An infinity of the classical theory of fluctuations in a nondegenerate electron gas

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The singularities of the fluctuations of the density and electric field in a nondegenerate ideal electron gas are discussed. Classical calculations for a point electron lead to an infinity in the spectral energy distribution of the Coulomb field. Measurements of the spectral energy distribution in the classical frequency region and the elimination of the singularity by incorporating quantum fluctuations are discussed. The space–time correlation function of the density fluctuations in a nondegenerate ideal gas is discussed.

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CONTENTS

1. INTRODUCTION

In this paper we will discuss some singularities of the space–time fluctuations in a nondegenerate ideal gas. It turns out that quantum fluctuations can be important in cases which appear to be completely classical.

2. INFINITY IN THE SPECTRAL ENERGY DISTRIBUTION OF THE COULOMB FIELD IN THE CLASSICAL THEORY

a) The classical theory of thermal radiation leads to the well-known "ultraviolet catastrophe," which Planck used as the starting point for the formulation of his quantum radiation law. The "catastrophic result," of course, is that if the energy is distributed uniformly among the various degrees of freedom the total energy of the transverse (electromagnetic) field, $W^{tr} = \int_{-\infty}^{+\infty} W_\omega^{tr} d\omega$, turns out to be infinite because of the infinite number of field oscillators. In a certain sense, there is a worse "catastrophe" in the classical theory of thermal fluctuations in an electron gas: In this case it is the spectral energy distribution (rather than the total energy) of the longitudinal (Coulomb) field, W_ω^l, which turns out to be infinite. This divergence of the spectral energy distribution of the Coulomb field follows from the assumption that the electron is a point particle, as we will see below. We believe it is interesting and instructive to analyze and remove this divergence (in some form or other).

b) We begin with a formal discussion of the equations of the standard theory of thermal fluctuations, which relate the correlation function of the electric field at the frequency ω, $\langle E_i(\mathbf{r}_1) E_i(\mathbf{r}_2)\rangle_\omega$, to the dielectric permittivity. For a medium with a dielectric permittivity $\varepsilon_{ij}(\omega) = \varepsilon(\omega)\delta_{ij}$ and with $\mu = 1$ we can write[1,2]

$$\left.\begin{aligned}
\langle E_i(\mathbf{r}_1) E_i(\mathbf{r}_2)\rangle_\omega &= \langle E_i^l(\mathbf{r}_1) E_i^l(\mathbf{r}_2)\rangle_\omega + \langle E_i^{tr}(\mathbf{r}_1) E_i^{tr}(\mathbf{r}_2)\rangle_\omega \\
&= \frac{\theta}{2\pi^3\omega}\int d\mathbf{k}\left[\frac{\operatorname{Im}\varepsilon(\omega)}{|\varepsilon(\omega)|^2} + 2k_0^4\frac{\operatorname{Im}\varepsilon(\omega)}{|k^2 - k_0^2\varepsilon(\omega)|^2}\right] e^{i\mathbf{k}\mathbf{R}}, \\
\langle E_i^l(\mathbf{r}_1) E_i^l(\mathbf{r}_2)\rangle_\omega &= \frac{4\theta}{\omega}\frac{\operatorname{Im}\varepsilon(\omega)}{|\varepsilon(\omega)|^2}\delta(\mathbf{R}),\quad \mathbf{R} = \mathbf{r}_1 - \mathbf{r}_2, \\
\theta &= \frac{\hbar\omega}{2}\operatorname{cth}\frac{\hbar\omega}{2T} = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{\exp(\hbar\omega/T) - 1},
\end{aligned}\right\} \tag{1}$$

where T is the temperature, $\hbar$ is Planck's constant, and $k_0 = \omega/c$. We write the correlation function for the electric field as the sum of the correlation functions of the longitudinal and transverse fields. We will see that the spectral energy distribution of the longitudinal field, $\langle (E^l)^2\rangle_\omega$, turns out to be infinite.

It is usually assumed, explicitly or implicitly,[1] that spatial dispersion, i.e., the nonlocal relationship between the current and the electric field [which, for example, causes the dielectric permittivity of the medium, $\varepsilon_{ij}(\omega, \mathbf{k})$, to vary with not only the frequency ω but also the wave vector[2] $\mathbf{k}$], should lead to a "smearing out" of the δ-function dependence of the correlation function on $\mathbf{r}_1 - \mathbf{r}_2$ and to the elimination of the divergence of $\langle (E^l)^2\rangle_\omega$.

c) But let us examine the correlation function of the Coulomb field in a classical electron gas. With spatial dispersion taken into account, the correlation function of the longitudinal field is[2,3]

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$$\langle E_i^l(\mathbf{r}_1)\, E_i^l(\mathbf{r}_2)\rangle_\omega = \frac{\theta}{2\pi^3\omega}\int d\mathbf{k}\,\frac{\operatorname{Im}\varepsilon^l(\omega,\ k)}{|\varepsilon^l(\omega,\ k)|^2}\,e^{i\mathbf{k}\mathbf{R}}, \tag{2}$$

where $\varepsilon^l(\omega, k)$ is the longitudinal dielectric permittivity of an isotropic medium with spatial dispersion taken into account. In the limit $\mathbf{R}=\mathbf{r}_1-\mathbf{r}_2\to 0$ we need to take into account only the collisionless losses in the dielectric permittivity (those losses which give rise to the Landau damping of the longitudinal waves in the electron gas). If $\omega\gg\omega_0$, where $\omega_0=\sqrt{4\pi e^2N/m}$, we can approximate $|\varepsilon^l(\omega, k)|$ by unity in the denominator of the integrand in (2). Substituting into (2) the well-known expression for $\operatorname{Im}\varepsilon^l(\omega, k)$ for a nondegenerate electron gas[1-3],

$$\operatorname{Im}\varepsilon^l(\omega,\ k)=\frac{\pi\omega_0^2m^2\omega}{k^3T\sqrt{2\pi mT}}\exp\left(-\frac{m\omega^2}{2k^2T}\right), \tag{3}$$

and assuming $\hbar\omega\ll T$ (when $\theta\approx T$) and $v_T/\omega R\gg 1$, we have[1)]

$$\langle E_i^l(\mathbf{r}_1)\, E_i^l(\mathbf{r}_2)\rangle_\omega=\frac{8}{\sqrt{\pi}}\frac{e^2N}{v_T}\ln\frac{v_T}{\omega R},\qquad \omega\gg\omega_0; \tag{4}$$

where $v_T=(2T/m)^{1/2}$ is the electron thermal velocity, and N is the number density of the electron gas. The correlation function of the longitudinal thermal field thus diverges logarithmically in the limit $R\to 0$, and the spectral energy distribution of the thermal fluctuations of the Coulomb field, $W_\omega^l=(8\pi)^{-1}\langle(E^l)^2\rangle_\omega$, turns out to be infinite. This infinity is the subject of the present paper.

The divergence of the total fluctuation energy of the Coulomb field, $W^l=\int_{-\infty}^{+\infty}W_\omega^l d\omega$, which follows from the assumption of point electrons, is a well-known result. For example, back in 1919 Holtsmark[4] wrote about a divergence in the second moment of the distribution of fluctuations in the electric field, $P(E)$ (the Holtsmark distribution[4-6]), which determines, for example, the shape of the emission line of an atom in the case of Stark splitting.

d) How would we actually measure the spectral energy distribution of the longitudinal electric field in an electron gas? We could use a small dipole antenna consisting of two straight conductors of length L. Then the square of the spectral emf induced in the antenna, $\mathscr{E}_\omega^2$, is a measure of W_ω^l.

$$\langle(E^l)^2\rangle_\omega=3\mathscr{E}_\omega^2L^{-2}. \tag{5}$$

The emf which arises in a small dipole "immersed" in an electron gas is actually the shot-effect emf due to the point electrons which pass near the dipole. This emf can be calculated by the standard methods,[7] so we will proceed immediately to the result[2)] (Ref. 7):

$$\mathscr{E}_\omega^2=\frac{4}{3\pi^{3/2}}\frac{\omega_0^2}{\omega^2}\frac{1}{\omega L}\left(\frac{\omega L}{v_T}\right)^3T\ln\frac{v_T}{\omega L}, \tag{5a}$$

$$8\pi W_\omega^l=\frac{8}{\sqrt{\pi}}\frac{e^2N}{v_T}\ln\frac{v_T}{\omega L},\qquad \omega\gg\omega_0. \tag{5b}$$

The averaging effect of a dipole of length L thus leads to a finite value for the measured spectral density W_ω^l. The result, however, is strongly dependent on the length of the antenna, increasing as the antenna is made shorter. This latter effect is a direct consequence of the assumption of point electrons.

3. QUANTUM FLUCTUATIONS OF THE COULOMB FIELD

a) The formal reason for the infinite value of W_ω^l is that the quantity $\operatorname{Im}\varepsilon^l(\omega, k)$ (the Landau damping) turns out to be too large for the Fourier field components with large values of k. At the same time the Landau damping is the only fundamental mechanism for a transfer of field energy to the particles at large values of k. Therefore, we would like to eliminate the divergence in W_ω^l while keeping the Landau-damping mechanism. There is in fact such a possibility; it involves taking into account the finiteness of the quantum of action, $\hbar$. In this case, momentum and energy conservation in the interaction of the $\mathbf{k}$ component of the field and the electrons automatically leads to a reduction of the damping at large values of $\mathbf{k}$. With $\hbar\neq 0$, the dielectric permittivity of a nondegenerate electron gas in equilibrium can be derived from the general equations (see, for example, Ref. 2):

$$\left.\begin{aligned}&\varepsilon^l(\omega,\ k)=1+\frac{\omega_0^2\beta^2}{2\sqrt{2\pi}\,\omega^2\gamma}[I_+(\beta+\gamma)-I_+(\beta-\gamma)],\\&\omega_0^2=\frac{4\pi e^2N}{m},\qquad \beta=\frac{\omega}{k}\sqrt{\frac{m}{T}},\qquad \gamma=\frac{\hbar k}{\sqrt{4mT}},\\&I_+(x)=2\sqrt{2\pi}\,F\left(\frac{x}{\sqrt{2}}\right)-i\pi e^{-x^2/2},\qquad F(x)=e^{-x^2}\int_0^x e^{u^2}du.\end{aligned}\right\} \tag{6}$$

For real ω and k, for example, one obtains

$$\operatorname{Re}\varepsilon^l(\omega,\ k)=1+\frac{\omega_0^2\beta^2}{\sqrt{2}\,\omega^2\gamma}\left[F\left(\frac{\beta+\gamma}{\sqrt{2}}\right)-F\left(\frac{\beta-\gamma}{\sqrt{2}}\right)\right], \tag{6a}$$

$$\operatorname{Im}\varepsilon^l(\omega,\ l)=\frac{2\pi\omega_0^2m^2}{\hbar k^3\sqrt{2\pi mT}}\operatorname{sh}\frac{\hbar\omega}{2T}\exp\left[-\frac{1}{2mT}\left(\frac{\hbar^2k^2}{4}+\frac{m^2\omega^2}{k^2}\right)\right]. \tag{6b}$$

The quantity $\operatorname{Im}\varepsilon^l(\omega, k)$ thus approaches zero exponentially, in both the limit $k\to 0$ and the limit $k\to\infty$. This is true in the case $k\to\infty$ because energy and momentum conservation in the absorption of a photon $(\omega, \mathbf{k})$, $\hbar\omega+\mathscr{E}=\mathscr{E}'$ and $\hbar\mathbf{k}+\mathbf{p}=\mathbf{p}'$ ($\mathscr{E}$ and $\mathbf{p}$ are the electron energy and momentum), simply leads to the result that the transitions occur from states with momenta $\mathbf{p}=\hbar\mathbf{k}/2$ to states with momenta $\mathbf{p}'=-\hbar\mathbf{k}/2$, whose populations are exponentially small at large values of k. Substituting (6b) into (2) and carrying out the integration with $\mathbf{r}_1=\mathbf{r}_2$ and $\omega\gg\omega_0$, we find

$$W_\omega^l=\frac{\omega_0^2m^2\theta}{\pi(2\pi)^{3/2}(mT)^{1/2}}\frac{\operatorname{sh}(\hbar\omega/2T)}{\hbar\omega}K_0\left(\frac{\hbar\omega}{2T}\right), \tag{7}$$

where $K_0(x)$ is the modified Hankel function.

We see that the incorporation of quantum effects leads to a finite value for the spectral energy distribution of the fluctuations of the Coulomb field in a plasma. Comparing Eqs. (7) and (4), we see that the role of a cutoff parameter in the equation for W_ω^l in the case $\hbar\omega\ll T$ (in the classical frequency region) is played by the de Bro-

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[1)] If $\omega\lesssim\omega_0$, we cannot assume $|\varepsilon|=1$ in calculating the correlation function. If $\omega\ll\omega_0$, by using $\varepsilon\approx 1+(\omega_0^2/k^2v_T^2)$, we find

$$\langle E_i^l(\mathbf{r}_1)\, E_i^l(\mathbf{r}_2)\rangle_\omega\approx\frac{8e^2N}{\sqrt{\pi}\,v_T}\ln\frac{v_T}{\omega_0R}.$$

[2)] The values given in Ref. 7 for the dipole resistance R due to the shot effect contain an incorrect numerical factor. The correct expression is

$$R=\frac{2}{3\pi^{1/2}}\frac{\omega_0^2}{\omega^2}\frac{1}{\omega L}\left(\frac{\omega L}{v_T}\right)^3\ln\frac{v_T}{\omega L}$$

[see Eq. (6.7) in Ref. 7].

glie wavelength $\lambda_T = 2\pi\hbar(2\pi mT)^{-1/2}$ for electrons at the thermal velocity v_T:

$$W_\omega^l \sim \ln\frac{T}{\hbar\omega} \sim \ln\frac{v_T}{\omega\lambda_T}. \quad (8)$$

Interestingly, in the quantum region ($\hbar\omega \gg T$) the value of W_ω^l is independent of the temperature. It is thus governed by quantum fluctuations of the electrons in this region; we will return to this point below.

b) Integration of Eq. (7) over the frequency leads to an infinite total energy for Coulomb field, but this is not surprising: it is a well-known fact that quantum effects cannot save us from a divergence due to the point nature of the electrons. Nevertheless, it is interesting to see that the divergence of the total energy is a consequence of the integration of the energy of the zero-point vibrations of the field, which are represented by the first term of the function

$$\theta(\omega, T) = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{\exp(\hbar\omega/T)-1}.$$

The energy of the Coulomb field associated with the zero-point vibrations is

$$W_0^l = \frac{\hbar}{2(2\pi)^4}\int\int d\omega\, d\mathbf{k}\,\frac{\operatorname{Im}\varepsilon^l(\omega, k)}{|\varepsilon^l(\omega, k)|^2}. \quad (9)$$

Substituting into (9) the quantum equation (6b), and approximating $|\varepsilon^l(\omega, k)|$ by unity in the denominator of the integrand (this simplification can have no effect on the nature of the convergence at large ω and k), we find

$$W_0^l = \frac{1}{2}\frac{4\pi e^2 N}{(2\pi)^3}\int\frac{d\mathbf{k}}{k^2}\Phi\left(\frac{k\lambda_T}{4\sqrt{\pi}}\right), \quad (10)$$

where

$$\Phi(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$

is the probability integral. We rewrite Eq. (10) in the form

$$W_0^l = \frac{1}{2}\frac{4\pi e^2 N}{(2\pi)^3}\int\frac{d\mathbf{k}}{k^2} - \frac{1}{2}\frac{4\pi e^2 N}{(2\pi)^3}\int\frac{d\mathbf{k}}{k^2}\left[1-\Phi\left(\frac{k\lambda_T}{4\sqrt{\pi}}\right)\right], \quad (11)$$

where the second integral converges, while the first is the spectral representation of the infinite self-energy of the electrons (cf. Refs. 3 and 8 and the discussion below). On the other hand, an integration of the second term in the function θ leads to a finite result:

$$W_T^l = \frac{4\pi e^2 N}{\pi\lambda_T}. \quad (12)$$

The energy W_T^l is the energy of the thermal Coulomb field without the energy of the zero-point vibrations.

When quantum effects are taken into account, the divergence of the total energy of the longitudinal field in an electron gas is thus manifested by the zero-point vibrations of the electric field. This circumstance means that the high-frequency behavior of the spectral energy distribution of the electric field is governed by the uncertainty relation. This point will be shown directly later on in this paper.

c) One of the most important quantities governing the thermodynamic properties of the electron gas is the energy of the Coulomb interaction of the charges:

$$W_{\text{int}}^l = \left\langle\frac{1}{2}\sum\frac{e^2}{|\mathbf{r}_i-\mathbf{r}_j|}\right\rangle, \quad (13)$$

where $\mathbf{r}_i$ and $\mathbf{r}_j$ are the radius vectors to the positions of the two electrons, and the angle brackets denote an averaging over the equilibrium electron distribution. This expression can be converted to the form (see, for example, Ref. 8)

$$W_{\text{int}}^l = \int\int W^l(\omega, \mathbf{k})\,d\omega\,d\mathbf{k} - \frac{1}{2}\frac{4\pi e^2 N}{(2\pi)^3}\int\frac{d\mathbf{k}}{k^2}. \quad (14)$$

As mentioned earlier, the last term in (14) is the infinite self-energy of the "bare" point charges. Clearly, in the classical region for a nondegenerate electron gas the expression in (14) should yield the well-known classical equation for the energy of the Coulomb interaction. As we saw above, however, when quantum effects are taken into account the integral $\int W_\omega^l\,d\omega$ diverges in the classical region only when zero-point vibrations are taken into account in the Planck equation. Then to find the classical expression for the Coulomb-interaction energy from (14) with the aid of the quantum expression for the dielectric permittivity (6), we must use the complete expression for the Planck function. Finally, in order to obtain a finite interaction energy W_{int}^l we must take into account the difference between $\varepsilon^l(\omega, k)$ and unity. These circumstances make it difficult to find W_{int}^l from the quantum equations. Nevertheless, such calculations can be carried out, and they are of definite interest.

We consider the quantity

$$M = \int_{-\infty}^{\infty}\operatorname{Im}\frac{1}{\varepsilon^l(\omega, k)}\operatorname{cth}\frac{\hbar\omega}{2T}\,d\omega. \quad (15)$$

To calculate it, we examine the integral

$$\oint\frac{d\omega}{\varepsilon^l(\omega, k)}\operatorname{cth}\frac{\hbar\omega}{2T}$$

over a closed contour around the upper half-plane of the complex variable ω, excluding the point $\omega = 0$. Within this contour, $\varepsilon^l(\omega, k)$ has no zeros and no poles. The integral along a semicircle of infinite radius, $|\omega| \to \infty$, is easily evaluated since $\varepsilon^l(\omega, k) \to 1$. We thus find (cf. Ref. 3)

$$M = \frac{2T}{\hbar}\left\{\pi\left[\frac{1}{\varepsilon^l(0, k)}-1\right] + 2\pi\sum_{n=1}^{\infty}\left[\frac{1}{\varepsilon^l(\omega_n, k)}-1\right]\right\}, \quad (16)$$

where $\omega_n = 2\pi i n T/\hbar$ and $\operatorname{Im}\varepsilon(\omega, k) = 0$.

Let us evaluate the sum in (16):

$$\varphi_1(k, \lambda_T) = \sum_{n=1}^{\infty}\frac{\varepsilon_n - 1}{\varepsilon_n}, \qquad \varepsilon_n = \varepsilon(\omega_n, k).$$

It would seem perfectly natural that we could find a good approximation to this sum by replacing ε_n by unity in the denominator of each term. Indeed, if we seek an expansion of the integral $\int\varphi_1(k, \lambda_T)d\mathbf{k}$ (the contribution of the sum being evaluated to the interaction energy), it will be of the form $a_{-1}\lambda_T^{-1} + a_0 + a_1\lambda_T + \ldots$. It can be shown that when we use the approximate value $\varphi(k, \lambda_T) = \sum_{n=1}(\varepsilon_n - 1)$ for the function $\varphi_1(k, \lambda_T)$ we find a value for the interaction energy integral which must be of the same as the actual value within terms of first and higher orders in λ_T. The subsequent calculations are thus essentially aimed at determing the coefficients a_{-1} and a_0, which we find by evaluating the sum $\varphi(k, \lambda_T)$. To find this sum, we use the following representation for $\varepsilon^l(\omega_n, k)$ at the points ω_n:

$$\varepsilon_n - 1 = \frac{\omega_0^2 m \exp(-\gamma^2/2)}{2k^2 T} \int_{-1}^{1} e^{\gamma^2 y^2/2} (-1)^n \cos \pi n y \, dy. \quad (17)$$

This equation, which holds for $n = 0, 1, 2, \ldots$, is easily found from the general expression in (6). Summing over n, and switching the order of integration and summation, we find an integral which we can easily evaluate by using

$$\sum_{n=1}^{\infty} (-1)^n \cos nx = -\frac{1}{2} + \pi \sum_{n=-\infty}^{\infty} \delta[x + (2n+1)\pi]. \quad (18)$$

As a result we find

$$\sum_{n=1}^{\infty} [\varepsilon(\omega_n, k) - 1] = -\frac{1}{\sqrt{2}\gamma} \frac{\omega_0^2 m}{k^2 T} F\left(\frac{\gamma}{\sqrt{2}}\right) + \frac{\omega_0^2 m}{2k^2 T}, \quad (19)$$

where the function $F(x)$ is defined in (6), and the second term obviously corresponds to the self-energy of the point charges. Integrating (16) over k, and using (1), (14), and (19), we thus find the final expression for the interaction energy:

$$W_{\text{int}}^{l} = -\frac{T}{\pi^{3/2}\lambda_T d^2} \int_0^{\infty} \frac{F(x)\,dx}{x} + \frac{T}{\pi^{3/2}\lambda_T d^2} \int_0^{\infty} \frac{x^2 F(x)\,dx}{x^3 + bF(x)}$$

$$= -\frac{bT}{\pi^{3/2}\lambda_T d^2} \int_0^{\infty} \frac{F(x)\,dx}{x[x^3 + bF(x)]}. \quad (20)$$

The second term arises in the integration of the first bracketed expression with $\varepsilon^l(0, k)$ in (16). Here also $b = (\lambda_T/4\sqrt{\pi}d)^2 \ll 1$, and $d = (T/\omega_0^2 m)^{1/2}$ is the Debye length. We easily find the leading term of the asymptotic expansion of the integral in (20) for $\lambda_T/d \to 0$:

$$\int_0^{\infty} \frac{F(x)\,dx}{x[x^3 + bF(x)]} \approx \frac{\pi}{2\sqrt{b}},$$

with $b \ll 1$. As a result we find the classical expression for the interaction energy of the charges (see, for example, Ref. 3):

$$W_{\text{int}}^{l} = -\frac{T}{8\pi d^3}. \quad (21)$$

4. SPACE-TIME CORRELATION FUNCTION OF THE COULOMB FIELD

We now consider the space–time correlation function of the fluctuational longitudinal electric field:

$$B^l(\mathbf{R}, t) = \langle E_l(\mathbf{r}_1, t_1) E_l(\mathbf{r}_2, t_2) \rangle$$

$$= \frac{2\hbar}{(2\pi)^3} \int\int \operatorname{cth} \frac{\hbar\omega}{2T} \frac{\operatorname{Im} \varepsilon^l(\omega, k)}{|\varepsilon^l(\omega, k)|^2} \exp(i\omega\tau + i\mathbf{kR})\, d\omega\, d\mathbf{k}, \quad (22)$$

where $\tau = t_2 - t_1$, $\mathbf{R} = \mathbf{r}_2 - \mathbf{r}_1$, and $R = |\mathbf{R}|$. For small values of R and τ we can again ignore the difference between $\varepsilon^l(\omega, k)$ and unity in the denominator of the integrand. Substituting $\operatorname{Im}\varepsilon^l(\omega, k)$ for (6b) into (22), and integrating over ω, we find

$$B^l(R, \tau) = \frac{2\omega_0^2 m}{\pi R} \int_0^{\infty} \frac{\sin x}{x} \exp\left(-\frac{x^2\tau^2 T}{4mR^2}\right) \cos \frac{\hbar x^2 \tau}{2mR^2}\, dx. \quad (23)$$

The integral in (23) is expressed in terms of the function $\Phi(z) = 2/\sqrt{\pi} \int_0^z e^{-t^2} dt$:

$$B^l(R, \tau) = \frac{\omega_0^2 m}{2R} \left[\Phi\left(\frac{R \exp(i\varphi/2)}{v_{\text{eff}}\tau}\right) + \Phi\left(\frac{R \exp(-i\varphi/2)}{v_{\text{eff}}\tau}\right)\right], \quad (24)$$

where $\xi = \tau_0/\tau$, $\tau_0 = \hbar/T$, $v_{\text{eff}}^2 = \sqrt{v_T^4 + v_q^4}$, $\varphi = \tan^{-1}\xi$, $v_T^2 = 2T/m$, and $v_q^2 = 2\hbar/\tau m$ is the square of the quantum velocity, which is a measure of the diffusive spreading of the wave packets in free motion with an energy $\mathscr{E} \approx \hbar/\tau$. This energy is governed by the energy–time uncertainty relation. The time τ_0 actually separates the regions of classical and quantum fluctuations (as of course follows from general considerations[10]): it is equal to the time required for an electron moving at the thermal velocity to traverse a distance equal to its de Broglie wavelength.

If $\tau \gg \tau_0$ ($\xi \ll 1$), the effective velocity at which the effect of the process propagates from one point to another is $v_{\text{eff}} \sim v_T$ ($v_q \ll v_T$), and from (24) we find an expression which does not contain $\hbar$:

$$B^l(R, \tau) = \frac{\omega_0^2 m}{R} \Phi\left(\frac{R}{v_T \tau}\right). \quad (25)$$

In the opposite case, $\tau \ll \tau_0$, we have the quantum-mechanical equation

$$B(R, \tau) = \frac{\omega_0^2 m}{R}\left[S\left(\frac{R}{v_q\tau}\right) + C\left(\frac{R}{v_q\tau}\right)\right], \quad (26)$$

$$S(x) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^x \sin^2 t\, dt, \qquad C(x) = \sqrt{\frac{2}{\pi}} \int_0^x \cos^2 t\, dt.$$

A slightly different expression for $B^l(R, \tau)$ can be found by requiring that $R/v_{\text{eff}}\tau \ll 1$. In this case we find from (24)

$$B^l(R, \tau) = \frac{2\omega_0^2 m^{3/2}}{(2\pi T)^{1/2}\tau} \left(\frac{\sqrt{1+\xi^2}+1}{1+\xi^2}\right)^{1/2}; \quad (27)$$

in other words, in the case $\tau \gg R/v_{\text{eff}}$ the correlation function is independent of R.

If $\xi \ll 1$, the classical equation follows from (27):

$$B^l(R, \tau) = \frac{2\omega_0^2 m^{3/2}}{(\pi T)^{1/2}\tau}. \quad (28)$$

This equation was derived in Ref. 11. If, on the other hand, $\xi \gg 1$, then (27) leads to

$$B^l(R, \tau) = \frac{2\omega_0^2 m^{3/2}}{(2\pi\hbar\tau)^{1/2}}. \quad (29)$$

For large distances ($R \gg v_{\text{eff}}\tau$), we find from (24)

$$B^l(R, \tau) \approx \frac{m\omega_0^2}{R}. \quad (30)$$

In other words, the correlation of the Coulomb field in this case is independent of τ, $\hbar$, and T, being governed by the Coulomb field of the stationary particles.

In summary, for $R = 0$ the correlation function becomes infinite, $B^l \to \infty$ in the limit $\tau \to 0$, in both the classical case, (28), and when quantum fluctuations are taken into account. This result contradicts the assertions in Ref. 12. In the quantum case, in contrast with the classical case, the singularity $B^l(R = 0, \tau \to 0)$ is integrable. Accordingly, the quantum characteristics also affect the spectral energy distribution of the longitudinal field even in the classical frequency region. In the quantum region, the correlation function B^l is independent of the temperature. It is actually governed by the quantum fluctuations of the electron energy, $\Delta\mathscr{E}$, which are governed by the energy–time uncertainty relation ($\Delta\mathscr{E}\tau \simeq \hbar$). For this reason, the spectral energy distribution of the longitudinal field (as mentioned earlier) is again in this case independent of the temperature and is the energy of the zero-point vibrations.

5. CORRELATION FUNCTION OF THE DENSITY FLUCTUATIONS IN A NONDEGENERATE IDEAL GAS

Finally, we consider the space–time correlation function $B_\rho(R, \tau)$ of the charge-density fluctuations in a nondegenerate electron gas. Since div $\mathbf{E} = 4\pi\rho$, we have

$$B_\rho(R, \tau) = \frac{2\hbar}{(2\pi)^3 (4\pi)^2} \int\int \operatorname{cth} \frac{\hbar\omega}{2T} \frac{k^2 \operatorname{Im} \varepsilon^l(\omega, k)}{|\varepsilon^l(\omega, k)|^2} \exp(i\omega\tau + i\mathbf{kR})\, d\omega\, d\mathbf{k}. \quad (31)$$

If we approximate $|\varepsilon^l(\omega, k)|$ by unity in (31), we find the charge-density correlation function for the electron gas without taking interaction into account. Then dividing B_ρ by the square of the electron charge, we find an equation for the correlation function of the density fluctuation in an arbitrary single-component nondegenerate ideal gas:

$$G(R, \tau) = \frac{1}{e^2} B_\rho(R, \tau) = \frac{N}{\pi^{3/2}\tau^3 v_{\text{eff}}^3} \exp\left(-\frac{R^2}{\sqrt{1+\xi^2}\,\tau^2 v_{\text{eff}}^2}\right) \cos\left(\frac{3}{2}\varphi - \frac{R^2\xi}{\sqrt{1+\xi^2}\,\tau^2 v_{\text{eff}}^2}\right). \quad (32)$$

This equation can of course be derived by a more direct method.[10,13]

In the case $\tau \gg \tau_0$, this equation converts into the familiar classical expression for the density fluctuations in an ideal gas[3]:

$$G(R, \tau) = \frac{N}{\sqrt{2}\pi^{3/2}(v_T\tau)^3} \exp\left(-\frac{R^2}{v_T^2\tau^2}\right). \quad (33)$$

It is easy to see the physical meaning of this equation by noting that the quantity $G(R, \tau)$ is proportional to the probability that the point R is reached by particles which emerge from the point $R = 0$ with a Maxwellian velocity distribution.

In the case $\tau \ll \tau_0$, we find the following quantum expression from (32):

$$G(R, \tau) = \frac{N}{2\pi^{3/2}(v_q\tau)^3} \exp\left(-\frac{2\pi R^2}{\lambda_T^2}\right)\left(\sin\frac{R^2}{v_q^2\tau^2} - \cos\frac{R^2}{v_q^2\tau^2}\right). \quad (34)$$

Hence, in the limit $\tau \to 0$, using the representation

$$\delta(\mathbf{R}) = \lim_{\alpha\to 0} \frac{2}{(2\pi\alpha)^{3/2}}\left(\sin\frac{R^2}{\alpha} - \cos\frac{R^2}{\alpha}\right) \quad (35)$$

for the three-dimensional δ-function, we find a δ-function correlation function for the simultaneous density fluctuations in a nondegenerate ideal gas.[10]

If $\tau \neq 0$, the correlation function in (34) is governed by the quantum diffusive expansion: the spreading of the wave packets from the point $R = 0$ to the point R at the velocity $v_q = \sqrt{2\hbar/\tau m}$, related to the uncertainty relation. The factor

$$\sin\frac{R^2}{v_q^2\tau^2} - \cos\frac{R^2}{v_q^2\tau^2}$$

determines the quantum interference effects during this spreading. The gas temperature enters (34) only through the thermal de Broglie wavelength λ_T. The correlation function decays at $R \gg \lambda_T$ because the gas particles—wave packets of dimensions $L \approx \lambda_T$—do not manage to undergo any significant spatial displacement during the time $\tau \ll \tau_0 = \lambda_T/v_T$.

Finally, we note that for a single particle (for one de Broglie wave) with a definite momentum $\mathbf{p}_0$ the space–time correlation function of the density fluctuations is

$$G^{(1)}(R, \tau) = \frac{1}{V_0} \frac{e^{3i\pi/4}}{\pi^{3/2}} \frac{\exp(-iR^2/v_q^2\tau^2) \exp[-i(\mathbf{p}_0\mathbf{R}/\hbar) - i(p_0^2\tau/2\hbar m)]}{v_q^3\tau^3} + \text{c.c.}, \quad (36)$$

where V_0, the volume in which the motion occurs, is assumed to be quite large. This equation is easily derived by a direct calculation, by considering the correlation-function operator $\frac{1}{2}[\hat{n}(\mathbf{x}, \mathbf{r}, 0)\,\hat{n}(\mathbf{x}, \mathbf{r}+\mathbf{R}, \tau) + \hat{n}(\mathbf{x}, \mathbf{r}+\mathbf{R}, \tau)\,\hat{n}(\mathbf{x}, \mathbf{r}, 0)]$, where $\hat{n}(\mathbf{x}, \mathbf{r}, t)$ is the operator representing the particle number density at the point with radius vector $\mathbf{r}$ at time t.

Averaging (36) using a Maxwell–Boltzmann distribution, we find Eq. (32) for a system of particles with an average number density N.

6. CONCLUSION

Let us review the basic point of this paper.

1. In the case of a purely classical calculation, the correlation function for the longitudinal (Coulomb) field is $\langle E_i^l(\omega, \mathbf{r}_1)\, E_i^l(\omega, \mathbf{r}_2)\rangle \sim \ln(v_T/\omega R)$, and it diverges in the limit $R \to 0$.

2. When the spectral energy distribution W_ω^l is measured by a classical "measuring instrument" of dimensions L, the measured quantity turns out to be finite, but the value found from the measurement varies with the size of the "measuring instrument."

3. A quantum calculation of the spectral energy distribution in the classical frequency region ($\hbar\omega \ll T$) leads to a finite value, $W_\omega^l \sim \ln(T/\hbar\omega)$.

4. The total energy of the Coulomb field, $W^l = \int W_\omega^l d\omega$ turns out to be infinite because of the point nature of the electron. The result of interest, however, is that this infinity in the total energy is due to the integration over the frequency of only the zero-point vibrations in the Planck function; that is, it is due to the quantum fluctuations.

5. The fluctuations in the Coulomb field are governed completely by the fluctuations in the electron density. The distinction between classical and quantum fluctuations is clearly seen in the space–time correlation function for the density fluctuations of an ideal gas, given above.

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