Paradoxes of the quantum theory of the Vavilov-Cerenkov and Doppler effects

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The Minkowski tensor gives $g^M = nu/c$ (I) for the momentum density g of a plane-wave electromagnetic field in a stationary medium, whereas the Abraham tensor gives $g^A = u/nc$ (II), where u is the energy density and n the refractive index. Expression (I) cannot be reconciled with $J = \mu v$ (III), where μ is the mass of the wave packet and v is its velocity if, according to Einstein, $\mu = E/c^2$, where J is the momentum and E the energy of the wave packet. On the other hand, the expression for the "pseudomomentum" $J^M = nE/c$ (IV), which follows from (I), is identical with the expression for the momentum of the quantum photon $J = nh\nu/c$ (*V*), whereas the formula the follows from (II), i.e., $J^A = E/nc$ (VI) is in agreement with the Einstein equation (III) but is in conflict with (V). Simple calculation for stationary medium and source shows that (IV) and (VI) can be reconciled if one takes into account the fact that, under certain assumptions, $J^M = J^A + \Delta J$ (VII), where ΔJ is the momentum communicated to the medium in the photon emission process. It is shown in this paper that, within the framework of the adopted assumptions and, probably, classical models generally, expression (VII) cannot be generalized to the case of a source moving relative to the medium. This result is in conflict with the conclusions reported by V. L. Ginzberg and V. A. Ugarov [Usp. Fiz. Nauk **118**, 175 (1976)] [Sov. Phys. Usp. **19**, 94 (1976)]. Moreover, it is shown that, if (VII) is introduced as a postulate for a source moving relative to the medium, one can satisfy at the same time both the quantum conditions and (V), on the one hand, and the fundamental Einstein relation (III), on the other.

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1. INTRODUCTION

There has recently been considerable renewed discussion on the pages of scientific periodicals of the old (in fact, 50-year-old) problem of the momentum-energy tensor of the electromagnetic field in electrically or magnetically polarized media.[1)]

This discussion started again after it was shown in a very simple and graphic way that the Minkowski expression for the momentum density of a field in a medium leads to a contradiction with such general propositions of classical and relativistic mechanics as the law of constant velocity of the "center of gravity," i.e., the center of mass of a system of particles. This is, in fact, essentially a contradiction of the basic Einstein formula $E = mc^2$.

It is now clear that the "correct" expression for the momentum density g of a field, or the momentum J of a wave packet, is the Abraham expression $g = u/nc$, where u is the energy density, n is the refractive index, and c is the velocity of light; correspondingly, $J = \mathcal{E}/nc$, where $\mathcal{E}$ is the energy of the wave train.

The word "pseudomomentum"[3] is generally accepted for the quantity $J = n\mathcal{E}/c$ which follows from the Minkowski tensor.

There has been extensive discussion in the literature about the physical significance of the concept of "pseudomomentum." Relatively little attention has, however, been devoted to the question of how the idea of a quantum of light in a medium, which is based on this concept, can be reconciled with the above-mentioned requirements of classical and relativistic mechanics.

It is, therefore, natural to enquire whether the quantum theory of the Cerenkov and Doppler effects, which was extensively treated in the late Forties and Fifties, requires a revision in the light of these developments.

1)For more detailed references relating to this question, see Skobel'tsyn[1] and Ginzburg and Ugarov.[2]

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In the ensuing discussion, we shall consider some aspects of the problem and will formulate some questions that will take us back to the fundamentals of the quantum theory of light. Bearing all this in mind, we recall, to begin with, the following well-known proposition: a *free* electron moving with constant speed in a straight line in empty space cannot emit radiation at the expense of its kinetic energy. This conclusion can be drawn from the theory of relativity by introducing an inertial reference frame in which the electron under consideration is at rest and in which, therefore, the energy necessary for the emission of a photon under the given conditions is zero. However, it is well known that an electron in a state of uniform motion *in a medium* will emit radiation, i.e., it will emit the Vavilov–Cerenkov radiation.

If we use the above reference frame (in which the electron is at rest), we have to consider the question: where is the source of the energy of the Cerenkov radiation? This question can be given a natural answer by supposing that the source of energy is, in fact, the kinetic energy of the medium which interacts with the electromagnetic field of the electric charge that is at rest in the medium.

This answer is undoubtedly correct in the "global," i.e., macroscopic, formulation of the problem. However, in the microscopic approach—in fact—in the quantum treatment of the phenomenon, we arrive at a different answer. The quantum description is based on relationships ensuing from the Minkowski tensor, and this leads to some paradoxical consequences.

In the above reference frame (in which the electron is at rest), the energy of a photon inside the "Cerenkov radiation cone"[2] is negative: the photon emitted under these conditions *does not receive* but, on the contrary, *gives up* energy (and momentum) to the electron that emits it. As we have already noted, this gives rise to a direct conflict with the fundamental Einstein formula $E=mc^2$.[3] A detailed discussion of the controversy arising from this is given below.

If we adopt the Minkowski expression $J=n\mathscr{E}/c=nh\nu/c$ for the momentum J of a photon in a medium, and write down the equations representing the conservation of momentum-energy by considering the emission of a photon when the electron momentum changes from $\mathbf{J}_1$ to $\mathbf{J}_2$, we obtain directly (but approximately) the following condition for the emission of the Cerenkov radiation:

$$n\beta\cos\theta = 1$$

and certain other relationships that are identical with the predictions of the classical theory of the effect (see, for sample, the discussion given by Ginzburg[4]).

The wave-mechanical picture demands identification of the wave vector (multiplied by $h/2\pi$) with the momentum vector of the quantum. The basic de Broglie relation must be satisfied, i.e.,

$$J=\frac{h}{\lambda} \tag{1.1}$$

where J is the momentum and λ is the de Broglie wavelength.

When (1.1) is applied to a photon in a vacuum, it is equivalent to

$$J=\frac{h\nu}{c}. \tag{1.2}$$

If, on the other hand, we consider a photon in a medium, then

$$\lambda=\frac{c}{n\nu}, \tag{1.3}$$

so that it follows then from (1.1) that

$$J=\frac{nh\nu}{c} \tag{1.4}$$

($\mathscr{E}$ is the photon energy, J is the momentum, ν is the frequency, and n is the refractive index).

As noted above, the quantum theory of the phenomena that we are considering was constructed on the basis of relationships ensuing from the Minkowski tensor.

The four-dimensional divergences of this tensor are zero. According to a general theorem, the components of a tensor of this kind can be used to construct the momentum-energy four-vector for any three-dimensional region of the field bounded by a closed surface (see Pauli,[5] pp. 93 and 130, and Møller,[6] p. 126).

In particular, the components of the momentum-energy four-vector constructed in this way for a wave train are identical with the above values of the energy ($\mathscr{E}$) and momentum (1.4) of the *quantum photon*.

Is it possible, however, to regard a "wave train" as a *model* of the photon and then treat the "quantum of light" as the subject of *direct* physical observation?

If we adopt this interpretation, we are unavoidably led to a contradiction with the principles of the theory of relativity, as already noted and as will be discussed again below.

The square of the momentum-energy four-vector corresponding to (1.4), i.e., its norm, is

$$\frac{n^2\mathscr{E}^2}{c^2}-\frac{\mathscr{E}^2}{c^2}=\frac{n^2-1}{c^2}\mathscr{E}^2>0. \tag{1.5}$$

Since the norm of this vector is positive (we have a *spatial* four-vector), the photon particle with which this vector is associated must be given an *imaginary rest mass*. (It is known that the norm of the momentum-energy four-vector of a particle is equal to $-\mu_0^2c^2$, where μ_0 is the rest mass of the particle.)

There is a similarity between the photon particles in a medium and the so-called tachyons, i.e., particles created during the last few years in the imagination of theoreticians occupied with the physics of high-energy

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[2]The Čerenkov radiation cone is seen in this context from the "proper reference frame" of the medium.

[3]More precisely, this is a conflict with the formula $J=mv$, where J is the momentum, m is the mass, and v is the velocity of this mass.

particles. In this connection, we recall the early suggestion put forward by de Broglie[7] that a photon in a medium should be assigned two "proper" masses, i.e., two rest masses. One of them appears in the formula giving the energy and the other in the expression for the momentum of the particle. The theory of pilot waves developed by de Broglie leads to the same conclusion.[4]

The idea of a photon in a vacuum is free of the above internal contradiction: the norm of the momentum-energy four-vector is equal to zero in this case.

Since the fifty-year-old Minkowski versus Abraham controversy appears to be finally moving toward the acceptance of the Abraham tensor (or, at any rate, toward the abandonment of the Minkowski tensor), we have to consider the problem of the fundamentals of the quantum theory of the Cerenkov and Doppler effects in a refracting medium.

In 1955, Hungarian physicists[8] put forward an interpretation that appeared to reconcile the Minkowski and Abraham expressions. The same idea was put forward again in 1972 by Costa de Beauregard.[9] Quite recently, an attempt at a justification of this idea was published by Ginzburg and Ugarov[2] in the present journal.

This idea can be summarized as follows.

If we consider the emission of a *wave train* by a source of light in a medium, we can easily see that the *very act of emission* can be accompanied by the transfer to the medium of the momentum J carried off by the radiation and a further recoil momentum ΔJ. Since we are considering the case of a medium *at rest* and a source *fixed in the medium*, then, subject to certain *assumptions*, the simple calculation given below shows that

$$J^{\mathrm{M}} = J^{\mathrm{A}} + \Delta J \tag{1.6}$$

where J is the photon momentum. We shall use the superscripts M and A to represent the Minkowski and Abraham quantities.

Let us take the x axis as the ray direction. The energy density u of the electromagnetic field is, in this case, a function of the argument $\nu[t-(xn/c)]$, where n is the refractive index, ν is the "carrier" frequency,

$$u \propto \varphi\left(\nu t - \frac{\nu x n}{c}\right), \tag{1.7}$$

and t is the time measured from the beginning of emission of the wave train.

The density f of the Abraham force for a plane wave is given by

$$f = -\frac{n^2-1}{4\pi c}\frac{\partial}{\partial t}|[\mathbf{E}\times\mathbf{H}]| = \frac{n^2-1}{cn}\frac{\partial u}{\partial t} \tag{1.8}$$

(see, for example, our previous paper[1a] and Chap. 3 of the present paper).

It may be assumed that (1.8) is also valid on the leading front of the wave train, where the electromagnetic field can be approximately represented by a sinusoidal wave with a damped (in space) amplitude satisfying the condition

$$\varphi = \text{const} \quad \text{at} \quad t \gg \frac{xn}{c} \text{ and } \varphi = 0 \quad \text{at} \quad t \leq \frac{xn}{c}.$$

Under these assumptions,

$$f = -\frac{n^2-1}{n^2}\frac{\partial u}{\partial x}, \tag{1.9}$$

$$P = \int_x^\infty f\,dx = \frac{n^2-1}{n^2}u, \tag{1.10}$$

where $u = \text{const}$ for $xn/c \ll t$ (where u is the energy density at a point with given x, averaged over a large number of periods). P is the pressure exerted by the electromagnetic field on the medium on the leading front of the wave train.

Let T be the time of emission (this is the time taken by the wave train to leave the source), so that, according to (1.10),

$$\Delta J = PT = \frac{n^2-1}{n^2}uT. \tag{1.11}$$

According to Abraham, we have

$$J^{\mathrm{A}} = \frac{\mathscr{E}}{nc}. \tag{1.12}$$

where $\mathscr{E} = ul$ is the energy of the wave train, J^{A} is its momentum, l is the length of the wave train, and the transverse cross section of the train is assumed to be equal to unity.

When the source is fixed, we have $l = (c/n)T$.

We thus see that $\mathscr{E} = (uc/n)T$, $uT = \mathscr{E}n/c$, and, according to (1.11) and (1.12),[5]

$$J^{\mathrm{A}} + \Delta J = \frac{\mathscr{E}}{nc} + \frac{n^2-1}{n^2}\frac{\mathscr{E}n}{c} = \frac{\mathscr{E}n}{c} = J^{\mathrm{M}}.$$

If we could generalize (1.6) to the case of a moving medium and a fixed source, or a source moving in a fixed medium (the two formulations are equivalent in the light of the principle of relativity[6]), then we would show that the treatment given on the basis of the Minkowski pseudomomentum is equivalent to the theory based on the Abraham formulas with the inclusion of (1.6).

Ginzburg and Ugarov[2] suggest that they have proved

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[4]In our previous paper,[1b] published in 1975, the reference to the de Broglie paper[7] was unfortunately omitted.

[5]If the source moves with velocity βc (for example, in the direction of emission of the wave train), and the values of u and T are specified to be as in the case of a fixed source, then ΔJ is independent of the velocity of the source, but

$$T = \frac{\mathscr{E}}{[(1/n)-\beta]\,cu}, \quad T \neq \frac{\mathscr{E}n}{uc},$$

and (1.6) is not satisfied. In the ensuing discussion, the points raised here will be discussed in detail in the general case, including that of a medium moving relative to the given reference frame.

[6]In the sense that the transition from one situation to the other is governed by the Lorentz transformation.

this proposition. We shall return to this question in Chap. 4. When we do, we shall require various relationships that follow from the expressions for the momentum-energy tensors and from the relativistic kinematics of wave packets in moving media.

We shall consider these relationships in detail in the following chapter. Our exposition will be somewhat overloaded with elementary derivations of the various auxiliary relationships, but we hope that this will contribute to the elucidation of this somewhat tangled problem and will clear up some of the errors that have crept into the literature.

2. MOMENTUM-ENERGY TENSORS OF THE ELECTROMAGNETIC FIELD IN A MEDIUM AND KINEMATICS OF WAVE PACKETS IN MOVING MEDIA

Henceforth, we shall be concerned with moving media and sources of light contained by them (both stationary and moving). To exclude side effects connected with the interaction between the source and the medium in the course of their relative motion, we shall follow Ginzburg and Frank[10] and suppose that the source of light is located, for example, in a plane slit cut in the medium by planes whose separation is small in comparison with the wavelength.

The "virtual" source of radiation can be regarded as a surface covered by a layer of oscillators. When these oscillators are synchronous, the beam of rays is emitted practically parallel to the direction of the normal to the emitter surface. It is, however, possible to specify the oscillator phase distribution (over the surface of the emitter) so that the beam of rays is emitted obliquely at some particular angle to the surface of the source.

The elastic stress tensor (P_{lm}) associated with the electromagnetic field was considered in our previous paper.[1a] Since we shall be concerned with limiting conditions, we consider two models of an idealized medium in which the permittivity is a constant independent of the parameters characterizing the state of the medium and of the electric field. We have thus excluded forces that depend on the derivative of the permittivity with respect to the density of the medium ("strictional forces"). In addition, we assume that dispersion can be neglected.

Omitting details for which we refer the reader to our previous paper,[1a] we shall now generalize the two-dimensional "space-time" scheme considered previously to the case of two spatial dimensions.

As in our previous paper,[1a] we shall consider the field of a plane-polarized electromagnetic wave. The field components corresponding to the situation shown in Fig. 1 are as follows:

$$\left.\begin{array}{ll} E_x = E\sin\theta, & H_x = 0, \\ E_y = E\cos\theta\cos\psi, & H_y = -H\sin\psi, \\ E_z = E\cos\theta\sin\psi, & H_z = H\cos\psi \end{array}\right\} \tag{2.1}$$

and so on. The azimuth ψ will be set equal to zero *only for the sake of brevity*.

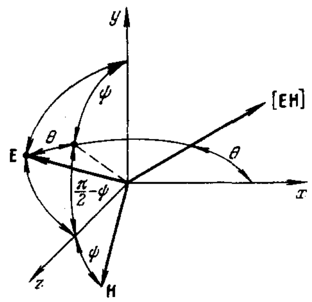

FIG. 1.

Details of the various derivations will be omitted from the analysis given below. The tensor P_{lm} can be written in the form (Skobel'tsyn[1a]):

$$\begin{array}{c} P_{lm} \\ -\frac{n^2-1}{n^2}u_0\times\left(\begin{array}{cc|c} \cos^2\theta & \sin\theta\cos\theta & 0 \\ \sin\theta\cos\theta & \sin^2\theta & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \end{array} \tag{2.2}$$

where u_0 is the electromagnetic energy density in the wave and P_{lm} is the mechanical stress tensor in the medium due to the electromagnetic field. We have shown (see, for example, Skobel'tsyn,[1a] Appendix 4) that the components of the Abraham tensor S_{lm} are given by the following table:

$$\begin{array}{c} S_{lm} \\ u_0\times\left(\begin{array}{cc|c} \cos^2\theta & \sin\theta\cos\theta & \frac{i}{n}\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta & \frac{i}{n}\sin\theta \\ \hline \frac{i}{n}\cos\theta & \frac{i}{n}\sin\theta & -1 \end{array}\right) \end{array} \tag{2.3}$$

Next, consider the sum T_{lm} of tensors S and P:

$$T_{lm} = S^{A}_{lm} + P_{lm}. \tag{2.4}$$

Transforming to the primed coordinates, the origin of which moves with velocity β relative to the frame x, t (stationary medium), we can transform S_{lm} and T_{lm} in accordance with the usual rules, i.e., assuming that

$$X'_{lm} = \alpha_{lk}\alpha_{ms}X_{ks},$$

where X_{ks} and X'_{lm} are the corresponding components of the tensors and α_{ks} are defined by the following matrix:

$$\begin{array}{c} \alpha_{ks} \\ \left(\begin{array}{cc|c} \gamma & 0\ \ 0 & i\beta\gamma \\ 0 & 1\ \ 0 & 0 \\ \hline -i\beta\gamma & 0\ \ 0 & \gamma \end{array}\right), \end{array}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}.$$

The final result is

$$\begin{array}{c} S'_{lm} \\ \gamma^2\frac{u_0}{n}\left(\begin{array}{cc|c} \begin{array}{c}\cos\theta(n\cos\theta-\beta)- \\ -\beta(\cos\theta-n\beta)\end{array} & \frac{\sin\theta(n\cos\theta-\beta)}{\gamma} & \begin{array}{c} i[\cos\theta(1-n\beta\cos\theta)- \\ -\beta(n-\beta\cos\theta)]\end{array} \\ \frac{\sin\theta(n\cos\theta-\beta)}{\gamma} & \frac{n\sin^2\theta}{\gamma^2} & i\frac{\sin\theta(1-n\beta\cos\theta)}{\gamma} \\ \hline \begin{array}{c} i[\cos\theta(1-n\beta\cos\theta)- \\ -\beta(n-\beta\cos\theta)]\end{array} & i\frac{\sin\theta(1-n\beta\cos\theta)}{\gamma} & -n+2\beta\cos\theta-n\beta^2\cos^2\theta \end{array}\right) \end{array} \tag{2.5}$$

$$P'_{lm}$$

$$-\frac{n^2-1}{n^2}\gamma^2 u_0 \times \left(\begin{array}{cc|c} \cos^2\theta & \frac{\sin\theta\cos\theta}{\gamma} & -i\beta\cos^2\theta \\ \frac{\sin\theta\cos\theta}{\gamma} & \frac{\sin^2\theta}{\gamma^2} & -i\beta\frac{\sin\theta\cos\theta}{\gamma} \\ \hline -i\beta\cos^2\theta & i\beta\frac{\sin\theta\cos\theta}{\gamma} & -\beta^2\cos^2\theta \end{array}\right). \tag{2.6}$$

The sum of the tensors (2.5) and (2.6) will be denoted by $T^{A}_{lm}: T^{A}_{lm} = S_{lm} + P_{lm}$,

$$T^{A}_{lm}$$

$$\gamma^2\frac{u_0}{n^2} \times \left(\begin{array}{cc|c} (n\beta-\cos\theta)^2 & \frac{\sin\theta(\cos\theta-n\beta)}{\gamma} & i(n-\beta\cos\theta)(\cos\theta-n\beta) \\ \frac{\sin\theta(\cos\theta-n\beta)}{\gamma} & \frac{\sin^2\theta}{\gamma^2} & i\frac{(n-\beta\cos\theta)\sin\theta}{\gamma} \\ \hline i(n-\beta\cos\theta)(\cos\theta-n\beta) & \frac{i\sin\theta(n-\beta\cos\theta)}{\gamma} & -(n-\beta\cos\theta)^2 \end{array}\right) \tag{2.7}$$

If we now replace the components S_{14} and S_{24} in the fourth column of (2.3) $[(i/n)\cos\theta$ and $(i/n)\sin\theta]$ by $in\cos\theta$ and $in\sin\theta$, and then use the general formulas for the transformation of the components of a tensor, we obtain the following table for the Minkowski tensor

$$T^{M}_{lm}$$

$$u_0\gamma^2 \times \left(\begin{array}{cc|c} \frac{(\cos\theta-n\beta)(n\cos\theta-\beta)}{n} & \frac{\sin\theta(n\cos\theta-\beta)}{n\gamma} & i\frac{(n-\beta\cos\theta)(n\cos\theta-\beta)}{n} \\ \frac{\sin\theta(\cos\theta-n\beta)}{\gamma} & \frac{\sin^2\theta}{\gamma^2} & i\frac{\sin\theta(n-\beta\cos\theta)}{\gamma} \\ \hline i\frac{(\cos\theta-n\beta)(1-n\beta\cos\theta)}{n} & i\frac{\sin\theta(1-n\beta\cos\theta)}{n\gamma} & i\frac{(\beta\cos\theta-n)(1-n\beta\cos\theta)}{n} \end{array}\right). \tag{2.8}$$

(The factor designated u_0 in front of each of the above tables is the energy density in the unprimed coordinate frame in which the medium is at rest.)

We now present the relationships that can be obtained from the above tensors.

The tensors T^{M} and T^{A} are tensors of the *closed* (field + medium) system:

$$\operatorname{div} T^{A}_{lm} = \operatorname{div} T^{M}_{lm} = 0. \tag{2.9}$$

The energy flux density for this case is given by

$$\Phi' = u'c^{**}, \tag{2.10}$$

where c^{**} is the rate of energy transfer. Simple considerations lead to the following requirement: c^{**} must transform like the velocity of a mass point.

Since, in accordance with (2.10),

$$c^{**}\cos\theta' = (c/i)S_{41}/u', \qquad c^{**}\sin\theta' = (c/i)\,S_{42}/u',$$

both tables, (2.7) and (2.8), yield the same result, namely:

$$c^{**} = \frac{c\sqrt{1-2n\beta\cos\theta+n^2\beta^2-\beta^2\sin^2\theta}}{n-\beta\cos\theta}. \tag{2.11}$$

At the same time, purely kinematic considerations lead to the expression for the phase velocity c^*.

This expression follows from the requirement that the phase of the wave must be invariant:

$$\varphi = \nu'\left(t' - \frac{x'\cos\alpha' + y\sin\alpha'}{c^*}\right) = \nu\left(t - \frac{x\cos\theta + y\sin\theta}{c/n}\right); \tag{2.12}$$

where α' and θ are the angles between the wave normal and the x axis in the primed and unprimed coordinate frames, respectively.

It follows from (2.12) and from the Lorentz transformations that

$$c^* = \frac{c(1-n\beta\cos\theta)}{\sqrt{n^2-2n\beta\cos\theta+\beta^2-n^2\beta^2\sin^2\theta}}, \tag{2.13}$$

and that the relationships defining α' and ν' are as follows:

$$\sin\alpha' = \frac{n\sin\theta\cdot c^*\sqrt{1-\beta^2}}{c(1-n\beta\cos\theta)}, \tag{2.14}$$

$$\cos\alpha' = \frac{c^*(n\cos\theta-\beta)}{c(1-n\beta\cos\theta)}, \tag{2.15}$$

$$\nu' = \nu\frac{1-n\beta\cos\theta}{\sqrt{1-\beta^2}}. \tag{2.16}$$

(See Moller,[6] p. 46 for detailed derivations). If we now consider a "wave train," we may assume that

$$\frac{J'}{\mathscr{E}'} = \frac{g'}{u'}, \tag{2.17}$$

where J' and $\mathscr{E}'$ are the momentum and energy of the wave train, respectively, $g' = (g_x'^2 + g_y'^2)^{1/2}$ is the momentum density, and u' is the energy density.

Correspondingly, from (2.7) and (2.8), we have

$$\frac{J'^{A}}{\mathscr{E}'^{A}} = \frac{\sqrt{1-2n\beta\cos\theta+n^2\beta^2-\beta^2\sin^2\theta}}{c(n-\beta\cos\theta)} \tag{2.18}$$

and

$$\frac{J'^{M}}{\mathscr{E}'^{M}} = \frac{\sqrt{n^2-2n\beta\cos\theta+\beta^2-n^2\beta^2\sin^2\theta}}{c(1-n\beta\cos\theta)}. \tag{2.19}$$

It is the consequences of these two tensors that lead to the controversy mentioned above, which, in the generalized form, follows from the ensuing comparison.

According to (2.18), (2.11), (2.13), and (2.16), it follows from (2.7) that

$$J'^{A} = \frac{\mathscr{E}'^{A}}{c^2}c^{**} \text{ — Einstein relation},$$

$$J'^{A} \neq \frac{\mathscr{E}'^{A}}{c^*} \neq \frac{h}{\lambda'} \text{ — violation of the de Broglie relation},$$

$$\mathscr{E}'^{A} \neq h\nu';$$

here,

$$\lambda' = \frac{c^*}{\nu'}, \qquad \nu' = \frac{\nu(1-n\beta\cos\theta)}{\sqrt{1-\beta^2}}, \tag{2.20}$$

where ν' is the Doppler frequency recorded by the observer relative to which the source and medium move with velocity $-\beta c$.

As a consequence of (2.19), (2.8), (2.13), (2.11), and (2.16), we obtain

$$J'^{M} = \frac{\mathscr{E}'^{M}}{c^*} = \frac{h\nu'}{c^*} = \frac{h}{\lambda'} \text{ — de Broglie relation},$$

and

$$J'^{M} \neq \frac{\mathscr{E}'^{M}}{c^2}c^{**} \text{ — violation of Einstein relation}.$$

Generalization of the results noted up to (1.6) will require the derivation of certain relationships that follow from the geometry or, more precisely, the kinematics of wave packets.

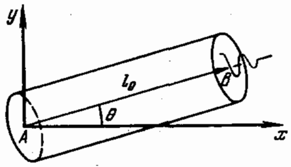

FIG. 2.

To derive these relationships, let us consider a wave packet (train) of given profile in the unprimed coordinate frame, and a medium at rest in this frame.

For simplicity, we shall suppose that the wave surfaces (or, as we shall call them, the planes of equal phase) are bounded by the surface of a cylinder of sufficiently large radius which is nevertheless small in comparison with the length of the wave train.

Moreover, we shall simplify the geometry by assuming that the sections cut by the planes of equal phase through the cylindrical volume Ω filled with the wave train are circular, and that these planes are perpendicular to the axis of the cylinder.[7] The transverse cross sections A and B of the cylinder Ω are shown in Fig. 2, and we regard them as the conditional boundaries of the wave packet. The centers A and B of these sections lie on the axis of the cylinder, which is at an angle θ to the x axis. Suppose that, at time $t = t' = 0$, the origins of both reference frames (x, t and x', t') coincide at the point A. In the x, t frame, the points B and A move with velocity whose x component is $c\cos\theta/n$.

In view of Fig. 2, and applying the Lorentz transformation to the coordinates of B, we obtain the following expressions:

$$\left.\begin{aligned} x_B &= l_0\cos\theta + t_B\frac{c\cos\theta}{n},\\ t_B &= \frac{t'+\beta x'_B}{\sqrt{1-\beta^2}},\\ x_B &= \frac{x'_B+\beta t'}{\sqrt{1-\beta^2}}. \end{aligned}\right\} \qquad (2.21)$$

Eliminating x_B and t_B from these equations, we have, after some simple intermediate steps,

$$x'_B = c\frac{\cos\theta - n\beta}{n-\beta\cos\theta}t' + \frac{nl_0\cos\theta\sqrt{1-\beta^2}}{n-\beta\cos\theta}, \qquad (2.22)$$

$$x'_A = c\frac{\cos\theta - n\beta}{n-\beta\cos\theta}t'. \qquad (2.23)$$

(The second equation is obtained from the first by putting $l_0 = 0$.)

An analogous procedure easily yields the expression for $y'_B = y_B$:

$$y'_B = c\frac{\sin\theta\sqrt{1-\beta^2}}{n-\beta\cos\theta}t' + \frac{n\sin\theta l_0}{n-\beta\cos\theta} \qquad (2.24)$$

and, correspondingly,

$$y'_A = c\frac{\sin\theta\sqrt{1-\beta^2}}{n-\beta\cos\theta}t'. \qquad (2.25)$$

From (2.22) and (2.24), we obtain the expressions for the velocity c^{**} and its x' and y' components:

$$c^{**}\cos\theta' = \frac{dx'_B}{dt'} = \frac{dx'_A}{dt'} = c\frac{\cos\theta - n\beta}{n-\beta\cos\theta}, \qquad (2.26)$$

$$c^{**}\sin\theta' = \frac{dy'_B}{dt'} = c\frac{\sin\theta\sqrt{1-\beta^2}}{n-\beta\cos\theta}, \qquad (2.27)$$

$$c^{**} = c\sqrt{\left(\frac{dx'_B}{dt'}\right)^2 + \left(\frac{dy'_B}{dt'}\right)^2} = c\frac{\sqrt{1-2n\beta\cos\theta + n^2\beta^2 - \beta^2\sin^2\theta}}{n-\beta\cos\theta}. \qquad (2.28)$$

We have thus verified the fact that kinematics[8] leads to the expression for c^{**} given by (2.28), which is identical with (2.11) obtained from the tensors (2.7) and (2.8).

We can now use (2.26)–(2.28) to determine the angle θ':

$$\cos\theta' = \frac{c(\cos\theta - n\beta)}{c^{**}(n-\beta\cos\theta)} = \frac{\cos\theta - n\beta}{\sqrt{1-2n\beta\cos\theta + n^2\beta^2 - \beta^2\sin^2\theta}}, \qquad (2.29)$$

$$\sin\theta' = \frac{c\sin\theta\sqrt{1-\beta^2}}{c^{**}(n-\beta\cos\theta)} = \frac{\sin\theta\sqrt{1-\beta^2}}{\sqrt{1-2n\beta\cos\theta + n^2\beta^2 - \beta^2\sin^2\theta}}. \qquad (2.30)$$

The velocity c^{**} is the velocity of the wave packet.

The planes of equal phase can be imagined as moving together with the packet, and the phase velocity c^* can be looked upon as a component of c^{**} along the normal to these planes. The angle χ between the directions of c^{**} and c^* is therefore given by

$$\cos\chi = \frac{c^*}{c^{**}}; \qquad (2.31)$$

The quantity $\cos\chi$ can be determined with the aid of (2.29), (2.30), (2.14), and (2.15). In accordance with (2.31), such calculations yield

$$\cos\chi = \cos\theta'\cos\alpha' + \sin\theta'\sin\alpha' = \frac{c^*}{c^{**}}. \qquad (2.32)$$

We now note the following consequence of the above relationship.

According to (2.13), $c^* = 0$ for $\cos\theta_0 = 1/n\beta$, i.e., for directions lying on the surface of the Cerenkov radiation cone ($\theta = \theta_0$). On the other hand, (2.32) shows that, in this "singular" case, $\cos\chi = 0$, i.e., *the direction of* c^{**} *is parallel to the planes of equal phase. This means that energy is transported parallel to the wave front.*

It is important to note that, when we speak of the Cerenkov radiation cone, we have in mind (here and henceforth) a source of radiation moving in a stationary medium with velocity β equal to the given velocity of the origin of the primed coordinates (relative to the medium). Figure 3 shows schematically the orientation of the wave packet ($A'B'$) in the x', t' frame and the directions of the vectors c^{**} and c^* in the following three cases: $\theta < \theta_0$; $\theta = \theta_0$; $\theta > \theta_0$.

It will be useful in the ensuing analysis to note the

[7] These simplifying assumptions can be removed. The direction of the axis of the wave-packet tube need not be parallel to the direction of the normal to the planes of equal phases. This is the situation when the medium is stationary in the given reference frame and the source moves relative to the medium.

[8] Another purely kinematic derivation can be based on the application of the Lorentz transformations to the Huygens construction for the wave surface. This is discussed by Møller, [6] p. 47.

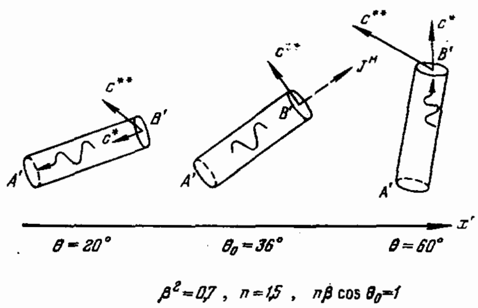

FIG. 3.

following property of radiation in the x', t' frame.

When the velocity β is greater than the velocity of light, i.e., $\beta > 1/n$, and the source of light is *stationary* in the x', t' frame, the only possible ray directions are those lying within a particular cone that opens in the direction of negative values of x. This can be verified, for example, as follows. Using (2.29), which defines $\cos\theta'$, we obtain (by solving the corresponding quadratic equation), the formula for the reverse transformation, i.e., the dependence of $\cos\theta$ on $\cos\theta'$. This formula is:

$$\cos\theta = \frac{n\beta(1-\cos^2\theta') \pm \cos\theta'\sqrt{(1-\beta^2)(n^2\beta^2\cos^2\theta' - \beta^2\cos^2\theta' - n^2\beta^2 + 1)}}{1-\beta^2\cos^2\theta'}. \tag{2.33}$$

When

$$\cos\theta_0' = -\sqrt{\frac{n^2\beta^2-1}{\beta^2(n^2-1)}}, \tag{2.34}$$

the expression under the square root is equal to zero. When $|\cos\theta'| < |\cos\theta_0'|$ $(|\theta'| < |\theta_0'|)$, $\cos\theta$ becomes imaginary.

Next, it follows from (2.33) that, when $\cos\theta_0'$ is given by (2.34), we have $\cos\theta_0 = 1/n\beta$, i.e., the directions on the surface of the cone of angle θ_0' correspond to the condition for Cerenkov radiation. The dependence of θ' on θ is not monotonic. For each value of θ' within the allowed cone, there are two values of θ, i.e., two directions, namely, one within the "Cerenkov cone" and the other outside this cone, in the x, t frame.

In the next section, we shall discuss in detail the relation $J'^{M} = J'^{A} + \Delta J'$, given by (1.6). To calculate $\Delta J'$, we require an expression for the *area* S' cut through a "wave train tube" by the planes of equal phase. This expression can again be obtained by applying simple kinematic relationships which we reproduce without proof for brevity. To obtain the required result, we must introduce a third reference frame, in addition to the above two, in which the direction of c^{**} coincides with that of the y' axis.

The three reference frames which we shall use in the following derivation will be indicated by the numbers I, II, III and the corresponding velocities β will also be labeled with subscripts I, II, III.

Frame III is the system just mentioned.

Frame II is the "primed set of coordinates" introduced above.

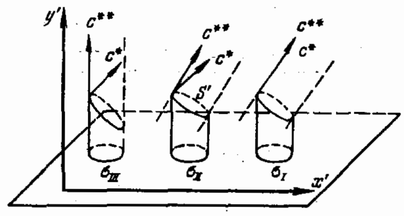

FIG. 4.

Frame I is the "unprimed set" in which the medium is stationary.

The formulas given below involve the velocities β_I and β_{II}. This notation is meant to indicate the velocity of the *third* system *relative* to I and II, respectively.

It is clear from (2.29) that

$$\beta_I = \frac{\cos\theta}{n}. \tag{2.35}$$

The rule for the addition of velocities yields

$$\beta_{II} = \frac{\beta-\beta_I}{1-\beta\beta_I} \tag{2.36}$$

and we recall that β is the velocity of the "primed set" determined by an observer in frame I. Hence, finally,

$$\sqrt{1-\beta_{II}^2} = \frac{\sqrt{1-\beta^2}\sqrt{1-\beta_I^2}}{1-\beta\beta_I}. \tag{2.37}$$

The remaining designations are illustrated in Fig. 4. (The letter σ represents the corresponding areas of the projections of the areas S onto the x, z plane.)

It is easily verified that the following results are valid:

$$\sigma_{III}\sqrt{1-\beta_I^2} = \sigma_I = \sigma_0, \tag{2.38}$$

$$\sigma_{III}\sqrt{1-\beta_{II}^2} = \sigma_{II} = \sigma'. \tag{2.39}$$

Hence, using (2.38), (2.39), (2.37), (2.35), and (2.14), we obtain, after some simple intermediate steps,

$$S' = S_0 \frac{\sin\theta}{\sin\alpha'} \frac{\sqrt{1-\beta^2}}{n-\beta\cos\theta} = S_0 \frac{c\,|1-n\beta\cos\theta|}{c^*(n-\beta\cos\theta)}. \tag{2.40}$$

This is valid when $n\beta\cos\theta \neq 1$.[9)]

A more direct and more detailed derivation is given in Appendix 1.

Here, however, we can use (2.15) and write down a further formula which we shall need later:

$$S'|\cos\alpha'| = S_0 \frac{c\,|1-n\beta\cos\theta|}{c^*(n-\beta\cos\theta)} \frac{c^*(n\cos\theta-\beta)}{c\,|1-n\beta\cos\theta|} = S_0 \frac{n\cos\theta-\beta}{n-\beta\cos\theta}. \tag{2.41}$$

We shall also need the expression for the volume Ω':

$$\Omega' = S'l'; \tag{2.42}$$

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9)The *size of the source* is taken into account implicitly in (2.40) by specifying S_0. When $n\beta\cos\theta = 1$ (and $S_0 = \infty$), the size of the source has to be taken explicitly into account. This exhibits the "inadequacy" of the macroscopic model when it is applied to the problems treated in this paper.

where l' is the distance in the primed coordinate frame between the planes of the cross sections S'_B and S'_A. It is required to determine the length l' for a given l_0 (between the same cross sections A and B) in the unprimed frame. The simplest way to obtain this is to use the invariance of the phase, which yields

$$\frac{l'}{\lambda'}=\frac{l_0}{\lambda_0} \tag{2.43}$$

where λ is the wavelength. Next,

$$\frac{\lambda'}{\lambda_0}=\frac{n\nu_0 c^*}{\nu' c}=\frac{\sqrt{1-\beta^2}}{1-n\beta\cos\theta}\frac{nc^*}{c}.$$

[Here, we have used the Doppler relation given by (2.16).] Thus,

$$l'=l_0\frac{\sqrt{1-\beta^2}}{1-n\beta\cos\theta}\frac{nc^*}{c}. \tag{2.44}$$

Equations (2.42), (2.40), and (2.44) yield[10]

$$\Omega'=S'l'=\Omega_0\frac{n\sqrt{1-\beta^2}}{n-\beta\cos\theta} \tag{2.45}$$

where $\Omega_0=S_0l_0$.

3. MOMENTUM OF A WAVE TRAIN IN A MOVING MEDIUM AND MOMENTUM TRANSFERRED TO THE MEDIUM WHEN A WAVE TRAIN IS EMITTED BY THE SOURCE

It was noted in the Introduction that attention had been drawn in the literature to the fact that (1.6) could, allegedly, be used to reconcile the Minkowski and Abraham concepts.

Thus, if the emission of a wave train by a source fixed in a *stationary medium* is considered classically, and use is made of a simplified model, simple calculations show that the sum of the momentum J, carried off by the emitted photon (according to Abraham's formula), and the additional momentum ΔJ communicated by the Abraham forces to the medium during the emission process itself, is equal to the momentum of the "Minkowski photon" (see the Introduction).

Henceforth, we shall discuss this situation (emitter stationary relative to the medium) in a more general form. In particular, we shall assume that the two objects (source and medium) move *relative to the observer* with equal arbitrary velocities $-\beta c$.

The above result is obtained if we calculate the momentum ΔJ due to the Abraham forces acting on the medium during the emission process within the volume of the radiated wave train. (The effective volume, by the way, is not the entire volume of the wave train but only a certain boundary zone near its leading front.) The "recoil" experienced by the source is the reaction of the radiation, assumed equal to the sum $J+\Delta J$. A more detailed discussion of the *assumptions* implicit in this and of the momentum flux balance will be given in Appendix 2.

General considerations already lead to the invariance or covariance of (1.6). This will be confirmed by direct calculations. However, we are concerned with the covariance under a transformation of the point of view of the observer (in the one case, fixed and, in the other, moving relative to the medium) subject to the condition that the above physically defined situation is maintained, i.e., the source is *fixed relative to the medium*.

Questions connected with the covariance of (1.6) are discussed in Appendix 3.

In our system of primed coordinates, the source (and medium) moves with velocity $-\beta$. When we calculate the momentum $\Delta J'$, the emission time determined by an observer in the primed set of coordinates will be set equal to τ'. If, as we shall suppose, the "clock hand" of the source indicates zero at the *beginning* of the emission process, then, at the *end* of the process, it will indicate the "proper time" $\tau=\tau'(1-\beta^2)^{1/2}$. During this time, the "head" of the wave train, which moves with velocity c/n (in the x, t frame), will traverse a distance (from the source that is at rest in this frame) given by

$$l_0=\tau\frac{c}{n}=\tau' c\frac{\sqrt{1-\beta^2}}{n} \tag{3.1}$$

(The propertime τ of the source is equal to the time t in the unprimed frame in which the source is at rest.) Let us suppose that the cross section S_0, cut by the plane of equal phases, is equal to unity, so that

$$\Omega_0=l_0 \tag{3.2}$$

and (2.45) and (3.1) give

$$\Omega'=l_0\frac{n\sqrt{1-\beta^2}}{n-\beta\cos\theta}=\frac{\tau' c(1-\beta^2)}{n-\beta\cos\theta}, \tag{3.3}$$

$$J'=g'\Omega'. \tag{3.4}$$

The expression given by (3.3) was derived on the assumption that the source was *at rest relative to the medium* (both in the x', t' and x, t frames).

We must now consider another expression for the volume of the wave train, which we shall denote by Ω'^* and which we shall require below. The point is that we are

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[10] By calculating the products $\Omega' g'$ and $\Omega' u'$, where g' and u' are given by (2.7) and (2.8), we obtain J' and $\mathscr{E}'$, respectively, where J' and $\mathscr{E}'$ are the momentum and energy of the given wave train, respectively. Substitution for g_x from (2.7) yields, for example,

$$\left.\begin{aligned} &J'_x=\Omega_0 u_0\frac{\cos\theta-n\beta}{cn\sqrt{1-\beta^2}}=\mathscr{E}_0\frac{\cos\theta-n\beta}{cn\sqrt{1-\beta^2}},\\ &\text{according to Abraham}\\ &\mathscr{E}'=\mathscr{E}_0\frac{n-\beta\cos\theta}{n\sqrt{1-\beta^2}}.\end{aligned}\right\} \tag{2.46}$$

Table (2.8) gives

$$\left.\begin{aligned} &J'_x=\frac{\mathscr{E}_0}{c}\frac{n\cos\theta-\beta}{\sqrt{1-\beta^2}},\\ &\text{according to Minkowski}\\ &\mathscr{E}'=\mathscr{E}_0\frac{1-n\beta\cos\theta}{\sqrt{1-\beta^2}}.\end{aligned}\right\} \tag{2.47}$$

These relationships correspond to the transformation formulas for the momentum-energy four-vector and (if we use the expressions for J_y not given here) the formulas (2.18) and (2.19) above.

concerned with a wave train emitted by a source that is *stationary in the* x', t' *frame*. The *medium is in motion* relative to this source. The volume Ω'^* can then be determined as follows.

Under the above conditions, the length of the wave train along the normal to the planes of equal phases in the x', t' frame is

$$l'_* = c^*\tau'.$$

In the unprimed (laboratory) system, the length l' of the wave train corresponds to the length l_0 given by (2.44). Consequently,

$$l_0 = l' \frac{c(1-n\beta\cos\theta)}{\sqrt{1-\beta^2}\, nc^*} = \tau' \frac{c\,|1-n\beta\cos\theta|}{\sqrt{1-\beta^2}\, n}. \tag{3.5}$$

Since, by hypothesis, $S_0 = 1$, it follows that $\Omega_0 = l_0$.

According to (2.45) and (3.5)

$$\Omega'^* = \tau' c \frac{|1-n\beta\cos\theta|}{n-\beta\cos\theta}. \tag{3.6}$$

We note that

$$\Omega' \neq \Omega'^*. \tag{3.7}$$

The x' component of the momentum $\Delta J'$ transferred in a time dt' is given by the following integral:

$$dt' \int_{x'}^{\infty} f'_x S' \cos\alpha'\, dx';\tag{3.8}$$

where f'_x denotes the component of the force density and $S' \cos\alpha' dx'$ is a volume element of the ray tube (see Fig. 5).[11] We are dealing with the density of the Abraham force, which can be obtained from the divergences of the tensors (2.5) and (2.6). By considering the integral given by (3.8), we can verify that the average values of this integral are time-independent. (We are concerned with averages over a small time interval dt which is, nevertheless, large in comparison with the period of the field oscillations.) It therefore follows from (3.8) that

$$\Delta J' = \tau' \int_{x'}^{\infty} f'_x S' \cos\alpha'\, dx';\tag{3.9}$$

where τ' is the time taken for the emission of the pulse of light.

The formulas given below will show that the force density averaged in this way is proportional to the partial derivative of the average value of the energy density u (or u') with respect to x (or x'), written as a function of the argument

$$\varphi = \nu' \left(t' - \frac{x'\cos\alpha' + y'\sin\alpha'}{c^*}\right). \tag{3.10}$$

[11]When $\beta < 1/n$, we must distinguish between the two cases corresponding to $\cos\theta > n\beta$ and $\cos\theta < n\beta$. The situation illustrated in Fig. 5 corresponds to the first of these two cases. The required integral is the same in both cases.

When $\beta > 1/n$ and $n\beta\cos\theta > 1$ for a source that is stationary in (x', t'), the quantity ΔJ is given by (3.9) with a minus sign.

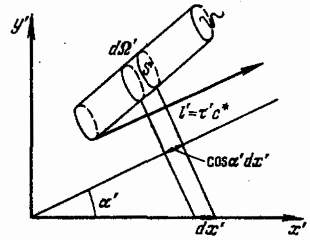

FIG. 5.

We are concerned with the mean value of u for points ("isochronous" points) on a given plane of equal phases. The required integral is independent of the form of this function provided, and we shall assume this, that the following conditions are satisfied:

$$\text{for } t' \gg \frac{x'\cos\alpha' + y\sin\alpha'}{c^*}, \ u' \text{ is independent of } x',\ y', \tag{3.11}$$

$$\text{for } t' < \frac{x'\cos\alpha' + y'\sin\alpha'}{c^*}, \ u' = 0. \tag{3.12}$$

We are assuming that the emission of light begins at time $t' = 0$. It follows from these conditions that the integral in (3.9) (at a given time t') includes contributions due to the variables x' within a certain finite interval $\Delta\varphi$ (and, correspondingly, $\Delta x'$ and $\Delta y'$), i.e., provided

$$0 < \nu' \left(t' - \frac{(x'-\Delta x')\cos\alpha' + (y'-\Delta y')\sin\alpha'}{c^*}\right) < \Delta\varphi \tag{3.13}$$

at

$$t' - \frac{x'\cos\alpha' + y\sin\alpha'}{c^*} = 0.$$

Evaluation of the divergences of the tensor involves the partial derivatives of the density with respect to three variables $(x, y, t$ and $x', y', t')$. As already noted, since we are concerned with a function of the argument φ, the derivatives with respect to y and t can be expressed in terms of definite functions, the argument of which involves only one derivative, $\partial/\partial x$ or $\partial/\partial x'$. This will be clear if we consider the following differential relationships:

$$\frac{\partial u}{\partial x} = \frac{du}{d\varphi}\frac{\partial\varphi}{\partial x}, \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial\varphi}\frac{\partial\varphi}{\partial y}, \quad \frac{\partial u}{\partial t} = \frac{du}{d\varphi}\frac{\partial\varphi}{\partial t}.$$

Next,

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial x}\frac{\sin\theta}{\cos\theta}, \quad \frac{\partial}{\partial t} = -\frac{\partial}{\partial x}\frac{1}{n\cos\theta}, \tag{3.14}$$

and, if we use the Lorentz transformation formulas,

$$\left.\begin{aligned} x &= (x' + \beta t')\gamma, \\ t &= (t' + \beta x')\gamma, \\ y &= y' \end{aligned}\right\} \tag{3.15}$$

(we have substituted $c = 1$), we have

$$\begin{aligned} \left(\frac{\partial}{\partial x'}\right)_{t'} &= \frac{\partial}{\partial x}\left(\frac{dx}{dx'}\right)_{t'} + \frac{\partial}{\partial t}\left(\frac{dt}{dx'}\right)_{t'}, \\ \left(\frac{\partial}{\partial t'}\right)_{x'} &= \frac{\partial}{\partial x}\left(\frac{dx}{dt'}\right)_{x'} + \frac{\partial}{\partial t}\left(\frac{dt}{dt'}\right)_{x'}. \end{aligned} \tag{3.16}$$

Hence:

$$\frac{\partial}{\partial x'} = \frac{\partial}{\partial x}\gamma + \frac{\partial}{\partial t}\beta\gamma = \frac{\partial}{\partial x}\gamma\frac{n\cos\theta - \beta}{n\cos\theta}, \tag{3.17}$$

$$\frac{\partial}{\partial y'} = \frac{\partial}{\partial y} = \frac{\partial}{\partial x'}\frac{n\sin\theta}{\gamma(n\cos\theta - \beta)}, \tag{3.18}$$

$$\frac{\partial}{\partial t'}=\frac{c(n\beta\cos\theta-1)}{n\cos\theta-\beta}\frac{\partial}{\partial x'}. \quad (3.19)$$

In (3.9), we now substitute

$$f'_x=-\operatorname{div}S'_{lm}=\operatorname{div}P'_{lm}, \quad (3.20)$$

where P_{lm} is the tensor defined by (2.6). [We recall that $T_{lm}=S_{lm}+P_{lm}$ and that $\operatorname{div}T_{lm}=0$, and this leads directly to (3.20).]

Using (3.17)–(3.19) and the table given in (2.6), we obtain the following expression:

$$f'_x=\operatorname{div}P'_{lm}=-\frac{n^2-1}{n^2}\frac{\partial u_0}{\partial x'}\left[\frac{\cos^2\theta}{1-\beta^2}+\frac{n\sin^2\theta\cos\theta}{n\cos\theta-\beta}-\frac{\beta\cos^2\theta(n\beta\cos\theta-1)}{(n\cos\theta-\beta)(1-\beta^2)}\right], \quad (3.21)$$

and then, after some simple rearrangement,

$$f'_x=-\frac{n^2-1}{n}\left(\frac{\partial u_0}{\partial x'}\right)_{t'}\frac{\cos\theta}{n\cos\theta-\beta}. \quad (3.22)$$

Let us suppose that the emission time for the wave train is $\tau'=1$. Substituting (3.22) and (2.41) in (3.8), and then setting $S_0=1$, we obtain

$$\Delta J'^{A}_x=-\int_{x'}^{\infty}\frac{n^2-1}{n}\frac{\cos\theta}{n\cos\theta-\beta}\frac{n\cos\theta-\beta}{n-\beta\cos\theta}\frac{\partial u_0}{\partial x'}dx'=\frac{n^2-1}{n}\frac{\cos\theta u_0}{n-\beta\cos\theta}. \quad (3.23)$$

According to (3.3) and (3.4) with $\tau'=1$,

$$J'^{A}_x=\frac{(1-\beta^2)c}{n-\beta\cos\theta}g'^{A}_x.$$

The table in (2.5) yields

$$J'^{A}_x=\frac{u_0}{n(n-\beta\cos\theta)}(\cos\theta-n\beta\cos^2\theta-n\beta+\beta^2\cos\theta). \quad (3.24)$$

Combining (3.23) and (3.24), and rearranging, we obtain

$$J'^{A}_x+\Delta J'^{A}_x=u_0\frac{n\cos\theta-\beta}{n}. \quad (3.25)$$

Evaluating J'^{M}_x in accordance with the table in (2.8) and substituting $g'_x=S_{14}/ic$, we obtain

$$J'^{M}_x=\Omega' g'^{M}_x=u_0\frac{(n\cos\theta-\beta)}{n}. \quad (3.26)$$

Comparison of (3.25) and (3.26) will show that

$$J'^{A}_x+\Delta J'^{A}_x=J'^{M}_x, \quad (3.27)$$

i.e., (1.6) is satisfied in the primed set of coordinates as well. We recall that *both* the medium and the source move with velocity $-\beta$ relative to the observer in this system. A similar argument will readily show that the y' component will also satisfy (3.27). Both sides of (3.27) contain the space components of a four-vector. The time components of this vector satisfy the analogous equation, which can be verified by similar calculations:

$$\mathscr{E}'^{A}+\Delta\mathscr{E}'=\mathscr{E}'^{M}; \quad (3.28)$$

where $\mathscr{E}'^{A}$ and $\mathscr{E}'^{M}$ are the energies of the given wave train (according to Abraham and Minkowski, respectively). The term $\Delta\mathscr{E}'$ is the energy communicated to the moving medium during the "injection" of the momentum $\Delta J'$ into it. It is given by the four-dimensional divergence of the fourth row of the tensor S'_{lm} multiplied by i/c and taken with the opposite sign.

Under the above conditions, the energy $\Delta\mathscr{E}'$ is negative: the energy of the electromagnetic field *increases*[12] upon injection of a positive momentum at the expense of the kinetic energy of the medium (the work done by the field forces is negative).

Our calculations have been based on the expressions for the components of the momentum-energy tensor. A much simpler way of obtaining the same result is to start with the *general relativistic* relationships for the components of the force and the force density.

A general proposition in relativistic mechanics is that the longitudinal (in the direction of the velocity β) component of a force acting on an element of volume $d\omega$ of a body is invariant under the Lorentz transformation. The "transverse" component, on the other hand, is transformed by multiplication by the factor $(1-\beta^2)^{1/2}$ (see Møller,[6] p. 59).

Hence, it follows that the force *densities* f'_x and f'_y are given by

$$f'_x=f_x\gamma,\quad f'_y=f_y. \quad (3.29)$$

According to Abraham,

$$f_x=-\frac{n^2-1}{n^2}\frac{\partial u_0}{\partial x}. \quad (3.30)$$

Using (3.17), we obtain immediately

$$f'_x=-\frac{n^2-1}{n}\frac{\partial u_0}{\partial x'}\frac{\cos\theta}{n\cos\theta-\beta}, \quad (3.31)$$

which is identical with (3.22).

We have thus verified that (1.6) remains valid under a transformation to the moving coordinate frame (x', t') provided the source of radiation is *at rest relative to the medium*. A consequence of this result is that, if the source is *stationary in the primed set of coordinates* and, therefore, *moves relative to the medium*, Eq. (1.6) is not satisfied.

In point of fact, it is clear from the derivation that, for *given* values of the density (u_0 or u') and *given* time τ' (which we have set equal to unity),[13] the quantity $\Delta J'$ is *independent* of whether the source emitting the radiation is moving (in the given reference frame) or is at rest because we are concerned with the force impulse acting on the wave front. Moreover, the volume Ω' of the wave train *depends*, as we have shown, on the state of motion of the source of these waves. If the source *moves* in the x', t' frame (*together with the medium*) with velocity $-\beta$, then, as we have shown

$$\Omega' g'^{A}_x+\Delta J'_x=J'^{M}_x. \quad (3.32)$$

Since, according to (3.7), $\Omega'^{*}\neq\Omega'$ for a *fixed* source,

[12] $\Delta\mathscr{E}=0$ in the primed set of coordinates.

[13] The values of these quantities are completely arbitrary. The value of the angle θ is assumed *given*.

and $\Delta J'$ is the same as in (3.32) (fixed source), it follows that (1.6) is not satisfied if the source is fixed in the (x', t') frame, i.e., if the source moves relative to the medium,

$$g_x'^{A}\Omega'^{*}+\Delta J_x'\neq J_x'^{M}.$$

This also applies to the other components (along y and t) if we consider the generalized, four-dimensional form of (1.6). In particular, in the primed frame, Eq. (3.28) for a *fixed source* and a *moving medium* is also *not* satisfied.

Finally, we note that, bearing in mind the expression for $\Delta J'$ (which, as we have shown, can be derived without writing out the tensor table), we can immediately determine $\Delta \mathscr{E}$ as well, since we must have $\Delta \mathscr{E}=-\beta c\,\Delta J_c'$ ($-\beta c$ is the velocity of the medium).

4. ON THE PAPER BY V. L. GINZBURG AND V. A. UGAROV[2]

Ginzburg and Ugarov attempt to show that the Minkowski and Abraham treatments as applied to the conservation of momentum and energy are equivalent. They claim to have proved the validity of (3.27) and (3.28) even for a *source moving relative to the medium*. Moreover, they state that this generalization of (1.6) follows from "general considerations."

The question is: How has the proof of the above proposition been carried out? In fact, certain particular terms involving the *spatial* divergences of functions of field variables are neglected in the formula for the density of the Abraham force f^{A}. Declaring, in general, that, when these relationships are written in integral form, the terms containing the spatial divergences cancel out, the above authors maintain that the solution of their (and our) problem can be obtained by starting from the following expression for the density of the ponderomotive force:

$$f'^{\Gamma}=\frac{\partial}{\partial t'}(g^{M}-g^{A})$$

[this is Eq. (46) in the paper by Ginzburg and Ugarov[2][14]]. In our notation, we have for a plane-wave field

$$f_x'^{\Gamma}=\frac{\partial}{\partial t'}(g_x^{M}-g_x^{A})=\frac{\partial}{\partial t'}\frac{u_0(x',t')(n^2-1)\cos\theta}{cn(1-\beta^2)}\neq f_x'^{A}, \qquad (4.1)$$

where $f_x'^{A}$ is given by (3.22).

If, instead of (3.22), we substitute (4.1), "truncated" as indicated above, into (3.9), and carry out all the calculations as in Chap. 3, it does, in fact, turn out that, when the *source is fixed in a moving medium* (in the x', t' frame), Eq. (1.6) is valid. This is what Ginzburg and Ugarov wish to prove. However, when the source is *moving* (in the x', t' frame) but is *stationary* relative to the medium, the result is different. The general considerations of Chap. 3, on the other hand, lead to the conclusion that calculations based on (3.9) show, in this case, that, when (4.1) is adopted as the expression for the force density, the equation given by (1.6) is not satisfied under the conditions under which relativistic covariance demands that it should be satisfied.[15]

This means that the results obtained by Ginzburg and Ugarov,[2] which appear to satisfy them, were, in fact, obtained by violating the general principles of relativistic mechanics that govern the choice of the expressions for the force (or density of force) in a moving medium.[16]

5. QUANTUM THEORY OF THE DOPPLER EFFECT AND CONCLUDING REMARKS

We shall now consider the derivation of the formulas of the quantum theory of the Doppler and the Cerenkov effects given by Ginzburg and Frank[10] (see also the paper by Frank[11]). It will be convenient to choose the reference frame in which the radiating atom (source) is at rest and the medium is moving. This simplifies the final expressions. If the expression for the energy $\mathscr{E}_{ph}'$ of a photon in this reference frame is already available (and this problem has been solved), transformation to the laboratory system is achieved simply by multiplication by a certain factor. Since we shall be concerned with the application of conservation laws based on the Minkowski theory, this factor is determined by the general rules for the transformation of the components of four-vectors, and is given by

$$\mathscr{E}_{ph}=\mathscr{E}_{ph}'\frac{\sqrt{1-\beta^2}}{1-n\beta\cos\theta}. \qquad (5.1)$$

We are assuming that, in its initial state, the radiating atom is stationary in the given reference frame. Its *rest mass* will be denoted by μ_1. The emission of a photon of energy $\mathscr{E}_{ph}'$ results in a change in the rest mass of the atom. The "proper mass" of the atom in the final state will be denoted by μ_2.

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[14]The quantity f'^{Γ} represents the $f_x'^{A}$ in Eq. (46) of the Ginzburg-Ugarov paper.[2]

[15]See Appendix 3 below for further details on the covariance conditions.

[16]Ginzburg and Ugarov[2] emphasize that integration with respect to time includes "...the period of time during which the wave train is already sufficiently 'separated from the emitter'" (p. 186 of original, 100 of translation). This explanation is surprising because once the wave train is "separated" from the emitter, the transfer of additional momentum to the medium ceases, i.e., as the wave train moves through the medium, it *transports* momentum equal to the product of the (Abraham) momentum density and the volume of the wave train. The momentum of the medium, on the other hand, remains constant.

Elastic forces must also be taken into account in the evaluation of ΔJ. Neglect of these forces will lead to errors that have probably influenced the results obtained by Ginzburg and Ugarov[2] (see the Appendix given below). Insofar as the calculations given in Chap. 3 of the present paper are concerned, the conditions formulated in Appendix 2 are particularly important in this connection. Ginzburg and Ugarov[2] cite the example of the "gas of heavy dust particles" discussed in our previous paper[1a] as confirmation of their results. In point of fact, Table (5.30) given in that paper is in direct conflict with these results.

We shall start by introducing the following simplified assumption: the photon recoil energy can be neglected in the given reference frame in comparison with the excitation energy.

In the above approximation, which is practically always acceptable for an atom, the energy balance equation can be written in the form

$$\mu_1 c^2 = \mu_2 c^2 + \mathscr{E}''_{\rm ph} \tag{5.2}$$

or

$$(\mu_1 - \mu_2) c^2 = \mathscr{E}'_{\rm ph}\,. \tag{5.3}$$

We shall consider the case of motion with speed in excess of the velocity of light and will assume that, in the laboratory system, the chosen direction of motion of the photon lies within the Cerenkov radiation cone. Since, according to Minkowski, the photon energy $\mathscr{E}'_{\rm ph}$ is then negative, Eq. (5.3) immediately predicts that $\mu_2 c^2 > \mu_1 c^2$. The energy $\mu_2 c^2$ of the atom after the emission is, therefore, greater than before emission: the result of emission is the excitation of the atom from a given energy level to a higher level. An observer at rest in the laboratory frame will conclude that the emission of light is accompanied by a partial conversion of the kinetic energy of the radiating atom into its excitation energy. Ginzburg has frequently emphasized (even quite recently[41]) this peculiar property of emission at velocities greater than that of the velocity of light.

This even leads to a possible application of this kind of effect. In particular, if we pass unexcited atoms traveling with velocities greater than the velocity of light through a "sieve" consisting of fine channels in a refracting medium, the emerging particles may, in principle, be found to be excited.

However, the above theory rests on assumptions that are in direct contradiction to the fundamentals of mechanics and electrodynamics.

This direct contradiction is apparent from the very fact that an electromagnetic field of negative energy is brought into play. Moreover, Eqs. (2.26) and (2.27) show that, within the Cerenkov radiation cone, the direction of the momentum of the Minkowski photon does not coincide with the energy-transport velocity c^{**}.

This is easily verified by comparing these formulas with Table (2.8).

In particular, for ray directions lying on the surface of the cone (for $n\beta\cos\theta_0 = 1$), the direction of the vector J'^M and the direction of the wave vector are perpendicular to the direction of c^{**}. This has already been noted [see (2.31) and (2.32)] and is shown in Fig. 3. Inside the cone, the component of the momentum along the x' axis is opposite in direction to the component of the velocity c^{**} along the same axis. Finally, if we consider the limiting case $\theta_0 = 0$ and $\beta = 1/n$, the photon energy is zero for nonzero momentum. These situations are in conflict with the fundamental Einstein formula $E = mc^2$.

Let us illustrate the foregoing by the following example, confining our discussion to the absorption of light, for simplicity, and assuming that $\beta = 1/n$.

We have the following situation in mind. A medium contains an absorbing sheet. In the laboratory reference frame, the wave train is moving in the direction of the absorber. In the reference frame introduced above, the wave train is at rest. The black sheet moves toward the wave train and absorbs it (Fig. 6). Absorption is accompanied by the transfer of the photon momentum to the medium whose motion is slowed down. The medium may be imagined to be a solid whose center coincides with the center of gravity (or center of mass) because the energy and mass of the photon are both zero. The slowing down of the medium due to the absorption of the photon gives rise to a change in its velocity, i.e., a slowing down of the center-of-mass system as a whole. It is easily verified that this violation of the law of motion of the center of mass (self-retardation) is also recorded by an observer in the laboratory reference frame.

The same violation of the velocity balance conditions (for the center of gravity) also occurs in the case of *emission* (emission of Minkowski photons). We also note that, if these violations are assumed for the *elementary* acts of emission, they must also be seen in emission by an ensemble of atoms on a macroscopic scale.

If in Eq. (5.3) we multiply $\mathscr{E}'_{\rm ph}$ by the factor (5.1), we obtain the Doppler effect formula

$$\mathscr{E}_{\rm ph} \approx \frac{(\mu_1 - \mu_2)\, c^2 \sqrt{1-\beta^2}}{1 - n\beta\cos\theta} = \frac{h\nu\sqrt{1-\beta^2}}{n\beta\cos\theta - 1} \tag{5.4}$$

for a moving source and an observer at rest relative to the medium.

If we abandon the above approximation and do not neglect the recoil energy, the energy balance equation in its rigorous form is

$$\mu_1 c^2 = \mu_2 c^2 \sqrt{1 + \left(\frac{\mathscr{E}_{\rm ph}}{\mu_2 c c^*}\right)^2} + \mathscr{E}'_{\rm ph}\,. \tag{5.5}$$

Here, we have taken into account the de Broglie relation $J'_{\rm ph} = \mathscr{E}'_{\rm ph}/c^*$, where $\mathscr{E}'_{\rm ph}$ and $J'_{\rm ph}$ are, respectively, the energy and momentum of the photon in the chosen system of primed coordinates.

This yields the following quadratic equation for $\mathscr{E}'_{\rm ph}$:

$$\mathscr{E}'^2_{\rm ph}\left(\frac{c^2}{c^{*2}} - 1\right) + 2\mu_1 c^2 \mathscr{E}'_{\rm ph} + (\mu_2^2 - \mu_1^2)\, c^4 = 0. \tag{5.6}$$

If we now use (2.13), we can easily show that

$$\frac{c^2}{c^{*2}} - 1 = \frac{(n^2-1)(1-\beta^2)}{(1 - n\beta\cos\theta)^2}\,. \tag{5.7}$$

Having solved (5.6) and having then transformed to the laboratory frame by multiplying $\mathscr{E}'_{\rm ph}$ by the factor given by (5.1), we obtain the result given by Ginzburg and

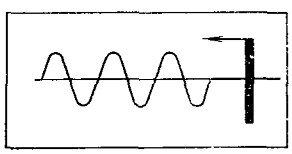

FIG. 6.

Frank[10] (in our case, for the excitation of a single atomic level).

If we are concerned with an *electron* traveling with a velocity greater than the velocity of light and, accordingly, put

$$\mu_2 = \mu_1 = m,$$

where m is the electron mass, (5.6) will directly lead to

$$\mathcal{E}_{\text{ph}} = h\nu = \frac{2mc^2(n\beta\cos\theta - 1)}{(n^2-1)\sqrt{1-\beta^2}}. \tag{5.8}$$

This formula can be read in different ways.

Reading left to right, we come to the conclusion that, since $h\nu$ is of the order of an electron volt (or even smaller) and mc^2 is of the order of 500 keV, then

$$n\beta\cos\theta = 1$$

which is the Cerenkov condition.

However, if we read this formula from right to left and assume that the theory is correct, we conclude that, when the condition $n\beta\cos\theta = 1$ is exactly satisfied, the frequency ν is zero and there is, therefore, no emission.

Jelley[12] refers to authors accepting the analogy with the Compton effect, and gives (5.8) in the following form:

$$\cos\theta = \frac{1}{n\beta} + \frac{\lambda_0}{\lambda}\frac{\sqrt{1-\beta^2}}{2\beta}\frac{n^2-1}{n^2}, \tag{5.9}$$

where $\lambda_0 = h/mc$ is the Compton wavelength.

For comparison, we can write down the Compton formula for the shift of a spectral line due to the scattering of light by a free electron. Of course, under real experimental conditions, this type of scattering cannot be observed. The formula is

$$\frac{\lambda_\theta}{\lambda} = 1 + \frac{\lambda_0}{\lambda}(1 - \cos\theta); \tag{5.10}$$

where θ is the scattering angle, λ_θ is the wavelength of light scattered at this angle, and λ is the wavelength of the primary radiation. The correction terms in (5.9) and (5.10) are of the same order. These quantum corrections are very small, but it is legitimate to ask whether these formulas are *exact* and whether the answer is different in these two cases.

The theory that leads to the quantum formula for the Compton effect has a prestige based on experimental confirmations whose number can hardly be imagined by now. Insofar as the quantum theory of the Cerenkov effect is concerned, there are no direct experimental confirmations of this theory and, at the same time, the theory is in conflict with the fundamental requirements of relativistic mechanics.

The quantum theory of the Cerenkov effect is based on the idea of a transition of an electron from one state of rectilinear motion to another state, and this transition is accompanied by the emission of radiation. It is difficult to imagine a specific experimental arrangement in which this kind of transition could be detected, even in principle. The picture used in the quantum model of the Cerenkov effect cannot be given a *direct* physical meaning. On the other hand, numerous experiments performed over many years enable us to claim that the *corpuscular* picture provides an adequate interpretation of the Compton effect. If this is so, we are justified in concluding that *any consequences* of the corpuscular model can, in principle, be *directly* confirmed by experiment.

Our final conclusion is that attempts at a quantum treatment of the above effects on the basis of simple quasiclassical models do not lead to satisfactory results.

This could have been foreseen.

Firstly, the model that we discussed in Chaps. 3 and 4 is highly simplified and, secondly, one could hardly expect to be able to describe a *quantum* mechanism of emission while remaining within the framework of *classical* ideas.

However, the interpretation of the "pseudomomentum" of a photon (or a quantum photon) as the resultant of two momenta, namely, the momentum of the photon and the "recoil momentum" received by the medium in the course of the emission of the photon, can be introduced as a postulate. The result is a set of relationships that is in agreement with the quantum equations obtained on the basis of the Minkowski "pseudomomentum." If we are considering covarient expressions, an important link in this scheme is the mechanism for the conversion of the kinetic energy $\Delta\mathcal{E}$ of the medium into the energy of the electromagnetic field on the wave front in the moving medium. To avoid model representations, and in the light of the ensuing analysis, it is better to speak of the conversion of the kinetic energy $\Delta\mathcal{E}$ into the energy of the electromagnetic field during the injection of mechanical momentum into the medium.

We shall assume, without proof, that

$$\mathcal{E}'^{\text{A}} + \Delta\mathcal{E}' = \mathcal{E}'^{\text{M}}.^* \tag{5.11}$$

From the tables given by (2.7) and (2.8), we have

$$\frac{\mathcal{E}'^{\text{M}}}{\mathcal{E}'^{\text{A}}} = \frac{u'^{\text{M}}}{u'^{\text{A}}} = \frac{(1-n\beta\cos\theta)\,n}{n-\beta\cos\theta},$$
$$\mathcal{E}'^{\text{M}} = \mathcal{E}'^{\text{A}}\frac{(1-n\beta\cos\theta)\,n}{n-\beta\cos\theta}. \tag{5.12}$$

Equation (5.11) then gives

$$\Delta\mathcal{E}' = -\mathcal{E}'^{\text{A}}\frac{(n^2-1)\beta\cos\theta}{n-\beta\cos\theta} = -\mathcal{E}'^{\text{M}}\frac{(n^2-1)\beta\cos\theta}{n(1-n\beta\cos\theta)}. \tag{5.13}$$

Since the velocity of the medium is $-\beta c$, we can now determine the component $\Delta J'_x$ of the injected momentum by analogy with the classical models, as follows:

$$\Delta J'_x = \frac{\Delta\mathcal{E}'}{-\beta c} = \frac{\mathcal{E}'^{\text{A}}(n^2-1)\cos\theta}{c(n-\beta\cos\theta)} = \frac{\mathcal{E}'^{\text{M}}(n^2-1)\cos\theta}{cn(1-n\beta\cos\theta)}. \tag{5.14}$$

We now use the (Einstein) relation

$$J'^{\text{A}} = \frac{\mathcal{E}'^{\text{A}}}{c^2}c^{**}. \tag{5.15}$$

Next, we make use of (2.26), (5.12), and (5.15) and write

$$J'_x = J'^{A}\cos\theta' = \mathscr{E}'^{A}\frac{\cos\theta - n\beta}{c(n-\beta\cos\theta)} = \mathscr{E}'^{M}\frac{\cos\theta - n\beta}{c(1-n\beta\cos\theta)\,n}. \tag{5.16}$$

Hence, combining (5.14) and (5.16), we obtain

$$J'^{A}\cos\theta' + \Delta J'_x = \mathscr{E}'^{M}\frac{n\cos\theta - \beta}{c(1-n\beta\cos\theta)}. \tag{5.17}$$

The sum $J'^{A} + \Delta J'$ will be called the "effective momentum." Finally, we introduce the de Broglie relation

$$\frac{\mathscr{E}'^{M}}{c^{*}} = J'^{M}, \tag{5.18}$$

and obtain

$$J'^{A}\cos\theta' + \Delta J'_x = J'^{M}\frac{c^{*}(n\cos\theta - \beta)}{c(1-n\beta\cos\theta)} = J'^{M}\cos\alpha'. \tag{5.19}$$

[This last equation was obtained with the aid of (2.15).] The "effective momentum" is, therefore, the same as the Minkowski pseudomomentum.[17]

The first part of (5.11) contains $\mathscr{E}'^{M}$ which turns out to be negative for a ray inside the Cerenkov radiation cone. According to Minkowski, $\mathscr{E}'^{M}$ is the energy of the field (wave packet). According to (5.11), the energy of the photon (wave packet) is $\mathscr{E}'^{A}$ and is *positive*. The quantity $\Delta\mathscr{E}'$ on the left-hand side of (5.11) is negative, and this is in agreement with the classical picture of the phenomenon.

The mechanism of the momentum transfer between the radiating source and the medium cannot, however, be described in terms of the above primitive model, or probably any other model based on purely classical ideas.[18]

It is clear from (5.13) that $\Delta\mathscr{E}'$, and consequently (for a given β) $\Delta J'$ as well, are uniquely determined when the photon energy ($\mathscr{E}'_A$ or $\mathscr{E}'_M$) is given, and are independent of whether or not the source is moving (in the given reference frame). This is the essential difference between the relationships obtained here and those ensuing from the above model in which the emission of a photon is looked upon as the emission of a wave train. This model shows that $\Delta J'$ and $\Delta\mathscr{E}'$ are determined when the radiation density (u' or u_0) and the time of emission are given. However, even when the radiation density and time of emission are given, the *energy* of the photon depends also on the "volume of the wave train" and, consequently, on whether the source is moving [see Eqs. (3.3) and (3.6)]. The dependence of $\Delta J'$ and $\Delta\mathscr{E}'$ on $\mathscr{E}'$ is not single-valued.

Apart from the factor $-\beta$, (5.13) is identical with (A.3.9) in Appendix 3, i.e., it agrees with the conclusions of Chap. 3 for a source that is *stationary* relative to the medium. However, according to (5.13)–(5.17), the dependence of $\Delta J'$ on $\mathscr{E}'$ is now assumed to remain in force even for a source *moving* relative to the medium and *stationary* in the reference frame x', t'. This ensures that the above "discrepancy" in relation to covariance becomes irrelevant. Since now the equations of momentum-energy balance have been reconciled with the Minkowski scheme, the hypothetical mechanism for the interaction between the source of radiation and the medium will satisfy the covariance conditions. This removes the conflict with relativity that was emphasized above.[19] For example, the paradox of a negative-energy electromagnetic field no longer arises. If we adopt (5.11), we should also have, in the corresponding reference frame, the equation given by (5.3)[20] which leads to the following consequence: the "de-excitation" of the atom within the Cerenkov radiation cone is regarded not as the release of energy but, rather, its absorption. However, this paradoxical effect in the new interpretation is described as the conversion of the kinetic energy of the medium into the energy of the electromagnetic field (emission of a photon) plus the excitation energy of the atom. (In the "system in which the medium is at rest," the deactivation of the atom within the Cerenkov cone also turns out to be associated with the excitation of the atom, and both are due to the *proper* kinetic energy of the atom.)

Thus, by introducing an *arbitrary* hypothesis, it is possible to establish a similarity between the description of the above effects and the picture that is a consequence of the quantum theory of these phenomena. The quantum language, using the idea of the "pseudomomentum" of the photon, can be used for an approximate description of the established features of the Cerenkov effect. The corpuscular representations lead to an (approximate) formula for the Doppler effect, which is consistent with kinematic requirements.

Moreover, the reconciliation between the idea of a quantum photon and the theory of relativity is achieved at the expense of abandoning certain fundamental quan-

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[17] It is readily verified that (5.19) is valid for other spatial components of J' and $\Delta J'$.

[18] Györgi[13] obtains (5.13) as the expression for the work A done by the field forces during the emission of energy $\mathscr{E}$ by a source stationary in x', t', allegedly a consequence of the classical model:

$$A = -\frac{\beta(n^2-1)\mathscr{E}_0\cos\theta}{n\sqrt{1-\beta^2}} = -\frac{\beta(n^2-1)\mathscr{E}'^{A}\cos\theta}{n-\beta\cos\theta}$$

However, rigorous evaluation of the work A done by the Abraham forces on the wave front, based on (3.22), leads to a different result (it has the *opposite sign* for $n\beta\cos\theta > 1$):

$$A = -\frac{(n^2-1)\mathscr{E}_0\beta\cos\theta\sqrt{1-\beta^2}}{(n-\beta\cos\theta)(1-n\beta\cos\theta)} \tag{5.20}$$

[19] We may also cite the paper by Watson and Jauch,[14] which is usually referred to in connection with questions involving the covariance of the quantum relationships that we are discussing: "...This generalization was obtained by subjecting the classical phenomenological field equations of Maxwell to the process of quantization. *In doing this it is no longer possible to maintain the principle of relativity.* This is only natural since a ponderable medium introduces automatically a preferred coordinate system, namely, the rest system of the medium..." (p. 126). We cannot agree with the conclusion that we have shown in italics nor with the subsequent argumentation.

[20] In this case, $\mathscr{E}'_{ph} = \mathscr{E}'^{M}_{ph}$.

tum relationships. The Planck–Einstein equations $\mathscr{E}' = h\nu'$ is not satisfied in moving media [see (2.20)] (here, $\mathscr{E}'$ is the energy of the *wave train* specified by the equation $\mathscr{E}_0 = h\nu_0$, where $\mathscr{E}_0$ and ν_0 represent the energy and frequency of the given wave plane in a stationary medium).

In relation to the justification of the predictions of specifically quantum effects, such as the self-excitation of an atom undergoing de-excitation through the emission of light within the Cerenkov cone, it must not be forgotten that this new variant of the theory is based on *arbitrary* assumptions. There may even be a definite contradiction between these assumptions and the original propositions on which the theory is based.

Quantum theory takes the ready-made formalism for the quantization of a field in vacuum and applies it to a medium, neglecting the Abraham forces and assuming that "the interaction with the medium is taken into account by the fact that $\varepsilon \neq 1$" (Ginzburg,[4b] p. 591).

This treatment was valid within the framework of the Minkowski ideas. However, the necessity has now arisen for reformulation of the theory, in which momentum transfer from the radiation to the medium (for a stationary medium) and the energy transfer to a moving medium must be admitted to play an important role.

The postulates leading to (5.11)–(5.19) are essentially consequences of the imposition of quantum conditions.

The very formulation of the quantization problem would, therefore, appear to require re-examination. The medium must evidently be regarded as a component part of a quantized system. At present, the quantum theory of the Vavilov–Cerenkov and Doppler effects must must be regarded as having only a heuristic significance.

ADDENDUM

In Footnote (16) it was suggested in connection with the discussion of the discussion of the paper by Ginzburg and Ugarov[2] that the neglect of elastic forces might have led to substantial errors in the conclusions reported by them. After the present manuscript was completed, it became apparent that this remark must be augmented and explained in greater detail.

Ginzburg and Ugarov[2] pointed out that, when they evaluated the integral,

$$\int f\,dt\,dv, \tag{Ad.1}$$

the integration with respect to time was carried out up to the point at which the wave train had "separated" from the emitter to a sufficient extent (Ginzburg and Ugarov,[2] p. 186 of original, 100 of translation).[21]

Let us consider the consequences to which this method leads in the case of a stationary medium (in the above set of unprimed coordinates).

We note that, in this case,

$$f^{\Gamma} = f^{\mathrm{A}}$$

where f is the force density; the notation is defined in Chap. 4.

According to Ginzburg and Ugarov[2] (p. 186 of original, 100 of translation)

$$f^{\Gamma} = \frac{\partial}{\partial t}(g^{\mathrm{M}} - \overline{g}^{\mathrm{A}}). \tag{Ad.2}$$

If the integration with respect to time is carried out as indicated above, then at the time t_1 corresponding to the upper limit of the integral, the wave train occupies some definite volume in the field (Ω) so that the momentum density outside this volume (g^{M} and G^{A}) is zero. This means that, outside this volume (Ω), integration with respect to time, according to (Ad.1) and (Ad.2), yields zero. Inside the volume, on the other hand, the integral with respect to time, given by (Ad.1), is equal to $(g^{\mathrm{M}} - g^{\mathrm{A}})\Omega = G^{\mathrm{M}} - G^{\mathrm{A}}$. Here, $\overline{g}$ is the mean value of the momentum density inside the volume Ω.

Consequently,

$$J + \Delta J = G^{\mathrm{A}} + (G^{\mathrm{M}} - G^{\mathrm{A}}) = G^{\mathrm{M}}.$$

If, on the other hand, the wave train has already broken off from the source, integration up to $t_2 > t_1$ yields the same result. This means that the wave train moving in the medium with velocity *c/n transports* momentum equal to G^{M}. Consequently, the magnitude of the Minkowski momentum, which is equal to $G^{\mathrm{M}} = g^{\mathrm{A}}\Omega$,[22] is, in fact, the magnitude of the *true* momentum of the wave train. There is then no necessity for introducing the concept of "pseudomomentum" (Ginzburg and Ugarov[2] prefer the term "quasimomentum," p. 187 of original, 100 of translation) but the contradictions noted here and elsewhere in the literature remain unresolved.

The whole point is that the derivation given by Ginzburg and Ugarov[2] ignores the effect of forces other than Abraham forces, e.g., elastic forces, on the given element of volume dV. It does not seem possible to take into account *in a general form* the effect of the forces due to the ambient medium on the given volume element dV. On the other hand, unless these forces are taken into account, the conclusion as given by Ginzburg and Ugarov[2] is devoid of the physical meaning ascribed to it by them (especially in the general form and in a moving medium).

By considering certain limiting conditions in an "*ideal*" dielectric medium (in which n is independent of the field), it is possible to obtain a result that is free of the above defect.

In our previous paper,[1] we discussed in detail the two opposite limiting cases of the ideally solid body and "dustlike" medium in which there were no elastic forces.

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[21] See Ginzburg and Ugarov,[2] Eq. (44).

[22] The momentum ΔJ is, in this case, *localized* within the volume of the wave train. In general, however, it is not only not localized, as just indicated, but cannot be localized at all: if the medium is a rigidly fixed solid, the momentum ΔJ is transferred to the earth.

Ginzburg and Ugarov refer to the latter model on their page 187 (of original, 100 of translation).

In the first of the two cases just mentioned, the Abraham forces at any given time are compensated (balanced) by elastic forces and the result of integration in (Ad.1) when these forces are taken into account is zero not only outside the volume of the wave train but also within this volume. As a result, the momentum *transported* by the wave train in the medium is given by the Abraham formula.

In the other limiting case, we have, as explained in detail in our previous paper,[1] the following property: the *rest* mass density of the medium depends on the field strength or (in the case of a plane wave) on the density u_0 of the electromagnetic energy [Skobel'tsyn,[1] Eq. (5.20)].

Because of this, the tensor for the system "field plus medium" is not the same as the Minkowski tensor.

In the case of a medium at rest, the total momentum density (in the "field plus medium" system) inside the volume Ω is g^M (this is referred to by Ginzburg and Ugarov[2] on their page 187, 100 of translation) but, after the transformation to the primed set of coordinates (moving medium), the momentum density g^T of the system (field plus medium) is no longer the same as the Minkowski momentum density, which should not happen if the conclusion of Ginzburg and Ugarov[2] were correct.

Under the conditions discussed in our previous paper[1] (direction of the x axis is parallel to the ray direction), the above momentum density is given by

$$g^{\mathrm{T}}=\frac{u_0}{c\,(1-\beta^2)}\,(n-\beta)\,(1-n\beta) \tag{Ad.3}$$

[Skobel'tsyn,[1], Table (5.30)].

Moreover, the Minkowski momentum density is given by

$$g^{\mathrm{M}}=\frac{u_0}{cn\,(1-\beta^2)}\,(n-\beta)^2 \tag{Ad.4}$$

[Skobel'tsyn,[1] Table (4.23), and Table (2.8) of the present paper with $\theta=0$].

It is *only* for $\beta=0$ that (Ad.3) and (Ad.4) become identical.

APPENDIX 1

We shall now give a more detailed derivation of (2.40) for the cross-sectional area S' of a wave-train tube cut by the plane of equal phases.

In Fig. 7, the straight line AB is the intersection of the plane of equal phases with the plane containing the x axis and the ray direction (wave normal) for a stationary medium (unprimed coordinates). AC is the intersection of the plane of equal phases and the x, y plane. The spherical triangles in the figure, used to define the angles, lie on the surface of a unit sphere ($R=1$). In the formulas given below to define the coordinates x_B, y_B, we consider a sphere of arbitrary radius ($R\neq 1$) equal to the circular cross section cut by the plane of equal phases through a tube of rays.

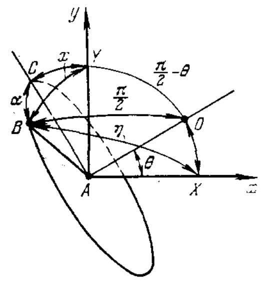

FIG. 7.

In the formulas written out below, η is the angle between the direction of the radius vector AB and the x axis. From the spherical triangle OBX (Fig. 7), we have

$$\cos\eta=-\sin\theta\cos\alpha; \tag{A.1.1}$$

where α is the angle between the planes containing the major circles OB and OC.

Again, from Fig. 7, we have

$$\cos\chi=\cos\theta\cos\alpha. \tag{A.1.2}$$

The first of the equations in (2.21) can now be written in the form

$$x_B=l_0\cos\theta+R\cos\eta=l_0\cos\theta-R\sin\theta\cos\alpha+t_B\frac{\cos\theta}{n}.$$

The formula defining y_B is

$$y_B=l_0\sin\theta+R\cos\theta\cos\alpha+t_B\frac{\sin\theta}{n}. \tag{A.1.3}$$

If we now apply the Lorentz transformation and repeat the procedure leading to (2.22)–(2.25), we obtain the following result:

$$x'_B-x'_A=-\frac{nR\sin\theta\cos\alpha\sqrt{1-\beta^2}}{n-\beta\cos\theta}, \tag{A.1.4}$$

$$y'_B-y'_A=\frac{R\cos\alpha\,(n\cos\theta-\beta)}{n-\beta\cos\theta} \tag{A.1.5}$$

where the origin of the x', y' set of coordinates moves with velocity β relative to the laboratory system x, t. Let us now introduce the coordinates x^*, y^* of points on the plane of equal phases, taking the x axis as the line of intersection between this plane and the x', y' plane, and the y^* axis parallel to the z' axis:

$$\begin{aligned} x^*&=\sqrt{(x_B-x'_A)^2+(y'_B-y'_A)^2},\\ y^*&=z'. \end{aligned} \tag{A.1.6}$$

We are considering values of x', y' at a given time t'. Using (A.1.4), (A.1.5), and (A.1.6), we obtain

$$x^*=R\cos\alpha\frac{\sqrt{n^2-2n\beta\cos\theta-n^2\beta^2\sin^2\theta+\beta^2}}{n-\beta\cos\theta}$$

or, according to (2.13),

$$x^*=R\cos\alpha\frac{(1-n\beta\cos\theta)\,c}{(n-\beta\cos\theta)\,c^*}, \tag{A.1.7}$$

$$y^*=z'=z=R\sin\alpha. \tag{A.1.8}$$

Hence,

$$\cos\alpha = \frac{x^*}{R\dfrac{(1-n\beta\cos\theta)c}{(n-\beta\cos\theta)c^*}}, \tag{A.1.9}$$

$$\sin\alpha = \frac{y^*}{R}. \tag{A.1.10}$$

We now put $R=1$. Equations (A.1.9) and (A.1.10) give

$$\frac{x^{*2}}{\dfrac{(1-n\beta\cos\theta)^2c^2}{(n-\beta\cos\theta)^2c^{*2}}} + y^{*2} = 1. \tag{A.1.11}$$

This is the equation of an ellipse (semiaxes a, b) whose area is

$$S' = \pi ab = \pi\frac{(1-n\beta\cos\theta)c}{(n-b\cos\theta)c^*} \tag{A.1.12}$$

or

$$S' = S_0\frac{(1-n\beta\cos\theta)c}{(n-\beta\cos\theta)c^*}, \tag{A.1.13}$$

where $S_0=\pi$ is the cross-sectional area in the laboratory frame. Equation (A.1.13) is identical with (2.40).

APPENDIX 2

We shall now specify more concretely the assumptions implied in the derivation of (1.6) (for the case of a stationary medium and a source stationary in the medium) and consider the virtual source of Chap. 2.

The time of emission τ of a wave train by this source will be taken to be long enough and the wave train itself will be assumed to be long enough to ensure that the emission process can be looked upon as "quasistationary" so that transient effects during the formation of the train and its breaking off from the source can be neglected.

On the leading wave front, in the transition zone, the time average of the force density f is not zero and produces a resultant force $F=[(n^2-1)/n^2]u$ on the medium (per unit cross-sectional area of the wave train), where u is the energy density. In the sinusoidal zone of the wave field, the time average of the force density is zero.

Near the surface of the source, the average force density (acting on the medium) is *assumed* to be zero. Under these *assumptions*, the reaction of the radiation, i.e., the force acting per unit area of the source per unit time must be equal to the resultant momentum loss which, according to Abraham, is

$$\frac{n^2-1}{n^2}u + \frac{u}{n^2} = u. \tag{A.2.1}$$

The second term in this equation, i.e., $u/n^2=(u/cn)c/n$, is the product of the density of the electromagnetic momentum by the velocity c/n at which this momentum is transported. The first term in (A.2.1) can be interpreted as the pressure of the radiation on the medium. It is assumed that the medium can be looked upon as an ideally solid body. The sum in (A.2.1) gives the total momentum flux transferred by the source to the medium and the field (per unit cross-sectional area per unit time) during the emission of the wave train.

According to Minkowski, when the total momentum "removed" from the source per unit area per unit time is calculated, one need only take into account the flux of the electromagnetic momentum, calculated as the product of the momentum density and the velocity of momentum transport. (The pressure of light on the medium is zero.) According to Minkowski, the density of this flux is $(nu/c)c/n=u$, and this leads to (A.2.1). Comparison of the components S_{lm} and T_{lm} (for $l, m = 1, 2$) of the momentum flux tensor defined by (2.5) and (2.8) with $\beta=0$, i.e., in the case of a medium at rest, will again lead to the conclusion that the momenta in the above two cases are the same. In the case of *a stationary medium*, these components are equal.

In the case of a moving medium, such simple considerations relating to the balance of momentum fluxes are no longer sufficient.

APPENDIX 3

We now consider a further (third) variant of the derivation of the relation

$$J'^{A} + \Delta J' = J'^{M} \tag{A.3.1}$$

for a source moving *together with the medium*.

We look upon the medium as a very massive, free (i.e., not fixed) body. The mass M of this body can be taken to be as large as convenient. When we consider the x components, the subscripts on J_x and ΔJ_x will be omitted.

It follows from the calculations given in Chap. 3 [Eq. (3.23)] that, when $\beta=0$.

$$\Delta J = \frac{n^2-1}{n^2}u_0\tau\cos\theta. \tag{A.3.2}$$

Let J_1 be the momentum of the medium prior to the injection of the momentum ΔJ, $J_2=J_1+\Delta J$, u_0 the energy density, and τ the wave-train emission time.

We shall assume that the conditions of the problem are the same as in Chap. 3. The emission time τ' seen by an observer in the x', t' set of coordinates will, as before, be set equal to unity. If $\tau'=1$ (the source moves together with the medium), then $\tau=(1-\beta^2)^{1/2}$ (in the laboratory frame) and

$$\Delta J = \frac{n^2-1}{n^2}u_0\cos\theta\sqrt{1-\beta^2}. \tag{A.3.3}$$

If l_0 is the length of the wave train along the normal to the planes of equal phases and (by assumption) $S_0=1$, and if the source is stationary relative to the medium, then $l_0=c/n=(1-\beta^2)^{1/2}c/n$,

$$u_0 = \frac{\mathscr{E}_0}{l_0} = \mathscr{E}_0\frac{n}{c\sqrt{1-\beta^2}}, \tag{A.3.4}$$

$$\Delta J = \frac{n^2-1}{nc}\mathscr{E}_0\cos\theta. \tag{A.3.5}$$

We now consider the set of primed coordinates and transform J (the momentum of the medium) in accordance with the general formulas of relativistic mechanics, *ignoring* the electromagnetic field of the photon and the associated mechanical momentum components:

$$J_1' = \frac{-M\beta c}{\sqrt{1-\beta^2}}, \tag{A.3.6}$$

where M is the mass of the body (medium).

To calculate J_2', we use the general formula for the transformation of the momentum component p_x:

$$p_x' = \frac{p_x - \beta(E/c)}{\sqrt{1-\beta^2}}, \tag{A.3.7}$$

where E is the energy of the body (medium) which can be set equal to Mc^2 (neglecting the kinetic energy communicated to the stationary medium during the injection of the momentum ΔJ).

According to (A.3.7) and (A.3.5),

$$J_2' = \frac{[(n^2-1)/cn]\,\mathscr{E}_0\cos\theta - \beta Mc}{\sqrt{1-\beta^2}}, \tag{A.3.8}$$

$$\Delta J' = J_2' - J_1' = \frac{n^2-1}{n}\,\frac{\mathscr{E}_0\cos\theta}{c\sqrt{1-\beta^2}} = \frac{n^2-1}{n^2}\cos\theta u_0. \tag{A.3.9}$$

The momentum J_{ph}' of the photon (its x component) is given by the formula for the transformation of the components of a four-vector:

$$J_{ph}' = \mathscr{E}_0\,\frac{\cos\theta - n\beta}{cn\sqrt{1-\beta^2}}. \tag{A.3.10}$$

The same result is obtained by substituting $p_x = \mathscr{E}_0\cos\theta/cn$ in (A.3.7) and replacing E with $\mathscr{E}_0$. Next, we obtain

$$J_{ph}' + \Delta J' = \mathscr{E}_0\,\frac{\cos\theta - n\beta + (n^2-1)\cos\theta}{cn\sqrt{1-\beta^2}} = \mathscr{E}_0\,\frac{n\cos\theta - \beta}{c\sqrt{1-\beta^2}} = J'^{M}. \tag{A.3.11}$$

Thus, the above simple derivation leads to the expression for $\Delta J'$ given by (A.3.9), and this is different from (3.23). However, the sum $J' + \Delta J'$ given by (A.3.11) is, as before, equal to J_M'.

To derive (3.27), we combine the J' of (3.23) with the expression for J' that follows from the table for the Abraham tensor given by (2.5). Here, on the other hand, (A.3.9) is combined with the magnitude of the momentum defined by (2.7) for the Abraham tensor, *supplemented by the mechanical components*.

In fact, the value of J' calculated from (A.3.10) was the same as that obtained from (2.46), in accordance with (2.7).

Both methods lead to the same result, where the sum $J' + \Delta J'$ in (A.3.11) is expressed as the sum of the components of four-vectors, whereas, in (3.27), the sum (equal to J'^M and, consequently, a component of a four-vector) is written as the sum of two terms, neither of which is a component of a four-vector.

The two methods of calculation correspond to two different variants of the treatment of mixed "electromechanical" components of the tensor for the system under consideration. The mechanical components of the tensor for the closed system (field plus medium) depend on the parameters of the electromagnetic field and may be more or less arbitrarily ascribed either to the field [the momentum energy tensor defined by the table in (2.7)] or to the medium.

In the derivation given in the present Appendix, we have chosen the first and, it would appear, the more natural variant.

If we were to choose the second variant of the "demarcation" between the mixed components, then, by applying the transformation formulas to the mechanical momentum, we could not *ignore* (as was done above) the electromagnetic field of the photon.

The mixed components that we are discussing are given by the table in (2.6). The component P_{14} of this tensor (with the factor $1/ic$) can be treated as the component of the mechanical momentum density.

In that case, the x component of the resultant mechanical momentum communicated to the medium during the emission of the photon can be written as the sum

$$\Delta J^* = \Delta J' + \frac{P_{14}}{ic}\,\Omega, \tag{A.3.12}$$

where $\Delta J'$ is the additional momentum calculated in the present Appendix, Ω is the volume of the wave train, and (by assumption) $S_0 = 1$.

To calculate ΔJ^* and $\Delta J'$, let us express them in terms of u_0 and τ. According to (A.3.9),

$$\Delta J' = \frac{n^2-1}{n^2}\,u_0\cos\theta.$$

Substituting for P_{14} from (2.6) in (A.3.12) and for Ω' from (3.3), we obtain

$$\Delta J^* = \frac{n^2-1}{n^2}\,u_0\cos\theta + \frac{n^2-1}{n^2}\,u_0\,\frac{\beta\cos^2\theta\,(1-\beta^2)}{(1-\beta^2)\,(n-\beta\cos\theta)} = \frac{n^2-1}{n}\,u_0\,\frac{\cos\theta}{n-\beta\cos\theta}. \tag{A.3.13}$$

which is identical with (3.23).

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