On the nature of turbulence

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The definition of turbulence and the differences between turbulence and random wave motions of liquids or gases are discussed. The Landau scheme of the generation of turbulence with increasing Reynolds number as a result of a sequence of normal bifurcations that creates a quasiperiodic motion is considered; several examples are discussed, including flow between rotating cylinders, convection at small Prandtl numbers, and the boundary layer at a flat plate. Results obtained in recent years from the ergodic theory and associated with the discovery of strange attractors in the phase spaces of typical dynamic systems are described. Flows with inverse bifurcations are considered, including the plane Poiseuille flow and Lorenz's example with idealized three-mode convection at large Prandtl numbers. In the latter case, the results of numerical calculations are analyzed and point to the existence of a strange attractor with the structure of a Cantor discontinuum; other examples of systems with strange attractors are also considered. It becomes clear that strange attractors in the phase spaces of systems with few modes may explain their nonperiodic behavior, but cannot explain why turbulence has a continuous spatial spectrum.

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CONTENTS

1. INTRODUCTION

According to existing conceptions the chastic, random appearance of turbulent liquid and gas flows is explained by the excitation of a very large number of degrees of freedom in these flows. As mechanical systems, such flows represent an aggregate of a very large number of vibrating and interacting oscillators. The point representing such a system in the corresponding phase space (whose number of dimensions is very large, but still finite in the case of flows in limited volumes) moves during the generation of turbulence along a path that makes an asymptotic approach to a certain limiting cycle that can be called a *quasiperiodic attractor*: here the time (t) functions that describe the turbulent fluctuations are *quasiperiodic*, i.e., they have the form $f(\omega_1 t, \ldots, \omega_n t)$, where n is very large, f has a period of 2π in each argument $\omega_k t$, and the frequencies ω_k with different subscripts k are generally not commensurable. This concept of developed turbulence was proposed already in 1944 by Landau[1] (see also § 27 of the Landau and Lifshitz book[2]). It was used in description of data on the generation of turbulence by Monin and Yaglom[3] and in § 2 of their book,[4] which appeared in 1965.

Several years ago, Ruelle and Takens[5] (see also the later paper of Ruelle[6]) advanced the hypothesis that *strange attractors*, i.e., sets that differ from stationary points and limiting cycles and are approached asymptotically, in sensitive dependence on the initial conditions, by certain phase paths of the flows, exist in the phase spaces of liquid or gas flows. Flows evolving on strange attractors, which are the ones that Ruelle and Takens call turbulent flows, are not quasiperiodic. The time functions that describe them are pseudorandom and have correlation functions that decay at infinity and continuous frequency spectra; at the same time apparently these flows may also have small numbers of excited degrees of freedom. A number of laboratory and numerical experiments on the generation of turbulence in various types of flows indicate the possibility that psendorandom fluctuations with a continuous frequency spectrum form abruptly as the Reynolds number rises, without preliminary development of a quasiperiodic flow or after the appearance of only a very small number of periodic components in the flow. This might be taken as evidence favoring the strange attractor hypothesis. It is to a review of these materials that the present paper is devoted.

2. THE DEFINITION OF TURBULENCE

Turbulence is a convenient name for a phenomenon that is observed in a very large number of liquid and gas flows involving eddies in nature and in engineering systems, in which the thermodynamic and hydrodynamic properties of the flows (velocity vector, temperature, pressure, impurity concentrations, density, sound velocity, electrical conductivity, refractive index, etc.) undergo random fluctuations created by the presence in the flows of numerous eddies of various sizes and, as a result, vary extremely irregularly in space and time, with broad frequency ranges corres-

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ponding to the Fourier components with fixed propagation vectors in the spatial distributions of these properties (i.e., there are no unique dispersion relations), while the phase shifts between the oscillations of different characteristics at fixed points in space vary randomly with the frequency of these oscillations.

According to this definition, the chief criterion of turbulence is the chaotic, random nature of the spatial and temporal variations of the thermohydrodynamic properties of the flow. However, it is not useful to refer to every flow of this kind as turbulent; for a number of purposes it may be necessary to distinguish turbulent flows from other types of random liquid and gas motions that exhibit some degree of regularity. Foremost among these other types of motions are waves that appear in a fluid because various restoring forces develop when fluid particles are displaced from their equilibrium positions in the fluid: the pressure force in acoustic oscillations of a compressible fluid, the force of gravity in oscillations of the free surface of a heavy liquid, surface tension in capillary waves on the free surface of a liquid, the buoyant force in internal gravity waves in a stratified fluid, the vortex part of the vertical Coriolis force component in meridional displacements of particles in a rotating spherical layer of liquid, etc.

The superposition of a large number of waves of one type or another with different propagation vectors and random amplitudes and phases may result in formation of a flow with highly irregular variations in space and time, but in many cases it can, in principle, be distinguished from turbulence by the properties of its elementary wave components—a definite (say, longitudinal or transverse) orientation of the particles displacements relative to the direction of the propagation vector, definite phase shifts between the oscillations of various characteristics of the elementary wave at a fixed point in space, or an oscillation frequency that is uniquely determined by the propagation vector (a so-called dispersion relation).

Flow vorticity plays a definite role in the mechanics of turbulence, making possible a cascade process in which small eddies are generated by large ones (if the large eddies are hydrodynamically unstable) and, as a consequence, a transfer of kinetic energy across the spectrum of scales of motion in the direction of smaller scales (for this reason, attempts to derive equations for the dynamics of turbulence from the equations of the kinetic theory of gases, in which in the lower approximations only potential random fluctuations of flow velocity appear,[7] are unsatisfactory). We have therefore defined turbulence as random fluctuations of the thermodynamic characteristics of *vortex* flows, thereby distinguishing it at the outset from any kind whatever of random irrotational i.e., *potential* flows (in which the velocity vector $\mathbf{u} = \nabla\Phi$ is the gradient of a certain scalar potential Φ) and, consequently, from all waves in an ideal fluid that are generated by potential forces, including all linear acoustic and surface waves and all nonlinear potential surface waves.

3. DIFFERENCES BETWEEN TURBULENCE AND WAVES

Because of the viscosity of water, real waves on the surface of an ocean show moderate vorticity (of the first order of smallness with respect to the slope of the waves in the boundary layer at the free surface, which is very thin for a clean surface and much thicker in the presence of a nearly incompressible surface film, and of the second order below the surface layer; see § 3.4 of Phillips' book[8]). The fluctuations created by the random field of these waves in the upper layer of the ocean differ from turbulence both in their small vorticity and in the dispersion and phase relations for the elementary wave components of which they consist. These wave-induced fluctuations are coherent with the surface waves themselves, and if these waves are registered they can be filtered out of the total-fluctuation records, at least approximately (a mathematical technique for this filtration has been developed by Benilov and Filyushkin[9]; see also Benilov's paper[10] and § 3.8 of the book by Monin, Kamenkovich, and Kort[11]).

Differentiating between internal waves in a stratified ocean and turbulence is a much more complex matter. Firstly, registration of internal waves separately from the total fluctuations is impossible. Secondly, internal waves are not potential waves—they are strongly rotational in vertical planes containing the direction of propagation of the wave, and also in horizontal planes in the range of low frequencies comparable to the inertial frequency (i.e., with the so-called Coriolis parameter $f = 2\Omega \sin\varphi$, where Ω is the angular velocity of the earth's rotation and φ is the geographic latitude). Thirdly, nonlinear effects may often be significant in internal-wave dynamics, and the internal waves may then be transformed into turbulence, as will be explained a little farther on.

In spite of these complications, the problem of distinguishing between the random field of these waves and turbulence can be solved in the case of *linear* internal waves because even the random field of linear internal waves exhibits a number of regularity properties that turbulence does not have. In fact, the random field $\zeta(\mathbf{x}, z, t)$ of vertical fluid particle displacements at a depth z in linear internal waves, which is statistically homogeneous in the horizontal coordinates $\mathbf{x}$ and stationary in time t, can be represented in the form

$$\zeta(\mathbf{x}, z, t) = \int e^{i(\mathbf{kx} - \omega t)} \zeta(k, z, \omega)\, dZ(\mathbf{k}, \omega), \quad (1)$$

where $\mathbf{k}$ are the horizontal propagation vectors, ω are the frequencies (in the range $f \leq \omega \leq N$ where f is the inertial frequency referred to above and N is the so-called Brunt-Vaisala frequency, which is defined by $N^2 = g d\rho_*/\rho_* dz$ where g is the acceleration of gravity and ρ_* is the potential density of the medium), $\zeta(k, z, \omega)$ are regular (nonrandom) functions of z that satisfy the linear internal-wave equation and one of the boundary conditions on the ocean surface $z = 0$ and on its bottom $z = H$:

$$\frac{d}{dz}\rho_*\frac{d\zeta}{dz}+k^2\rho_*\frac{N^2-\omega^2}{\omega^2-f^2}\zeta=0,$$
$$z=0:\quad(\omega^2-f^2)\frac{d\zeta}{dz}+gk^2\zeta=0,\qquad z=H:\quad \zeta=0. \tag{2}$$

Also substituting $\zeta(k,z,\omega)$ into the second boundary condition, we obtain for a fixed horizontal wave number k an equation for ω that has a denumerable number of roots—the eigenfrequencies $\omega=\omega_n(k)$ of the various internal-wave modes $n=1, 2, \ldots$. Therefore, the random spectral measure $Z(\mathbf{k},\omega)$ in (1) is concentrated on dispersion surfaces $\omega=\omega_n(k)$:

$$dZ(\mathbf{k},\omega)=\sum_n\delta[\omega-\omega_n(k)]\,d\omega\,dZ_n(\mathbf{k}). \tag{3}$$

The velocity components u, v, and w and the fluctuations of the pressure p and density ρ in the field of the linear internal waves are represented by formulas of the type (1) with $\zeta(k,z,\omega)$ replaced respectively by the functions

$$u=\frac{k_1\omega-ik_2f}{k^2}\frac{d\zeta}{dz},\quad v=\frac{k_2\omega+ik_1f}{k^2}\frac{d\zeta}{dz},\quad w=-i\omega\zeta,$$
$$p=\frac{\omega^2-f^2}{k^2}\rho_0\frac{d\zeta}{dz},\quad \rho=\frac{\omega^2-f^2}{k^2}\frac{\rho_0}{c_0^2}\frac{d\zeta}{dz}-\frac{\rho_0N^2}{g}\zeta, \tag{4}$$

where k_1 and k_2 are the Cartesian components of the propagation vector and $\rho_0(z)$ and $c_0(z)$ are the unperturbed density and sound velocity.

Thus, to determine the nature of the measured fluctuations it is necessary first of all when the function $\rho_*(z)$ has been measured (approximated, say, by some convenient analytic expression) to use (2) to find the eigenfrequencies $\omega_n(k)$ and the corresponding eigenfunctions $\zeta_n(k,z)=\zeta[k,z,\omega_n(k)]$ orthonormalized, for example, to the total energy of the internal waves, i.e., satisfying the condition $\zeta_m\circ\zeta_n=\delta_{mn}$, where the functional scalar product is given by

$$\zeta_m\circ\zeta_n=\frac{\omega_m\omega_n}{\sqrt{\omega_m^2-f^2}\sqrt{\omega_n^2-f^2}}\left[g(\rho_0\zeta_m\zeta_n)_{z=0}+\int_0^H(N^2-f^2)\zeta_m\zeta_n\rho_*\,dz\right]. \tag{5}$$

Then, for example, expansion of the measured field $\zeta(\mathbf{x},z,t)$ in terms of the functions $e^{i\mathbf{kx}}\zeta_n(k,z)$ will enable us to verify that the dispersion relation for linear internal waves are satisfied, while expansions of the measured fields in terms of the functions $\exp\{i[\mathbf{k}\cdot\mathbf{x}-\omega_n(k)]t\}$ enable one to verify relations (4).

The phase differences between the oscillations of any two hydrodynamic characteristics a and b at a fixed point in space can be estimated by calculating their cross-correlation function $B_{ab}(\tau)=\overline{a(t+\tau)b(t)}$ (here and everywhere below, the overbar indicates the statistical mean value, i.e., the mathematical expectation), representing its Fourier transform (the cross spectrum) in the form $C_{ab}(\omega)-iQ_{ab}(\omega)$ (where C_{ab} is the cospectrum and Q_{ab} the quadrature spectrum) and, finally, constructing the spectrum of the phase shift

$$\varphi_{ab}(\omega)=\mathrm{arctg}\,\frac{Q_{ab}(\omega)}{C_{ab}(\omega)}.$$

For linear internal waves it should agree with that calculated from formulas (4) (for example, at $\omega\gg f$, the oscillations of u, v, ζ, p, ρ, and T are phase-shifted by $\pi/2$ with respect to those of w), while no phase-shift regularities would be expected in turbulence.

In weakly nonlinear internal waves, the dispersion relations and phase shifts turn out to be slightly spread out around the values predicted by the linear theory. In the nonlinear case, interactions take place between internal waves with different three-dimensional propagation vectors $\varkappa_1$, and $\varkappa_2$ which, firstly, are resonant interactions in which the resultant wave with propagation vector $\varkappa=\varkappa_1\pm\varkappa_2$ has a frequency $\omega(\varkappa)=\omega(\varkappa_1)\pm\omega(\varkappa_2)$ and the typical time of such an interaction is of the order of $\tau\sim(\varkappa_1w_1)^{-1/2}(\varkappa_2w_2)^{-1/2}\gg N^{-1}$ and, secondly, are nonresonant interactions, which give rise to so-called forced modes, i.e., internal waves with propagation vectors $\varkappa=\varkappa_1\pm\varkappa_2$ and frequencies $\omega=\omega(\varkappa_1)\pm\omega(\varkappa_2)$ that do not satisfy the dispersion relation [i.e., $\omega\neq\omega(\varkappa)$]. The amplitudes of the forced modes are small when $\tau\gg N^{-1}$, but when $\tau\sim N^{-1}$ they are comparable to the amplitudes of the original waves and may be large, and the interactions of these forced modes with one another and with free internal waves will generate a spectrum of vortex type oscillations that do not satisfy any definite dispersion relation, i.e., a turbulence spectrum. According to a proposal of Miropol'skiĭ and Filyushkin,[12] the interaction time can be estimated from the formula $\tau=[k^3E(k)]^{-1/2}$ where $E(k)$ is the spectral density of the kinetic energy of the fluctuations per unit mass. $E(k)$ can be regarded as the spectrum of the interacting internal waves when $\tau\gg N^{-1}$, and as a turbulent spectrum when $\tau\lesssim N^{-1}$.

4. NORMAL BIFURCATIONS AND LANDAU TURBULENCE

Information as to how turbulence arises can assist us greatly in understanding its nature. We introduce several concepts for discussion of this problem: degrees of freedom and the phase space of the fluid flow, decomposing the latter into elementary components whose states are characterized by the values of a small number of parameters and the sum of whose energies equals the energy of the flow as a whole. Mathematically, this reduces to the expansion of the velocity field in the volume occupied by the fluid in a suitable orthogonal system of functions of the space points. The coefficients of this expansion will serve as the generalized coordinates of the flow. The number of these coordinates that are capable of varying in time will be the number of degrees of freedom of the flow. The set of values of all of the generalized coordinates, which forms a point in a certain multidimensional space known as the phase space of the flow, will be a complete characteristic of the instantaneous state of the flow. The evolution of the flow is represented in phase space by a certain line—the phase path of the flow; it will consist of a single point for a stationary flow and will form a closed line (cycle) for a periodic flow.

Let us examine the process by which turbulence arises as a result of stability loss by an initial laminar (stationary) flow $\mathbf{u}_0(\mathbf{x})$ when perturbations are superimposed on it. The velocity field corresponding to a *small* (infinitesimal) perturbation can be found as the solution of the *linearized* equations of hydrodynamics, which has the form

$$\mathbf{u}'(\mathbf{x}, t) = A(t)\,\mathbf{f}_0(\mathbf{x}),\ A(t) = e^{\lambda t},\ \lambda = \gamma \pm i\omega_1. \tag{6}$$

We then find that at small Reynolds numbers $\mathbf{Re} = LU/\nu$ (L and U are the length and velocity scales typical for the laminar flow being analyzed and ν is the kinematic molecular viscosity) all eigenvalues λ of the linearized equations have negative real parts $\gamma < 0$, so that all small perturbations (6) are damped in time and, consequently, the laminar flow is stable under small perturbations. As **Re** increases, however, the real parts γ of some of the eigenvalues increase and we find a *critical* value $\mathbf{Re}_{1cr}$ of the Reynolds number at which one of the eigenvalues $\lambda(\mathbf{Re})$ of the linearized equations first crosses the imaginary axis in the complex λ-plane, i.e., $\gamma(\mathbf{Re}_{1cr}) = 0$; the corresponding perturbation (6) will neither damp nor increase with time, i.e., it will be *neutral*. At $\mathbf{Re} > \mathbf{Re}_{1cr}$ there will exist eigenvalues λ with positive real parts $\gamma > 0$, i.e. perturbations (6) that increase with time, so that the laminar flow being analyzed will be unstable under small perturbations.

The Hopf bifurcation theorem,[13] according to which there exists a single-parameter family of closed flow phase paths for values of **Re** in a certain neighborhood of $\mathbf{Re}_{1cr}$, is important for further inferences as to the behavior of perturbations (Hopf proved this theorem for dynamic systems of a rather general type; its applicability to fluid dynamics was demonstrated by Brushlinskaya,[14] see also the papers by Sattinger,[15] Ruelle and Takens,[5] Joseph and Sattinger,[16] and Chen and Joseph[17]). We shall first discuss in detail the case of "normal bifurcation," in which a family of closed phase paths exists for $\mathrm{Re} > \mathbf{Re}_{1cr}$. They are then limiting cycles to which time-periodic flows correspond. Landau[1] described the transition from an unstable small perturbation (6) to a periodic flow (see also the book by Landau and Lifshitz[2]). While the perturbation (6) is small, its amplitude $A(t)$ satisfies the equation

$$\frac{d|A|^2}{dt} = 2\gamma|A|^2, \tag{7}$$

but at finite $|A|$ the right-hand side of this equation must be supplemented with subsequent terms of its expansion in powers of A and A^* (where the asterisk denotes the complex conjugate). Here it is helpful to exclude the high-frequency oscillations in (6) (those with frequencies $|\omega_1| \gg \gamma)|$ by smoothing in time (over a period τ from the range $2\pi/|\omega_1| \ll \tau \ll 1/\gamma$); then terms of the third degree drop out and only the term proportional to $|A|^4$ remains of the terms of fourth degree, and to this accuracy we obtain in place of (7) the Landau expansion in the form

$$\frac{d|A|^2}{dt} = 2\gamma|A|^2 - \delta|A|^4. \tag{8}$$

Let us now consider the case $\delta > 0$ (we shall return later to the opposite case $\delta < 0$). In this case the solution of (8) has the form

$$|A(t)| = \frac{A_0^2A_\infty^2}{A_0^2 + (A_\infty^2 - A_0^2)\,e^{-2\gamma t}},\quad A_\infty = \sqrt{\frac{2\gamma}{\delta}}, \tag{9}$$

so that with a small initial value A_0, the amplitude $|A(t)|$ first rises exponentially (as $A_0e^{\gamma t}$, in accordance with the linear theory) and then more slowly, tending as $t \to \infty$ to a finite value A_∞ that does not depend on A_0 and is proportional to $\sqrt{\mathbf{Re} - \mathbf{Re}_{1cr}}$ at small values of the radicand (since as $\mathbf{Re} \to \mathbf{Re}_{1cr}$ we have $\gamma \sim \mathbf{Re} - \mathbf{Re}_{1cr}$ and $\delta \neq 0$). At small $\mathrm{Re} - \mathrm{Re}_{1cr} > 0$, therefore, the perturbation (6) tends with increasing t to a periodic oscillation $\mathbf{u}_1(\mathbf{x}, t)$ with a certain finite amplitude and an arbitrary phase that is determined not by fixed external conditions, but by the random initial phase of the perturbation, and is therefore a degree of freedom of the limiting flow.

Another critical value of the Reynolds number, Re_{2cr}, may be reached as **Re** increases; at this point, there is a second bifurcation, and the periodic flow $\mathbf{u}_0(\mathbf{x}) + \mathbf{u}_1(\mathbf{x}, t)$ becomes unstable under any perturbation of the form $e^{\lambda t}\mathbf{f}_1(\mathbf{x}, t)$, where $\mathbf{f}_1$ is a periodic function of t with period $2\pi/\omega_1$ and the eigenvalue λ has an imaginary part $\pm i\omega_2$. At small $\mathbf{Re} - \mathrm{Re}_{2cr}$, this perturbation will increase with time to a finite limit—a quasiperiodic oscillation with two periods $2\pi/\omega_1$ and $2\pi/\omega_2$ and two degrees of freedom (oscillation phases).

According to Landau's hypothesis, more and more normal bifurcations will occur as **Re** increases further, and as t increases the phase path of the flow will approach a limiting regime with a corresponding quasiperiodic flow $\mathbf{u}[\mathbf{x}, \varphi_1(t), \ldots, \varphi_n(t)]$ that has a period of 2π with respect to each of the arguments $\varphi_k(t) = \omega_k t + \alpha_k$. This limiting cycle will occupy a phase- region and corresponds to all possible sets of initial phases $\alpha_1, \ldots \alpha_n$, and the phase path wound onto it will, with time, pass through practically all points of this region (indeed, at the time $t_n = 2\pi n/\omega_1, n = 0, 1, 2, \ldots$ at which the phase $\varphi_1(t)$ assumes the value α_1, the phase $\varphi_2(t)$ of any other oscillation will assume values $(2\pi n\omega_2/\omega_1) + \alpha_2, n = 0, 1, 2, \ldots$ that contain, after reduction to the interval $(0, 2\pi)$, numbers as close as we please to any preassigned number from this interval, since the frequencies ω_1 and ω_2 are, generally speaking, not commensurable). It is the quasiperiodic flow $\mathbf{u}[\mathbf{x}, \varphi_1(t), \ldots \varphi_n(t)]$ with a very large number of degrees of freedom n, which is ergodic in this sense, that is the developed turbulence according to Landau (we note, however, that here the time correlation functions of the velocities do not, generally speaking, tend to zero at infinity). Hopf[18] constructed a mathematical example of this kind.

5. FLOWS WITH NORMAL BIFURCATIONS

A number of laboratory and numerical experiments on the generation of turbulence in a Couette flow between rotating cylinders, convection at small Prandtl numbers, in the boundary layer near a flat plate, in the mixing zone between flows with unequal velocities, in the wake of a fluid flow around a cylinder, and in multilayer models of atmospheric circulation has to a certain degree confirmed Landau's hypotheses concerning the development of quasiperiodic flow, but only a few successive bifurcations could be detected in these experiments, and then the flow essentially abruptly became highly irregular in time (with a continuous frequency spectrum), although the wave-number spectrum still remained discrete and turned continuous apparently only after a further increase in Reynolds number.

The evolution of a flow between coaxial rotating cylinders, which was measured in greatest detail by Gollub and Swinney,[19] is a striking example of a bifurcation sequence. In their experiment, they measured the radial velocity $u_r(t)$ in the middle of the gap between an inner rotating cylinder with a radius $r_1 = 2.224$ cm and an outer nonrotating cylinder with an inside radius $r_2 = 2.540$ cm (the measurements were made by an optical method, using the scattering of light by a volume with a diameter of about 5% of the gap width; see the paper by Gollub and Freilich[20]). Here it is convenient to express the velocity of the inner cylinder in units of $\mathbf{R}^* = \mathbf{Re}/\mathbf{Re}_{cr}$, where $\mathbf{Re} = 2\pi r_1(r_2 - r_1)/\nu\tau$, where τ is the period of rotation and $\mathbf{Re}_{cr} = 2501$ is the critical Reynolds number for transition to the nonperiodic regime. The flow was laminar at $\mathbf{R}^* < 0.051$. At $\mathbf{R}^* = 0.051$ there was a transition to a new stationary flow (of the form (6) with $\omega_1 = 0$; the effect is called *stability succession*)—toroidal *Taylor vortices* with a wavelength of 0.79 cm along the cylinder axis.

The bifurcation observed at $\mathbf{R}^* = 0.064$ resulted in the formation of the first periodic regime—standing or traveling transverse expansion-compression or bending waves on the Taylor vortices (four waves on the circumference), with the dimensionless frequency $f_1^* = f_1\tau = 1.30$ (this frequency and six of its harmonics can be seen in the spectrum of Fig. 1a against a background of instrumental white noise). The second periodic regime appeared at $\mathbf{R}^* = 0.54 \pm 0.01$—low-frequency modulation of the transverse waves with frequency f_2 (see spectrum in Fig. 1b), which decreases as $\mathbf{R}^*$ increases further, to zero at $\mathbf{R}^* = 0.78 \pm 0.04$ (this time with a simultaneous increase of the real continuous spectrum instead of the instrumental spectrum). At this point a third periodic regime appeared with the dimensionless frequency $\mathbf{f}_3^* = \mathbf{f}_3\tau = 0.87(2\mathbf{f}_1^*/3)$ (see spectrum in Fig. 1c).

Just before the transition to the nonperiodic regime ($\mathbf{R}^* = 0.982$, Fig. 1d), the discrete peaks on the spectrum still contained 90% of the energy of the oscillations, and the fluctuation correlation function of $u_r(t)$ was of periodic undamped form. Immediately after the transition ($\mathbf{R}^* = 1.022$, Fig. 1e), the discrete peaks had vanished, leaving a continuous spectrum with a broad maximum B (which was noticeable even before the transition) containing 60% of the energy, and the correlation function had become damped. The bifurcation at $\mathbf{R}^* = 1$ was abrupt, reversible, and without hysteresis within the resolution limits of the experiment $\delta\mathbf{R}^* = 0.01$, i.e., $\delta\mathbf{Re} = 25$ (although it is

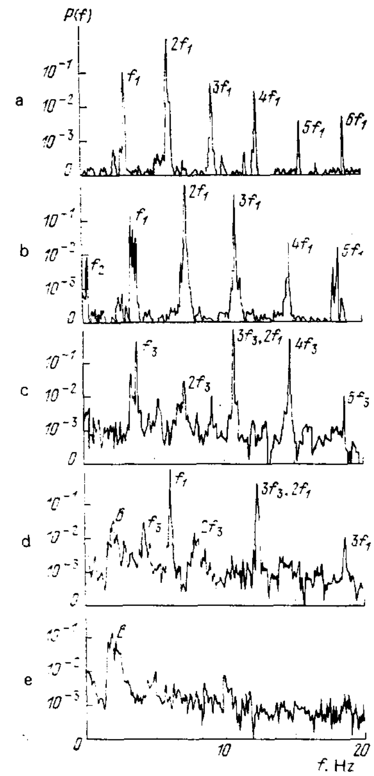

FIG. 1. Spectra of radial velocity component $P(f)$ ($cm^2 sec^{-1} Hz^{-1}$) in Couette flow between rotating cylinders at various Reynolds numbers.[19] a) $\mathbf{R}^* = \mathbf{Re}/\mathbf{Re}_{cr} = 0.504$; b) $\mathbf{R}^* = 0.595$; c) $\mathbf{R}^* = 0.841$; d) $\mathbf{R}^* = 0.982$; e) $\mathbf{R}^* = 1.022$.

naturally not possible to exclude condensation of the bifurcation points in this interval on the $\mathbf{Re}$ axis).

A similar sequence of normal bifurcations was observed in the experiments of Willis and Deardorff[21] on the generation of convection in a layer of liquid or gas heated from below with a small Prandtl number $\mathbf{Pr} = \nu/\chi$, where χ is the kinematic molecular thermal conductivity (air with $\mathbf{Pr} = 0.71$ was used in these experiments; we recall also the experiments of Krishnamurti[22,23] with air and mercury, Rossby[24] with mercury, Ahlers[25] with classical liquid helium ($\mathbf{Pr} = 0.86$) and Moller and Riste[26] with a liquid crystal). It is known (see § 2.7 of the book by Monin and Yaglom[4]) that centrifugal forces in a curvilinear flow whose velocity of rotation decreases with increasing distance from the center of curvature and buoyant forces in convection have a similar (destabilizing) effect on the flow.

In the case of appearance of convection, just as in the flow between rotating cylinders, there is first a *stability succession*—at a certain value $\mathbf{Ra}_{1cr}$ of the Rayleigh number $\mathbf{Ra} = \alpha g H^3 \delta T/\nu\chi$ (α is the thermal expansion coefficient of the medium, H is the layer thickness, and δT is the temperature difference between its upper and lower boundaries), a new stationary flow arises out of the quiescent state, taking the form either of two-dimensional rolls periodic in the horizontal coordinate and resembling toroidal Taylor vortices (if the variations of the material properties of the medium (α, ν, χ) with height in the layer—chiefly those due to temperature—are negligibly small, see Whitehead[27]) or of hexagonal Benard cells (if the material properties depend on temperature, see Schluter, Lortz and Busse,[28] and Busse.[29] Here we shall discuss the stability of two-dimensional rolls, taking the direction of their axes as the y axis and describing their stream function ψ and the deviation θ of the temperature from the linear profile in the (x,z) plane by three modes (Busse[30] demonstrated that in the case of infinitesimal perturbations, the amplitudes of the other modes are small quantities of higher order; this is true in particular at small Pr), which are conveniently written in the form

$$\frac{k_1}{\pi}\left(1+\frac{k_1^2}{\pi^2}\right)^{-1}\varkappa^{-1}\psi = X\sqrt{2}\sin\frac{k_1 x}{H}\sin\frac{\pi z}{H},$$
$$\frac{k_1^2}{\pi^4}\left(1+\frac{k_1^2}{\pi^2}\right)^{-3}\frac{g\alpha H^3}{\nu\chi}\theta = Y\sqrt{2}\cos\frac{k_1 x}{H}\sin\frac{\pi z}{H} - Z\sin\frac{2\pi z}{H} \qquad (10)$$

(these modes correspond to the case in which both boundaries of the layer are free surfaces, but this limitation is evidently immaterial, see, for example, the paper by Palm, Ellingsen, and Gjevik[31]). It was established in experiments with air[21] that transverse oscillations of the rolls—standing waves or waves that travel along their axes (which are approximately in phase and have relatively constant amplitudes everywhere except in the neighborhood of the boundaries, i.e., are rather insensitive to the boundary conditions)—appear when $\mathbf{Ra}_{2cr}$ is approximately three times $\mathbf{Ra}_{1cr}$. These oscillations were calculated in the framework of the linear theory by Busses[30] and of the nonlinear theory by McLaughlin and Martin[32] (who also discussed the strange-attractor hypothesis in general terms). They began by computing an eight-mode motion containing nonstationary rolls (10) and one harmonic along the y axis, for which $\mathbf{Ra}_{2cr}$ was determined analytically, and by constructing the Landau expansion (8), showing that $\delta > 0$ in it, i.e. the case is one of normal bifurcation.

Secondly, they computed, this time numerically, a 39-mode motion containing nonstationary rolls (10) and four harmonics along the y axis with parameters similar to those of Ahlers' liquid helium experiments.[25] Here it was convenient to express the temperature difference causing the convection in units of $\mathbf{R}^* = (k_1^2/\pi^6)\times[1+(k_1^2/\pi^2)]^{-3}\mathbf{Ra}$ and it was assumed in the calculations that $k_2/\pi = 0.1k_1/\pi = 0.072$ and $\mathbf{Pr} = 1$; here $\mathbf{R}^*_{2cr} \approx 1.25$. Calculations with $\mathbf{R}^* = 1.4$ and 1.45 yielded periodic and slighly nonperiodic regimes, respectively, while $\mathbf{R}^* = 1.5$ and 1.55 again produced a periodic regime (due to the disappearance of the mean motion, which contributes to an increase in perturbations with high wave numbers), and $\mathbf{R}^* = 1.6$ gives a sharply nonperiodic regime (Ahlers observed the transition to it at $\mathbf{R}^* = 2.18$). However, *if the fourth harmonic is left* out calculations with $\mathbf{R}^* = 1.6$, 2 and even 20 gave periodic regimes.

A similar sequence of bifurcations has also been observed in the generation of turbulence in boundary layers, e.g., in the experiments of Klebanoff, Tidstrom, and Sargent[33] on the flow over a smooth flat plate. In this case, the Reyonlds number increases along the x axis in the downstream direction, so that when a vibrator was used at a fixed distance x_{1cr} from the leading edge of the plate (which corresponds to $\mathbf{Re}_{1cr}$) to set up a neutral perturbation—a two-dimensional Tollmien-Schlichting wave propagating along the flow—, the experimentors observed sequential bifurcations at various distances from the vibrator.

Initially, the amplitude of the twodimensional wave increased downstream. Then, at a certain value $\mathbf{Re}_{2cr}$, a secondary three-dimensional wave was superimposed on it with periodic variations in the transverse y direction generated by longitudinal eddies with axes along the flow, which brought about a sudden redistribution of the intensity of the longitudinal pulsation of u' in the y direction. This secondary wave had a group velocity along x that was close to the phase velocity of the primary wave. It built up very rapidly downstream (Fig. 2) and became nonlinear. This resulted, first of all, in space-time focusing of the secondary wave packet on the crest of the primary wave and, secondly, in the disappearance of secondary-wave segments with positive longitudinal-velocity anomalies $u' > 0$, leaving only the segments with negative anomalies—pulses with $u' < 0$ (Landahl[34] offered a theoretical explanation of these effects of nonlinearity). First, a single pulse per cycle of the vibrator was observed, then two downstream, and so forth (for this see also the observations of Kovasznay, Komoda, and Vasudeva,[35] and turbulence apparently arose after the regime with four pulses per cycle.

It was possible to observe the small-scale flow forms that lead to the generation of turbulence in the boundary layer near a smooth solid wall by using precision methods to make water flows visible—chains of micron-size hydrogen bubbles generated periodically by a voltage on a platinum wire and microinjections of a dye (Kline *et al.*[36-44]; see also Corino and Brodkey's experiments[45] with visualization of a trichloroethylene flow in a glass tube by the use of suspended micron-size magnesium oxide particles).

It was established that streams of decelerated liquid of widths $\delta y^+ = 10-30$ and a transverse spacing of about $\Delta y^+ \sim 100$ form in the viscous sublayer under the influence of the above longitudinal eddies in the troughs of the transverse waves (longitudinal-velocity minima) at heights of the order of $z^+ = 2.5-10$ (the plus superscript identifies lengths measured in units of ν/u_*, where u_* is the "friction velocity" at the wall) and

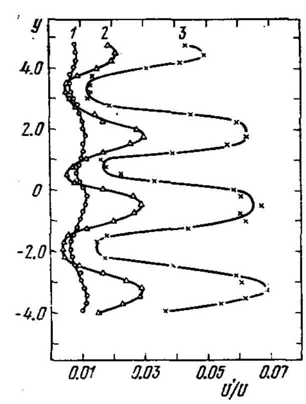

FIG. 2. Downstream growth of longitudinal-velocity fluctuations in secondary wave in boundary layer over a flat plate.[33] 1—$x = 7.6$ cm; 2—$x = 15.2$ cm; 3—$x = 19$cm.

FIG. 3. Hydrogen-bubble field (topview) generated by a transverse filament in a boundary layer at a flat plate (upper edge of photograph, flow from top to bottom) at height $z^+=4.5$.[39]

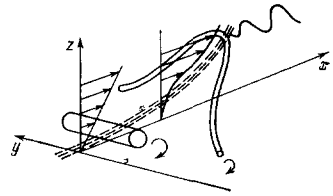

FIG. 5. Formation of a horseshoe vortex.[44]

move downstream, floating slowly upward under the influence of the longitudinal eddies (see the example of the hydrogenbubble field in Fig. 3, which was generated by a horizontal transverse filament at a height $z^+=4.5$). Such a stream behaves as a miniature boundary layer and separates and moves upward into the rapidly flowing liquid under the action of the negative pressure gradient created by the transverse-axis eddy passing above it (concerning which see below), creating a curvature with an inflection point on the instantaneous velocity profile (in Fig. 4, see the example of bubble isochrones generated by a vertical filament). Then oscillations appear on the stream at heights $z^+=8-12$, and soon its end "explodes" (mostly at heights $z^+=10-30$ and at distances $\delta x^+=1000-1500$ from the point of separation), creating an extremely irregular small-scale motion. The frequency of the "explosions" per unit width of the flow is $F=(u_*^3/\nu^2)F^+$, where $2\pi F^+\Delta y^+\sim 0.06$. It was established[41] that practically all the turbulent energy production $-\overline{u'w'}d\bar{u}/dz$ occurs in the "explosions."

According to Offen and Kline,[44] and eddy with a transverse axis and the sign corresponding to the main-flow vorticity $d\bar{u}/dz$ is bound to the stream of decelerated liquid in a reference system that moves with the velocity of the main flow at the height at which the stream originates (Fig. 5). When the stream separates, the central part of this eddy floats upward and moves away downstream, so that the eddy is stretched out and acquires a horseshoe shape (a longitudinal section through a "leg" of such a "horseshoe" is seen in the isochrones of Fig. 4). Overtaking on the upper side the decelerated stream next in the downstream direction, the top of the "horseshoe" produces a negative pressure gradient on it and causes it to separate (Fig. 6, b and c) with formation of a new "horseshoe." Superposition of two "horseshoes" causes them to combine to form a single larger eddy, but much more often the eddy lines do not combine, but intersect, creating "explosions" and breaking up into smaller eddies; both "randomization" and transfer of turbulent energy along the spectrum of scales takes place in this manner.

A sequence of bifurcations was observed with increasing Reynolds number also in the experiments of Miksad,[46-47] who introduced (using a loudspeaker) vibrations of fixed frequency into the zone between air flows of various velocities (from two fans) and detected the successive appearance of new discrete frequencies, and then a continuous spectrum. Similar phenomena are also observed in the wake of a fluid flow past a cylinder (see, for example, the description in Chap. 41 of the book by Feynman, Leighton, and Sands[48]): a *stability succession* takes place at $\mathbf{Re}\sim 10$, and a pair of stationary vortices forms behind the cylinder; at $\mathbf{Re}>40$ they began to separate one by one from the cylinder, are replaced by new eddies, and move off downstream, forming a Karman vortex chain; at $\mathbf{Re}>100$ the vortices are replaced by rapidly turbulized regions of successively separating boundary layers; at $\mathrm{Re}>10^5$ the boundary layers are turbulized even before separation, the separation point moves downstream, the turbulent wake narrows, and the resistance drops (resistance crisis); at $\mathbf{Re}\sim 10^6$, the turbulent wake broadens and the resistance rises; finally, at $\mathbf{Re}\sim 10^7$ the wake begins to oscillate as a single entity. Finally, we mention the numerical experiments of Lorenz[49] with a two-layer twelve-mode model of the general circulation of the earth's atmosphere, in which, with two and three degrees of freedom, per-

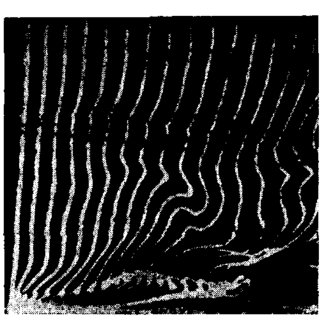

FIG. 4. Bubble isochrones (i.e., lines of constant time; side view) generated in boundary layer at a flat plate by a vertical filament (left edge of photograph, flow from left to right).[41]

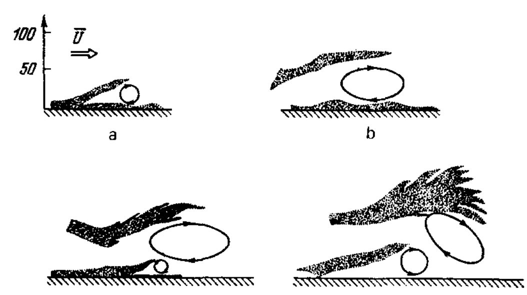

FIG. 6. Evolution of two successive streams of decelerated fluid.[44] a) Separation of first stream; b) start of interaction with second stream; c) separation of second stream; d) "explosion."

iodic motions were obtained with two and three periods, respectively, while four degrees of freedom give a nonperiodic pseudorandom motion.

6. THE HYPOTHESIS OF STRANGE ATTRACTORS

The equations of fluid dynamics are mathematically complex and thus far they have yielded only a very few concrete conclusions as to the nature of the phase paths of a viscous fluid, which describe the evolution of its states in time at various Reynolds numbers (one of the important results that have been obtained can be found in the paper by Brusklinskaya that was cited above[14]). However, significant indications of existing possibilities can be extracted from an analysis of the phase paths of dynamic systems of various general forms (see, for example, Smale's review[50] and Nitecki's book[51]). Even dynamic systems with phase spaces M of a small number of dimensions may be of interest for fluid dynamics, since the phase paths of a fluid lie in finite-dimensional subspaces of phase space after a small number of normal bifurcations.

The concept of the *nonwandering* phase point—a point any neighborhood of which intersects a certain phase path at least twice—is highly useful for inferences as to the nature of the phase paths of a dynamic system. The simplest particular cases are *stationary* points, which correspond to stationary solutions of the dynamic equations, and *periodic* points, which lie on closed paths and correspond to solutions that are periodic in time. A stationary point x of the transformation f of phase space M after a fixed time t is called *hyperbolic* if the bounded linear operator Df that serves as the differential of the transformation f at this point [which maps the subspace T_x tangent to the space M at the point x onto the tangential subspace $T_{f(x)}$ at the point $f(x)$] is hyperbolic, i.e., its spectrum does not intersect the unit circle; a periodic point of the transformation f is called hyperbolic if it is a hyperbolic stationary point of a certain degree of the transformation f. A *stable manifold* of a stationary hyperbolic point x is a set of points p of phase space for which the sequence $f^m(p)$ of iterations of the transformation f converges to x as $m\to\infty$; a stable manifold with respect to f^{-1} is called an *unstable manifold* of the point x with respect to f. Intersection points of stable and unstable manifolds that differ from the point x itself are called *homoclinic points* (Poincare first encountered such points in the three-body problem).

Kupka and Smale (see[51]) proved that dynamic systems (on a smooth compact manifold) all of whose periodic points are hyperbolic are *typical* in the sense that they form in the space of all possible dynamic systems a so-called *Baire* subset that can be represented as a denumerable intersection of everywhere open dense subsets (we note, incidentally, that this definition of typicality is perhaps only conventional, since a set of nontypical systems may have a nonzero measure). Dynamic systems in which the set of nonwandering points consists only of a finite number of stationary points and closed paths, with all periodic points being hyperbolic (and the stable and unstable manifolds corresponding to any two of these points being transversal), are called Morse-Smale systems. Peixoto showed that these properties are typical for dynamic systems whose phase space is a circle. However, these properties are found to be nontypical for dynamic systems with phase spaces of higher dimension. The existence of an infinite number of periodic points is then possible in the transformation f after a fixed time t, and the set Ω of nonwandering points may contain a Cantor discontinuum (i.e. a nowhere dense closed set without isolated points).

A model of such a system is obtained if all two-sided sequences $\{\ldots, a_{-1}, a_0, a_1, \ldots\}$ of n elements $a=1, 2, \ldots, n$ are taken as the points of the phase space (open sets will consist of sequences with fixed elements at a finite number of positions). Such a phase space is a Cantor discontinuum; it can be brought into one-to-one and mutually continuous correspondence with the set of numbers from the segment $[0,1]$ whose ternary expansions do not contain ones. We take the so-called "Bernoulli shift" $a_k\to a_{k-1}$ as the transformation f. Sequences consisting of repeating blocks of finite length will be periodic points of this transformation. They are everywhere dense in phase space, so that Ω coincides with the entire space and is, therefore, a Cantor discontinuum. Smale showed that in a broad class of typical dynamic systems, every homoclinic point belongs to a certain closed subset Λ of the set of nonwandering points Ω that is invariant under the transformation f and is a Cantor discontinuum, with a certain power f^m of the transformation f being topologically equivalent to the Bernoulli shift on Λ.

Of special interest for the problems considered in the present paper are cases in which Cantor subsets Λ of sets of nonwandering points Ω are *attractors*, i.e., have neighborhoods such that the phase paths appearing in them asymptotically approach Λ; attractors that differ from stationary points and closed paths are called *strange*. No exact proofs of the existence of strange attractors in the phase space of viscous liquid or gas flows have as yet been obtained, but this hypothesis can be recognized as plausible if the presence of strange attractors is generally *typical* for dynamic systems. The most general result in this area at the present time[51] is Robbin's theorem, according to which any dynamic system (on a compact manifold M without an edge) is *structurally stable* (this means that such systems form an open set in the space of all possible dynamic systems) if the transformations f of the phase space M that it brings about satisfy "axiom A", i.e., for each f the set of nonwandering points $\Omega(f)$ is *hyperbolic* and the set of periodic points is dense in Ω (and if, further, each stable manifold intersects each unstable manifold transversally). A set Λ will here be called *hyperbolic* if the ensemble $T_\Lambda(M)$ of tangential subspaces T_x can be decomposed on it continuously and invariantly with respect to Df into the sum of a set E^s that is compressed by the operator Df and a set E^u that is stretched by this operator (a finite hyperbolic set is simply a finite set of hyperbolic points; the Cantor set is hyperbolic in the model with the Bernoulli shift).

Let us now consider a situation in which all the generalized coordinates of the dynamic system ω are quasiperiodic functions of time with m fixed noncommensurable periods; then the phase paths of system ω are found to lie in a certain m-dimensional torus T^m in phase space and the system itself can be represented as a constant vector field on this torus. Ruelle and Takens[5] demonstrated that for $m \geq 3$ there exists in each set of dynamic systems obtained from ω by small perturbations (i.e., in every small neighborhood of ω, with, of course, the "small neighborhood" concept precisely defined) an open subset of dynamic systems that are not Morse-Smale systems, namely, with $m = 3$ there are systems on $T^3 = T^2 \times T^1$ such that the transformations of the two-dimensional torus T^2 that they induce have sets of nonwandering points that contain a Cantor set, while for $m \geq 4$ there are systems that contain strange attractors in their phase spaces.

In particular, at $m = 4$ there is in every small neighborhood of ω an open subset of dynamic systems ω' with strange attractors of the following type. Let Σ be a three-dimensional subset in T^4 that is intersected transversally by the phase paths of an ω' system. We define the mapping of $P(x)$ of the subset Σ onto itself (called a Poincare *succession mapping*) as the point of the next intersection of Σ by the phase path originating from the point x of this subset. We can then take systems ω' for which $P(x)$ maps the interior U of a two-dimensional torus embedded in Σ onto itself in such a way that $P(U)$ is the interior of the single-loop torus embedded in U that appears in Fig. 7. The circle S, which is a cross section of the solid U, is then transformed into two circles $P(S)$ within S. The next iteration $P^2(S)$ gives two small circles within each of the circles $P(S)$, and so forth. The intersection of all iterations $P^n(S)$ gives a Cantor point set in S, so that the intersection of all iterations $P^n(U)$ is a Cantor line set (a so-called one-dimensional Williams solenoid), while the dynamic system ω' itself has in its four-dimensional phase space a strange attractor that is a local Cantor set of two-dimensional surfaces.

The Ruelle-Takens theorem indicates that the appearance of strange attractors in the phase spaces of dynamic systems after a few normal bifurcations (as few as four or even three, not counting stability successions) should be a typical effect (in the sense indicated in the formulation of the theorem). Whether or not liquid and gas flows have such typical properties remains to be clarified, both analytically taking into account the specific form of the hydrodynamic equations and experimentally by following successive bifurcations in the loss of stability by laminar flows.

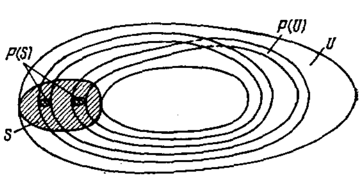

FIG. 7. Mapping of succession $P(U)$ of the interior U of a two-dimensional torus onto itself giving rise to a strange attractor.

7. FLOWS WITH INVERSE BIFURCATIONS

Let us return now to the discussion of Sec. 4 and examine the case of "inverse bifurcation," in which the single-parameter family of closed phase paths predicted by the Hopf bifurcation theorem[13] appears even at $\mathrm{Re} < \mathrm{Re}_{1cr}$. In this case, the coefficient δ in the second term of the Landau expansion (8) for the smoothed squared amplitude $|A|^2$ of the flow velocity field perturbation $\mathbf{u}'(\mathbf{x},t) = A(t)\,\mathbf{f}_0(x)$ must be negative (the case $\delta < 0$, discussion of which was postponed in Sec. 4), while the coefficient $\gamma \sim \mathrm{Re} - \mathrm{Re}_{1cr}$ will be negative for $\mathrm{Re} < \mathrm{Re}_{1cr}$ and positive for $\mathrm{Re} > \mathrm{Re}_{1cr}$. Here Eq. (8) is found to be suitable for study of the behavior of the perturbations $\mathbf{u}'$ in the range $\mathrm{Re} < \mathrm{Re}_{1cr}$, in which it assumes the form

$$\frac{d|A|^2}{dt} = -2|\gamma||A|^2 + |\delta||A|^4. \qquad (11)$$

It is clear from this that at $\mathrm{Re} < \mathrm{Re}_{1cr}$ the limiting cycle that exists in the phase space is unstable because, firstly, phase paths lying within it are wound onto a stationary point (or, in other words, perturbations with small amplitudes $|A| < A_1 = \sqrt{2|\gamma|/|\delta|}$ decay with time) and, secondly, phase paths lying outside of this limiting cycle unwind from it and lead into other regions of the phase space (i.e., perturbations with finite amplitudes $|A| > A_1$ build up in time, so that at $\mathrm{Re}_{1cr} > \mathrm{Re} > \mathrm{Re}_{Acr} = \mathrm{Re}_{1cr} - \alpha^2 |A|^2$ the motion becomes unstable under finite perturbations with amplitudes $|A| > A_1$).

The limiting cycle contracts with increasing $\mathrm{Re} < \mathrm{Re}_{1cr}$, and vanishes after Re passes through the value Re_{1cr}. At $\mathrm{Re} > \mathrm{Re}_{1cr}$, Eq. (8) (with the coefficients $\gamma > 0$, $\delta < 0$) has the solution

$$|A(t)|^2 = \frac{A_0^2 A_1^2}{(A_0^2 + A_1^2)\,e^{-2\gamma t} - A_0^2}, \quad A_1 = \sqrt{\frac{2\gamma}{|\delta|}}, \qquad (12)$$

which increases without limit after a finite time $t = (1/2\gamma)\ln[1 + (A_1^2/A_0^2)]$, but it is clear that Eq. (8) fails even before this and must be supplemented by succeeding terms of the Landau expansion. The available examples indicate that in cases of inverse bifurcation, motions with $\mathrm{Re} > \mathrm{Re}_{1cr}$ apparently quickly become nonperiodic; it is possible that there are strange attractors with phase paths wound onto them in the phase spaces of these flows.

One of the most thoroughly studied examples of viscous fluid flows with inverse bifurcation is perhaps the plane-parallel channel flow (see the reviews in § 2.9 of the American edition of Monin and Yaglom's book[52] and Stuart's paper[53]). Here for the laminar flow (the so-called plane Poiseuille flow, which has a parabolic velocity profile) the linear theory predicts an instability region $\gamma > 0$ in the plane of the Reynolds number Re and the dimensionless longitudinal perturbation wave numbers k, as indicated by the solid line in Fig. 8; we note that as Re increases, both branches of this "neutral curve" asymptotically approach the

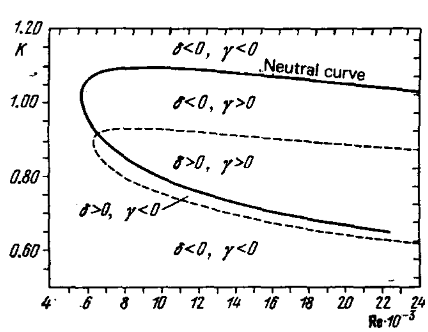

FIG. 8. Instability regions of plane Poiseuille flow. The solid line bounds the instability region of infinitesimally small perturbations ($\gamma > 0$); the dashed line bounds the region $\delta > 0$.[54]

axis of abscissas $k = 0$. The smallest critical Reynolds number (calculated from the maximum velocity and the half-width of the channel) on this curve has a value of about 5800. However, both the early experimental data of Davies and White[55] and Tilman's recent data, which are cited by Stuart,[53] indicate appearance of turbulence in the plane Poiseuille flow at much smaller values $\mathrm{Re} \sim 1000-2500$, so that we must suspect inverse bifurcation and instability with respect to finite-amplitude perturbations.

Indeed, calculations made by several authors and cited in the above reviews[52,53] predicted a negative sign for the coefficient δ in the Landau expansion (8) and, consequently, instability under finite perturbations at values of $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr,min}} \approx 2500-2900$, in satisfactory agreement with the experimental data. More precisely, Pekeris and Shkoller[54] and Reynolds and Potter[56] calculated values of δ for various k and Re. The "neutral curve" $\delta(k, \mathrm{Re}) = 0$ from the first of these papers is indicated by the dashed line in Fig. 8; it and the "neutral curve" $\gamma(k, \mathrm{Re}) = 0$ of linear stability theory divide the (k, Re) plane into four regions with different combinations of signs of the coefficients γ and δ. Reynolds and Potter[56] also obtained similar results for a combined plane Poiseuille-Couette flow with a velocity profile $U(y) = (4 - a)y - (4 - 2a)y^2$, $0 \leq y \leq 1$ (where $a = 0$ for the Poiseuille flow and $a = 2$ for the Couette flow).

It is curious that the Poiseuille flow in a round pipe (with parabolic velocity profile) and the plane Couette flow (with a linear velocity profile) should behave totally differently with respect to perturbations; they are apparently stable to arbitrary infinitesmal perturbations (i.e., $\mathrm{Re}_{1\mathrm{cr}} = \infty$), so that normal bifurcations cannot occur in them. However, experimental data indicate that these flows are unstable under finite perturbations; it appears that "neutral surfaces" that bound the instability region exist for them in the three-dimensional space (k, Re, A). For the plane Couette flow, this surface was calculated approximately by Kuwabara[57] and Ellingsen, Gjevik, and Palm,[58] and for the Poiseuille flow in a round pipe by Davey and Nguyen.[59] However, we note that when there is a rod or string on the axis of the pipe (on whose surface the flow velocity should be zero), both the region of linear instability and the region of inverse bifurcation reappear in the (k, Re) plane of the flow in the resulting "coaxial pipe."

Another interesting ezample of a system with inverse bifurcation is the idealized three-mode roll convection in a layer of fluid with a *large* Prandtl number, which can be described by formulas (10). If we neglect the interactions with all other modes, the hydrodynamic equations in the Boussinesq approximation yield the following equations for the dimensionless amplitudes X, Y, and Z of these three modes;

$$\left.\begin{array}{l} X' = -\sigma X + \sigma Y, \\ Y' = rX - Y - XZ, \\ Z' = -bZ + XY, \end{array}\right\} \qquad (13)$$

where the prime indicates the dimensionless-time derivative with respect to $\pi^2 H^{-2} \times [1 + (k_1^2/\pi^2)]\chi t$, $\sigma = \mathrm{Pr}$, $b = 4[1 + (k_1^2/\pi^2)]^{-1}$ and $r = \mathrm{Ra}/\mathrm{Ra}_{1\mathrm{cr}}$, with $\mathrm{Ra}_{1\mathrm{cr}} = \pi^4(k_1/\pi)^{-2}[1 + (k_1^2/\pi^2)]^3$. The smallest $\mathrm{Ra}_{1\mathrm{cr}}$ is obtained at $k_1 = \pi/\sqrt{2}$ and equals $(27/4)\,\pi^4 \approx 657.5$, as established long ago by Rayleigh; the resulting value $b = 8/3$ will be used below in analyzing solutions of Eqs. (13). More general equations for finite-mode two-dimensional convection, abridged in accordance with a recipe proposed by Lorenz,[60] were derived and numerically integrated by Saltzman,[61] and it was found that in some cases all of the unknown functions except X, Y, and Z tend to zero with increasing time, while the quantities X, Y, and Z vary nonperiodically in time. This is apparently what moved Lorenz[62] to make a special study of Eqs. (13), in the course of which he found them to have the surprising properties that we shall discuss in the next section.

The three-dimensional space (X, Y, Z) is the phase space of system (13). When Z is replaced by $Z_1 = Z - r - \sigma$, Eqs. (13) become a *system of the hyprodynamic type*[63]; in consequence of these equations, the sum $X^2 + Y^2 + Z_1^2$, which attains rather large values, should decrease with time. Therefore all phase paths remain within a certain bounded region at large times. Further, their divergence $(\partial X'/\partial X) + (\partial Y'/\partial Y) + (\partial Z'/\partial Z)$ has a constant negative value $-(\sigma + b + 1)$, so that every small phase volume decreases with time and all the paths tend to a certain subset of zero volume. At $r < 1$, the system (13) has one stationary point $O = (0, 0, 0)$, and this point is stable (it is an attractor). At $r > 1$, this stationary point becomes unstable and two more stationary points appear: $C = (\sqrt{b(r-1)}, \sqrt{b(r-1)}, r - 1)$ and $C' = (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r - 1)$, which are equally valid because system (13) does not change on the transformation $(X, Y, Z) \to (-X, -Y, Z)$. For $\sigma < b + 1$, the points C and C' are stable, but if $\sigma > b + 1$ they are stable at $1 < r < r_{\mathrm{cr}} = \sigma(\sigma + b + 3)(\sigma - b - 1)^{-1}$, and become unstable for $r > r_{\mathrm{cr}}$. This latter case will be of special interest to us.

We acknowledge that the three-mode system (13) no longer corresponds to any real convection at large Prandtl ($\sigma > b + 1$) and Rayleigh ($r > r_{\mathrm{cr}}$) numbers. The experiments of Willis and Deardorff[21] with silicone oil ($\sigma = 57$) and Krishnamurti[22,23] with water ($\sigma = 6.7$) and other liquids with large Prandtl numbers showed that nonstationarity is manifested not in roll motions (10), but in the appearance of convective filaments that grow out of the thermal boundary layer at the lower boundary of the liquid. While it loses its hydrodynamic content at these parameter values, the system (13) nevertheless remains physically interesting: it differs from the lasing equations only in the scales of measurement of the variables X, Y, and t and in the origin from which Z is reckoned (see, for example, Haken's paper[64]).

According to the linear theory, neutral (neither growing nor decaying, i.e. purely periodic infinitesmal perturbations of the stationary roll motions (10) described by the phase point C turn out to be possible at $r = r_{cr}$, and their frequencies ω_{cr} are determined by the formula $\omega_{cr}^2 = 2b\sigma(\sigma+1)(\sigma-b-1)^{-1}$; for the X coordinate, this neutral perturbation can be written in the form $\delta X = A\cos\omega_{cr}t$. At values of r just below r just below r_{cr}, a small nonlinear correction with a principal term of the order of A^2 that contains a nonperiodic term and the harmonic of frequency $2\omega_{cr}$ is added to the perturbation of this kind, and A becomes a slowly varying function of the time. McLaughlin and Martin[32] derived the Landau equation (8) for the squared perturbation amplitude $|A|^2$ with zeroth-order accuracy in $\sqrt{r-1}-\sqrt{r_{cr}-1}$. The values $\gamma = (b/2\sqrt{\sigma})[\sqrt{r-1} - \sqrt{r_{cr}-1}]$, $\delta = -37/72\sigma$ were obtained for its coefficients with accuracy of the order of σ^{-1}, proving the existence of inverse bifurcation.

8. THE LORENZ ATTRACTOR AND OTHER EXAMPLES

Lorenz[62] integrated Eq. (13) numerically with $b = 8/3$ and $\sigma = 10$ (in this case $r_{cr} = 470/19 \approx 24.74$), using the slightly supercritical value $r = 28$. It was found that each path sooner or later arrives in the neighborhood of one of the stationary points C or C', describes a few unwinding loops around it, and moving away to a considerable distance from it, crosses into the neighborhood of the other one of these points, and so forth; the sequence of these transitions is of irregular nature and sensitively dependent on the initial data (see, in Fig. 9, an example of a path of this type, for which we are indebted to M. I. Rabinovich, who integrated Eqs. (13) on an analog computer). Lorenz surmised that all these paths fill a two-dimensional infinite-sheeted

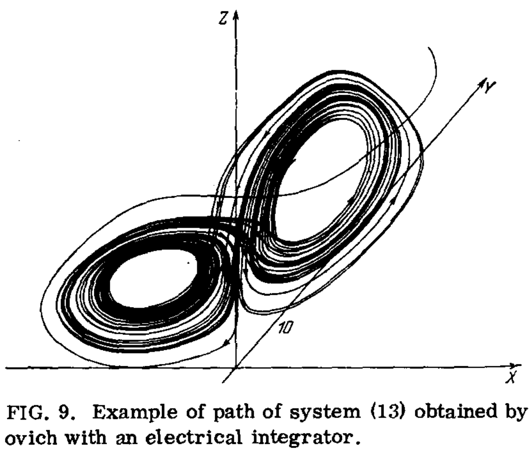

FIG. 9. Example of path of system (13) obtained by M. I. Rabinovich with an electrical integrator.

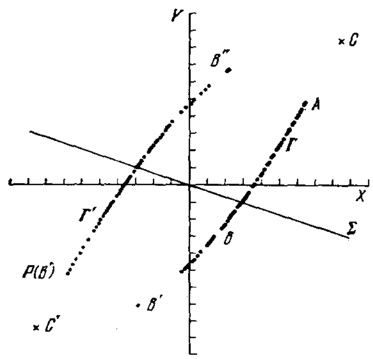

FIG. 10. Iterations of Poincare mapping of plane $Z = 27$ according to Lanford's calculations (from Ruelle's paper[66]).

surface whose intersection with a certain straight line is a Cantor point set (since the phase paths cannot intersect, it would appear that the phase space must have no fewer then three dimensions for such an attractor to exist; however, Plykin[65] constructed an example of a structurally stable transformation f of a two-dimensional phase space whose set of nonwandering points consists of a one-dimensional Williams solenoid and four stationary points).

Lanford, whose results are described in a recent paper by Ruelle,[66] applied numerical integration methods to Eqs. (13) to calculate a series of iterations of the Poincare mapping $P(M)$ of the plane $Z = 27$, in which the stationary points C and C' lie—see Fig. 10, which we have taken from Ref. 66. The iterations $P^n(M_0)$ of one of the points of this plane lie on the arcs Γ and Γ'; the line Σ that intersects them consists of points that do not return to the $Z = 27$ plane (paths that pass through points of Σ recede to a stationary point O and from a two-dimensional *stable manifold* of this point). In one iteration, the arc Γ is transformed as follows: arc AB is stretched into AB' while BB' becomes Γ'; the point B on the same side of Σ as C becomes B', and the one on the other side becomes B'' (points B' and B'' belong to one-dimensional *unstable manifolds* of point O, to so-called *separatrices*). Arc Γ' is similarly transformed.

The phase-space transformations brought about by system (13) contain contraction in one direction and expansion in the other (hyperbolicity), which is what causes the sensitive dependence on the initial data; however, Ruelle[66] observes that system (13) still does not, apparently, possess complete hyperbolicity in the sense of "axiom A" because the uniformity of hyperbolicity that is required by this axiom in disturbed, to judge from the available calculations, by the ability of the paths to pass arbitrarily close to O and be detained there for an arbitrarily long time.

Henon and Pomeau[67] integrated Eqs. (13) on an analog computer, putting $b = 8/3$ and $\sigma = 10$ and varying r from 28 to values above 200; at $r \approx 220$ they observed a bifurcation that resulted in replacement of the strange attractor by a limiting cycle, and they made a numerical study of formation of the attractor by iterations of the Pointare mapping of a $Z = \text{const}$ plane, assuming for the latter the analytic model $P = CB$ [which is possibly not adequate for Eqs. (13)], where

$$\left.\begin{array}{llll} BX=2X, & BY=\frac{Y}{2} & \text{for} & 0\leqslant X<\frac{1}{2}, \\ BX=2X-1, & BY=\frac{Y+1}{2} & \text{for} & \frac{1}{2}\leqslant X\leqslant 1, \\ CX=X, & & & \\ CY=\frac{1}{8}+\frac{Y}{2} & & \text{for} & 0\leqslant Y<\frac{1}{2}, \\ CY=\frac{3}{8}+\frac{Y}{2} & & \text{for} & \frac{1}{2}\leqslant Y\leqslant 1. \end{array}\right\} \quad (14)$$

This model mapping P has a strange attractor consisting of seqments $0 < X < 1$, $Y = (1/8)\sum_0^\infty \alpha_n/4^n$, where the α_n are random sequences of the numbers 1 and 5. Another model of the Poincare mapping was also proposed in this paper (see also Henon[68]); it consists of the quadratic bending $X_1 = X$, $Y_1 = Y + 1 - aX^2$, compression along the axis of abscissas $X_2 = bX_1$, $Y_2 = Y_1$, and reorientation of the axes, $X_3 = Y_2$, $Y_3 = X_2$, which eventually gives the "Cremona transformation"

$$X_{n+1}=Y_n+1-aX_n^2, \quad Y_{n+1}=bX_n, \quad (15)$$

which has a constant Jacobian($-b$) and is the canonical form of quadradic transformations with a constant Jacobian. Numerical execution of $n = 5\cdot 10^6$ iterations of the transformation (15) applied to one of the points (X, Y) with $a = 1.4$ and $b = 0.3$ outlined, with various details, a strange attractor in the (X, Y) plane that consisted of a set of lines of, to all appearances, Cantor structure.

On the basis of results of numerical integration and material from the general theory of dynamic systems, Afraĭmovich, Bykov, and Shil'rikov[69] indicate the following sequence of bifurcations for system (13) with $b = 8/3$ and $\sigma = 10$:

1) For $1 \leqslant r < r_1 \approx 13.92$, there are three stationary points O, C, and C', where O is a saddle point with a two-dimensional stable manifold W (which consists of paths meeting at O) and two one-dimensional unstable manifolds—the separatrices G and G' (paths emanating from O and tending to C and C', respectively, as $t\to\infty$).

2) For $r = r_1$, the separatrices become doubly asymptotic to the saddle point O and periodic saddle-point motions L and L' are generated from their loops, with the simultaneous appearance of a one-dimensional invariant set Ω_1 of Cantor structure (including a denumerable set of periodic saddle-point motions), which, however, is not an attractor. This pattern persists for $r_1 < r < r_2 \approx 24.06$, with $G \to C'$ and $G' \to C$ $(t\to\infty)$.

3) For $r = r_2$, the separatrices G and G' tend as $t\to\infty$ to closed paths L' and L instead of to C' and C, and Ω_1 is replaced by a two-dimensional invariant set —a Lorenz attractor Ω_2, whose region of attraction is bounded by the stable manifolds of the periodic motions L, L' (so that the excitation of stochasticity is hard). For $r_2 < r < r_3 \approx 24.74$, it is stable, as are the points C and C' (and includes O, G, and G' and is therefore not structurally stable); in it the periodic motions are everywhere dense (they can vanish as r varies only by sticking in the loops of the separatices), and the paths are dispersed exponentially on it; the set Ω_2 is not hyperbolic.

4) For $r = r_3$, the periodic motions L, L' contract to the points C and C' and the latter lose stability, so that for $r_3 < r < r_4 \approx 220$ the Lorenz attractor is the only stable limiting set.

5) As r decreases from r_4 to r_2, the phase point remains in the Lorenz attractor; it loses stability at $r = r_2$, and for $r < r_2$ it leaves the neighborhood of the attractor and tends to C or C'—hysteresis of this kind is typical for systems with inverse bifurcation.

Strange attractors similar to those discussed here apparently exist in the phase spaces of a number of model dynamic systems that are described by simple systems of equations (see, for example, the papers by Rabinovich *et al.*[70-72]). It was possible to prove stochasticity rigorously in the example of Pikovskiĭ and Rabinovich.[72] This example is that of a self-excited negative-conductance oscillator and a tunnel diode, which can be described in terms of dimensional variables by the equations

$$\left.\begin{array}{l} X'=Y-\delta Z, \\ Y'=-X+2\gamma Y+\alpha Z, \\ \mu Z'=X-f(Z), \end{array}\right\} \quad (16)$$

where $\mu \ll 1$ and $f(Z) = Z - Z^3$ or is of similar form. As $\mu\to 0$, the system phase space degenerates into two half-planes $Z = -1$, $X < 1$ and $Z = 1$, $X > -1$ with crossing of the paths from one to the other only on the half-lines $S^-(X = -1, Z = -1, Y > -\delta)$ and $S^+(X = 1, Z = 1, Y < \delta)$. Investigation of the motion reduces to analysis of the Poincare mapping of the set $S = S^- + S^+$, which is defined by $P_1(S) = e^{2\pi\varkappa}S$ for paths lying in only one-half plane and by $P_2(S) = 2\delta - \omega(\text{ctg}\,\tau + \varkappa)$, $S = (\omega/\sin\tau)e^{-\varkappa\tau}$ for paths that cross to the other half-plane (where $\omega = \sqrt{1-\gamma^2}$ and $\varkappa = \gamma/\omega$). For a fixed point of S, iterations of the Poincare mapping form a sequence $P_1^{m_1}P_2^{n_1} \times P_1^{m_2}P_2^{n_2}\ldots$, where m_i and n_i are integers; two symmetric attractors exist in the phase space for paths in which all n_i are even, and one attractor for the other paths. Since $|\partial P/\partial S| > 1$, the Poincare mapping has no stable stationary points. Moreover, it satisfies the ergodicity conditions (established by Kosyakin and Sandler,[73] so that here the attractors do not contain stable stationary points or limiting cycles and are ergodic.

A number of systems of the hydrodynamic type, model equation systems from relativistic cosmology and gasdynamics, and the perturbation-theory equations for fully integrable systems whose solutions admit separatrix approximation and exhibit stochastic properties were studied in a series of papers by Bogoyavlenskiĭ and Novikov.[74-76] Finally, we note the quasistochastic magnetic field polarity reversals in the Bullard two-disk dynamo, in which the dimensionaless currents x_1, x_2 and the rotation speeds of the disks y_1 and y_2 are described by the equations

$$x_1'=-\mu_1x_1+x_2y_1, \quad x_2'=-\mu_2x_2+x_1y_2, \quad y_1'=y_2'=1-x_1x_2 \quad (17)$$

(see, for example, Chap. 10 of the book of Ref. 77).

9. DISCUSSION

a) It was tacitly assumed only ten years ago that only stationary points and closed or quasiperiodic orbits could be attractors for the phase paths of dynamic systems. Irregularity ("stochasticity") of the behavior of such systems could be brought about either by introducing randomness into their initial data, or by applying random external disturbances to them, or, finally, by increasing the complexity of the limiting orbit to correspond to excitation of a very large number of degrees of freedom of the system.

The discovery of strange attractors forces us to abandon these intuitive assumptions, which simplify reality too drastically. We then find that the structural complexity of certain strange attractors, which fall far short of filling the phase space (have smaller dimensionality) and contain Cantor discontinua in some of their sections, by no means implies that they are pathological cases (abnormal exceptions). To the contrary, it has been shown that the presence of such strange attractors is a *typical* phenomenon, in a certain precisely defined sense of the adjective. Thus it has now been established that most dynamic systems are capable of generating pseudorandom functions of the time without introduction of randomness into the initial data or application of random external disturbances, and without excitation of a very large number of degrees of freedom.

However, it is still unknown whether any liquid or gas flows exhibit such properties.

b) Contrary to the proposal of Ruelle and Takens,[5] it appears that a liquid or gas flow that evolves on a strange attractor (if there is such an attractor) cannot yet be called turbulent: the definition of turbulence (Sec. 2) includes the requirement that the thermohydrodynamic characteristics of the flow vary irregularly *in space*, i.e., that they be described by a large number of spatial modes (or, empirically, have continuous spatial spectra).

When turbulence is imposed on a strong averaged flow, small-scale segments of the spatial spectra on the streamlines are found to be similar to the corresponding segments of the frequency spectrum of the fluctuations at fixed points of these streamlines (Taylor's "frozen turbulence" hypothesis), so that a flow with continuous frequency spectra and a discrete spatial spectrum (one with few modes) is not yet turbulence. From this point of view, roll convection evolving on a Lorenz attractor in a fluid with a large Prandtl number—convection that has a highly regular and simple three-mode spatial structure (10)—is, of course, not turbulence, just as the other flows with a spatial structure involving only a few modes that were mentioned above. Thus, multidimensionality is required of the turbulent attractor, and the notion of turbulence as a system with a very large number of excited degrees of freedom should remain in force.

This also leaves in force the problem of the development in time of the spatial spectrum of the turbulence or, in other words, of the sequence of bifurcations that increase the dimensionality of the turbulent attractor. This problem has not yet been investigated within the framework of the strange-attractor hypothesis. Hardly any experimental data are as yet available. In this context, we mention only the information in section 5 on "explosions" at intersections of horseshoe vortices in the boundary layer near a smooth flat plate, which result in the formation of an extremely irregular small-scale motion in which pratically all the turbulent energy in this flow is produced.

c) The experimental evidence in favor of the Ruelle-Takens hypothesis of arrival at the strange attractor from a four-dimensional torus (after four normal bifurcations) appears very shaky. It seems clear that the nonperiodic motion in Gollub and Swinney's experiments on the evolution of a Couette flow between rotating cylinders actually appeared after the fourth bifurcation (not counting stability successions), but the transition to this regime occurred not from a four-dimensional or even a three-dimensional torus, but from a closed orbit on a two-dimensional torus. It appears to be important to stress that the actual (not the instrumental) continuous spectrum appeared here and then built up progressively already after the second bifurcation, and that the fourth bifurcation manifested itself rather in the vanishing of discrete spectral lines than in the appearance of a continuous spectrum (see Fig. 1).

In McLaughlin and Martin's numerical experiments on the evolution of roll convection in a fluid with a small Prandtl number, an essentially nonperiodic motion actually arose after a four-period motion, not immediately, but as a new bifurcation after a considerable increase in the Rayleigh number. In the experiments of Klebanoff, Tidstrom and Sargent on the evolution of a flow in the boundary layer near a flat plate, turbulence apparently appeared after the sixth bifurcation (after the appearance of four nonlinear Landahl pulses per vibrator cycle on a secondary three-dimensional wave). In the evolution of the wake of a fluid flow around a cylinder, turbulence first appears after the second bifurcation (not counting a stability succession), in rapidly turbulized regions of boundary layers that separate successively from the cylinder; the further evolution of this flow bears little resemblance to a sequence of normal bifurcations.

We note that to distinguish experimentally between the sudden appearance of nonperiodic motion with a continuous spectrum and a sudden crowding of a sequence of bifurcation points on the Reynolds-number axis is, of course, impossible; this can be done only within the limits of resolution of the experiment with reference to **Re** (which, for example, was δ**Re** = 25 in Gollub and Swinney's experiments). Further, the possibility of empirical verification of the exponential damping of correlation-functions, which is characteristic of the strange attractor, and its absence in the case of quasiperiodic attractors, is limited by the finite time of the measurements and by their discreteness. Finally, yet another possibility for empirical differentiation of motions on quasiperiodic and strange attractors—the sensitive dependence on the initial data on arrival at a strange attractor with few modes—may be lost after a large number of bifurcations, which increase the dimensionality of the attractor.

d) The strange attractor example closest to fluid dynamics, which was first studied by Lorenz, first of all, does not, as we have already noted, correspond to any real flow; secondly, because of its few modes, there is no direct relation to turbulence; thirdly, it has been investigated only numerically (among other things, this turned up an original feature of system (13) in which it does not resemble turbulence —a bifurcation at very large Ra, which results in replacement of the strange attractor by a limiting cycle); fourthly, according to Ruelle, it is apparently not quite hyperbolic, so that the value of this unique example from the standpoint of the general theory of dynamic systems is somewhat compromised.

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