The four-dimensional group velocity

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The advantages of using the 4-dimensional group velocity are demonstrated. After first giving its purely kinematic definition for an arbitrary wave field, the discussion turns to packets of electromagnetic waves in electrically and magnetically anisotropic media which possess space-time dispersion and are smoothly nonuniform in space and slowly varying in time. In particular, a simple expression is obtained for the energy-momentum 4-tensor in terms of the group-velocity 4-vector. Finally, it is shown how the 4-dimensional notation simplifies the derivation of the conditions of orthogonality and conservation of the adiabatic invariant. A note by M. L. Levin which follows this paper contains a brief account of the basis results of W. R. Hamilton's investigations relating to the velocity of wave motion.

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CONTENTS

1. INTRODUCTION

Group velocity is one of those physical concepts to which for various reasons one returns again and again, although it might seem that everything that could be said about it has long ago been elucidated and written. This is apparently so because group velocity is one of the fundamental concepts of the kinematics (and not only of the kinematics) of wave motion and is essential for the propagation of waves of arbitrary physical character, including "matter waves" in quantum mechanics.

The concept of group velocity was first introduced by Hamilton,[1] long before Stokes and Rayleigh (see the paper by M. L. Levin in this issue). Its subsequent history, associated with the names of Stokes[2] and Rayleigh[3] (who, for a long time, were credited with introducing the concept of group velocity) and with the classical papers of Sommerfeld[4] and Brillouin,[5] is sufficiently well known. We shall not list here the numerous subsequent works, a few of which will be mentioned later.

As is well known, group velocity characterizes the motion of a wave packet in a medium with dispersion when one considers distances which are not too large, i.e., under the condition that the packet retains its "individuality"—its shape and dimensions. Within this limitation, the group velocity of the packet "as a whole" is analogous to the velocity of a body in classical mechanics. But under these same conditions, i.e., when it is at all meaningful to employ the concept of group velocity, the group velocity in the case of a medium which is sufficiently smoothly non-uniform and/or which varies sufficiently slowly in time need not remain constant, i.e., it may describe not only a purely uniform and rectilinear motion of the packet.

When there is anisotropy due to either the structure of bodies (crystals or gyrotropic media) or their motion (for example, in the case of aberration or a moving dispersive medium[6]), the group velocity plays the role of the so-called ray velocity.

In other phenomena, it is of interest to consider the relationship between group velocity and the dynamical characteristics of the wave field in question, as well as the idea of a four-dimensional (relativistically invariant) generalization of the concept of group velocity itself. There exists a rather voluminous literature on both of these questions, and the aim of the present methodological paper is merely to present certain additional arguments (within the framework of the special theory of relativity).

The four-dimensional generalization of group velocity, being a purely kinematic problem, is considered for an arbitrary wave field (Sec. 2), and the role which the 4-vector of this velocity plays in dynamics is illustrated in the case of electromagnetic waves. In Sec. 3 we briefly recall the four-dimensional form of Maxwell's equations and consider the motion of a wave packet in the general case of an electrically and magnetically anisotropic medium which possesses space-time dispersion and is also smoothly non-uniform in space and slowly varying in time. We then turn to certain dynamical relations for the electromagnetic field (Sec. 4) and show in particular that the energy-momentum 4-tensor is related in a very simple way to the group-velocity 4-vector. In Sec. 5 we discuss the problem of an adiabatic invariant (orthogonality conditions for the equations of the first approximation) and present some concluding remarks.

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2. THE FOUR-DIMENSIONAL GROUP VELOCITY

In using the linear approximation to describe any wave field in a uniform and time-independent medium, use is often made of the Fourier expansion, i.e., the expansion in plane harmonic waves $\exp(i\varkappa_n x^n)$.[1] The equations for the field in question then lead to the so-called dispersion equation, which is a relationship among the components of the 4-vector $\varkappa$:

$$\Delta(\varkappa) = 0. \tag{1}$$

In the absence of absorption, the real solutions $\varkappa = K$ of Eq. (1) specify those plane harmonic waves with wave 4-vector $K^j = \{\mathbf{k}, \omega/c\}$ which can propagate in the given uniform, time-independent, and source-free medium—the so-called proper waves. Only these solutions (which are real in the absence of absorption) will be considered here.

A wave group (or packet) is some superposition of proper waves with similar values of K. If the group is described by the integral

$$f(\mathbf{r}, t) = \int g(\varkappa)\, e^{i[\varkappa\mathbf{r} - \omega(\varkappa)t]} d^3\varkappa,$$

where the frequency ω is taken from (1) as a function of $\varkappa$, and $g(\varkappa)$ is non-zero only in a sufficiently small neighborhood of $\varkappa = \mathbf{k}$, then by retaining only the linear term of the expansion

$$\omega(\varkappa) = \omega(\mathbf{k}) + \frac{\partial\omega(\mathbf{k})}{\partial\mathbf{k}}(\varkappa - \mathbf{k}) + \ldots,$$

we obtain

$$f(\mathbf{r}, t) = f_0(\mathbf{r}, t)\, e^{i[\mathbf{kr} - \omega(\mathbf{k})t]},$$

in which

$$f_0(\mathbf{r}, t) = \int g(\varkappa)\, e^{i(\varkappa - \mathbf{k})(\mathbf{r} - \mathbf{u}t)} d^3\varkappa \equiv \mathcal{F}(\mathbf{r} - \mathbf{u}t),$$

where

$$\mathbf{u} = \frac{\partial\omega(\mathbf{k})}{\partial\mathbf{k}}. \tag{2}$$

Thus, in this approximation, the group represents a quasi-monochromatic quasi-plane wave with a "carrier" $\exp(i[\mathbf{k}\cdot\mathbf{r} - \omega(\mathbf{k})t])$ "inscribed" in a smooth "envelope" $\mathcal{F}(\mathbf{r} - \mathbf{u}t)$ moving with a group velocity $\mathbf{u}$. It is in this approximation, with no allowance for deformations of the envelope during the course of its motion, that it makes sense to introduce the group velocity.

The group velocity admits a four-dimensional generalization, which has been introduced in a number of papers on electrodynamics. In these papers, a 4-vector U was defined by means of the energy relations.[7,8] However, it seems more natural to define the kinematic concept of a velocity without resorting to dynamical quantities. This can be done as follows.

In the 4-space of $\varkappa$, the real solutions of the dispersion equation (1) are described by a three-dimensional (in general, multi-sheeted) hypersurface, part of which is shown in Fig. 1, where for obvious reasons the number of dimensions is reduced by unity. The group consists of proper waves with sufficiently similar values of K, i.e., it is described by a Fourier integral

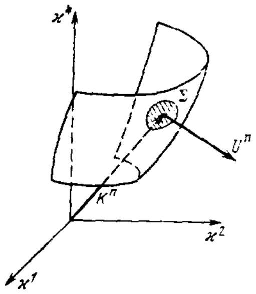

FIG. 1.

$$f(x) = \int g(\varkappa)\, e^{i\varkappa_n x^n} d^4\varkappa, \tag{3}$$

in which the integration actually extends only over a sufficiently small sector Σ of one of the sheets of the dispersion surface $\Delta(\varkappa) = 0$.[2]

We define U^n as a 4-vector normal to the dispersion surface at the point K_n:

$$U^n = A\frac{\partial\Delta(\varkappa)}{\partial\varkappa_n}\bigg|_{\Delta=0}, \tag{4}$$

and we choose the scalar factor $A(\varkappa)$ so that $U^n U_n = -c^2$, as in the case of solid bodies. To avoid writing all subsequent equations with the symbols $|_{\Delta=0}$ to indicate that the solution $\varkappa_n = K_n$ of the dispersion equation (1) must be substituted for $\varkappa_n$ after differentiating with respect to $\varkappa_n$, we shall simply write K_n instead of $\varkappa_n$, bearing in mind that this notation has the meaning described.

According to (4), the vector U^n has components

$$U^n = A\left\{\frac{\partial\Delta}{\partial K_\alpha}, \frac{\partial\Delta}{\partial K_4}\right\} = A\frac{\partial\Delta}{\partial K_4}\left\{\frac{\partial\Delta/\partial K_\alpha}{\partial\Delta/\partial K_4}, 1\right\}.$$

But applying the dispersion equation (1) and the definition (2) of the three-dimensional group velocity $\mathbf{u}$, we find that its components are given by

$$u_\alpha = -c\frac{\partial\varkappa_4}{\partial\varkappa_\alpha}\bigg|_{\varkappa=K} = c\frac{\partial\Delta/\partial K_\alpha}{\partial\Delta/\partial K_4},$$

so that

$$U^n = A\frac{\partial\Delta}{\partial K_4}\left\{\frac{\mathbf{u}}{c}, 1\right\}.$$

The normalization condition then gives

$$U^n U_n = A^2\left(\frac{\partial\Delta}{\partial K_4}\right)^2\left(\frac{u^3}{c^2} - 1\right) = -c^2,$$

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[1] Greek indices take on the values 1, 2, 3 and Latin indices the values 1, 2, 3, 4; lower and upper Latin indices denote, as usual, covariant and contravariant components, which are related to one another in the usual way by means of the pseudo-Euclidean metric tensor.

[2] This means that the function $g(\varkappa)$ contains a factor $\delta[\Delta(\varkappa)]$. On a small region Σ.

$$\Delta(\varkappa) = \Delta(K) + \frac{\partial\Delta}{\partial\varkappa_n}\bigg|_{\Delta=0}(\varkappa_n - K_n) + \ldots,$$

and, since $\Delta(K) \equiv 0$, we find to first order in $(\varkappa_n - K_n)$ that

$$\delta[\Delta(\varkappa)] = \delta\left[\frac{\partial\Delta}{\partial\varkappa_n}\bigg|_{\Delta=0}(\varkappa_n - K_n)\right].$$

The vector $(\partial\Delta/\partial\varkappa_n)_{\Delta=0}$ is directed along the normal to Σ at the point K_n, so that the integration extends only over those $(\varkappa_n - K_n)$, which lie in the region Σ (orthogonal to the normal).

and hence

$$A\frac{\partial\Delta}{\partial K_4}=\frac{c}{\sqrt{1-(u^2/c^2)}}.$$

Consequently, the components of the group 4-velocity are the same as those of the 4-velocity of a body:

$$U^n=\left\{\frac{\mathbf{u}}{\sqrt{1-(u^2/c^2)}},\ \frac{c}{\sqrt{1-(u^2/c^2)}}\right\}. \tag{5}$$

We note that in the case of degeneracy, i.e., when there is some multiplicity in the solution $\varkappa_n=K_n$ of the dispersion equation, the derivatives $\partial\Delta/\partial\varkappa_n$ vanish at $\varkappa_n=K_n$ together with $\Delta(\varkappa)$. With the normalization $U^nU_n=-c^2$, the scalar factor A in Eq. (4) then obviously tends to infinity at the point K_n, so that the components U^n of the group velocity are again determined by Eq. (5).[3)]

It is well known[10] that the 3-velocity $\mathbf{u}$ in an isotropic medium is directed along the wave vector $\mathbf{k}$ and, in the absence of absorption, is smaller than the velocity of light in vacuum. In an anisotropic medium, however, the vectors $\mathbf{u}$ and $\mathbf{k}$ are not collinear. Nevertheless, since the group velocity still determines the velocity of propagation of a signal in this case (in media with no absorption), it must be less than c. The components of the group-velocity 4-vector are then real.

3. ELECTROMAGNETIC WAVE PACKET. THE DISPERSION EQUATION

Combining the electric and magnetic field intensities $\mathbf{E}$ and $\mathbf{B}$ and the inductions $\mathbf{D}$ and $\mathbf{H}$ into the antisymmetric 4-tensors F_{lm} and H^{jk} with components

$$F_{\alpha\beta}=e_{\alpha\beta\gamma}B_\gamma,\quad F_{\alpha4}=F_\alpha,$$
$$H_{\alpha\beta}=e_{\alpha\beta\gamma}H_\gamma,\quad H_{\alpha4}=D_\alpha,$$

where $e_{\alpha\beta\gamma}$ is the completely antisymmetric unit tensor ($e_{123}=1$), and introducing the current-density 4-vector J^i with components $J^i=\{\mathbf{j}, c\rho\}$, Maxwell's equations can be written in the form

$$\frac{\partial F_{jk}}{\partial x^l}+\frac{\partial F_{kl}}{\partial x^j}+\frac{\partial F_{lj}}{\partial x^k}=0, \tag{6}$$

$$\frac{\partial H^{jk}}{\partial x^k}=\frac{4\pi}{c}J^j. \tag{7}$$

As is well known, this system of equations must be supplemented by the constituent equations, which relate the tensors H^{jk} and F_{lm}. We shall consider quasi-uniform and quasi-stationary media. To describe the fields and the medium itself, it is convenient to introduce in this case, in addition to the coordinates $x\equiv\{x^j\}$, another set of "slow coordinates" $\xi\equiv\{\xi^j\}=\{\mu x^j\}$, where the dimensionless parameter μ of order $(ka)^{-1}$ (a is the scale of the nonuniformities) is sufficiently small ($\mu\ll1$). It should be noted that the fact that all four coordinates ξ are of the same order in μ by no means implies that a medium which is spatially quasi-uniform, for example, is necessarily also quasi-stationary in its time variation. However, it is clear that such an asymmetry is possible only in one coordinate system. In all other systems, in accordance with the Lorentz transformation, there is a special type of time dependence in the same order in μ, namely a drift of "frozen" nonuniformities. It is natural to assume that all the slow coordinates are of the same order in the parameter μ even in the general case in which there are no "frozen" nonuniformities.

In the general case of a space-time nonlocality in the relations between the inductions and field intensities, the linearized constituent equations can be written in the form

$$H^{jk}(x)=\int\varepsilon^{jklm}(\xi,x')F_{lm}(x-x')\,d^4x'. \tag{8}$$

The non-uniform and non-stationary character of the medium is taken into account here by the fact that the permeability 4-tensor $\varepsilon^{jklm}(\xi,x')$ depends not only on x'^j but also on the slow coordinates $\xi^j=\mu x^j$.[4)]

By virtue of the anisymmetry of the tensors H^{jk} and F_{lm}, it follows from (8) that the tensor ε^{jklm} must be antisymmetric in the first pair of indices (jk) and can always be taken to be antisymmetric also in the second pair of indices (lm):

$$\varepsilon^{jklm}=-\varepsilon^{kjlm}=-\varepsilon^{jkml}. \tag{9}$$

When $\mu=0$, the medium becomes uniform and time-independent. In the inertial coordinate system associated with this medium, the components of the 4-vector ε^{jklm} can be expressed in terms of the components of the dielectric permeability and inverse magnetic permeability 3-tensors $\varepsilon_{\alpha\beta}(x')$ and $\gamma_{\alpha\beta}(x')=\mu^{-1}_{\alpha\beta}(x')$, respectively, as follows[11]:

$$\varepsilon^{4\alpha\beta\gamma}(0,x')=\varepsilon^{\alpha\beta\gamma4}(0,x')=0,\quad 2\varepsilon^{4\alpha\beta4}(0,x')=\varepsilon_{\alpha\beta}(x'),$$
$$2\varepsilon^{\alpha\beta\gamma\nu}(0,x')=e_{\alpha\beta\tau}e_{\gamma\nu\sigma}\gamma_{\tau\sigma}(x'). \tag{10}$$

In saying that ε^{jklm} is a 4-tensor, we are of course assuming that it is determined in an arbitrary inertial coordinate system by the usual equations of the Lorentz transformation.

In what follows, we shall have to consider the Fourier transform (with respect to the "fast" coordinates x') of the kernel $\varepsilon^{jklm}(\xi,x')$:

$$\tilde{\varepsilon}^{jklm}(\xi,\varkappa)=\int\varepsilon^{jklm}(\xi,x')\,e^{-i\varkappa_s x'^s}d^4x'. \tag{11}$$

We note at once that if the medium is not absorptive, i.e., if the 3-tensors $\tilde{\varepsilon}_{\alpha\beta}(\varkappa)$ and $\tilde{\gamma}_{\alpha\beta}(\varkappa)$ are Hermitian ($\tilde{\varepsilon}_{\alpha\beta}=\tilde{\varepsilon}^*_{\beta\alpha}$ and $\tilde{\gamma}_{\alpha\beta}=\tilde{\gamma}^*_{\beta\alpha}$), then it follows from (10) that the 4-tensor $\tilde{\varepsilon}^{jklm}(0,\varkappa)$ satisfies the condition

$$\tilde{\varepsilon}^{jklm}(0,\varkappa)=\tilde{\varepsilon}^{*lmjk}(0,\varkappa). \tag{12}$$

We turn now to the propagation of a wave packet, i.e., a group of proper waves ($J^j=0$), in a medium described

[3)]It is perhaps worth mentioning that the argument in the envelope of the packet $\mathscr{F}(\mathbf{r}-\mathbf{u}t)$, after introducing the 4-velocity U^n, can be written in the form of the 4-vector $x^j-U^j\tau$, where $\tau=t\sqrt{1-(u^2/c^2)}$ is the proper time of the packet. The fourth component of this 4-vector is equal to zero.

[4)]In a non-dispersive medium, the kernel $\varepsilon^{jklm}(\xi,x')$ takes the form $\varepsilon^{jklm}(\xi)\delta(x')$. If the medium is also uniform and stationary, then $H^{jk}=\varepsilon^{jklm}F_{lm}$, where the tensor ε is constant. The fourth-rank permeability 4-tensor (the DM tensor) for such a medium was first introduced by Mandel'shtam and Tamm[9] (originally, in the framework of the special theory of relativity). A minor difference is that they introduced the tensor s_{lmjk} which is the inverse of ε^{jklm} in the sense that it can be used to express the field intensities in terms of the inductions: $F_{lm}=s_{lmjk}H^{jk}$.

by the constituent equations (8). We seek a solution of Maxwell's equations in this case in the form of a quasi-monochromatic quasi-plane wave

$$F_{lm}=\mathcal{F}_{lm}(\xi)\,e^{i\psi(x)}, \tag{13}$$

where $\mathcal{F}_{lm}(\xi)$ are slowly varying amplitudes, and $\psi(x)$ is a phase which is quasi-linear in the fast coordinates and whose gradient represents a local wave 4-vector that depends only on the slow coordinates ξ:

$$K_j(\xi)=\frac{\partial\psi}{\partial x^j}. \tag{14}$$

We note that the definition (14) has the immediate consequence that

$$\frac{\partial K_j}{\partial\xi^s}=\frac{\partial^2\psi}{\partial\mu x^s\,\partial x^j}=\frac{\partial^2\psi}{\partial x^s\,\partial\mu x^j}=\frac{\partial K_s}{\partial\xi^j},$$

and if we write the derivatives with respect to the slow coordinates using the more concise notation

$$\partial_s\equiv\frac{\partial}{\partial\xi^s}, \tag{15}$$

this becomes

$$\partial_s K_j=\partial_j K_s. \tag{16}$$

Let us first see what the constituent equations (8) imply for fields of the type (13). Substituting (13) into (8), we begin by expanding $\mathcal{F}_{lm}$ and ψ in powers of x', retaining the terms up to first order in μ:

$$\begin{aligned}&\mathcal{F}_{lm}(\xi-\xi')\exp[i\psi(x-x')]\\&=[\mathcal{F}_{lm}(\xi)-\mu x'^s\partial_s\mathcal{F}_{lm}(\xi)]\exp\left[i\psi(x)-iK_s(\xi)x'^s+\tfrac{1}{2}i\mu x'^n x'^s\partial_s K_n(\xi)\right]\\&\approx\left[\mathcal{F}_{lm}-\mu x'^s\partial_s\mathcal{F}_{lm}+\tfrac{1}{2}i\mu\mathcal{F}_{lm}x'^n x'^s\partial_s K_n(\xi)\right]\exp[i\psi(x)-iK_s x'^s].\end{aligned}$$

Then it follows from (8) that

$$H^{jk}=\mathcal{H}^{jk}(\xi)\,e^{i\psi(x)}, \tag{17}$$

where

$$\mathcal{H}^{jk}=\tilde{\varepsilon}^{jklm}\mathcal{F}_{lm}-i\mu\left(\frac{\partial\tilde{\varepsilon}^{jklm}}{\partial K_s}\partial_s\mathcal{F}_{lm}+\frac{1}{2}\mathcal{F}_{lm}+\frac{1}{2}\mathcal{F}_{lm}\frac{\partial^2\tilde{\varepsilon}^{jklm}}{\partial K_n\,\partial K_s}\partial_s K_n\right), \tag{18}$$

and where we have made use of the Fourier transform (11). Since we shall henceforth always encounter only this transform $\tilde{\varepsilon}^{jklm}(\xi,K)$, we shall omit the tilde and write simply ε^{jklm}, referring to this quantity as the permeability 4-tensor and bearing in mind that it is a function of the slow coordinates ξ and the local wave vector $K(\xi)$.

For the main problems in which we are interested, it is sufficient to consider only the zeroth approximation in the small parameter μ. In this approximation, (18) has the form

$$\mathcal{H}^{jk}=\varepsilon^{jklm}\mathcal{F}_{lm}, \tag{19}$$

and in general the slow coordinates ξ appear quasi-statically in all quantities and relations simply as parameters on which ε^{jklm} and thus also the solutions $\varkappa_n=K_n(\xi)$ of the dispersion equation depend. In other words, in the zeroth approximation everything takes the same form as in a uniform time-independent medium.[5]

Having established (17) and (19), we turn now to Maxwell's equations with a view to obtaining the dispersion equation.

Assuming that there are no sources ($J^j=0$) and substituting (13) and (17) into (6) and (7), we find the zeroth approximation

$$\mathcal{F}_{jk}K_l+\mathcal{F}_{kl}K_j+\mathcal{F}_{lj}K_k=0, \tag{20}$$

$$\mathcal{H}^{jk}K_k=0, \tag{21}$$

from which it follows, incidentally, that[6]

$$\mathcal{F}_{jk}\mathcal{H}^{*jk}=0. \tag{22}$$

By (19), Eq. (21) takes the form

$$\varepsilon^{jklm}K_k\mathcal{F}_{lm}=0. \tag{23}$$

Since the eight equations (20) and (23) for the six quantities $\mathcal{F}_{lm}$ are certainly not independent, it is already expedient at this stage to go over from the 4-vector F_{lm} to the potential 4-vector $A^m=\{\mathbf{A},\varphi\}$ by putting

$$F_{lm}=\frac{\partial A_m}{\partial x^l}-\frac{\partial A_l}{\partial x^m}. \tag{24}$$

The equations (6) are then satisfied identically, and by substituting the expression $A_m=\mathcal{A}_m(\xi)e^{i\psi(x)}$ for the 4-potential and (13) for F_{lm} into Eq. (24) we obtain in the zeroth approximation the following relation between the slow amplitudes $\mathcal{F}_{lm}$ and $\mathcal{A}_m$:

$$\mathcal{F}_{lm}=i\,(K_l\mathcal{A}_m-K_m\mathcal{A}_l). \tag{25}$$

Using this relation, Eq. (23) takes the form

$$S^{jm}\mathcal{A}_m=0, \tag{26}$$

where we have introduced the second-rank tensor

$$S^{jm}=\varepsilon^{jklm}K_kK_l. \tag{27}$$

The antisymmetry of ε^{jklm} in the indices (jk) implies that $K_jS^{jm}\equiv 0$, i.e., there exists a linear relationship between the rows of the matrix S^{jm}, so that

$$\det\|S^{jm}\|\equiv 0. \tag{28}$$

Thus the determinant of the system of equations (25) is equal to zero, so that these equations always have a non-zero solution for $\mathcal{A}_m$, although this of course does not mean that the field intensities $\mathcal{F}_{lm}$ are also non-zero.

In order to render the conditions that $\mathcal{A}_m$ and $\mathcal{F}_{lm}$ are non-zero equivalent, we subject the 4-potential to the invariant Lorentz gauge by putting $\partial A^m/\partial x^m=0$, so that the amplitudes $\mathcal{A}_m$ in the zeroth approximation satisfy the condition

$$K_m\mathcal{A}^m=0. \tag{29}$$

Equation (25) then implies that

$$K^m\mathcal{F}_{lm}=-iK^mK_m\mathcal{A}_l.$$

As we are considering the field in a medium and not in a vacuum, i.e., $K^mK_m=k^2-\omega^2/c^2\neq 0$, it is obvious that it is not possible for all the amplitudes $\mathcal{F}_{lm}$ to vanish with $\mathcal{A}_l\neq 0$.

With the condition (29), which indicates that from

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[5] As can be seen from (18), for example, the terms of first order in μ already contain derivatives with respect to ξ. We shall require the first approximation only in Sec. 5.

[6] The contraction of (20) with $\mathcal{H}^{*jk}$ gives $\mathcal{F}_{jk}\mathcal{H}^{*jk}K_l+\mathcal{F}_{kl}\mathcal{H}^{*jk}K_j+\mathcal{F}_{lj}\mathcal{H}^{*jk}K_k=0$. By (21) and the antisymmetry of $\mathcal{H}^{jk}$, the last two terms are equal to zero (recall that we are considering only real K_n).

among all the non-trivial solutions of (26) we are selecting those that are orthogonal to the 4-vector K_m, we can write (26) in the form[7)]

$$L^{jm}\mathscr{A}_m = 0, \quad L^{jm} = S^{jm} + K^j K^m. \tag{30}$$

The non-trivial solutions of these equations now automatically entail non-zero fields $\mathscr{F}_{lm}$. The condition for the existence of non-trivial solutions is the dispersion equation

$$D\,(K,\,\xi) = \det \| L^{jm} \| = \det \| S^{jm} + K^j K^m \| = 0.$$

Using (28), we can transform the determinant D to the form

$$D = 2K^m K_m (3SS_l^m S_m^l - 2S_l^m S_m^p S_p^l - S^3), \tag{31}$$

where $S \equiv S_l^l$ and $S_l^m = g_{ln} S^{nm}$ (g_{ln} is the metric tensor).

Thus the dispersion equation for the anisotropic medium with space-time dispersion under consideration can be written in the four-dimensional form

$$\Delta\,(K) = 3SS_l^m S_m^l - 2S_l^m S_m^p S_p^l - S^3 = 0, \tag{32}$$

where the smoothly non-uniform and slowly non-stationary character of the medium are taken into account by the dependence on the slow coordinates $\xi = \mu x$ of the permeability tensor which appears in S_l^m [see (27)].

According to (27), the expression (32) for $\Delta(K)$ has the form

$$\Delta\,(K) = \Delta^{abcdef}\,(K)\,K_a\,\ldots\,K_f,$$

where the coefficients $\Delta^{abcdef}(K)$ are sums of triple products of the components of $\varepsilon^{jklm}(K)$. Applying the operator $K_s(\partial/\partial K_s)$ to this expression and using (32), we obtain

$$K_s \frac{\partial \Delta\,(K)}{\partial K_s} = K_s \frac{\partial \Delta^{abcdef}}{\partial K_s} K_a \ldots K_f + 6\Delta = K_s \frac{\partial \Delta^{abcdef}}{\partial K_s} K_a \ldots K_f.$$

This implies that if there is no dispersion, i.e., if ε^{jklm} and hence also Δ^{abcdef} is independent of K, then the vectors K and U are mutually orthogonal:

$$K_s U^s = 0. \tag{33}$$

In terms of three-dimensional quantities, this condition can be written in the form

$$\mathbf{ku} = \omega,$$

or, if we introduce the phase velocity $\mathbf{v} = \omega\mathbf{k}/k^2$ and the angle α between $\mathbf{v}$ and $\mathbf{u}$, in the form

$$u \cos \alpha = v.$$

As can be seen from its derivation, this relation, which is well known in the optics of non-dispersive crystals (see, e.g., Ref. 12), remains valid in the presence of not only electric but also magnetic anisotropy (with an arbitrary relative orientation of the axes of the 3-tensors $\varepsilon_{\alpha\beta}$ and $\gamma_{\alpha\beta} = \mu^{-1}_{\alpha\beta}$).

We also give an expression for U^s in terms of the tensor S_m^l [Eq. (27)], obtained from the definition (4) by differentiating the expression for $\Delta(K)$ directly with respect to K_s:

$$U^s = 3A\left(P\frac{\partial S}{\partial K_s} - 2P_m^l \frac{\partial S_l^m}{\partial K_s}\right),$$

where

$$P_m^l \equiv S_m^n S_n^l - SS_m^l, \quad P \equiv P_l^l = S_l^m S_m^l - S^2.$$

Since the dispersion equation (32) itself can be written in terms of the auxiliary tensor P_m^l in the form

$$\Delta\,(K) = PS - 2P_m^l S_l^m = 0,$$

by taking the expression for P from this relation we can represent U^s in the form

$$U^s = 6A\frac{P_m^l}{S}\left(S_l^m \frac{\partial S}{\partial K_s} - S\frac{\partial S_l^m}{\partial K_s}\right) = -6ASP_m^l \frac{\partial}{\partial K_s}\frac{S_l^m}{S}. \tag{34}$$

4. THE ENERGY-MOMENTUM TENSOR

We turn now to dynamical quantities, i.e., quantities which are bilinear in the field. As is well known, the contraction of the tensor F_{jl} with the current-density vector J^l gives the force-density 4-vector f_j:

$$f_j = \frac{1}{c} F_{jl} J^l, \tag{35}$$

with components $f^j = \{\rho\mathbf{E} + (1/c)(\mathbf{j}\times\mathbf{B}), (1/c)\,\mathbf{j}\cdot\mathbf{E}\}$. Thus the space-like part (f_α) is the density of the Lorentz force acting on the free charges and currents, and the time-like component (f_4) is equal to the power density generated by the field (divided by c). By eliminating J^l from (35) by means of (7) and then making use of the equations (6), we can express f_j entirely in terms of the field intensities F_{lm} and the inductions H^{jk} in different but equivalent forms. One of these equivalent representations is

$$f_j = \frac{1}{4\pi}\frac{\partial}{\partial x^k}\left(F_{jl}H^{lk} + \frac{1}{4}\delta_j^k F_{lm}H^{lm}\right) + \frac{1}{16\pi}\left(H^{lm}\frac{\partial F_{lm}}{\partial x^j} - F_{lm}\frac{\partial H^{lm}}{\partial x^j}\right), \tag{36}$$

where δ_j^k is the unit tensor. This general expression is of course independent of the form of the constituent equations. We shall apply it to the case of free waves in which we are interested ($J^l = 0$, and accordingly $f_j = 0$) and to the propagation of a group of such waves in the case of the constituent equations (18), but first we make some preliminary remarks.

If we were considering Maxwell's equations together with the equations of motion of the medium, taking into account the effect of the field on the medium, we would have to require conservation of the total (mechanical + electromagnetic) energy and momentum in a closed system with no dissipation. But we are not considering the mechanics of the medium, and we are assuming that its space-time variations or motions are specified. Thus, we need consider only the electromagnetic energy and momentum of the system {medium + field}. Obviously, in a transparent, uniform, and time-independent medium which contains no free currents or charges, i.e., in the case in which $f_j = 0$ and the three-dimensional permeability tensors are independent of $\xi = \mu x$ and are Hermitian, the electromagnetic energy and momentum must be conserved in a closed system. This means that the equations

[7)]We could have introduced an arbitrary scalar C as a factor in front of the product $K^j K^m$ in (30), but this would simply give a factor C in (31).

$$\frac{1}{4\pi}\frac{\partial}{\partial x^k}\left(F_{jl}H^{kl}-\frac{1}{4}\delta_j^k H^{lm}F_{lm}\right)-\frac{1}{16\pi}\left(H^{lm}\frac{\partial F_{lm}}{\partial x^j}-F_{lm}\frac{\partial H^{lm}}{\partial x^j}\right)=0 \tag{37}$$

under these conditions must take the form of the continuity equations

$$\frac{\partial T_j^k}{\partial x^k}=0, \tag{38}$$

in which it is natural to regard the tensor T_j^k, which is bilinear in the field, as the energy-momentum tensor of the system {field + medium}. Of course, the equations (38) themselves do not determine T_j^k uniquely, but only up to a term R_j^k for which $\partial R_j^k/\partial x^k \equiv 0$. In the presence of absorption and/or space-time variations of ε^{jklm}, the equations (37) must contain not only terms of the form $\partial T_j^k/\partial x^k$ but also additional terms which are not reducible to this form but which vanish in the case of a non-absorbing, uniform, and stationary medium. Bearing this in mind, we apply Eq. (37) to a wave packet, writing the real field intensities and inductions in the form

$$F_{lm}=\frac{1}{2}(\mathcal{F}_{lm}e^{i\psi}+\mathcal{F}^*_{lm}e^{-i\psi}),\quad H^{lm}=\frac{1}{2}(\mathcal{H}^{lm}e^{i\psi}+\mathcal{H}^{*lm}e^{-i\psi}). \tag{39}$$

Substituting this into (37), making use of (14) and the fact that $\partial/\partial x^k=\mu\partial/\partial\xi^k\equiv\mu\partial_k$ for the slow amplitudes (which depend on $\xi=\mu x$), and averaging over a wavelength or period of high frequency (terms containing $\exp(\pm i2\psi)$ vanish after taking this average), we obtain the equation

$$\begin{aligned}\mu\partial_k\Big(\frac{\mathcal{F}_{jl}\mathcal{H}^{*kl}+\mathcal{F}^*_{jl}\mathcal{H}^{kl}}{16\pi}-\delta_j^k\frac{\mathcal{F}_{lm}\mathcal{H}^{*lm}+\mathcal{F}^*_{lm}\mathcal{H}^{lm}}{16\pi}\Big)\\+\frac{\mu}{64\pi}(\mathcal{F}_{lm}\partial_j\mathcal{H}^{*lm}-\mathcal{H}^{*lm}\partial_j\mathcal{F}_{lm}+\mathcal{F}^*_{lm}\partial_j\mathcal{H}^{lm}-\mathcal{H}^{lm}\partial_j\mathcal{F}^*_{lm})\\+\frac{iK_j}{32\pi}(\mathcal{F}^*_{lm}\mathcal{H}^{lm}-\mathcal{F}_{lm}\mathcal{H}^{*lm})=0.\end{aligned} \tag{40}$$

Let us now assume that the medium possesses weak absorption, i.e., that the permeability tensor ε^{jklm} contains a small anti-Hermitian component (of order μ), so that[8)]

$$\varepsilon^{jklm}=\alpha^{jklm}+i\mu\beta^{jklm}. \tag{41}$$

In this form, the tensors α^{jklm} and β^{jklm} are obviously Hermitian, i.e.,

$$\alpha^{*jklm}=\alpha^{lmjk},\quad \beta^{*jklm}=\beta^{lmjk}. \tag{42}$$

Let us substitute Eq. (41) and the constituent equation (18) into (40). In doing so, it is necessary to take into account the correction of order μ in (18) only in calculating the last term of (40), and the remaining terms, which already contain the factor μ, can be evaluated by simply substituting the zeroth approximations, i.e., $\varepsilon^{jklm}=\alpha^{jklm}$ and (19). After simple manipulations, Eq. (40) reduces to the form

$$\partial_k T_j^k=\frac{1}{32\pi}\Big(2K_j\beta^{lmpq}-\hat{\partial}_j\alpha^{lmpq}+K_j\hat{\partial}_s\frac{\partial\alpha^{lmpq}}{\partial K_s}\Big)\mathcal{F}^*_{lm}\mathcal{F}_{pq}, \tag{43}$$

where

$$\begin{aligned}T_j^k=\frac{1}{16\pi}\Big[\mathcal{F}^*_{jl}\mathcal{H}^{kl}+\mathcal{F}_{jl}\mathcal{H}^{*kl}-\frac{\delta_j^k}{4}(\mathcal{F}^*_{lm}\mathcal{H}^{lm}+\mathcal{F}_{lm}\mathcal{H}^{*lm})\\+\frac{K_j}{2}\mathcal{F}^*_{lm}\mathcal{F}_{pq}\frac{\partial\alpha^{lmpq}}{\partial K_k}\Big],\end{aligned} \tag{44}$$

and $\hat{\partial}_s$ is the derivative with respect to the argument ξ^s which appears explicitly, i.e., for the function $f(\xi,K(\xi))$ we have

$$\partial_s f=\hat{\partial}_s f+\frac{\partial f}{\partial K_n}\partial_s K_n.$$

As we have already pointed out, these equations always involve the amplitudes $\mathcal{F}_{lm}$ and $\mathcal{H}^{jk}$ taken from the equations of the zeroth approximation. We could have omitted the terms containing δ_j^k in (40) and (44), since (22) holds in the zeroth approximation.

The right-hand side of (43) vanishes as soon as we consider the case of a uniform and stationary medium (α^{lmpq} is independent of ξ) with no absorption ($\beta^{lmpq}=0$), and the conservation laws for the tensor T_j^k then come into effect. Thus we can regard T_j^k as the energy-momentum tensor. We note that the last term in (44) is purely dispersive: if α^{lmpq} is independent of the components of the wave vector K, this term vanishes.[9)]

We shall now show that the energy-momentum tensor in the form (44) can be written in a very compact form in terms of the group 4-velocity U. First of all, we use Eqs. (19) (where $\varepsilon^{jklm}=\alpha^{jklm}$) and (25) to express all the field intensities and inductions in (44) in terms of the 4-potential $\mathcal{A}$. It is easy to see that by using (29) and (42) the result is reduced to the form

$$T_j^k=-\frac{K_j}{8\pi}\frac{\partial}{\partial K_k}(\alpha^{lpqm}K_pK_q\mathcal{A}^*_l\mathcal{A}_m).$$

By virtue of (27) and the expression (30) for L^{lm}, this can also be written in the form

$$T_j^k=-\frac{K_j}{8\pi}\frac{\partial S^{lm}\mathcal{A}^*_l\mathcal{A}_m}{\partial K_k}=-\frac{K_j}{8\pi}\frac{\partial L^{lm}\mathcal{A}^*_l\mathcal{A}_m}{\partial K_k}, \tag{45}$$

and by using the hermiticity of S^{lm} and L^{lm} and Eqs. (26) and (30) we can take $\mathcal{A}_m\mathcal{A}^*_l$ outside the derivative with respect to K_k, i.e., we obtain the alternative expression

$$T_j^k=-\frac{K_j}{8\pi}\frac{\partial S^{lm}}{\partial K_k}\mathcal{A}^*_l\mathcal{A}_m=-\frac{K_j}{8\pi}\frac{\partial L^{lm}}{\partial K_k}\mathcal{A}^*_l\mathcal{A}_m. \tag{46}$$

For $\varkappa_n\neq K_n$, we now write the nonhomogeneous equations

$$L^{lm}(\varkappa)\mathcal{A}_m=B^l(\varkappa)D(\varkappa), \tag{47}$$

where $B^l(\varkappa)$ is an arbitrary 4-vector which does not vanish in the case $\varkappa_n=K_n$, when the equations (47) reduce to the homogeneous equations (30). Since $D(\varkappa)\neq0$, there exists a unique solution of (47), namely

$$\mathcal{A}_m=\frac{D_{mp}}{D}B^pD=D_{mp}B^p,$$

where D_{mp} denotes the third-order determinants given

8) In this case, the roots $K_j=\partial\psi/\partial x^j$ of the dispersion equation (32) are complex (like the phase ψ), but their imaginary parts are of order μ. Since the difference in the last term of (40) is of order μ, allowance for the imaginary parts of K_j would give a correction of order μ^2. We have therefore immediately written $2K_j$ instead of $K_j+K_j^*$ in (40), assuming that the K_j are real. For the same reason, we also take ψ in (39) to be real.

9) Equation (44) for T_j^k, corresponding to the presence of space-time dispersion in the medium, was obtained by Gertsenshteĭn, who first introduced the term "spatial dispersion" for the case in which the permeabilities of the medium depend not only on $\omega(\mathbf{k})$ but also on the wave vector $\mathbf{k}$ itself.[14]

by the adjoints of the elements L^{mp}, so that $L^{lm}D_{mp} = \delta^l_p D$. The bilinear form $L^{lm}\mathcal{A}^*_l\mathcal{A}_m$ is then equal to the expression

$$L^{lm}\mathcal{A}^*_l\mathcal{A}_m = L^{lm}D_{mp}B^pD^*_{lq}B^{*q} = D^*_{pq}B^pB^{*q}D(\varkappa),$$

or, using (31), to

$$L^{lm}\mathcal{A}^*_l\mathcal{A}_m = 2\varkappa_r\varkappa^r D^*_{pq}B^pB^{*q}\Delta(\varkappa) \equiv \Phi(\varkappa)\Delta(\varkappa). \tag{48}$$

If the solution $\varkappa_n = K_n$ of the dispersion equation is not degenerate, i.e., if not all the adjoints $D_{pq}(\varkappa)$ vanish together with $D(\varkappa)$ for $\varkappa_n = K_n$, then the scalar $\Phi(\varkappa) = 2\varkappa_r\varkappa^r D^*_{pq}B^pB^{*q}$ is non-zero for $\varkappa_n = K_n$. Consequently, differentiating (48) with respect to $\varkappa_k$ and then putting $\varkappa_k = K_k$, we obtain, according to (4),

$$\frac{\partial L^{lm}\mathcal{A}^*_l\mathcal{A}_m}{\partial K_k} = \frac{\Phi(K)}{A(K)}U^k \equiv \tilde{\Phi}(K)U^k. \tag{49}$$

Thus, writing the tensor (45) not in the mixed form but in the completely contravariant form (i.e., raising the index j), we have

$$T^{jk} = -\frac{\tilde{\Phi}(K)}{8\pi}K^jU^k. \tag{50}$$

In the case of a multiple root K_n, when all the $D_{pq}(K) = 0$, it is clearly always possible to choose the 4-vector $B(\varkappa)$ in such a way that its components become infinitely large as $\varkappa_n \to K_n$, with a non-zero value of $\tilde{\Phi}(K)$ at the point K_n. Thus the result (50) remains valid in the case of degeneracy.

Consequently, the electromagnetic energy-momentum tensor of the system {medium + field} is proportional to the dyad formed from the two 4-vectors—the wave velocity and the group velocity.

It is well known that in a non-absorbing medium the three-dimensional energy density w, energy flux S_α, momentum density g_α, and Maxwell stress tensor $\theta_{\alpha\beta}$ coincide with the following components of the energy-momentum 4-tensor:

$$w = T^{44},\quad S_\alpha = cT^{4\alpha},\quad g_\alpha = \frac{1}{c}T^{\alpha 4},\quad \theta_{\alpha\beta} = T^{\alpha\beta}. \tag{51}$$

To determine the scalar factor in (50) which depends on the field intensity, we can make use of any of the components T^{jk},[10] but it is best to take the component T^{44} because the energy density is non-zero for any structure of the field. Dividing (50) by $T^{44} = w = -(\tilde{\Phi}(K)/8\pi)K^4U^4$, substituting $K^4 = \omega/c$, and taking U^4 from (5), we find

$$T^{jk} = \frac{w}{\omega}\sqrt{1-\frac{u^2}{c^2}}\,K^jU^k. \tag{52}$$

In the three-dimensional notation, it follows from (51) and (52) that

$$S_\alpha = wu_\alpha,\quad \theta_{\alpha\beta} = g_\alpha u_\beta,\quad g_\alpha = \frac{w}{\omega}k_\alpha. \tag{53}$$

The first equation, which relates the energy flux to the three-dimensional group velocity, has been known for a long time.[15] The second equation was obtained in Ref. 11 for the case of media with no spatial dispersion. In Ref. 13 both relations were generalized to the case in which there is dispersion. Thus (52) unifies the relations (53) in a four-dimensional form.

By contracting T^{jk} with either K_j or U_k, we can introduce two 4-vectors, which we write with a definite normalization, namely

$$I^k = \frac{K_jT^{jk}}{K_qK^q},\quad G^j = \frac{T^{jk}U_k}{U^qU_q}. \tag{54}$$

By (5), (52), and (53), the components of these vectors are as follows:

$$I^j = \frac{w}{\omega}\{\mathbf{u}, c\} = \frac{1}{\omega}\{\mathbf{S}, wc\},$$
$$G^j = \frac{w}{\omega}\sqrt{1-\frac{u^2}{c^2}}\left\{\mathbf{k}, \frac{\omega}{c}\right\} = \sqrt{1-\frac{u^2}{c^2}}\left\{\mathbf{g}, \frac{w}{c}\right\}. \tag{54a}$$

The vector I is related to the density of the adiabatic invariant of the field (see Sec. 5), and G is related to the density of momentum and energy. The tensor T^{jk} itself can be written in terms of these vectors in two forms:

$$T^{jk} = G^jU^k = K^jI^k, \tag{55}$$

and the 4-vector of the total energy and momentum of the wave packet, $P^j = \int T^{j4}dV\,(dV = dx^1dx^2dx^3)$, can accordingly be expressed in the form

$$P^j = \int G^jU^4dV = \int K^jI^4dV = \left\{c\int \mathbf{g}dV,\ \int w\,dV\right\}.$$

As is already clear from the original expression (44) for T^{jk}, and in particular from the expression (52) in terms of K and U, the electromagnetic energy-momentum tensor of the system {field + medium} is *non-symmetric* in the general case. Moreover, the values of all its components at the point ξ are *local in the field*, i.e., they are local functions of the field intensities $\mathcal{F}$ and inductions $\mathcal{H}$ at *this* point.

It is easy to show that any tensor $T^{jk}(\xi)$ can always be symmetrized without altering the values of its divergence $\partial_k T^{jk}$. This can be done by adding to $T^{jk}(\xi)$ a tensor of the form $\partial_l R^{jkl}(\xi)$, where R^{jkl} is a third-rank tensor which is antisymmetric in the indices (kl) and bilinear in $\mathcal{F}$ and $\mathcal{H}$. In the case of a uniform, stationary, and non-absorbing medium, the conservation laws $\partial_k T^{jk} = 0$ are still valid, but we lose *locality in the field* for the components of the tensor T^{jk}, since they then contain terms involving integrals with respect to ξ^j of bilinear functions of $\mathcal{F}$ and $\mathcal{H}$. Should we, under these circumstances, require that T^{jk} be symmetric and thereby abandon the usual (local in the field) relation between T^{jk} and the quantities $\mathcal{F}$ and $\mathcal{H}$?

The symmetry condition might indeed be justified for the total energy-momentum tensor of an autonomous (and non-linear) system {field + medium} which depends not only on the electromagnetic field but also on the mechanical state of the medium. But we have considered only the approximation of specified variations (motions) of the medium, i.e., we have neglected the inverse effect of the field on the motion of the medium. In this formulation of the problem, there is no physical basis for attempting to make T^{jk} symmetric.

It is of interest to determine under what conditions the tensor (52) is nevertheless symmetric. Obviously, a necessary condition for its symmetry is the collinearity of K and U:

$$K^j = aU^j,$$

[10]However, the trace $T = T^j_j$ is inconvenient, since, according to (33), it vanishes in the absence of dispersion.

i.e., the medium must be isotropic and certainly dispersive [otherwise K and U would be orthogonal; see (33)]. In the three-dimensional form, the foregoing equality implies the collinearity of **k** and **u** and the following equations for k, u, and ω:

$$k=\frac{au}{\sqrt{1-(u^2/c^2)}}, \quad \frac{\omega}{c}=\frac{ac}{\sqrt{1-(u^2/c^2)}}.$$

Eliminating $a/\sqrt{1-(u^2/c^2)}$ from these equations, we obtain

$$k=\frac{\omega}{c^2}u.$$

Since the phase velocity is $v=\omega/k$, this equation can also be written in the form $vu=c^2$. Substituting $u=d\omega/dk$ into this relation, we find

$$\frac{\omega}{k}\frac{d\omega}{dk}=c^2,$$

and hence $\omega^2=k^2c^2+b$, where $b=\text{const}$. Consequently, if we introduce the index of refraction $n=c/v$, we arrive at the dispersion law

$$n^2(\omega)=1-\frac{b}{\omega^2},$$

where $b\geq 0$, since $u/c=n\leq 1$. Thus an isotropic medium with a "Langmuir" dispersion law is the only one in which the tensor T^{jk} [Eq. (52)] is symmetric and, as is easy to show, has the form

$$T^{jk}=\frac{w}{c^2}\left(1-\frac{u^2}{c^2}\right)U^jU^k.$$

5. THE FIRST APPROXIMATION. FURTHER REMARKS

We have considered above only the zeroth approximation in the small parameter μ, which characterizes the nonuniformity and nonstationarity of the medium and the degree of absorption. It is not difficult to write all the equations with allowance for the terms of first order in μ. For the corrections $\mathcal{A}_m^{(1)}$ in the expansion $\mathcal{A}_m=\mathcal{A}_m^{(0)}+\mu\mathcal{A}_m^{(1)}+\ldots$ (for brevity, we have omitted the index 0 and continue to do so in what follows), we obtain the system of nonhomogeneous equations

$$L^{jm}\mathcal{A}_m^{(1)}=i\left[\partial_k(\alpha^{jklm}K_l\mathcal{A}_m)+\alpha^{jklm}K_k\partial_l\mathcal{A}_m+\frac{\partial\alpha^{jklm}}{\partial K_s}K_k\partial_s(K_lA_m)\right.$$
$$\left.+\frac{1}{2}\frac{\partial^2\alpha^{jklm}}{\partial K_s\,\partial K_n}K_kK_l\mathcal{A}_m\partial_sK_n-\beta^{jklm}K_kK_l\mathcal{A}_m\right]\equiv iX^j(\mathcal{A}). \tag{56}$$

As is well known, the condition for the consistency of the system of equations (56) with zero determinant is that the right-hand side $X^j(\mathcal{A})$, after substituting into it the general solution $\mathcal{A}_m$ of the homogeneous equations (30), should be orthogonal to each of the linearly independent particular solutions of the transposed equations (30). The number of such particular solutions depends on the degree of degeneracy, i.e., on the multiplicity of the root K_n of the dispersion equation. By virtue of the hermiticity of L^{jm}, the solutions of the transposed system are simply the complex conjugates of those of the system (30).

Let K_n be a simple root, so that the solution $\mathcal{A}_j^*$ of the transposed system is unique, apart from a scalar factor. The orthogonality condition then reduces to the single (complex) scalar equation

$$X^j(\mathcal{A})\mathcal{A}_j^*=0. \tag{57}$$

Separating the real and imaginary parts of (57), we obtain two real conditions, which, after allowing for the hermiticity of α^{jklm} and β^{jklm}, can be written in the form

$$\partial_k(\alpha^{jklm}K_l\mathcal{A}_m\mathcal{A}_j^*)+\partial_l(\alpha^{jklm}K_k\mathcal{A}_m\mathcal{A}_j^*)+\frac{\partial\alpha^{jklm}}{\partial K_s}\partial_s(K_kK_l\mathcal{A}_m\mathcal{A}_j^*)$$
$$+\frac{\partial^2\alpha^{jklm}}{\partial K_s\,\partial K_n}K_kK_l\mathcal{A}_m\mathcal{A}_j^*\,\partial sK_n-2\beta^{jklm}K_kK_l\mathcal{A}_m\mathcal{A}_j^*=0, \tag{58}$$

$$A_j^*\left[\partial_k(\alpha^{jklm}K_l\mathcal{A}_m)+\alpha^{jklm}K_k\partial_l\mathcal{A}_m+\frac{\partial\alpha^{jklm}}{\partial K_s}K_k\partial_s(K_l\mathcal{A}_m)\right]$$
$$-\mathcal{A}_m\left[\partial_l(\alpha^{jklm}K_k\mathcal{A}_j^*)+\alpha^{jklm}K_l\partial_k\mathcal{A}_j^*+\frac{\partial\alpha^{jklm}}{\partial K_s}K_l\partial_s(K_k\mathcal{A}_j^*)\right]=0. \tag{59}$$

We make the substitution $\partial_k\equiv\partial_s\delta_s^k=\partial_s(\partial K_k/\partial K_s)$ (and similarly for ∂_l) in (58). Using (46), the condition (58) then reduces to the form

$$-8\pi\partial_s\left(\frac{K_pT^{ps}}{K_qK^q}\right)=\left[\hat{\partial}_s\left(\frac{\partial\alpha^{jklm}}{\partial K_s}\right)+2\beta^{jklm}\right]K_kK_l\mathcal{A}_m\mathcal{A}_j^*. \tag{60}$$

Using (25) and the hermiticity of α^{jklm} and β^{jklm}, it is easy to see that the product $K_kK_l\mathcal{A}_m\mathcal{A}_j^*$ on the right-hand side of (60) can be replaced by $-1/4\mathcal{F}_{lm}\mathcal{F}_{jk}^*$. As to the 4-vector being differentiated on the left-hand side of (60), this is nothing other than the vector I^s introduced above [see (54) and (54a)]. Consequently, (60) takes the form

$$\partial_sI^s=\frac{1}{32\pi}\left[\hat{\partial}_s\left(\frac{\partial\alpha^{jklm}}{\partial K_s}\right)+2\beta^{jklm}\right]\mathcal{F}_{lm}\mathcal{F}_{jk}^*. \tag{61}$$

In a medium with no dispersion (α^{jklm} is independent of K) and no absorption ($\beta^{jklm}=0$), the equality (61) is the continuity equation for the density of the adiabatic invariant w/ω (see Ref. 16). In the general case, the condition for the conservation of the adiabatic invariant is the vanishing of the tensor in the square brackets in (61):

$$\hat{\partial}_s\left(\frac{\partial\alpha^{jklm}}{\partial K_s}\right)+2\beta^{jklm}=0.$$

Under the same conditions, but for a stationary medium and for $\omega=\text{const}$, (61) reduces to the continuity equation for the energy density w.

The second orthogonality condition (59) determines the law of variation of the additional advance of the phase of the wave group. This advance is important only in a non-uniform gyrotropic medium and is discussed in detail (in the three-dimensional form) in Ref. 17.

We shall not consider the case of polarization degeneracy (a double root K_n of the dispersion equation), which was studied in Ref. 18 for an isotropic and spatially non-uniform medium and in Ref. 19 for a medium with weak anisotropy, for which the so-called "quasi-isotropic" approximation is applicable.

As has already been stressed earlier, the concept of group velocity makes sense only in the case of sufficiently small curvature of the region of integration Σ on the dispersion hypersurface, when at least the terms containing second derivatives in the expansion

$$\Delta(\varkappa)\approx\frac{\partial\Delta(K)}{\partial K_n}(\varkappa_n-K_n)+\frac{1}{2}\frac{\partial^2\Delta(K)}{\partial K_n\,\partial K_m}(\varkappa_n-K_n)(\varkappa_m-K_m)+\ldots$$

can be neglected. This in turn imposes a limitation on the length of the 4-interval or the proper time interval τ of the packet within which the envelope of the packet has not yet undergone appreciable change. These conditions are well known and have been studied in detail, for example, in Ref. 20 in the case of a uniform and time-independent medium.

A medium which is smoothly non-uniform and slowly varying in time is subject to not only this dispersive spreading of the packet, but also to another mechanism: distortion of the wave fronts (bending of the rays and the Doppler effect) as a result of the space-time nonhomogeneities. This "nonhomogeneity" deformation of the packet obviously has its characteristic dimensions, which become infinitely large as $\mu \to 0$.

Of course, in the presence of both types of deformation of the packet, these two mechanisms interact, but in principle they are independent. The first (dispersive) mechanism is still present in a uniform medium, while the second ("nonhomogeneity") mechanism is not necessarily associated with the presence of dispersion.

In conclusion, we would like to say a few words about the four-dimensional form of the fundamental equations and the corresponding form of all the conclusions. This formalism, which has been employed in very many works, was applied, in particular, in Refs. 11 and 13. The results of Refs. 11 and 13 were derived in the three-dimensional form in a more recent paper[21] in the case of a dispersive, electrically anisotropic medium. In this connection, the authors of Ref. 21 expressed the following opinion: "As to the form of the notation and the derivation of the relations, this is naturally a matter of taste and habit. However, it seems to us that the foregoing [authors' note: i.e., three-dimensional] derivation is so simple that the transition to the relativistic equations, rarely used in macroscopic electrodynamics, is more complicated than the proof itself and is therefore no longer expedient."

Of course, tastes differ, but this point of view regarding the expediency of the four-dimensional notation can hardly be justified. Four-dimensional tensors and operations using them are extremely concise and most adequately reflect the relativistic invariance of the relations in which we are interested. They could perhaps have been regarded as unusual at the beginning of the century. One might criticize Refs. 11 and 13 not for the fact that they employ the four-dimensional notation, but for the fact that they do not apply it sufficiently consistently, as they do not explicitly introduce the group 4-velocity. It seems to us that this is the reason why it is of methodological interest to introduce this velocity together with the four-dimensional description of anisotropy and space-time dispersion.

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