the constant a in the equation of motion (1). In the language of modern physics, this is the velocity of the precursor.

In the case of the Lagrange chain with the initial conditions (3), which Hamilton studied in greatest detail, estimates were given for the widths of the precursor,[1)] the transition region, and the front of proper wave motion that propagates with the group velocity. In particular, the width of the front is of order

$$p\sqrt{at\sin\frac{\pi}{\lambda}},$$

where λ is the wavelength, and p is a sufficiently large number such that

$$\frac{1}{\pi}\int_{-\infty}^{+\infty}\cos\left(\Phi\pm\frac{\psi^2}{p^2}\right)\frac{\sin\psi}{\psi}d\psi\approx\cos\Phi.$$

He also considered the "diffusion" of a finite wave packet in such a lattice.

Unfortunately, these estimates did not appear in published works during Hamilton's lifetime. But even what was published significantly anticipated the results of the classical works of Brillouin and Sommerfeld and their successors.

In conclusion, we mention that the words in the title of this note are taken from a paper of Hamilton[3] in which he speaks of the group velocity as the "velocity wherewith light conquers darkness."

[1)]By choosing initial conditions of the type (3), which contain a constant overall displacement, Hamiltonian was able to draw a clear distinction between the precursor and the proper wave propagation.

[1]L. I. Mandel'shtam, "Lectures on some problems in the theory of oscillations (1944)" (in Russian), in: Polnoe sobranie trudov (Complete Collected Works), Vol. 5, Izd. Akad. Nauk SSSR, Moscow, 1950.

[2]W. R. Hamilton, "Researches on the dynamics of light," Proc. Roy. Irish Acad. 1, 267 (1839).

[3]W. R. Hamilton, "Researches on vibration connected with the theory of light," *ibid.*, p. 341.

[4]T. Stiks, Teoriya plazmennykh voln (Theory of Plasma Waves), Atomizdat, Moscow, 1965.

[5]J. W. Strutt (Lord Rayleigh), Volnovaya teoriya sveta (Wave Theory of Light), Gostekhizdat, Moscow-Leningrad, 1940 (This book consists of Russian translations of scientific works by Lord Rayleigh).

[6]The Mathematical Papers of Sir William Rowan Hamilton, Vol. II (Dynamics), Cambridge Univ. Press, 1940.

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