Generalized coherent states and some of their applications

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The review is devoted to an analysis of definite overcomplete non-orthogonal state systems that are connected with irreducible representations of Lie groups—the so called systems of generalized coherent states. These systems, which the author is the first to propose, are generalizations of Glauber's coherent-state system and arise in natural fashion in physical problems that have dynamic symmetry. They permit a considerable simplification of the solution of the quantum problem by reducing it to a simpler "classical" problem. The review deals with the properties of generalized-coherent-state systems connected with the simplest Lie groups.

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INTRODUCTION

In theoretical physics one usually uses complete sets of orthornormal states in the Hilbert space. For example, in quantum electrodynamics the calculations are based on the stationary states $|n\rangle$ of the field Hamiltonian $\mathscr{H}$. These states correspond to the presence of an integral number n of field quanta, i.e., they satisfy the equation

$$\mathscr{H}|n\rangle = \hbar\omega\left(n+\frac{1}{2}\right)|n\rangle.$$

These states form a complete orthonormal set, which is usually used as a basis for expanding all the field states. Since all electrodynamic calculations are actually based on expanding in powers of the field strengths, the number of photons involved in the calculations is usually very small.

On the other hand, in the classical limit of quantum electrodynamics the quantum numbers are not only very large, but also very indeterminate. For example, if a harmonic oscillator vibrates in a state with a relatively well determined phase one must attribute to it a large quantum number n, which is not precisely determined ($\Delta n \Delta\varphi \geqslant 1$). The coherent quantum states of the electromagnetic field, i.e., states in which the phase of the field is precisely determined, are states in which the occupation number n is in principle undetermined. In such cases it becomes very laborious to calculate expectation values with the aid of n-quantum states.

States with an undetermined number of photons, which arise naturally in treating the correlation and coherence properties of the field, are called coherent states.[1)]

Such a state is characterized by a complex number $\alpha = |\alpha|\exp(i\varphi)$ and its expansion in n-quantum states has the form

$$|\alpha\rangle = e^{-(1/2)|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{I. 1}$$

A coherent state $|\alpha\rangle$ describes a nonspreading wave packet for an oscillator, the quantity $|\alpha|$ specifying the amplitude of the oscillations, and φ, their phase.

An unusual property of this set is that the coherent states are not orthogonal to one another; moreover, the set of coherent states is overcomplete, i.e., it contains more states than are necessary for the expansion of an arbitrary state. Hence the standard methods cannot be used when working with such states; however, it is possible to develop a suitable apparatus that makes it possible to expand an arbitrary state in coherent states and to use such states to describe operators—the density matrix, for example.

The set of coherent states has a number of remarkable properties. For example, by expanding the field in such states we can easily pass to the classical limit while always remaining within the quantum region.

This is due to the fact that the coherent states minimize the Heisenberg uncertainty relation $\Delta p\Delta q \geqslant \hbar/2$ (for these states, $\Delta p \Delta q = \hbar/2$) and are therefore the quantum states whose properties are most like to those of classical states. Coherent states are closely associated with the boson field and therefore with the boson creation and destruction operators a^+ and a. We recall that these operators, together with the unit operator I, satisfy the well known commutation relations

1)A set of coherent states was first used by Schrödinger[1] in 1926 to describe nonspreading wave packets of an oscillator. The concept of coherent state was introduced by Glauber,[2] who showed that the use of a set of coherent states makes it possible to give an adequate quantum description of a coherent beam of laser light. The properties of this set of states is treated in detail in Refs. 3–5, where there will also be found references to many original papers touching on this problem.

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$$[a, a^+] = I, \quad [a, I] = [a^+, I] = 0 \tag{I.2}$$

and therefore generate a three-parameter Lie algebra. The group W_1 corresponding to the Lie algebra is the group of transformations of the form $T(g) = \exp(it) \cdot \exp(\alpha a^+ - \bar{\alpha} a)$, $g = (t, \alpha)$ and is called the Heisenberg-Weyl group. *Then the coherent state* $|\alpha\rangle$ *is obtained* by the action of the operator $T(g)$ on the vacuum state $|0\rangle$[2]:

$$T(g)|0\rangle = e^{it}|\alpha\rangle, \quad |\alpha\rangle = e^{\alpha a^+ - \bar{\alpha} a}|0\rangle. \tag{I.3}$$

The method of coherent states accordingly works especially well when the Heisenberg-Weyl group is a dynamical symmetry group of the Hamiltonian concerned. Such, for example, is the problem of a quantum oscillator acted on by a variable external force. In this case the Heisenberg equations of motion coincide with the corresponding equations for the classical quantities. Then the coherent state remains coherent as the system evolves, while the motion on the α plane of the point corresponding to the coherent state is described by the classical equations. This makes it possible greatly to simplify the quantum problem by reducing it to the simpler classical problem.

Such states are well suited to describe a system of interacting particles whose low-energy excitations are boson modes and have large occupation numbers. Such excitations behave, in a certain sense, classically. Thus, the coherent states play the part of classical fields, which describe a set of many bosons as a whole, just as the classical electromagnetic field describes the classical limit of quantum electrodynamics. Hence it is not surprising that coherent states have been widely used in the last decade, not only in quantum optics and radiophysics, but also in a number of other branches of physics—in the theory of superfluidity, for example. Such states are also used in the Heisenberg model of ferromagnetism to describe spin waves, in quantum electrodynamics to describe the cloud of soft photons around charged particles, and in nonlinear field theories to obtain an approximate quantum description of localized states (solitons).

However, the Heisenberg-Weyl group is not universal, and we frequently encounter other dynamical symmetry groups. Hence the question arises whether there exist sets of states having similar properties for other Lie groups. A positive answer to this question was given in Ref. 6, where a set of generalized coherent states was constructed and investigated for an arbitrary Lie group.[3]

The construction of a set of generalized coherent states is based on the relation

$$|\psi_g\rangle = T(g)|\psi_0\rangle, \tag{I.4}$$

where $T(g)$ is a representation of the group G, and $|\psi_0\rangle$ is a fixed vector in the vector space of the representation $T(g)$. Then $\{|\psi_g\rangle\}$ is the set of generalized coherent states.

The theory turns out to be meaningful when $T(g)$ is an irreducible unitary representation and the corresponding representation space is a Hilbert space. It is obvious that the set of coherent states is invariant under the operators $T(g)$, or in other words, that an operator $T(g)$ takes one coherent state into another. This is a characteristic property of a set of generalized coherent states defined in accordance with Ref. 6.

Since the properties of the set of ordinary coherent states have been thoroughly reviewed elsewhere,[3–5] the present review will be devoted to the discussion of sets of generalized coherent states associated with the simplest groups other than the Heisenberg-Weyl group.

Generalized coherent states have all the properties of ordinary coherent states. In some cases they are the quantum states whose properties are the closest to those of classical states, and which therefore facilitate the passage from the classical case to the quantum case in the most natural manner.

In the case of the representation $T^j(g)$ of the group of rotations of three dimensional space (j is a nonnegative integer or half-integer), for example, if we choose the state $|j, -j\rangle$ (or $|j, j\rangle$) with the minimal (maximal) projection of the angular momentum onto the z axis as the state $|\psi_0\rangle$, we obtain the set of coherent spin states first discussed by Radcliffe.[8] Such a state $|\zeta\rangle$, like an ordinary coherent state, is specified by a complex number ζ, and its expansion in the set $\{|j, \mu\rangle\}$ with a definite projection of the angular momentum onto the z axis has the form

$$|\zeta\rangle = (1 + |\zeta|^2)^{-j} \sum_{\mu=-j}^{j} \sqrt{\frac{(2j)!}{(j+\mu)!\,(j-\mu)!}}\, \zeta^{j+\mu} |j, \mu\rangle. \tag{I.5}$$

We note that the ζ plane is the stereographic projection of the two-dimensional sphere $S^2 = \{\mathbf{n} : \mathbf{n}^2 = 1\}$ and in this case plays the same role as the α plane for the oscillator, i.e., it is the spin phase space.

Using this set, Lieb[50] estimated the partition function for a quantum spin system. Such states have also been used[27,51,52] in the so-called Dicke model[53] for the interaction of radiation with matter consisting of two-level molecules. It is just these states that can be used to describe the phase transition from an ordinary state to a superradiance state,[54] in which the intensity of the radiation is proportional to the square of the number of molecules in the system.

As was shown in Ref. 55, the generating function for the Clebsch-Gordan coefficients for the three-dimensional rotation group can be derived very simply with the aid of the coherent spin states.

In this review the coherent-state method is illustrated

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[2] That is what one calls the vector that satisfies the condition $a|0\rangle = 0$.

[3] An earlier attempt was made to generalize the concept of coherent state in a different manner,[7] but the proposed method is not applicable to all Lie groups and, in particular, it is not suited for compact groups. Moreover, the set of states constructed in Ref. 7, unlike that constructed in Ref. 6, is not invariant under the operators of a representation of the group. The special case of the group of rotations of three-dimensional space was treated in Ref. 8; the set of states constructed there is the same as the corresponding set of coherent states constructed in Ref. 6.

by two examples: 1) the motion of spin in a varying magnetic field, and 2) the relaxation to thermodynamic equilibrium of a particle with spin in a magnetic field. In the second example, the coherent spin states make it possible to reduce the equation for the density matrix to the Fokker-Planck equation on the two-dimensional sphere $S^2=\{\mathbf{n}\cdot\mathbf{n}^2=1\}$.

In Secs. A and B of Chap. 1 of the review we consider the set of coherent states for the three-dimensional Lorentz group—the group $SO(2,1)$, which as is known, is isomorphic to the group $SU(1,1)$ of second-order matrices that leave the form $|z_1|^2-|z_2|^2$ invariant. This group has several series of irreducible unitary representations: the principal, discrete, and supplementary series. It is accordingly possible to construct several sets of coherent states associated with this group.

In the review we shall consider only sets of coherent states associated with representations of the discrete series and shall give especial attention to the sets of coherent states associated with those discrete-series representations that can be realized with the aid of boson creation and destruction operators. Such representations are specified by a number k, $0<k<\infty$, and the corresponding coherent states $|\zeta\rangle$ are specified by a complex number ζ ($|\zeta|<1$). The expansion of a coherent state in the orthonormal basis $\{|k,\mu\rangle : \mu=k+m,\ m=0,1,2,\ldots\}$ has the form

$$|\zeta\rangle=(1-|\zeta|^2)^k\sum_{m=0}^{\infty}\sqrt{\frac{\Gamma(m+2k)}{m!\,\Gamma(2k)}}\,\zeta^m\,|k,k+m\rangle. \tag{I.6}$$

In this case a coherent state is specified by a point on the unit sphere, which is the phase space for the problem and can be treated as a Lobachevskii plane.

We note that in addition to the discrete-series representations of the three-dimensional Lorentz group and other noncompact groups (i.e., groups having an infinite invariant volume), there exist continuous (principal) series of representations. The corresponding sets of coherent states have been thoroughly investigated in Refs. 11 and 12. For the Lorentz group, these coherent states accomplish the transformation from the hyperboloid to the cone that was first discussed by Shapiro.[31] We shall not discuss these coherent states in this review.

The coherent states associated with representations of the discrete series will be used in Sec. C of Chap. 2 to solve two problems. The first problem—the parametric excitation of a quantum oscillator—has been thoroughly treated in Refs. 30 and 72–75. Its solution can be simplified by using the set of coherent states for the discrete series of $SU(1,1)$. As the second problem, we use coherent states to treat Bogolyubov's model[76] of an almost ideal Bose gas.

We note that the coherent states discussed in this section are convenient to use in solving problems in which one must find the spectrum and wave functions of a Hamiltonian that is quadratic in boson creation and destruction operators. Such coherent states arise, for example, when treating the production of pairs of spinless particles in a uniform alternating electric field or in the gravitational field of an expanding universe.[63]

It is shown in Refs. 63 and 69 that the problem of the production of pairs of spin-S particles leads to coherent states associated with the group $SU(2S+1,2S+1)$ for integral S and with the group $SU(2(2S+1))$ for half-integral S. In the exceptional case $S=1/2$, there arise coherent states associated with the group $SO(5)$.[71] We note that the problem of the production of pairs of spin-zero and spin-1/2 particles has been thoroughly treated in Refs. 63–71.

Coherent states have also proved to be useful in treating a number of mathematical problems in the theory of the representations of Lie groups. However, these problems will not be discussed in this review. We note only that a class of sets of generalized coherent states associated with a large class of representations of Lie groups has properties similar to those of the sets discussed in the first part of the review. In particular, this is the case for all representations of compact semisimple groups and for the discrete-series representations of semisimple Lie groups. The general theory of such sets of coherent states will be found in Ref. 10 for the case of the discrete series, and in Refs. 11 and 12 for the case of the fundamental (continuous) series.

1. PROPERTIES OF A SET OF GENERALIZED COHERENT STATES

A. The ordinary set of coherent states and its relation to the Heisenberg-Weyl group

In this section we discuss the properties of a definite overcomplete and nonorthogonal set of states—the set of the so-called ordinary coherent states.[4)] After establishing the relation of this set of coherent states to the Heisenberg-Weyl group[13] we derive the most important properties of this set by group-theory methods. For simplicity we shall consider only the case of one degree of freedom, the case of a finite number of degrees of freedom bringing with it only unimportant technical complications. The reader interested in applications can go to Sec. A of Chap. 2 immediately after reading this section.

[4)] In the coordinate representation, the ordinary coherent states describe nondispersing wave packets for an oscillator and were discussed from this point of view in 1926 by Schrödinger.[1] Somewhat later von Neumann in his well-known book[14] considered an important subset of coherent states associated with the division of phase space into regular cells and used it to analyze the measurement process. After a 30-year break, the properties of sets of coherent states again began to be investigated (see Refs. 15-18). We note the important paper of Glauber[2] in which the concept of coherent state was introduced and it was shown that coherent states provide an adequate apparatus for a quantum description of a coherent beam of laser light. A detailed treatment of the properties of the ordinary set of coherent states for a finite number of degrees of freedom will be found, together with references to many other papers on this subject, in Refs. 2–5. The case of an infinite number of degrees of freedom was treated in Refs. 17 and 18, and in many articles published in Communications in Mathematical Physics.

A coherent state $|\alpha\rangle$ is usually defined (see, e.g., Refs. 2–4) as an eigenstate of a boson destruction operator:

$$a|\alpha\rangle = \alpha|\alpha\rangle. \tag{1.1}$$

From this it follows that such a state exists for any complex α and that its expansion in the n-quantum states $|n\rangle = (n!)^{-1/2}(a^+)^n|0\rangle$ (which form the usual orthonormal basis; here $|0\rangle$ is the vacuum state: $a|0\rangle = 0$) has the form

$$|\alpha\rangle = e^{-(1/2)|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{1.2}$$

Thus, an ordinary coherent state is completely specified by a complex number α. Further properties of the set of coherent states can be derived from this. However, one cannot construct a set of coherent states for an arbitrary Lie group in this manner.[5)] In constructing the set of coherent states we shall therefore follow the general scheme of Ref. 6. Following this approach, we can easily pass to the construction of a set of generalized coherent states for an arbitrary Lie group, as will be evident from what follows (see Secs. B and C).

Let us begin by establishing the relation of the ordinary set of coherent states to the so-called Heisenberg-Weyl group. We recall that the boson creation and destruction operators $a^+ = (\hat{q} - i\hat{p})/\sqrt{2\hbar}$ ($\hbar$ is Planck's constant and $\hat{q}$ and $\hat{p}$ are the coordinate and momentum operators) together with the unit operator I satisfy the Heisenberg commutation relations

$$[a, a^+] = I, \quad [a, I] = [a^+, I] = 0 \tag{1.3}$$

and therefore generate a Lie algebra W_1. This is the Heisenberg-Weyl algebra.

A general element of this algebra has the form

$$tI + i(\bar{\alpha}a - \alpha a^+), \tag{1.4}$$

where t is a real number and α is a complex number. From this it follows that the operators

$$T(t, \alpha) = e^{it}D(\alpha), \quad D(\alpha) = e^{\alpha a^+ - \bar{\alpha}a} \tag{1.5}$$

form a group. To find the multiplication law for the operators $D(\alpha)$ we use the well-known identity (see, e.g., Ref. 3)

$$e^Ae^B = e^{(1/2)[A,B]}e^{A+B}, \tag{1.6}$$

which is valid when the conditions

$$[A[A, B]] = [B[A, B]] = 0. \tag{1.7}$$

are satisfied. From this we obtain

$$D(\alpha)D(\beta) = e^{i\,\mathrm{Im}(\alpha\bar{\beta})}D(\alpha+\beta). \tag{1.8}$$

In this way we obtain the three-parameter group W_1 that was first treated by H. Weyl[13]—the Heisenberg-Weyl group. An element g of this group is specified by a real number t and a complex number α:

$$g = (t, \alpha), \tag{1.9}$$

the multiplication law for W_1 having the form

$$(s, \alpha)(t, \beta) = (s + t + \mathrm{Im}(\alpha\bar{\beta}), \alpha + \beta). \tag{1.10}$$

The operators $T(g) = T(t, \alpha)$ act in the Hilbert space $\mathscr{H}$ and form an irreducible unitary representation of the group W_1. According to von Neumann's theorem[19] any two irreducible unitary representations $T_1(g)$ and $T_2(g)$ of W_1 that satisfy the condition $T_1(t, 0) = T_2(t, 0)$ are unitarily equivalent. This means that there exists a unitary operator U such that $T_1(g) = UT_2(g)U^+$. In other words, the group W_1 essentially admits just one irreducible unitary representation.

Now let us turn to the construction of a set of coherent states associated with the group W_1. We select any fixed vector $|\psi_0\rangle$ in the Hilbert space $\mathscr{H}$. Acting on this vector with the operators $T(g)$, we obtain a set $\{|\alpha\rangle\}$ of states

$$|\alpha\rangle = D(\alpha)|\psi_0\rangle, \tag{1.11}$$

which constitutes a set of coherent states.[6)]

In the case of an ordinary set of coherent states ($|\psi_0\rangle = |0\rangle$) it is not difficult to obtain an expansion of a coherent state in an orthonormal basis. To do this we make use of the identity (1.6) to write the operator $D(\alpha)$ in the normal form:

$$D(\alpha) = e^{\alpha a^+ - \bar{\alpha}a} = e^{-(1/2)|\alpha|^2}e^{\alpha a^+}e^{-\bar{\alpha}a}. \tag{1.12}$$

From this we obtain the following expression for an or-

[5)]For some Lie groups one can define a set of states by formulas of the type of (1.1).[7] However, this procedure cannot be used for all Lie groups and, in particular, it is unsuitable for compact groups. In addition, the set of states discussed in Ref. 7 is not invariant under the operators of a representation of the group.

[6)]By choosing the vacuum state $|0\rangle$, i.e. the state that satisfies the condition $a|0\rangle = 0$, for $|\psi_0\rangle$ we obtain the ordinary set of coherent states. As we shall see later on, many of the properties of the set of ordinary coherent states remain valid for a general set of coherent states.

We see that a coherent state is specified by a complex number α, i.e., it depends on two parameters, while an element $g = (t, \alpha)$ of the group W_1 is determined by three parameters. This decrease in the number of parameters is associated with the fact that there exist transformations $T(h)$ that do not alter the state $|\psi_0\rangle$: $T(h)|\psi_0\rangle$. The set $\{h\}$ of the elements of W_1 that have this property obviously forms a subgroup H of W_1. We call this subgroup the stationary subgroup for the state $|\psi_0\rangle$. It is not difficult to see that in the present case the stationary subgroup for any state $|\psi_0\rangle$ consists of elements of the form $h = (t, 0)$. From this it follows that both the operators $T(g)$ and $T(gh)$ take $|\psi_0\rangle$ to the same state. But the elements of the form gh with g fixed and h ranging over the entire subgroup H form a rest class of the group G by the subgroup H (an element of the factor space $X = G/H$ of the group G by the subgroup H). Hence a coherent state is defined by a point of the factor space $X = G/H$, which in the present case is the plane of the complex variable α.

dinary coherent state:

$$|\alpha\rangle = e^{-(1/2)|\alpha|^2} e^{\alpha a^+}|0\rangle, \tag{1.13}$$

from which expansion (1.2) follows.

Now let us consider the transformation of the operators of the algebra W_1 by the elements of the group W_1. It is not difficult to show that

$$\begin{aligned} T(t, \alpha)\, a T^+(t, \alpha) &= D(\alpha)\, a D^+(\alpha) = a - \alpha I, \\ T(t, \alpha)\, a^+ T^+(t, \alpha) &= D(\alpha)\, a^+ D^+(\alpha) = a^+ - \bar{\alpha} I. \end{aligned} \tag{1.14}$$

On applying the first of relations (1.14) to the ordinary coherent state $|\alpha\rangle$, we obtain (1.1). In the present case, therefore, the definitions of coherent states via formulas (1.11) and (1.1), respectively, are equivalent.

The set of coherent states thus obtained has a number of remarkable properties. Let us look at some of them:

1) An operator $T(g) = T(t, \alpha)$ of the representation takes one coherent state into another:

$$T(t, \alpha)|\beta\rangle = e^{i\varphi}|\beta + \alpha\rangle, \quad \varphi = t + \operatorname{Im}(\alpha\bar{\beta}). \tag{1.15}$$

2) The set of coherent states is complete. This follows from the irreducibility of the representation $T(g)$. In fact, since the operator $T(g)$ takes one coherent state into another, the set of coherent states constitutes an invariant subspace of the Hilbert space $\mathcal{H}$. In view of the irreducibility of the representation, the linear cover of the vectors of this subspace must coincide with the entire Hilbert space $\mathcal{H}$, and this means that the set of coherent states is complete.

3) The coherent states of the set are not orthogonal:

$$\langle\alpha|\beta\rangle = e^{i\varphi}\langle\psi_0|D(\beta - \alpha)|\psi_0\rangle, \quad \varphi = \operatorname{Im}(\bar{\alpha}\beta). \tag{1.16}$$

We note that the phase φ is equal to twice the area of the triangle whose vertices are the points 0, α, and β:

$$\varphi = 2A(0, \alpha, \beta). \tag{1.17}$$

Formula (1.16) simplifies for a set of ordinary coherent states:

$$\begin{aligned} \langle\alpha|\beta\rangle &= \exp\left[-\tfrac{1}{2}(|\alpha|^2 + |\beta|^2 - 2\bar{\alpha}\beta)\right], \\ |\langle\alpha|\beta\rangle|^2 &= \exp(-|\alpha - \beta|^2). \end{aligned} \tag{1.18}$$

4) We obtain an important identity—the so-called expansion of unity.[7] Let us consider the operator

$$\hat{A} = \int d^2\alpha\, |\alpha\rangle\langle\alpha|, \tag{1.19}$$

where $d^2\alpha = d\alpha_1 d\alpha_2$, $\alpha = \alpha_1 + i\alpha_2$, and $|\alpha\rangle\langle\alpha|$ is the projection operator onto the state $|\alpha\rangle$.

It is not difficult to verify that this operator commutes with all the operators $T(g)$. By Schur's lemma, this operator is therefore a multiple of the unit operator:

$$\hat{A} = c\hat{I}. \tag{1.20}$$

We obtain the following expression for the normalizing constant by averaging both sides of this equation over the state $|\beta\rangle$:

$$c = \int |\langle\alpha|\beta\rangle|^2 d^2\alpha = \int e^{-|\alpha-\beta|^2} d^2\alpha = \pi. \tag{1.21}$$

Now the expansion unity for an ordinary set of coherent states takes the form

$$\int |\alpha\rangle\langle\alpha|\, d\mu(\alpha) = \hat{I}, \quad d\mu(\alpha) = \frac{1}{\pi} d^2\alpha. \tag{1.22}$$

5) We note that the α plane is the phase space for the problem under consideration; this follows from (1.1) and the formulas $\alpha = (q + ip)\sqrt{2\hbar}$ and $a = (\hat{q} + i\hat{p})/\sqrt{2\hbar}$. Now the expansion of unity in the variables q and p takes the "quasiclassical" form ($|\alpha\rangle \equiv |p, q\rangle$):

$$\int \frac{dp\, dq}{2\pi\hbar} |p, q\rangle\langle p, q| = \hat{I}. \tag{1.22'}$$

6) It is not difficult to show (see Ref. 5, for example) that all the coherent states of a given set (unlike the states of an orthonormal set) give the same value for the quantity $\Delta p \Delta q$ in Heisenberg's uncertainty relation

$$\Delta p\, \Delta q \geqslant \frac{\hbar}{2}, \tag{1.23}$$

this value being the same as the value $(\Delta p)_0(\Delta q)_0$ given by the state $|\psi_0\rangle$:

$$(\Delta p)_\alpha (\Delta q)_\alpha = (\Delta p_0)(\Delta q)_0. \tag{1.24}$$

We note again that the ordinary coherent states minimize the Heisenberg uncertainty[8]

$$(\Delta p)_\alpha (\Delta q)_\alpha = \frac{\hbar}{2} \tag{1.25}$$

and are therefore quantum states that are as much like classical states as possible.

7) The set $\{|p, q\rangle\}$ of coherent states is overcomplete. This means that it has complete proper subsets. Let us consider the following important class of subsets. We divide the phase space (p, q) into regular cells of area S and select a coherent state at the center of each of them. We thus obtain the subset $\{|\{p, q\}_{mn}\rangle\}$ previously considered by von Neumann.[14]

The problem of the completeness of such subsets is fully solved in Refs. 20 and 21.[9] It turns out that:

a) When $S < 2\pi\hbar$ this subset is overcomplete and remains so when a finite number of states are removed;

b) when $S > 2\pi\hbar$ the subset is not complete;

c) when $S = 2\pi\hbar$ the subset is complete; it remains complete when one state is removed but becomes incomplete when any two states are removed.

These results confirm the fundamental importance of

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[7]This relation was first derived in another manner by Klauder.[15]

[8]But these are not the only states that minimize the Heisenberg uncertainty.

[9]The problem of how overcomplete the set is when $S = 2\pi\hbar$ was not discussed in Ref. 21.

the division of the phase space into Planckian cells.

8) We can expand an arbitrary state $|\psi\rangle$ in coherent states with the aid of the expansion of unity (1.22), obtaining

$$|\psi\rangle = \int d\mu(\alpha)\,\psi(\bar{\alpha})\,|\alpha\rangle, \text{ where } \psi(\bar{\alpha}) = \langle\alpha|\psi\rangle. \qquad (1.26)$$

The function $\psi(\bar{\alpha})$ may be called the symbol of the state vector $|\psi\rangle$. Then, as is not difficult to see, we have

$$\langle\psi_1|\psi_2\rangle = \int \bar{\psi}_1(\bar{\alpha})\,\psi_2(\bar{\alpha})\,d\mu(\alpha). \qquad (1.27)$$

9) For the ordinary set of coherent states, we have

$$\psi(\bar{\alpha}) = e^{-(1/2)|\alpha|^2}\tilde{\psi}(\bar{\alpha}), \qquad (1.28)$$

when $\tilde{\psi}(\bar{\alpha})$ is an entire function of the variable $\bar{\alpha}$. Equation (1.27) now takes the form

$$\langle\psi_1|\psi_2\rangle = \int \bar{\tilde{\psi}}_1(\bar{\alpha})\,\tilde{\psi}_2(\bar{\alpha})\,e^{-|\alpha|^2}\,d\mu(\alpha). \qquad (1.27')$$

This representation of the Hilbert space is called the Fock-Bargmann representation.[22,16]

Now we present an expression for the symbols of the usual orthonormal basis $|n\rangle = (n!)^{-1/2}(a^+)^n|0\rangle$. It is not difficult to see that

$$\tilde{\psi}_n(\bar{\alpha}) = \frac{\bar{\alpha}^n}{\sqrt{n!}}, \qquad (1.29)$$

i.e., we obtain a much simpler expression for $\tilde{\psi}_n(\bar{\alpha})$ than the expression $\psi_n(q) = (\pi\hbar)^{-1\cdot 4}(2^n n!)^{-1/2}H_n(q/\sqrt{\hbar})$ in the coordinate representation.

10) The coherent states are also suitable for describing operators. With the aid of these states we can obtain for each operator a certain function that completely determines it. We shall call this function the symbol of the operator.

Let $\hat{A}$ be an operator. With this operator we associate the functions

$$A(\bar{\alpha}, \beta) = \langle\alpha|\hat{A}|\beta\rangle \qquad (1.30)$$

and

$$Q_A(\alpha) = A(\bar{\alpha}, \alpha) = \langle\alpha|\hat{A}|\alpha\rangle. \qquad (1.31)$$

The function (1.30) fully determines the operator $\hat{A}$. In fact, it follows from the expansion of unity that

$$\hat{A} = \int A(\bar{\alpha}, \beta)|\alpha\rangle\langle\beta|\,d\mu(\alpha)\,d\mu(\beta). \qquad (1.32)$$

The result φ of operating with $\hat{A}$ on the state vector $|\psi\rangle$ ($\hat{A}|\psi\rangle = |\varphi\rangle$) is given by the formula

$$\varphi(\bar{\alpha}) = \int A(\bar{\alpha}, \beta)\,\psi(\bar{\beta})\,d\mu(\beta). \qquad (1.33)$$

In the case of the ordinary set of coherent states it can be shown (see Ref. 5 for example) that even the function $Q_A(\alpha)$ fully determines the operator $\hat{A}$. We note that in some cases the operator $\hat{A}$ can also be expressed in the form

$$\hat{A} = \int P_A(\bar{\alpha}, \alpha)\,|\alpha\rangle\langle\alpha|\,d\mu(\alpha). \qquad (1.34)$$

Such a representation of an operator was discussed earlier by Glauber[2] and Sudarshan[24] and was later investigated in detail by Berezin.[23] Following Berezin we shall call the functions $Q_A(\alpha)$ and $P_A(\alpha)$ the covariant and contravariant symbols, respectively, of the operator $\hat{A}$. These two symbols are related by the formula

$$Q_A(\alpha) = \int |\langle\alpha|\beta\rangle|^2 P_A(\beta)\,d\mu(\beta), \qquad (1.35)$$

which takes the following simpler form for the ordinary set of coherent states:

$$Q_A(\alpha) = \int e^{-|\alpha-\beta|^2}P_A(\beta)\,d\mu(\beta). \qquad (1.36)$$

This integral equation has a smoothing kernel; hence the function $Q_A(\alpha)$ is determined if $P_A(\alpha)$ is given, but the converse is not always true: there may exist operators that have Q representations but not P representations.

Now let us establish the relation between the P and Q symbols of an operator and its so-called normal and antinormal forms. We recall that the normal or Wick form of an operator is an expansion of the operator in creation and destruction operators of the form

$$\hat{A} = \sum A_{mn}\,(a^+)^m a^n, \qquad (1.37)$$

in which the creation operators $(a^+)^m$ are written to the left of the destruction operators a^n. It is not difficult to see that

$$Q_A(\bar{\alpha}, \alpha) = \langle\alpha|A|\alpha\rangle = \sum A_{mn}\bar{\alpha}^m\alpha^n. \qquad (1.38)$$

Thus, the expansion of the function $Q_A(\bar{\alpha}, \alpha)$ in the powers of $\bar{\alpha}$ and α yields the coefficients A_{mn} in the normal form of the operator $\hat{A}$.

Similarly, the expansion of the function $P_A(\bar{\alpha}, \alpha)$ in powers of α and $\bar{\alpha}$,

$$P_A(\bar{\alpha}, \alpha) = \sum A_{mn}^{(1)}\alpha^m\bar{\alpha}^n \qquad (1.39)$$

yields the coefficients $A_{mn}^{(1)}$ in the antinormal (anti-Wick) form of the operator $\hat{A}$:

$$\hat{A} = \sum A_{mn}^{(1)}a^m(a^+)^n. \qquad (1.40)$$

It is not difficult also to obtain the following very useful formulas for the trace (Spur) of the operator $\hat{A}$ from relations (1.31) and (1.34):

$$\mathrm{Sp}\,\hat{A} = \int P_A(\alpha)\,d\mu(\alpha) = \int Q_A(\alpha)\,d\mu(\alpha). \qquad (1.41)$$

The trace of an operator is thus expressed as an integral of its symbols over the entire phase space.

As an example of the use of formula (1.41), let us calculate the trace of the operator $D(\gamma)$. It is not difficult to see that $D(\gamma)$ has the symbols

$$P(\alpha) = e^{(1/2)|\gamma|^2}e^{\gamma\bar{\alpha}-\bar{\gamma}\alpha}, \quad Q(\alpha) = e^{-(1/2)|\gamma|^2}e^{\gamma\bar{\alpha}-\bar{\gamma}\alpha}. \qquad (1.42)$$

From this we obtain

$$\operatorname{Sp} D(\gamma)=e^{-(1/2)|\gamma|^2}\int e^{2i\operatorname{Im}(\gamma\bar{\alpha})}d\mu(\alpha)=\pi\delta^2(\gamma)=\pi\delta(\gamma_1)\delta(\gamma_2),\quad \gamma=\gamma_1+i\gamma_2. \tag{1.43}$$

In particular, the following important relations follow from (1.43):

$$\frac{1}{\pi}\operatorname{Sp}[D(\alpha)D^{-1}(\beta)]=\delta^2(\alpha-\beta), \tag{1.44}$$

$$\hat{F}=\int d\mu(\eta)\operatorname{Sp}(D(\eta)\hat{F})D^{-1}(\eta). \tag{1.45}$$

It is also easy, using the symbols, to obtain an explicit expression for the matrix elements of the operator $D(\gamma)$. Indeed, we obtain the following expression for the function G from Eq. (1.42):

$$G(\bar{\alpha},\beta;\gamma)=e^{(|\alpha|^2+|\beta|^2)/2}\langle\alpha|D(\gamma)|\beta\rangle=e^{-(1/2)|\gamma|^2}e^{\bar{\alpha}\beta+\bar{\alpha}\gamma-\beta\bar{\gamma}}. \tag{1.46}$$

On the other hand, this expression is equal to

$$G(\bar{\alpha},\beta;\gamma)=\sum_{m,n}\frac{\bar{\alpha}^m}{\sqrt{m!}}\frac{\beta^n}{\sqrt{n!}}D_{mn}(\gamma). \tag{1.47}$$

Thus, the function $G(\bar{\alpha},\beta;\gamma)$ is a generating function for the matrix elements of $D(\gamma)$. It remains to expand expression (1.46) in a power series in $\bar{\alpha}$ and β. This leads to Schwinger's formula[25] [10)]

$$D_{mn}(\gamma)=\begin{cases}\sqrt{\frac{m!}{n!}}e^{-|\gamma|^2/2}(-\bar{\gamma})^{n-m}L_m^{(n-m)}(|\gamma|^2), & n\geqslant m,\\ \sqrt{\frac{n!}{m!}}e^{-|\gamma|^2/2}(\gamma)^{m-n}L_n^{(m-n)}(|\gamma|^2), & n\leqslant m,\end{cases} \tag{1.48}$$

where the $L_n^{(k)}(x)$ are Laguerre polynomials.

We note that formula (1.48) was obtained earlier, but in a somewhat different form, by Feynman.[26]

B. Coherent states for the group of rotations of three-dimensional space (coherent spin states)

The set of coherent states for the group of rotations of three dimensional space[11)]—the group $SO(3)$—was first treated by Radcliffe.[8] The properties of the set of such states were investigated in Refs. 8, 6, and 27. In this section we shall follow the general scheme of Ref. 6 (cf. the preceding section). The applications of coherent spin states will be discussed in Sec. B of Chap. 2.

First we recall a few well-known facts.

Let us consider a particle of spin j. Then the states $|j,\mu\rangle$ with a definite projection μ $(-j\leqslant\mu\leqslant j)$ of the spin onto the x_3 axis form a basis in the space of the irreducible unitary representation $T^j(g)$ of the three-dimensional rotation group $SO(3)$. The infinitesimal operators $J_\pm=J_1\pm iJ_2$ and $J_0=J_3$ of the $T^j(g)$ representation satisfy the standard commutation relations

$$[J_0,J_\pm]=\pm J_\pm,\quad [J_-,J_+]=-2J_0. \tag{1.49}$$

In accordance with Ref. 6, we obtain the desired set of coherent states by operating on a fixed vector $|\psi_0\rangle$, which we shall take as the vector $|j,-j\rangle$, with the operators of $T^j(g)$.

As is well known, an operator of $T^j(g)$ can be expressed in the form $T^j(g)=\exp(-i\varphi J_3)\exp(-i\theta J_2)\times\exp(-i\psi J_3)$. Then it follows that a coherent spin state is specified by a unit vector $\mathbf{n}=(\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)$[12)]:

$$|\mathbf{n}\rangle=e^{i\alpha(\mathbf{n})}e^{-i\varphi J_3}e^{-i\theta J_2}|\psi_0\rangle. \tag{1.50}$$

It is convenient to choose the phase factor $\exp[i\alpha(\mathbf{n})]$ so that we have

$$|\mathbf{n}\rangle=D(\mathbf{n})|\psi_0\rangle, \tag{1.51}$$

where

$$D(\mathbf{n})=e^{i\theta(\mathbf{mJ})}, \tag{1.52}$$

$\mathbf{m}$ being a unit vector perpendicular to $\mathbf{n}$ and to $\mathbf{n}_0=(0,0,1)$, i.e., $m=(\sin\varphi,-\cos\varphi,0)$.

The operator $D(\mathbf{n})$ can also be written in another form, analogous to (1.5):

$$D(\mathbf{n})=D(\alpha)=e^{\alpha J_+-\bar{\alpha}J_-},\quad \alpha=-\frac{\theta}{2}e^{-i\varphi}. \tag{1.53}$$

Although the operators $D(\mathbf{n})$ do not form a group, their multiplication law can be expressed in the form

$$D(\mathbf{n}_1)D(\mathbf{n}_2)=D(\mathbf{n}_3)e^{i\Phi(\mathbf{n}_1,\mathbf{n}_2)J_0}. \tag{1.54}$$

It can be shown by direct calculation that the quantity Φ in (1.54) is equal to the area $A(\mathbf{n}_0,\mathbf{n}_1,\mathbf{n}_2)$ of the geodesic triangle with vertices at the points $\mathbf{n}_0$, $\mathbf{n}_1$, and $\mathbf{n}_2$ on the sphere:

$$\Phi(\mathbf{n}_1,\mathbf{n}_2)=A(\mathbf{n}_0,\mathbf{n}_1,\mathbf{n}_2). \tag{1.55}$$

As in Sec. A of Chap. 1, this indicates that the set of coherent states thus constructed is quasiclassical.

The operator $D(\alpha)$ can be written in the "normal" form:

$$D(\alpha)=e^{\zeta J_+}e^{\beta J_0}e^{\gamma J_-}, \tag{1.56}$$

where

$$\zeta=-\operatorname{tg}\frac{\theta}{2}e^{-i\varphi},\quad \beta=-2\ln\cos|\alpha|=\ln(1+|\zeta|^2),\quad \gamma=-\bar{\zeta}. \tag{1.57}$$

We also give the "antinormal" form of this operator:

$$D(\alpha)=e^{\gamma J_-}e^{-\beta J_0}e^{\zeta J_+}, \tag{1.58}$$

where the quantities ζ, β, and γ are defined in Eqs. (1.57).

[10)]A derivation of this formula, as well as some properties of the Laguerre polynomials, will be found in the appendix to Ref. 5.

[11)]The group of rotations of three-dimensional space is the most thoroughly studied of all compact Lie groups. It is locally isomorphic to $SU(2)$—the group of second order unitary matrices of determinant unity.

[12)]This parametrization of the set of coherent states is in accordance with the general assertion that a coherent state is determined by a point of the factor space G/H, where H is the stationary subgroup of the state $|\psi_0\rangle$; in this case $H=SO(2)$ and G/H is the two-dimensional sphere $S^2=\{\mathbf{n}:\mathbf{n}^2=1\}$.

Since the quantities ζ, β, and γ in formulas (1.56) and (1.58) are independent of j, it is sufficient to verify these formulas for the case $j=1/2$, in which $\mathbf{J}=\sigma/2$, σ_1, σ_2, and σ_3 being the Pauli matrices.

By operating on $|\psi_0\rangle$ with $D(\alpha)$ written in the form (1.56), we obtain another representation (another parametrization) of the coherent spin states (cf. formula (1.13)):

$$|\zeta\rangle=(1+|\zeta|^2)^{-j}e^{\zeta J_+}|\psi_0\rangle. \tag{1.59}$$

We note that the transformation from the variables θ and φ to the variable ζ has a simple geometric meaning: it is the stereographic projection from the south pole $\mathbf{n}=(0,0,-1)$ of the sphere onto the plane $\zeta=\xi+i\eta$ followed by a reflection in the η axis.

On expanding the exponential and using the relation

$$|j,\mu\rangle=\sqrt{\frac{(j-\mu)!}{(2j)!\,(j-\mu)!}}\,(J_+)^{j+\mu}|j,-j\rangle, \tag{1.60}$$

we obtain an expansion of a coherent state in the orthonormal basis (cf. formula (1.2)):

$$|\zeta\rangle=(1+|\zeta|^2)^{-j}\sum_\mu\sqrt{\frac{(2j)!}{(j+\mu)!\,(j-\mu)!}}\,\zeta^{j+\mu}|j,\mu\rangle, \tag{1.61}$$

or

$$\begin{gathered}|\zeta\rangle=\sum_\mu u_\mu|j,\mu\rangle,\\ u_\mu=\sqrt{\frac{(2j)!}{(j+\mu)!\,(j-\mu)!}}\left(-\sin\frac{\theta}{2}\right)^{j+\mu}\left(\cos\frac{\theta}{2}\right)^{j-\mu}e^{-i(j+\mu)\varphi}.\end{gathered} \tag{1.62}$$

We also note that a coherent spin state is an eigenstate of the operator $(\mathbf{n}\cdot\mathbf{J})$:

$$(\mathbf{nJ})\,|\,\mathbf{n}\rangle=-j\,|\,\mathbf{n}\rangle. \tag{1.63}$$

Equation (1.63) determines a coherent spin state to within a phase factor $e^{i\alpha}$; it follows from the equation $J_0|\mathbf{n}_0\rangle=-j|\mathbf{n}_0\rangle$ with $\mathbf{n}_0=(0,0,1)$ and the relation

$$D(\mathbf{n})\,J_0D^{-1}(\mathbf{n})=(\mathbf{nJ}). \tag{1.64}$$

The set of coherent spin states has all the properties of the ordinary set of coherent states (see Sec. A of Chap. 1). We list them without proof.

1) The operators $T^j(g)$ take one coherent state into another:

$$T^j(g)\,|\,\mathbf{n}\rangle=e^{i\Phi(\mathbf{n},\,g)}\,|\,\mathbf{n}_g\rangle, \tag{1.65}$$

where

$$\Phi(\mathbf{n},\,g)=jA(\mathbf{n}_0,\,\mathbf{n},\,\mathbf{n}_g). \tag{1.66}$$

2) The set of coherent spin states is complete.

3) The coherent states are not orthogonal to one another:

$$\begin{gathered}\langle\mathbf{n}_1|\mathbf{n}_2\rangle=e^{i\Phi(\mathbf{n}_1,\,\mathbf{n}_2)}\left(\frac{1+\mathbf{n}_1\mathbf{n}_2}{2}\right)^j,\\ \langle\xi|\eta\rangle=[(1+|\xi|^2)(1+|\eta|^2)]^{-j}(1+\bar{\xi}\eta)^{2j},\end{gathered} \tag{1.67}$$

where[13]

$$\Phi(\mathbf{n}_1,\,\mathbf{n}_2)=jA(\mathbf{n}_0,\,\mathbf{n}_1,\,\mathbf{n}_2). \tag{1.68}$$

4) The coherent spin states minimize the Heisenberg uncertainty: the equality sign holds in the relation

$$\langle J_1^2\rangle\langle J_2^2\rangle\geqslant\frac{1}{4}\langle J_3\rangle^2 \tag{1.69}$$

for the state $|\mathbf{n}_0\rangle$.

Correspondingly, the uncertainty relation is minimized for the state $|\mathbf{n}\rangle$

$$\langle\tilde{J}_1^2\rangle\langle\tilde{J}_2^2\rangle=\frac{1}{4}\langle\tilde{J}_3\rangle^2, \tag{1.70}$$

where

$$\tilde{J}_k=D(\mathbf{n})\,J_kD^{-1}(\mathbf{n}). \tag{1.71}$$

5) The following "expansion of unity" is valid:

$$\frac{2j+1}{4\pi}\int d\mathbf{n}\,|\,\mathbf{n}\rangle\langle\mathbf{n}\,|=\hat{I},\qquad d\mathbf{n}=\sin\theta\,d\theta\,d\varphi. \tag{1.72}$$

When the coherent states are parametrized by a point on the ζ plane we have

$$\int d\mu_j(\zeta)\,|\,\zeta\rangle\,\langle\zeta\,|\;=\hat{I}, \tag{1.73}$$

where

$$d\mu_j(\zeta)=\frac{2j+1}{\pi}\,\frac{d^2\zeta}{(1+|\zeta|^2)^2}. \tag{1.74}$$

6) Using these formulas we can expand an arbitrary state in coherent states:

$$|\psi\rangle=\sum c_\mu\,|\,j,\,\mu\rangle,\quad|\psi\rangle=\int d\mu_j(\zeta)\,\psi(\bar{\zeta})\,|\,\zeta\rangle, \tag{1.75}$$

where

$$\begin{gathered}\psi(\bar{\zeta})=\langle\zeta\,|\,\psi\rangle=(1+|\zeta|^2)^{-j}\tilde{\psi}(\bar{\zeta}),\\ \tilde{\psi}(\bar{\zeta})=\sum c_\mu\sqrt{\frac{(2j)!}{(j+\mu)!\,(j-\mu)!}}\,(\bar{\zeta})^{j+\mu};\end{gathered} \tag{1.76}$$

here $\tilde{\psi}(\bar{\zeta})$ is a polynomial of degree $m\leqslant 2j$ in $\bar{\zeta}$. On using formula (1.60) we see that $|\psi\rangle$ can also be written in the form

$$|\psi\rangle=\tilde{\psi}\left(\frac{1}{j+1-J_0}J_+\right)|\psi_0\rangle.$$

It follows from these formulas, in particular, that the following formula is valid for any function $f(\zeta)$ of the form $P_m(\zeta)/(1+|\zeta|^2)^j$, where $P_m(\zeta)$ is an arbitrary polynomial of degree $m\leqslant 2j$:

$$\int d\mu_j(\eta)\,f(\eta)\,\langle\eta\,|\,\xi\rangle=f(\xi). \tag{1.77}$$

We note that these functions $f(\xi)$ constitute the Hilbert space of states of a spin-j particle.

7) Now we give the expressions for the infinitesimal operators in the coherent-state representation:

$$\langle\zeta|J_0|\zeta\rangle=-j\frac{1-|\zeta|^2}{1+|\zeta|^2},\qquad\langle\zeta|J_+|\zeta\rangle=+2j\frac{\bar{\zeta}}{1+|\zeta|^2} \tag{1.78}$$

or

$$\langle\mathbf{n}\,|\,J\,|\,\mathbf{n}\rangle=-j\mathbf{n}. \tag{1.79}$$

[13] It is not accidental that Φ is given by formula (1.68). It is associated with the fact that the sphere $S^2=\{\mathbf{n}:\mathbf{n}^2=1\}$ can be regarded as the phase space for spin and that the coherent spin states are quasiclassical states.

8) We note that in the limit of large j the coherent spin states go over into ordinary coherent states. To see this one need only make the subsitution

$$J_+ \to (2j)^{1/2}a^+, \quad \zeta \to (2j)^{-1/2}\alpha \tag{1.80}$$

and let j tend to infinity. For example,

$$|\zeta\rangle \to \lim_{j\to\infty}\left(1+\frac{|\alpha|^2}{2j}\right)^{-j} e^{\alpha a^+}|\psi_0\rangle = |\alpha\rangle. \tag{1.81}$$

9) The coherent-state representation is suitable for representing operators—the density matrix ρ for a spin-j particle, for example. The density matrix is fully determined by either $P(\mathbf{n})$ or $Q(\mathbf{n})$ in accordance with the formulas

$$\rho = \int d\mu_j(\mathbf{n})\, P(\mathbf{n})\,|\mathbf{n}\rangle\langle\mathbf{n}|, \quad d\mu_j(\mathbf{n}) = \frac{2j+1}{4\pi}\, d\mathbf{n}, \tag{1.82}$$

$$Q(n) = \langle\mathbf{n}|\rho|\mathbf{n}\rangle, \tag{1.83}$$

and the expansions of these functions in series of spherical functions Y_{lm} contain only terms with $l \leqslant 2j$. For example,

$$P(\mathbf{n}) = \sum c_{lm} Y_{lm}(\mathbf{n}). \tag{1.84}$$

From this we obtain the expansion $\rho = \sum c_{lm}\hat{P}_{lm}$ for the density matrix, in which

$$\hat{P}_{lm} = \int d\mu_j(\mathbf{n})\, Y_{lm}(\mathbf{n})\,|\mathbf{n}\rangle\langle\mathbf{n}|. \tag{1.85}$$

On performing the integration in this formula we obtain

$$\langle j\nu'|\hat{P}_{lm}|j\nu\rangle = \sqrt{\frac{2l-1}{4\pi}}\,\langle j\nu';\, lm|j\nu\rangle\langle j-j;\, l0|j-j\rangle, \tag{1.86}$$

in which $\langle j\nu'; lm|j\nu\rangle$ is a Clebsch-Gordan coefficient.

C. Coherent states for the three-dimensional Lorentz group

The three-dimensional Lorentz group[14] $SO(2,1)$ has several series of irreducible unitary representations: the principal, discrete, and supplementary series. One can accordingly construct several sets of coherent states associated with this group.

Here we consider only sets of coherent states associated with representations of the discrete series. Of these, we shall consider in most detail the sets associated with the representations of the discrete series that can be realized with the aid of boson creation and destruction operators.

Applications of coherent states to a number of problems, for example to the parametric excitation of a quantum oscillator and to the superfluidity of an almost ideal Bose gas, will be discussed in Sec. C of Chap. 2.

First we recall some properties of $SO(2,1)$ and its representations. It will be convenient to discuss $SU(1,1)$, which is locally isomorphic to it. An element g of this group is a matrix of the form

$$g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad |\alpha|^2 - |\beta|^2 = 1. \tag{1.87}$$

$SU(1,1)$ has two so-called discrete series of representations, T^+ and T^-. It is enough to consider only one of them, T^+ for example, since all the results to be obtained transfer automatically to the other. The representations of the discrete series are infinite dimensional, but they have much in common with the finite dimensional representations of the $SU(2)$. For example, a basis vector $|m\rangle$ in the space of such a representation can be specified by an integer m that ranges from zero to infinity.

The Lie algebra of $SU(1,1)$ is generated by three operators K_1, K_2, and K_0, which have the following commutation relations:

$$[K_1, K_2] = -iK_0, \quad [K_2, K_0] = iK_1, \quad [K_0, K_1] = iK_2. \tag{1.88}$$

Here, as in the case of $SU(2)$, it is convenient to introduce the new generators

$$K_\pm = K_1 \pm iK_2, \tag{1.89}$$

which satisfy the following commutation relations:

$$[K_0, K_\pm] = \pm K_\pm, \quad [K_-, K_+] = 2K_0. \tag{1.90}$$

It is not difficult to verify that the operator

$$\hat{C}_2 = K_0^2 - K_1^2 - K_2^2 = K_0^2 - \frac{1}{2}(K_+K_- + K_-K_+) \tag{1.91}$$

is invariant (the Casimir operator), i.e., it commutes with all the operators K_i. By Schur's lemma, such an operator reduces for an irreducible representation to a multiple of the unit operator:

$$\hat{C}_2 = k(k-1)\hat{I}. \tag{1.92}$$

Thus, a representation of $SU(1,1)$ is specified by a single number k; for the discrete series this number assumes the values $k = 1,\ 3/2,\ 2, \ldots$.

An important difference between $SU(2)$ and $SU(1,1)$ must be noted: $SU(2)$ is simply connected, and $SU(1,1)$ is not. That $SU(2)$ is simply connected means that any closed path in it can be continuously deformed into a point.

On the other hand, it can be shown that a path in $SU(1,1)$ corresponding to a rotation in the x_1x_2 plane through an angle $2\pi n$, where n is an integer, cannot be continuously deformed into a point. This means that $SU(1,1)$ has infinite connectivity. It is known that for a multiply connected group G one can obtain a corresponding simply connected group $\tilde{G}$, the so-called universal covering group of G, by taking enough examples of G (these examples are called sheets) and joining them in the appropriate manner.

[14] $SO(2,1)$ is the group of "rotations" of three-dimensional pseudo-Euclidian space. It is locally isomorphic to $SU(1,1)$, to the symplectic group $Sp(2,R)$, and to the group $SL(2,R)$ of real second-order matrices with unit determinant. Detailed treatments of the representations of $SU(1,1)$ will be found in Refs. 28 and 29.

In the present case the universal covering group $\widetilde{SU(1,1)}$ contains an infinite number of sheets. The representations of this group are also specified by a number k, but now k varies continuously from zero to infinity: $0<k<\infty$.

Returning to the representations of the discrete series, we note that the basis vectors $|k,\mu\rangle$ for the space of the representation $T^k(g)$ are specified by a number μ, which is an eigenvalue of the operator K_0:

$$K_0\,|\,k,\ \mu\rangle=\mu\,|\,k,\ \mu\rangle,\quad \mu=k+m, \tag{1.93}$$

where m is a non-negative integer.

It will be convenient to choose the vector $|k,k\rangle$ as the fixed vector $|\psi_0\rangle$. In accordance with Ref. 6, we obtain the desired set of coherent states by operating on the fixed vector with the operators of $T^k(g)$.

The further constructions are analogous to those in Sec. B of Chap. 1.

As is well known, an operator of $T^k(g)$ can be represented in the form $T^k(g)=\exp(-i\varphi K_0)\exp(-i\tau K_2)\times\exp(-i\psi K_0)$. From this it follows that a coherent state is specified by a pseudo-Euclidian unit vector[15]:

$$\mathbf{n}=(\mathrm{ch}\,\tau,\ \mathrm{sh}\,\tau\cos\varphi,\ \mathrm{sh}\,\tau\sin\varphi),\quad \mathbf{n}^2=n_0^2-n_1^2-n_2^2=1,$$
$$|\,\mathbf{n}\rangle=e^{i\alpha(\mathbf{n})}e^{-i\varphi K_0}e^{-i\tau K_2}\,|\,\psi_0\rangle. \tag{1.94}$$

It is convenient to choose the phase factor $\exp[i\alpha(\mathbf{n})]$ so that

$$|\,\mathbf{n}\rangle=D\,(\mathbf{n})\,|\,\psi_0\rangle,$$

where

$$D\,(\mathbf{n})=e^{i\tau(\mathbf{mK})}, \tag{1.95}$$

in which $\mathbf{m}$ is a unit vector perpendicular to the vectors $\mathbf{n}$ and $\mathbf{n}_0=(1,0,0)$, i.e., $\mathbf{m}=(0,\sin\varphi,-\cos\varphi)$.

We give another form, analogous to (1.53), for the operator $D(\mathbf{n})$:

$$D\,(\mathbf{n})=D\,(\alpha)=e^{\alpha K_+-\bar{\alpha}K_-},\quad \alpha=-\frac{\tau}{2}e^{-i\varphi}. \tag{1.96}$$

We note that the operators $D(\mathbf{n})$ do not form a group, but their multiplication law can be written in the form

$$D\,(\mathbf{n}_1)\,D\,(\mathbf{n}_2)=D\,(\mathbf{n}_3)\,e^{i\varphi(\mathbf{n}_1,\,\mathbf{n}_2)K_0}. \tag{1.97}$$

It can be shown by direct calculation that φ is equal to the area $A(\mathbf{n}_0,\mathbf{n}_1,\mathbf{n}_2)$ of a geodesic triangle on the hyperboloid, whose vertices are at the points $\mathbf{n}_0$, $\mathbf{n}_1$, $\mathbf{n}_2$:

$$\varphi\,(\mathbf{n}_1,\ \mathbf{n}_2)=A\,(\mathbf{n}_0,\ \mathbf{n}_1,\ \mathbf{n}_2). \tag{1.98}$$

As in Sec. B of Chap. 1, this indicates that the set of coherent states so constructed is in a certain sense quasiclassical.

One can also write $D(\mathbf{n})$ in the "normal" form:

$$D\,(\mathbf{n})=e^{\zeta K_+}e^{\beta K_0}e^{\gamma K_-}, \tag{1.99}$$

where

$$\zeta=-\mathrm{th}\,\frac{\tau}{2}\,e^{-i\varphi},\quad \beta=-2\ln\mathrm{ch}\,\frac{\tau}{2}=+\ln(1-|\zeta|^2),\quad \gamma=-\bar{\zeta}. \tag{1.100}$$

We also give the "antinormal" form of this operator:

$$D\,(\alpha)=e^{\gamma K_-}e^{-\beta K_0}e^{\zeta K_+}, \tag{1.101}$$

in which the quantities ζ, β, and γ are defined by Eqs. (1.100).

We note that the quantities ζ, β, and γ occurring in formulas (1.99) and (1.101) are independent of k, so that it is sufficient to verify these formulas for the case in which $K_0=\sigma_3/2$, $K_1=i\sigma_1/2$, and $K_2=i\sigma_2/2$, where σ_1, σ_2, and σ_3 are the Pauli matrices.

By operating on $|\psi_0\rangle$ with $D(\alpha)$ written in the form (1.99) we reach a different representation (a different parametrization) of the coherent states:

$$|\,\zeta\rangle=(1-|\,\zeta\,|^2)^k\,e^{\zeta K_+}\,|\,k,\ k\rangle. \tag{1.102}$$

We note that the transformation from the variables τ and φ to the variable ζ has a simple geometric meaning: it is the stereographic projection from the south pole $\mathbf{n}=(-1,0,0)$ of the hyperboloid onto the unit sphere $\zeta=\xi+i\eta$, $|\zeta|<1$ followed by a reflection in the η axis.

On expanding the exponential and using the relation

$$|\,k,\ k+m\rangle=\sqrt{\frac{\Gamma\,(2k)}{m!\,\Gamma\,(m+2k)}}\,(K_+)^m\,|\,k,\ k\rangle, \tag{1.103}$$

we obtain the expansion of a coherent state in the orthonormal basis:

$$|\,\zeta\rangle=(1-|\,\zeta\,|^2)^k\sum_{m=0}^{\infty}\sqrt{\frac{\Gamma\,(m+2k)}{m!\,\Gamma\,(2k)}}\,\zeta^m\,|\,k,\ k+m\rangle. \tag{1.104}$$

We also note that a coherent spin state is an eigenstate of the operator

$$(\mathbf{nK})=n_0K_0-n_1K_1-n_2K_2,\quad (\mathbf{nK})\,|\,\mathbf{n}\rangle=k\,|\,\mathbf{n}\rangle. \tag{1.105}$$

Equation (1.105) follows from the equation $K_0|\mathbf{n}_0\rangle=k|\mathbf{n}_0\rangle$ ($\mathbf{n}_0=(1,0,0)$) and the relation

$$D\,(\mathbf{n})\,K_0D^{-1}\,(\mathbf{n})=(\mathbf{nK}). \tag{1.106}$$

Equation (1.105) determines the coherent states within a phase factor $e^{i\alpha}$.

The set of coherent states thus obtained has all the properties of the set of coherent spin states (see Sec. B of Chap. 1). We shall only list three of them.

1) The coherent states are not orthogonal to one another:

$$|\,\langle\mathbf{n}_1\,|\,\mathbf{n}_2\rangle\,|^2=\left[\frac{1+(\mathbf{n}_1\mathbf{n}_2)}{2}\right]^{-k}. \tag{1.107}$$

[15]This parametrization of the set of coherent states is in accordance with the general proposition that a coherent state is determined by a point of the factor space G/H, which in this case is the two-dimensional hyperboloid $H^2=\{\mathbf{n}:\mathbf{n}^2=n_0^2-n_1^2-n_2^2=1\}$.

2) For $k>1/2$ we have the expansion of unity:

$$\int d\mu_k(\zeta)|\zeta\rangle\langle\zeta|=\hat{I}, \tag{1.108}$$

where

$$d\mu_k(\zeta)=\frac{2k-1}{\pi}\frac{d^2\zeta}{(1-|\zeta|^2)^2}. \tag{1.109}$$

3) The generating function for the matrix elements of an operator of $T^k(g)$ has the form

$$G(\bar{\xi}, \eta; g)=\sum T^k_{k+m,\,k+n}(g)\,u_m(\bar{\xi})\,u_n(\eta)=(\bar{d}\bar{\xi}\eta+\beta\bar{\xi}+\bar{\beta}\eta+\bar{\alpha})^{-2k}, \tag{1.110}$$

where $u_m(\xi)=\xi^m[\Gamma(m+2k)/m!\,\Gamma(2k)]^{1/2}$.

Now let us consider the representations of $SU(1,1)$ that can be realized by means of operators that are quadratic in the boson creation and destruction operators a^+ and a.

Let us consider the following three operators:

$$K_+=\frac{1}{2}(a^+)^2,\quad K_-=\frac{1}{2}a^2,\quad K_0=\frac{1}{4}(aa^++a^+a). \tag{1.111}$$

Calculation shows that these operators satisfy the commutation relations (1.90). On calculating the Casimir operator (1.91) for the operators (1.111) we obtain

$$C_2=-\frac{3}{16}=k(k-1). \tag{1.112}$$

The second of these equations, which determines k, has two solutions: $k=1/4$ and $k=3/4$.

It is not difficult to see that the states $|n\rangle=(n!)^{-1/2}\times(a^+)^n|0\rangle$ with even n constitute a basis in the space of the irreducible unitary representation T^k with $k=1/4$ and that the corresponding states $|n\rangle$ with odd n constitute a basis for the space of the representation T^k with $k=3/4$.

The matrix elements for representations of $SU(1,1)$ can be expressed in terms of the hypergeometric function.[28] A simpler expression has been found[30] for the representations T^k ($k=1/4$ and $3/4$) considered here:

$$T^k_{mn}(\tau)=\sqrt{\frac{n_<!}{n_>!}}\left(\operatorname{ch}\frac{\tau}{2}\right)^{-1/2}P^{|m-n|/2}_{(m+n)/2}\left[\frac{1}{\operatorname{ch}(\tau/2)}\right], \tag{1.113}$$

where P^m_n is the associated Legendre function.

Now let us consider the representations of $SU(1,1)$ in the discrete series that can be realized with the aid of a pair of boson operators a_+ and a_-. Let us consider the three quadratic operators

$$K_+=a^+_+a^+_-,\quad K_-=a_+a_-,\quad K_0=\frac{1}{2}(a^+_+a_++a^+_-a_-+1). \tag{1.114}$$

Calculation shows that these operators satisfy the commutation relations (1.90). Calculating the Casimir operator (1.91) for the generators (1.111) we obtain

$$C_2=-\frac{1}{4}+\frac{1}{4}(a^+_+a_+-a^+_-a_-)^2. \tag{1.115}$$

Thus, for the states $|m,n\rangle=(m!\,n!)^{-1/2}(a^+_+)^m(a^+_-)^n|0,0\rangle$ with $m-n=n_0=$ const, we have $C_2=k(k-1)=$ const and $k=(1+|n_0|)/2$; hence the states $\{|n+n_0,n\rangle\}$ constitute a basis for the discrete-series irreducible unitary representation T^k of $SU(1,1)$, where

$$k=\frac{1}{2}(1+|n_0|). \tag{1.116}$$

The matrix elements for these representations are known, and in the simplest case, when the initial state is the vacuum state, we have

$$\langle n,n|T^{1/2}(g)|0,0\rangle|^2=(1-\rho)\rho^n,\qquad \rho=\frac{|\beta|^2}{|\alpha|^2}. \tag{1.117}$$

Here we have considered sets of coherent states associated with discrete-series representations of $SU(1,1)$. The general case has been treated in Ref. 10. Besides these representations, $SU(1,1)$ and other noncompact groups (groups whose invariant volume is infinite) have continuous (or principal) series of representations. The corresponding sets of coherent states have been studied in detail in Refs. 11 and 12. For the Lorentz group these coherent states realize the transformation from the hyperboloid to the cone that was first discussed by Shapiro.[31] We shall not discuss these sets of coherent states here, however, because of lack of space.

2. APPLICATIONS OF GENERALIZED COHERENT STATES

As was mentioned in the Introduction, the apparatus of generalized coherent states is especially effective for problems in which the Hamiltonian has a dynamical symmetry group G. More precisely, in the case that we shall consider *the Hamiltonian is linear in the generators* X_k of an irreducible unitary representation $T(g)$ of the corresponding Lie algebra:

$$\mathscr{H}=\sum h^k X_k. \tag{2.1}$$

The operators X_k satisfy the standard commutation relations

$$[X_k, X_l]=C^m_{kl}X_m, \tag{2.2}$$

in which the C^m_{kl} are the so-called structure constants. We note that the operators X_k transform under the operators $T(g)$ according to the associated representation of G:

$$T(g)X_kT^{-1}(g)=A^l_k(g)X_l. \tag{2.3}$$

Accordingly, the Hamiltonian $\mathscr{H}$ transforms under $T(g)$ into $\tilde{\mathscr{H}}$, where $\tilde{\mathscr{H}}$ is given by formula (2.1) with

$$\tilde{h}^k=A^k_l(g)h^l. \tag{2.4}$$

We shall consider three types of problems.

a) Hamiltonian (2.1) is independent of time and it is required to find its spectrum and eigenfunctions. To simplify the problem we may use the unitary transformation

$$T(g)\mathscr{H}T^{-1}(g)=\tilde{\mathscr{H}}, \tag{2.5}$$

in which $\tilde{\mathscr{H}}$ is given by (2.1) with the $\tilde{h}^k$ from (2.4).

Having made this transformation, we can reduce $\mathcal{H}$ to a simpler form and then find its spectrum and eigenfunctions $|\tilde{\psi}_n\rangle$. The eigenfunctions $|\psi_n\rangle$ are now given by the formula

$$|\psi_n\rangle = T^{-1}(g)\,|\tilde{\psi}_n\rangle. \tag{2.6}$$

Thus, if $|\tilde{\psi}_0\rangle$ is selected as the fixed vector, the state $|\psi_0\rangle$ will be a generalized coherent state.

b) Hamiltonian (2.1) is time dependent but tends to appropriate limits as $t\to\pm\infty$ fast enough to assure the existence of the corresponding asymptotic states $|\psi_\pm\rangle$.

Here the time evolution operator $U(t, t_0)$ for the system has the form $T(g(t))$:

$$U(t,\ t_0) = T(g), \tag{2.7}$$

and there exists an S matrix:

$$S = U(+\infty,\ -\infty) = T(g_0). \tag{2.8}$$

In this case the transition probability from the state $|m\rangle$ at $t\to-\infty$ to the state $|n\rangle$ at $t\to+\infty$ is given by the square of the matrix element T_{mn}:

$$W_{nm} = |T_{nm}(g_0)|^2. \tag{2.9}$$

c) Hamiltonian (2.1) is a periodic function of time:

$$\mathcal{H}(t+T) = \mathcal{H}(t). \tag{2.10}$$

In this case there exist states for which

$$|\psi_\varepsilon(t+T)\rangle = e^{-i\varepsilon T/\hbar}\,|\psi_\varepsilon(t)\rangle, \tag{2.11}$$

the so-called states of definite quasienergy.[16] Then the time evolution operator $U(t, t_0)$ has the property

$$U(t_0+T,\ t_0) = T(g_0) = e^{-iT\tilde{\mathcal{H}}/\hbar}, \tag{2.12}$$

in which $\tilde{\mathcal{H}}$ has the form (2.1). Thus, the spectrum of $\tilde{\mathcal{H}}$ gives the quasienergy spectrum of the problem.

Now let us consider some specific examples.

A. The ordinary set of coherent states

Many studies have been devoted to the application of the ordinary set of coherent states to the solution of various physical problems (see the reviews in Refs. 3–5). In addition to these reviews we may also mention the following studies.

In Refs. 36 and 37, coherent states were used to investigate the condensation phenomena in a system of interacting bosons. It was shown in Ref. 38 with the aid of coherent states that in a definite class of field theories there exists a classical limit for the quantum mechanical correlation functions. Coherent states were used in Ref. 39 to prove the virial theorem for liquid helium, and in Ref. 40 to describe multiple production of particles at high energies. Finally, in Refs. 41–43 (also see Ref. 44) coherent states were used to obtain a quasiclassical description of localized states (solitons) in nonlinear field theories. In Ref. 41 such states were used to clarify certain properties of the recently discovered ψ mesons.

In this section the ordinary set of coherent states will be employed in the solution of two problems.

1) *A quantum oscillator acted on by a variable external force.*[17] The time development of the system considered is determined by the Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = (\mathcal{H}_0+\mathcal{H}_1)\,|\psi(t)\rangle, \tag{2.13}$$

in which

$$\mathcal{H}_0 = \hbar\omega\left(a^+a+\frac{1}{2}\right),\quad \mathcal{H}_1 = -f(t)\,q = -f(t)\sqrt{\frac{\hbar}{2\omega}}\,(a+a^+). \tag{2.14}$$

By making the transformation $|\psi(t)\rangle = \exp[-(i/\hbar)\mathcal{H}_0 t] \times |\tilde{\psi}(t)\rangle$ to the interaction representation we get rid of the term $\mathcal{H}_0$ in the Schrödinger equation. For the new wave function $|\tilde{\psi}(t)\rangle$ we obtain the equation

$$i\hbar\frac{d}{dt}|\tilde{\psi}(t)\rangle = \tilde{\mathcal{H}}_1(t)\,|\tilde{\psi}(t)\rangle, \tag{2.15}$$

in which

$$\tilde{\mathcal{H}}_1(t) = e^{i\mathcal{H}_0 t/\hbar}\mathcal{H}_1 e^{-i\mathcal{H}_0 t/\hbar} = -f(t)\sqrt{\frac{\hbar}{2\omega}}\,(ae^{-i\omega t}+a^+e^{i\omega t}). \tag{2.16}$$

It is convenient to rewrite Eq. (2.15) in the form

$$\frac{d}{dt}|\tilde{\psi}(t)\rangle = (\beta(t)\,a^+ - \bar{\beta}(t)\,a)\,|\tilde{\psi}(t)\rangle, \tag{2.17}$$

where

$$\beta(t) = \frac{i}{\sqrt{2\hbar\omega}}\,f(t)\,e^{i\omega t}. \tag{2.18}$$

Since the Hamiltonian $\tilde{\mathcal{H}}_1$ is linear in the operators of the Lie algebra W_1, the time evolution operator $\tilde{S}(t)$, defined by $|\tilde{\psi}(t)\rangle = \tilde{S}(t)|\tilde{\psi}(0)\rangle$, is an operator of the representation of the group W_1, i.e.,

$$\tilde{S}(t) = T(g(t)) = e^{-i\varphi(t)}D(\gamma(t)). \tag{2.19}$$

It follows from this, in particular, that if the initial state is coherent, it will remain coherent for all time. Thus, there exists a solution $|\tilde{\psi}(t)\rangle$ of the form

$$|\tilde{\psi}(t)\rangle = e^{-i\varphi(t)}\,|\alpha(t)\rangle. \tag{2.20}$$

In particular, the expectation value of the operator a in this state is given by

$$\langle\tilde{\psi}|\,a\,|\tilde{\psi}\rangle = \alpha(t). \tag{2.21}$$

On differentiating this relation by t and using Eq. (2.17), we obtain

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[16] Nikishov and Ritus[32] introduced the concept of four-dimensional quasimomentum, the quasienergy being its fourth component. Zel'dovich[33,34] and Ritus[35] used states of definite quasienergy to treat atomic systems in the field of an electromagnetic wave.

[17] We note that Feynman[26] and Schwinger[25] solved this problem earlier by a different method.

$$\dot{\alpha}=\beta, \quad \alpha(t)=\alpha_0+\int_0^t \beta(t')\,dt'. \tag{2.22}$$

Further, from Eq. (2.17) we find that in the limit $\Delta t \to 0$ we have

$$|\tilde{\psi}(t+\Delta t)\rangle = D(\beta(t)\,\Delta t)\,|\tilde{\psi}(t)\rangle. \tag{2.23}$$

Substituting $|\tilde{\psi}(t)\rangle$ from (2.20) into this equation and using relation (1.15), we obtain the equation

$$\dot{\varphi}=\operatorname{Im}(\bar{\beta}\alpha)=\operatorname{Im}(\dot{\bar{\alpha}}\alpha). \tag{2.24}$$

We note that Eq. (2.22) is the classical equation of motion for an oscillator acted on by an external force, and that it follows from Eq. (2.24) that $\varphi(t)$ is equal to twice the area swept out by the radius vector during the motion of the phase point in the phase space, i.e.,

$$\varphi(t)=\frac{1}{\hbar}\int_{q_0}^{q_t} p\,dq,$$

and thus has a simple quasiclassical meaning.

The situation is especially simple if the force $f(t)$ tends to zero rapidly enough as $t\to\pm\infty$. In this case the limits $\alpha_\pm$ and $\varphi_\pm$ exist and it is meaningful to speak of the transition probability from the state $|m\rangle$ at $t\to-\infty$ to the state $|n\rangle$ at $t\to+\infty$. This transition probability is given by the formula

$$W_{mn}=|\langle m|S|n\rangle|^2=|\langle m|D(\gamma)|n\rangle|^2, \tag{2.25}$$

and it follows from formula (1.48) that

$$W_{mn}=\frac{n_<!}{n_>!}\,|\gamma|^{2|m-n|}e^{-|\gamma|^2}|L_{n_<}^{|m-n|}(|\gamma|^2)|^2. \tag{2.26}$$

Now let us consider the second example.

2) *Relaxation of a quantum oscillator to thermodynamic equilibrium*. A quantum oscillator in thermodynamic equilibrium at temperature T is described by the following density matrix:

$$\rho=(1-e^{-\beta})\,e^{-\beta a^+a}, \quad \beta=\frac{\hbar\omega}{kT}. \tag{2.27}$$

From this it is not difficult to obtain the following expressions for the symbols of the density matrix:

$$P(\alpha)=\frac{1}{\nu}e^{-|\alpha|^2/\nu}, \tag{2.28}$$

$$Q(\alpha)=\frac{1}{\nu+1}e^{-|\alpha|^2/(\nu+1)}, \tag{2.29}$$

in which ν, which is equal to the average number $\bar{n}$ of quanta, is given by Planck's formula

$$\nu=(e^{\hbar\omega/kT}-1)^{-1}. \tag{2.30}$$

The time evolution of a quantum oscillator in thermal contact with a thermostat at temperature T is described by the following equation, which was derived by Shen[45] and has been thoroughly investigated in Refs. 46 and 47[18)]:

$$\dot{\rho}=-\frac{\gamma}{2}[(\nu+1)(a^+a\rho-2a\rho a^+ +\rho a^+a)+\nu(aa^+\rho-2a^+\rho a+\rho aa^+)] \tag{2.31}$$

in which the constant γ determines the rate at which the oscillator approaches thermodynamic equilibrium ($\gamma>0$). On substituting the expressions for ρ in terms of the symbols $P(\alpha)$ and $Q(\alpha)$ into Eq. (2.31) we obtain the following equations[46,47]:

$$\dot{P}=+\frac{\gamma}{2}\frac{\partial}{\partial\alpha_i}(\alpha_i P)+\frac{\gamma}{4}\nu\,\Delta P, \tag{2.32}$$

$$\dot{Q}=+\frac{\gamma}{2}\frac{\partial}{\partial\alpha_i}(\alpha_i Q)+\frac{\gamma}{4}(\nu+1)\,\Delta Q, \tag{2.33}$$

$$\Delta=\frac{\partial^2}{\partial\alpha_1^2}+\frac{\partial^2}{\partial\alpha_2^2}.$$

We note that the equation for P is the same as the equation for the Brownian motion (in phase space) of a classical oscillator (see Ref. 49). Thus, we have again reduced a quantum problem to a classical one with the aid of coherent states.

B. The set of coherent spin states

We recall that the set of coherent spin states was introduced by Radcliffe[8] and was thoroughly investigated in Refs. 6, 8, and 27. These states have been employed to estimate the partition function for a system of quantum spins.[50] Such states have also been used[27,51,52] in the so-called Dicke model[53] for the interaction of radiation with matter to describe the superradiant state. Such a state was found experimentally by Gross *et al.*[54]

We note, further, that with the aid of coherent spin states one can very simply obtain an expression for the generating function for the Clebsch-Gordan coefficients of the rotation group (see Ref. 55).

Here we shall consider two problems.

1) *Motion of spin in a variable magnetic field*. Let us consider a neutral spin-j particle with magnetic moment μ in a varying magnetic field $\mathbf{H}(t)$. The time variation of the state of such a system is determined by the Schrödinger equation

$$i\frac{d}{dt}|\psi(t)\rangle=-\mathbf{a}(t)\,\mathbf{J}\,|\psi(t)\rangle=(AJ_+ +\bar{A}J_- +BJ_0)\,|\psi(t)\rangle, \tag{2.34}$$

where

$$\mathbf{a}=\frac{\mu}{j}\mathbf{H}, \quad A=-\frac{1}{2}(a_1-ia_2), \quad B=-a_3. \tag{2.35}$$

Concerning the vector $\mathbf{H}(t)$ we assume only that it tends rapidly enough to definite limits as $t\to\pm\infty$ to assume the existence of the corresponding asymptotic states $|\psi^\pm\rangle$

It has long been known (see, e.g., Refs. 56–58) that the problem of a particle of arbitrary spin j can be reduced to the simpler problem of the motion of a spin-1/2 particle. This reduction can be achieved in an especially simple manner with the use of coherent spin states.

We shall seek a solution to Eq. (2.34) in the form

$$|\psi(t)\rangle=e^{-i\varphi(t)}\,|\zeta(t)\rangle. \tag{2.36}$$

18)Equation (2.31) for the case $\nu=0$ ($T=0$) is essentially contained in Landau's classical paper[48] in which the density matrix was first introduced.

Then Eq. (2.34) takes the form

$$i\frac{d}{dt}|\zeta(t)\rangle=(\mathcal{H}(t)-\dot{\varphi})|\zeta(t)\rangle. \qquad (2.37)$$

On the other hand, from the explicit formula for the coherent states we have

$$\zeta J_+ + |\zeta\rangle=(j+J_0)|\zeta\rangle, \qquad (2.38)$$

$$J_-|\zeta\rangle=\zeta(j-J_0)|\zeta\rangle, \qquad (2.39)$$

$$\frac{d}{dt}|\zeta(t)\rangle=\left[\frac{-j}{1+|\zeta|^2}\frac{d}{dt}(1+|\zeta|^2)\right]|\zeta(t)\rangle+\frac{\dot{\zeta}}{\zeta}(J_0+j)|\zeta(t)\rangle. \qquad (2.40)$$

From this we obtain equations for the quantities $\zeta(t)$ and $\varphi(t)$[19]:

$$i\dot{\zeta}=A-\bar{A}\zeta^2+B\zeta, \qquad (2.41)$$

$$\frac{1}{j}\dot{\varphi}=-i\frac{\dot{\zeta}}{\zeta}+i\left[\frac{1}{1+|\zeta|^2}\frac{d}{dt}(1+|\zeta|^2)\right]+\frac{A}{\zeta}+\bar{A}\zeta. \qquad (2.42)$$

We note that from (2.41) we can obtain the equation

$$i\frac{d}{dt}(1+|\zeta|^2)=(A\bar{\zeta}-\bar{A}\zeta)(1+|\zeta|^2), \qquad (2.43)$$

with the aid of which we can obtain

$$\dot{\varphi}=j(\bar{\zeta}A+\zeta\bar{A}-B). \qquad (2.44)$$

Thus, the problem of finding the wave function reduces to the simpler problem of solving Eqs. (2.41) and (2.44).

We note that if vectors of the unit sphere are used as parameters, Eq. (2.41) takes the form

$$\dot{\mathbf{n}}=-[\mathbf{a}(t),\mathbf{n}]. \qquad (2.45)$$

Thus, the ζ plane (or the unit sphere S^2) plays the part of the phase space of the classical dynamical system.

In the simplest case, when $A(t)\to 0$ and $B(t)\to$ const as $t\to+\infty$, we have $|\zeta(t)|^2\to\rho=$ const, as is evident from (2.41). From this we immediately obtain the following expression for the transition probability from the initial state $|0\rangle=|j,-j\rangle$ to the final state $|m\rangle=|j,-j+m\rangle$:

$$W_m=\frac{(2j)!}{m!(2j-m)!}\frac{\rho^m}{(1+\rho)^{2j}}. \qquad (2.46)$$

The general formula for the transition probability has the form

$$W_{mn}=|d^j_{\mu\nu}(\theta)|^2, \qquad (2.47)$$

in which $\mu=m-j$, $\nu=n-j$, $\rho=\tan(\theta/2)$, and the $d^j_{\mu\nu}(\theta)$ are the known matrix elements of the representation $T^j(g)$.

Now let us consider the second example.

2) *Relaxation to thermodynamic equilibrium of a particle with spin in a magnetic field.* A particle with spin in thermodynamic equilibrium at temperature T in a magnetic field $\mathbf{H}=(0,0,H)$ is described by the density matrix

$$\rho=\frac{\mathrm{sh}(\beta/2)}{\mathrm{sh}[j+(1/2)]\beta}e^{\beta J_0}, \quad \beta=\frac{\mu H}{kT}. \qquad (2.48)$$

From this we obtain the following expressions for the symbols of the density matrix:

$$Q(\mathbf{n})=\left(\mathrm{ch}\frac{\beta}{2}+\mathrm{sh}\frac{\beta}{2}\cos\theta\right)^{2j}, \qquad (2.49)$$

$$P(\mathbf{n})=\mathrm{ch}\frac{\beta}{2}+3\,\mathrm{sh}\frac{\beta}{2}\cos\theta \quad \text{for} \quad j=\frac{1}{2}. \qquad (2.50)$$

The time evolution of such a system in thermal contact with a thermostat of temperature T is described by the following equation, which was derived in Ref. 59:

$$\dot{\rho}=-\frac{\gamma}{2}\{(\nu+1)(J_+J_-\rho-2J_-\rho J_++\rho J_+J_-)+\nu(J_-J_+\rho-2J_+\rho J_-+\rho J_-J_+)\}, \qquad (2.51)$$

in which ν is given by Planck's formula (2.30). For a thermostat at zero temperature, $\nu=0$ and the equation takes the form

$$\dot{\rho}=\frac{\gamma}{2}\{[J_-,\rho J_+]+[J_-\rho,J_+]\}. \qquad (2.52)$$

We shall assume for simplicity that $P(\mathbf{n})=P(\theta,\varphi)$ depends only on θ. Then substituting expression (1.82) for ρ into Eq. (2.52) we reach the equation for $P(\theta,t)$ that was derived and studied by Narducci *et al.*[60–62]:

$$\frac{\partial}{\partial t}(\sin\theta\cdot P(\theta,t))=\frac{\partial}{\partial\theta}\left[\left(j\sin\theta+\frac{\sin\theta}{2(1+\cos\theta)}\right)\sin\theta\cdot P(\theta,t)\right]$$
$$+\frac{\partial^2}{\partial\theta^2}\left[\frac{1-\cos\theta}{2}\sin\theta\cdot P(\theta,t)\right]. \qquad (2.53)$$

But Eq. (2.53) is just the Fokker-Planck equation for the function $f(\theta,t)=\sin\theta\cdot P(\theta,t)$ on the sphere $S^2=\{\mathbf{n}:\mathbf{n}^2=1\}$. This equation contains a "displacement" factor that leads to the motion of the distribution f as a whole on the surface of the sphere. In addition, the distribution expands (or contracts) owing to the diffusion coefficient $D(\theta)=(1-\cos\theta)/2$, which is maximum at $\theta=\pi$ and vanishes at $\theta=0$. As a result of the combined effect of the "displacement" factor and the diffusion term, the distribution on the sphere expands and its maximum shifts toward the point $\theta=0$. As $t\to+\infty$ the density matrix tends to $\rho=|j,-j\rangle\langle j,-j|$. We note that the following equation can be obtained for the position $\theta=\theta_{\max}$ of the maximum of the distribution:

$$\frac{d}{dt}\theta_{\max}=-j\sin\theta_{\max}. \qquad (2.54)$$

C. The set of coherent states for the group $SU(1,1)$

As was noted in Sec. C of Chap. 1, the group $SU(1,1)$ has several series of irreducible unitary representations. Accordingly, there are several series of sets of coherent states associated with it. The coherent states associated with the principal-series representations of $SU(1,1)$[11] represent the transformation from a hyperboloid to a cone in three-dimensional pseudo-Euclidian space and will not be discussed here for lack of space. Here we shall consider only certain sets of coherent states associated with the so-called discrete-series representations of $SU(1,1)$.

[19]Majorana[56] obtained an equation equivalent to Eq. (2.41) by a different method.

These states are convenient for use in solving certain problems in which one must find the spectrum and wave functions of a Hamiltonian that is quadratic in the boson creation and destruction operators. As was shown in Ref. 63, for example, such coherent states arise in treating the production of pairs of zero-spin particles in a uniform varying electric field or in the gravitational field of the expanding universe.[20]

In this section we shall discuss two problems.

1) *Parametric excitation of a quantum oscillator.* This problem has been thoroughly discussed elsewhere.[30,72–75] In solving this problem here we shall use a set of coherent states for a discrete-series representation of $SU(1,1)$ (see Chap. 1, Sec. C). The system of interest, a quantum oscillator with a variable frequency, is described by the Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle=\mathscr{H}(t)|\psi(t)\rangle, \tag{2.55}$$

in which

$$\mathscr{H}(t)=\frac{p^2}{2}+\frac{\omega^2(t)}{2}q^2. \tag{2.56}$$

Expressing the coordinate and momentum operators in terms of the boson creation and destruction operators, we obtain

$$\begin{aligned}\mathscr{H}(t)&=-\frac{\hbar}{4}(1-\omega^2(t))(a^2+a^{+2})\\&+\frac{\hbar}{4}(\omega^2(t)+1)(aa^++a^+a)=\hbar(AK_++\bar{A}K_-+BK_0).\end{aligned} \tag{2.57}$$

Now using Eqs. (1.111), we rewrite $\mathscr{H}(t)$ in the form

$$\mathscr{H}(t)=\hbar\Omega_0(t)K_0-\hbar\Omega_1(t)K_1, \tag{2.58}$$

where

$$\Omega_0=1+\omega^2(t),\quad \Omega_1=1-\omega^2(t), \tag{2.59}$$

and K_0 and K_1 are the generators of the discrete-series representation of $SU(1,1)$ with $k=1/4$ and $k=3/4$.

Thus, our Hamiltonian is linear in the generators of the Lie algebra of the group $SU(1,1)$. Hence there exists a solution of the Schrödinger equation of the form

$$|\psi(t)\rangle=e^{(-i\varphi(t))}|\zeta(t)\rangle,\qquad |\zeta|<1, \tag{2.60}$$

where $|\zeta\rangle$ is a coherent state with $k=1/4$ or $3/4$. Then, in analogy with what was done in the preceding section, we substitute (2.60) into the Schrödinger equation (2.55) and obtain the following equations for ζ and φ:

$$\dot{\zeta}=A+\bar{A}\zeta^2+B\zeta, \tag{2.61}$$
$$\dot{\varphi}=k(A\bar{\zeta}+\bar{A}\zeta+B). \tag{2.62}$$

We note that in this case the ζ plane is a Lobachevskiĭ plane and is the phase space for the problem. Equation (2.61) describes the motion of a classical system (oscillator) on the phase plane, and the quantum state $|\zeta(t)\rangle$ follows the classical motion precisely. The phase factor $\varphi(t)$ is precisely equal to the area in the Lobachevskiĭ metric swept out by the radius vector in its motion. Both of these circumstances are associated with the fact that in this case the "quasiclassical approximation" leads to the exact solution.

Although this result is valid for arbitrary time variations of the frequency $\omega(t)$, two special cases are of physical interest.

a) The case in which $\omega(t)$ tends rapidly enough to definite limits as $t\to\pm\infty$. In this case the asymptotic states $|n\rangle_\pm$ at $t\to\pm\infty$ exist and it is meaningful to speak of the transition probability W_{mn} from state $|m\rangle_-$ to state $|n\rangle_+$. We shall assume for simplicity that the two limiting Hamiltonians at $t\to\pm\infty$ ($\mathscr{H}_+$ and $\mathscr{H}_-$) are the same. Then

$$W_{mn}=|\langle m|T(g_0)|n\rangle|^2,\quad g_0=\begin{pmatrix}\alpha&\beta\\ \bar{\beta}&\bar{\alpha}\end{pmatrix},\quad |\alpha|^2-|\beta|^2=1. \tag{2.63}$$

Then using Eq. (1.113) for $\langle m|T|n\rangle$, we obtain the final result from Ref. 30:

$$W_{mn}=\frac{n_<!}{n_>!}\sqrt{1-\rho}\,|P^{|m-n|/2}_{(m+n)/2}(\sqrt{1-\rho})|^2,\quad \rho=\frac{|\beta|^2}{|\alpha|^2}. \tag{2.64}$$

b) The case in which $\omega(t)$ is a periodic function of time: $\omega(t+T)=\omega(t)$. In this case there exist solutions of the Schrödinger equation having definite quasienergy, i.e., states for which

$$|\psi_\varepsilon(t+T)\rangle=e^{(-i\varepsilon T/\hbar)}|\psi(t)\rangle. \tag{2.65}$$

We shall be interested in the quasienergy spectrum. To find it we consider the time evolution operator $U(t,t_0)$ of the system, defined by $U(t,t_0)|\psi(t_0)\rangle=|\psi(t)\rangle$, and use it to form the unitary operator

$$S(t_0)=U(t_0+T,\,t_0). \tag{2.66}$$

Since this operator is unitary it can be expressed in the form

$$S=e^{-iT\tilde{\mathscr{H}}/\hbar}, \tag{2.67}$$

where $\tilde{\mathscr{H}}$ is a Hermitian operator. The spectrum of this operator is just the quasienergy spectrum.

In our case S is an operator for a finite transformation of the group $SU(1,1)$, while the operator $\tilde{\mathscr{H}}$ belongs to a representation of the Lie algebra of this group and therefore has the form

$$\tilde{\mathscr{H}}=\hbar(\Omega_0K_0-\Omega_1K_1-\Omega_2K_2). \tag{2.68}$$

There are three different cases depending on the form of the vector $\Omega=(\Omega_0,\Omega_1,\Omega_2)$.

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[20]The problem of the production of pairs of spin-zero particles has been thoroughly treated in Refs. 63–68, as well as in a number of other papers. The pair-production problem for spin-1/2 particles has been treated in Refs. 69–71. It has been shown[63,69] that the dynamical symmetry group for the pair-production problem for spin-S particles is $SU(2S+1,\ 2S+1)$ for integer S and $SU(2(2S+1))$ for half-integer S. The case $S=\frac{1}{2}$ is exceptional; for this case the symmetry group is $SO(5)$.[71]

1) $\Omega^2=\Omega_0^2-\Omega_1^2-\Omega_2^2>0, \quad \Omega_0>0.$ (2.69)

Here we can use the unitary transformation $\tilde{\mathscr{H}} \to \tilde{\mathscr{H}}' = U \tilde{\mathscr{H}} U^+$ to transform the operator $\tilde{\mathscr{H}}$ to the form

$$\tilde{\mathscr{H}}' = \hbar\Omega K_0. \tag{2.70}$$

In this case the quasienergy spectrum is discrete, is bounded below, and has the form

$$\varepsilon_n = \hbar\Omega\,(k+n). \tag{2.71}$$

The ground state of such a Hamiltonian is a coherent state associated with the discrete-series representation T^k of $SU(1,1)$.

1′) Suppose that

$$\Omega^2>0, \quad \Omega_0<0. \tag{2.72}$$

In this case we have $\tilde{\mathscr{H}}' = -\hbar\Omega K_0$. Here the quasienergy spectrum is discrete and bounded above:

$$\varepsilon_n = -\hbar\Omega\,(k+n).$$

2) Suppose that $\Omega_0^2-\Omega_1^2-\Omega_2^2=-\lambda^2<0$. Then $\tilde{\mathscr{H}}$ can be reduced to the form

$$\tilde{\mathscr{H}}' = -\hbar\lambda K_1. \tag{2.73}$$

In this case the quasienergy spectrum is continuous and fills the entire line $-\infty<\varepsilon<+\infty$. In the classical case this corresponds to unstable motion.

3) But if $\Omega^2=0$ and $\Omega_0>0$, we have $\mathscr{H}=\hbar\Omega_0(K_0-K_1)$ and the spectrum is continuous and fills the half-line $0<\varepsilon<\infty$.

3′) Finally, if $\Omega^2=0$ and $\Omega_0<0$ we have

$$\tilde{\mathscr{H}}' = -\hbar\Omega_0\,(K_0-K_1). \tag{2.74}$$

Here the spectrum is also continuous, and it fills the half-line $-\infty<\varepsilon<0$. For the classical oscillator, cases 3) and 3′) represent the boundaries of the instability region. Now let us consider the next example.

2) *The superfluidity of an almost-ideal Bose gas*. As Bogolyubov showed,[76] this problem reduces to that of finding the spectrum and wave functions of a Hamiltonian that is quadratic in the boson creation and destruction operators. In the same paper[76] Bogolyubov indicated how the problem could be solved by diagonalizing the Hamiltonian with the aid of the linear canonical transformation that has come to be called Bogolyubov's canonical transformation.

The set of linear canonical transformations for this problem forms a certain group, and indeed a direct product of $SU(1,1)$ groups. Then the ground state of the Hamiltonian turns out to be a coherent state associated with a certain representation of this group.

Let us first consider a simplified superfluidity model.[77] Let the system consist of N weakly interacting bosons, and let it be described by the following Hamiltonian:

$$\mathscr{H} = \sum_k \varepsilon_k a_k^+ a_k + \frac{1}{2}\sum_{k,p,q} V_k a_{p+k}^+ a_{q-k}^+ a_p a_q, \quad \varepsilon_k = \frac{k^2}{2m}. \tag{2.75}$$

We shall consider only the first three states, so that p, q, and k assume only the values $(-1, 0, +1)$, and shall assume that $\varepsilon_{\pm1}=\varepsilon$, $V_{\pm1}=V$, and $\varepsilon_0=V_0=0$.[77,78] Then Hamiltonian (2.75) takes the form

$$\mathscr{H} = \varepsilon\,(a_+^+a_+ + a_-^+a_-) + V\,[a_0^+a_0\,(a_+^+a_+ + a_-^+a_-) + a_0^2 a_+^+ a_-^+ + a_0^{+2} a_+ a_-], \tag{2.76}$$

in which we have used the notation $a_\pm = a_{\pm1}$. For $V=0$ the ground state would consist of N particles with zero energy. We shall assume that for a weakly interacting system the zero-energy state will be macroscopically filled (so that we can assume the operators a_0 and a_0^+ to be c-numbers equal to $\sqrt{N_0}$, where $N_0=\langle a_0^+a_0\rangle$). This is the physical assumption that leads to the superfluid character of the model. Thus, the reduced Hamiltonian has the form

$$\mathscr{H}_{\rm red} = (\varepsilon+N_0V)\,(a_+^+a_+ + a_-^+a_-) + N_0V\,(a_+^+a_-^+ + a_+a_-). \tag{2.77}$$

We see that the reduced Hamiltonian is linear in the operators of the representation $T^{1/2}$ of the Lie algebra of the group $SU(1,1)$:

$$\mathscr{H}_{\rm red} = 2N_0V\left(\mu K_0 + K_1 - \frac{1}{2}\mu\right), \quad \mu = 1+\frac{\varepsilon}{N_0V}. \tag{2.78}$$

Thus, our problem reduces to the previously solved problem of the spectrum and eigenfunctions of the operator

$$\mathscr{H}_\Omega = \Omega_0K_0 - \Omega_1K_1 - \Omega_2K_2 = \Omega\mathrm{K}, \tag{2.79}$$

where

$$\Omega_0 = 2N_0V\mu, \quad \Omega_1 = -2N_0V, \quad \Omega_2 = 0. \tag{2.80}$$

In our case $\mathscr{H}$ is given by formulas (2.79) and (2.80), and to simplify it it is sufficient to consider the "rotation" $R(\theta)=\exp(-iK_2\theta)$ about the x_2 axis:

$$\begin{aligned} K_1' &= R(\theta)\,K_1R^{-1}(\theta) = \operatorname{ch}\theta\cdot K_1 + \operatorname{sh}\theta\cdot K_0,\\ K_0' &= R(\theta)\,K_0R^{-1}(\theta) = \operatorname{ch}\theta\cdot K_0 + \operatorname{sh}\theta\cdot K_1, \end{aligned} \tag{2.81}$$

so that

$$R(\theta)\,\mathscr{H}R^{-1}(\theta) = 2N_0V\left[K_0\,(\mu\operatorname{ch}\theta - \operatorname{sh}\theta) + K_1\,(\operatorname{ch}\theta - \mu\operatorname{sh}\theta) - \frac{\mu}{2}\right]. \tag{2.82}$$

Since $|\tanh\theta|<1$, we can use the rotation of K_1 or K_0, depending on the sign of the potential V. Thus,

1) if $V<0$ (an attractive potential), then $\mu<1$ and we take $\tanh\theta=\mu$:

$$R\mathscr{H}R^{-1} = -2N_0V\left(K_1\operatorname{sech}\theta + \frac{\mu}{2}\right), \; 2N_0|V|>\varepsilon; \tag{2.83}$$

2) if $V>0$ (a repulsive potential), then $\mu>1$, and we take $\coth\theta=\mu$:

$$R\mathscr{H}R^{-1} = 2N_0V\left(K_0\operatorname{cosech}\theta - \frac{\mu}{2}\right). \tag{2.84}$$

From this it follows that the energy spectrum is continuous in case 1). Case 2) is of greater physical interest; in this case the energy spectrum is discrete. It follows from (2.84) that

$$E_n = (2n+1+|\Delta|)\,E - N_0V - \varepsilon, \tag{2.85}$$

where $E=[2\varepsilon N_0 V+\varepsilon^2]^{1/2}$ and Δ is an eigenvalue of the operator $a_+^\dagger a_+ - a_-^\dagger a_-$. Then the eigenvectors have the form

$$|\psi_n\rangle = R^{-1}(\theta)\,|n\rangle = \frac{1}{n!}\,(b_+^\dagger b_-^\dagger)^n\,|\psi_0\rangle, \tag{2.86}$$

where

$$|\psi_0\rangle = R^{-1}(\theta)\,|0\rangle, \quad b_+^\dagger = R^{-1}(\theta)\,a_+^\dagger R(\theta). \tag{2.87}$$

Thus, the eigenvectors are coherent states associated with the discrete-series representations T^k of $SU(1,1)$. In the simplest case, in which $\Delta=0$ and $k=1/2$, we have

$$|\psi_0\rangle = \sum_m (-1)^m \operatorname{sech}\frac{\theta}{2}\left(\operatorname{th}\frac{\theta}{2}\right)^m |m\rangle, \tag{2.88}$$

or

$$|\psi_0\rangle = \sqrt{1-t^2}\,e^{-ta_+^\dagger a_-^\dagger}\,|0\rangle, \tag{2.89}$$

where

$$t=\operatorname{th}\frac{\theta}{2} = -\frac{E_0}{N_0 V}.$$

We note that transformation (2.87) is a linear canonical transformation of the operators $a_+^\dagger$ and a_-:

$$\begin{aligned} b_+^\dagger &= u a_+^\dagger + v a_-, \\ b_- &= u a_- + v a_+^\dagger. \end{aligned} \tag{2.90}$$

This transformation was first used to solve the problem of the superfluidity of an almost ideal Bose gas by Bogolyubov in his well-known paper, Ref. 76.

Now let us consider the almost ideal Bose gas described by Hamiltonian (2.75). Using the Bogolyubov approximation $a_0=a_0^\dagger\approx\sqrt{N_0}$ as before, we can write $\mathscr{H}$ in the form

$$\mathscr{H} = \frac{1}{2}N_0^2 V_0 + \sum_k (\varepsilon_k + N_0 V_k + N_0 V_0)\,a_k^\dagger a_k + \frac{1}{2}N_0 \sum_k V_k\,(a_k^\dagger a_{-k}^\dagger + a_k a_{-k}), \tag{2.91}$$

where the summation is taken over all values of k except the value $k=0$, and there remain only second order terms in the a_k and $a_k^\dagger$. In this approximation we have $N=N_0+\sum a_k^\dagger a_k$ and our Hamiltonian can be written in the form

$$\mathscr{H} = \frac{1}{2}N^2 V_0 + \sum (\varepsilon_k + NV_k)\,a_k^\dagger a_k + \frac{N}{2}\sum V_k\,(a_k^\dagger a_{-k}^\dagger + a_k a_{-k}). \tag{2.92}$$

Let us introduce the operators

$$\begin{aligned} K_1^{(q)} &= \frac{1}{2}\,(a_q^\dagger a_{-q}^\dagger + a_q a_{-q}), \\ K_2^{(q)} &= -\frac{i}{2}\,(a_q^\dagger a_{-q}^\dagger - a_q a_{-q}), \\ K_0^{(q)} &= \frac{1}{2}\,(a_q^\dagger a_q + a_{-q}^\dagger a_{-q} + 1). \end{aligned} \tag{2.93}$$

We see that these operators generate the Lie algebra of the group $SU(1,1)_{(q)}$, and the Hamiltonian is a linear combination of generators of this algebra:

$$\mathscr{H} = \sum_q \oplus\, NV_q\left(K_1^{(q)} + \mu_q K_0^{(q)} - \frac{\mu_q}{2}\right) + \frac{1}{2}N^2 V_0, \qquad \mu_q = 1 + \frac{\varepsilon_q}{NV_q}. \tag{2.94}$$

As before, we have a continuous spectrum in the case in which $V_{(q)}<0$ and $|\mu_q|<1$, while in the case $V_q>0$ or $\mu_q>1$ the Hamiltonian can be simplified with the aid of the unitary transformation

$$R=\prod_q \otimes R(\theta_q), \qquad R(\theta_q) = e^{-iK_2^{(q)}\theta_q}, \qquad \theta_q = \operatorname{cth}\mu_q, \tag{2.95}$$

so that

$$RHR^{-1} = \sum_q \oplus\left(\operatorname{csech}\theta_q K_0^{(q)} - \frac{\mu_q}{2}\right) NV_q + \frac{1}{2}N^2 V_0. \tag{2.96}$$

The Casimir operators have the form

$$C_q = K_0^{(q)2} - K_1^{(q)2} - K_2^{(q)2} = \frac{1}{4}\,(\Delta_q^2 - 1), \tag{2.97}$$

where the integrals of motion $\Delta_q = a_q^\dagger a_q - a_{-q}^\dagger a_{-q}$ are the differences between the numbers of particles in states with opposite momenta. Since the energy spectrum must be bounded below the only allowed representation is this one:

$$\prod_q \otimes T^{K_q}, \qquad K_q = \frac{1}{2} + \frac{1}{2}\Delta_q. \tag{2.98}$$

Now we see that the energy spectrum has the form

$$E(n_1, \ldots, n_l, \ldots) = \sum\left(n_l + \frac{1}{2} + \frac{1}{2}|\Delta_l|\right)E_l + \text{const}, \tag{2.99}$$

where $E_1=[2\varepsilon_1 NV_1+\varepsilon_1^2]^{1/2}$. The ground-state wave function for the case $\Delta_1=0$ has the form

$$|\Psi_0\rangle = \left\{\prod_l \sqrt{1-t_l^2}\exp\left[\sum_m(-t_m a_m^\dagger a_{-m}^\dagger)\right]\right\}|0\rangle, \tag{2.100}$$

where $t_1=\tanh(\theta_1/2)$.

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