Covariance of Maxwell equations and comparison of electrodynamic systems

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Usp. Fiz. Nauk **121**, 525–538 (March 1977)

As is well known, Maxwell equations in vacuum are covariant under a transformation of the Lorentz group. The same equations in a material medium have a more general symmetry group and retain their form under arbitrary nondegenerate linear transformations of space-time variables to which, naturally, are added definite rules for recalculating fields and material characteristics of the medium. This enables a formal correspondence to be established between solutions of physically different electrodynamic problems related by linear transformations of the space-metric and of the tensor characteristics of the media. Such a correspondence (comparison) turns out to be useful, in particular, for investigating the propagation of electromagnetic waves in the presence of external gravitational fields, and also in systems with moving inhomogeneities or sources. Some examples of comparisons are examined and the results obtained in the course of this are indicated. A discussion is given of the relationship of the comparison method to the transformations of special and general theories of relativity.

PACS numbers: 04.40.+c

CONTENTS

1. INTRODUCTION

The opinion is quite widespread that the Lorentz transformations

$$x' = \gamma (x - Vt), \quad y' = y, \quad z' = z, \quad t' = \gamma (t - Vc^{-2}x), \qquad (1.1)$$

where $\gamma=[1-(V^2/c^2)]^{-1/2}$ are distinguished among other transformations of coordinates and time (for example, the classical Galilean transformations) by the fact that, in contrast to the latter, they (and only they) leave Maxwell equations invariant. However, it is well known that the Maxwell equations

$$\begin{aligned} &\operatorname{rot}\mathbf{E}=-\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \quad \operatorname{rot}\mathbf{H}=\frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}+\frac{4\pi}{c}\mathbf{j},\\ &\operatorname{div}\mathbf{D}=4\pi\rho, \qquad \operatorname{div}\mathbf{B}=0 \end{aligned} \qquad (1.2)$$

can be written in 4-tensor form without making specific the relation between the field vectors in matter.[1-4] And this means not only that they are Lorentz-invariant (which is usually emphasized in physical literature) but that they are also invariant with respect to arbitrary nondegenerate linear transformations of space-time variables (affine covariance).

In other words, if along with the coordinates one also recalculates fields and sources according to an appropriate law (as is done, in particular, also in the special theory of relativity (STR)), equations (1.2) will retain their form under arbitrary linear transformations including the Galilean ones. Naturally to each such transformation (i.e., to each system of 4-coordinates) there will correspond in such a procedure appropriate material equations for the medium.

From a formal point of view Lorentz transformations are distinguished only by the fact that in vacuum they preserve the form of material equations for the medium ($\mathbf{D}=\mathbf{E}$, $\mathbf{B}=\mathbf{H}$) and this physically corresponds to the relativistic postulate of the invariance of the velocity of light. For fields in matter the Lorentz transformations no longer have such an advantage, and, as will be shown later, they are therefore not the optimum ones for solving a number of problems. In other words, the group of invariant transformations of the original system of equations (1.2) turns out to be wider than the symmetry group of the wave equation obtained from (1.2) (the

D'Alembertian operator).[1]

But the assertion widespread in physics literature concerning the uniqueness of the invariant properties of the Lorentz group is far from always accompanied by a clear indication of whether one has in mind fields specifically in vacuum or in an arbitrary material medium (cf., for example, [1,3,6–8]).[2]

Of course, according to their physical meaning Lorentz transformations differ in principle from other linear transformations of variables, since the latter no longer correspond to a transition from one inertial reference system to another. Nevertheless, to each invariant transformation there formally corresponds a definite prescription for the recalculation of fields and of material characteristics of the medium, and this enables one to speak of *comparison* of different electrodynamic systems. Lorentz transformations can be regarded as a particular case of such a comparison when the same physical system is described from the point of view of two different inertial observers.

While relativistic transformations are the usual method for solving many electrodynamic problems, other invariant transformations find almost no application. But there exist definite classes of problems for which non-Lorentz transformations turn out to be more convenient. The calculation of fields in media with moving inhomogenieties or sources[11–16] can serve as an example.

As is well known, the use of a Lorentz transformation from the system K where the medium is at rest and is described by the simplest material equations

$$\mathbf{D}=\varepsilon\mathbf{E},\quad \mathbf{B}=\mu\mathbf{H} \tag{1.3}$$

(ε, μ = const), to the system K' comoving with inhomogeneities or sources moving with velocity V leads to the material relations of Minkowski[2–4][3]

$$\begin{aligned}\mathbf{D}'+c^{-1}[\mathbf{V}\times\mathbf{H}']&=\varepsilon(\mathbf{E}'+c^{-1}[\mathbf{V}\times\mathbf{B}']),\\ \mathbf{B}'-c^{-1}[\mathbf{V}\times\mathbf{E}']&=\mu(\mathbf{H}'-c^{-1}[\mathbf{V}\times\mathbf{D}']).\end{aligned} \tag{1.4}$$

But the use of sensibly chosen non-Lorentz transformations enables us to reduce the problem to the investigation of fields in media with simpler material equations than (1.4) with the corresponding formulas remaining applicable also to motion with velocity greater than that of light ($V>c$). Such systems are physically entirely realizable and in recent times have attracted considerable interest (cf., for example, [13,17]). The possibility of comparing the effect on electromagnetic fields of a homogeneous gravitational field and of a dielectric medium[4,18] can serve as a well-known example of quite a different kind.

In connection with this the attempt undertaken below to discuss the special features of non-Lorentz transformations and to illustrate the applications of the method of comparison to the solution of some problems appears to be useful.

2. MAXWELL EQUATIONS IN NONORTHOGONAL 4-COORDINATES AND INTRODUCTION OF COMPARABLE SYSTEMS

It is well known that from the geometrical point of view Lorentz transformations represent unitary transformations (rotations) in orthogonal (pseudo-Euclidean) 4-space. In such a case the space metric tensor g_{ik} which defines the corresponding line element[4]

$$ds=\sqrt{g_{ik}\,dx^i\,dx^k},$$

remains unchanged ("Galilean")

$$g_{ik}=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}. \tag{2.1}$$

But we are considering arbitary nondegenerate linear transformations of 4-coordinates of the form

$$x'^i=\alpha^i_kx^k,\quad x^l=\tilde{\alpha}^l_nx'^n, \tag{2.2}$$

such that $\alpha^i_k\tilde{\alpha}^k_n=\delta^i_n$, where δ^i_n is the Kronecker symbol, $\operatorname{Det}\alpha^i_k\neq 0$.

In the language of general theory of relativity (GTR) such transformations denote the introduction of nonorthogonal (oblique) 4-coordinate systems described by a metric tensor with constant components which, however, differ from (2.1). The transformation law for the covariant components[5] g_{ik}, corresponding to (2.2), has the form

$$g'_{ik}=\tilde{\alpha}^m_i\tilde{\alpha}^l_kg_{ml}. \tag{2.3}$$

In such a case in GTR, as is well known, the covariance of equations (1.2) under arbitrary transformations of coordinates is assured; but, of course, usually electrodynamics in GTR is formulated only for a vacuum, while we are here interested in fields in material me-

[1]In this connection we note the remark of Minkowski[5] that the Lorentz-covariance of the equations of electrodynamics is a mathematical fact which is essentially based "on the form of the differential equation for the propagation of waves with the velocity of light."

[2]It is possible that the emergence of the misunderstanding indicated above was aided by the fact that Poincaré,[9] having formulated the principle of convariance of the equations of motion, introduced the Lorentz transformations specifically as transformations which do not alter the equations of electrodynamics. However, he[9] (just as Einstein did in the first papers on the STR[10]) examined the case of vacuum, and the possibility of recalculating the material equations of the medium was excluded.

[3]Both here and later quantities related to the system K' will be denoted by a prime.

[4]Here and below Greek subscripts take on the values 1, 2, 3, and Latin subscripts take on the values 0, 1, 2, 3.

[5]In order to emphasize the difference between the covariant and contravariant components in nonorthogonal coordinates we recall their explicit geometric definitions for vectors[19]:

$$\mathbf{A}=A_i\mathbf{e}^i,\quad A^i=(\mathbf{A}\mathbf{e}^i),$$

where $\mathbf{e}^i$ are unit vectors along the coordinate axes.

dia. However, the problem of comparison under discussion here differs essentially from that which is solved with the aid of the well-known apparatus of GTR. The point is that in GTR in going over to other independent variables x'^i the space-time metric is also uniquely transformed in accordance with formula (2.3). But in our case the auxiliary system is physically, generally speaking, not in any way connected with the initial one, and the metric tensor for it can be chosen arbitrarily. This circumstance broadens the class of comparable systems and enables us to obtain formulas for the transformation of fields and sources leaving the system (1.2) invariant which are simpler and more convenient for the solution of specific problems. In particular, the metric of the auxiliary system can be retained in the form (2.1), and this will be utilized in Sec. 3.

It is well known (cf., for example, the review[20]), that within the framework of the STR a transition to the vector-potential description is often convenient for the introduction of 4-tensor quantities. In an arbitrary nonorthogonal coordinate system in the presence of a material medium the connection between the potential and the field characteristics becomes more complicated, and the advantages of such a description are to a large extent lost. We shall therefore take as our point of departure the antisymmetric tensors of the displacement D^{ik} and of the field B^{ik} the contravariant components of which can be expressed in terms of the components of the corresponding vectors in 3-space in the following manner:

$$D^{\alpha\beta} = -\eta^{\alpha\beta\gamma}H_\gamma, \quad D^{\alpha 0} = (g_{00})^{-1/2} D^\alpha, \tag{2.4}$$

$$B^{\alpha\beta} = \eta^{\alpha\beta\gamma}E_\gamma, \quad B^{\alpha 0} = (g_{00})^{-1/2} B^\alpha; \tag{2.5}$$

here $\eta^{\alpha\beta\gamma}$ is the antisymmetric unit tensor in 3-space, the metric tensor for which is defined in agreement with[18] by

$$h_{\alpha\beta} = -g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}}. \tag{2.6}$$

At the same time $\eta^{\alpha\beta\gamma} = h^{-1/2} e^{\alpha\beta\gamma}$, where $e^{\alpha\beta\gamma}$ is the antisymmetric unit tensor defined by the value of $e^{123} = 1$, $h = \mathrm{Det}\, h_{\alpha\beta}$.

Introducing the 4-vector for the current in the usual manner[18]:

$$I^k = (g_{00})^{-1/2} (c\rho, j^k), \tag{2.7}$$

Eqs. (1.2) for the quantities D^{ik}, B^{ik} can be written in the form

$$\frac{\partial D^{ik}}{\partial x^i} = \frac{4\pi}{c} I^k, \quad \frac{\partial B^{ik}}{\partial x^i} = 0, \tag{2.8}$$

which is invariant with respect to the transformations (2.2).[6)]

[6)]The covariant form of Maxwell equations in the form (2.8) (cf., also[11]) is preferable for the case of material media than the form utilized in[2], since, in contrast to the latter, it preserves explicitly the property of the duality of the initial equations (1.2), i.e., their symmetry with respect to the replacements $\mathbf{E} \to \mathbf{H}$, $\mathbf{B} \to -\mathbf{D}$ in a medium without sources.

In accordance with the general rules for tensor transformation the contravariant components D^{ik} are recalculated according to the formulas

$$D'^{ik} = \alpha^i_l \alpha^k_n D^{ln} \tag{2.9}$$

(similarly for B^{ik}), while the transition from contravariant components to covariant components in a given system of coordinates is performed with the aid of the metric tensor, for example,

$$D_{0\beta} = D^{ik} g_{0i} g_{\beta k} = \sqrt{g_{00}}\, D_\beta - \eta^{\alpha\gamma\delta} g_{0\alpha} g_{\beta\gamma} H_\delta. \tag{2.10}$$

Formulas (2.4)–(2.10) enable us to find completely the fields and the material equations in the auxiliary electromagnetic system K', if they are known in the initial system K, and conversely. Thus, if in $K: D^{ik} = \varepsilon^{ik}_{lm} B^{lm}$, then the components of the material tensor in the system K' with a metric defined in accordance with (2.3) are equal to

$$\varepsilon'^{ik}_{lm} = \alpha^i_j \alpha^k_s \tilde{\alpha}^n_l \tilde{\alpha}^p_m \varepsilon^{js}_{np}. \tag{2.11}$$

It is not difficult to establish with the aid of formulas (2.4), (2.5) the form of the tensor $\hat{\varepsilon}$ for a specific medium. For example, for an anisotropic dielectric described in 3-coordinates by the permittivity tensor $\varepsilon^{\alpha\beta}$ and the scalar magnetic permeability μ, the nonvanishing components of the antisymmetric material 4-tensor $\hat{\varepsilon}$ are equal to

$$\varepsilon^{0\alpha}_{\beta\gamma} = -\frac{1}{2}\sqrt{\frac{h}{g_{00}}}\, e_{\beta\gamma\mu}\varepsilon^{\alpha\mu}, \quad \varepsilon^{\alpha\beta}_{0\nu} = \frac{1}{2\mu}\sqrt{\frac{g_{00}}{h}}\, e^{\alpha\beta\gamma} h_{\gamma\nu}, \tag{2.12}$$

with, as follows from (2.6): $g_{00} h = -g = -\mathrm{Det}\, g_{ik}$. The particular case of vacuum is obtained from (2.12) when $\varepsilon^{\alpha\beta} = h^{\alpha\beta}$, $\mu = 1$, including the case of Cartesian coordinates when the metric tensor for 4-space is chosen in the form (2.1): $\varepsilon^{\alpha\beta} = \delta^\alpha_\beta$.

Thus, the initial problem of finding fields in an electrodynamic system K with given sources and definite material equations of the medium can be compared to another (auxiliary) system where the same initial equations (1.2) are to be solved, but in a medium with different material relations defined by formulas (2.4), (2.5), (2.9) and by appropriately recalculated sources (2.7). The solutions of these two problems will be of the same type and can be obtained one from the other by a simple recalculation using formulas of the form (2.9).

We note that the possibilities of comparison can be extended as a result of the similarity principle for electromagnetic systems: if D^{ik} and B^{ik} satisfy equations (2.8) with sources I^k, then the quantities $k_D D^{ik}$, $k_B B^{ik}$ are solutions of the same equations with the sources $k_D I^k$, where k_D, k_B are arbitrary constant multipliers.

3. STR AND GTR TRANSFORMATIONS AS PARTICULAR CASES OF COMPARABLE SYSTEMS

We consider the relation of the method of comparison to the transformations of STR and GTR in the language of the general formulas set out in Sec. 2. If the matrix

α_k^i in (2.2) is equal to

$$\alpha_k^i = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.1}$$

where $\beta = V/c$, then the space metric in K' in accordance with (2.3) remains Galilean and the transformation (2.9) brings the material equations (1.3) into formulas (1.4), i.e., the comparison procedure corresponds to the Lorentz–Minkowski transformations (STR) when for the comparison system we consider the same medium, but taken in another inertial reference system.

For arbitrary α_k^i a comparison can be carried out in two ways. 1) The choice of the metric in K' in accordance with (2.3) means that the comparison system is the same medium but described not in terms of orthogonal but in terms of oblique 4-coordinates. 2) Any other choice of the metric makes the comparison procedure more formal and means that in the auxiliary problem we are, generally speaking, considering a medium quite different from the one in K which even need not necessarily be physically realizable (for example, it can correspond to negative values of density; cf., Sec. 5). This circumstance is not an obstacle to the application of the comparison method—it is sufficient that the auxiliary system should be a simpler one for making calculations.

In other words, in the former case the comparison is carried out by replacing the coordinate systems for one and the same medium (GTR), while in the latter case (which, as we hope to show in Sec. 5, is of greater interest for applications) it consists of the choice of a suitable medium while retaining in K' possibly simpler metric relations.

The correspondence between these two approaches can be explicitly demonstrated using the example quoted above of comparing electrodynamic problems in a static gravitational field and in a dielectric medium with a given permittivity.

Suppose that in K we consider an electromagnetic field in vacuum in the presence of a static gravitational field. Such a system is described by the metric 4-tensor g_{ik} in which all the components $g_{0\alpha} = 0$, [18] and by the material 4-tensor ε_{lm}^{ik} whose nonvanishing components are in accordance with (2.12) equal to

$$\begin{aligned} \varepsilon_{\beta\gamma}^{0\alpha} &= -\frac{1}{2}\sqrt{\frac{h}{g_{00}}}\, e_{\beta\gamma\mu} h^{\alpha\mu}, \\ \varepsilon_{0\nu}^{\alpha\beta} &= \frac{1}{2}\sqrt{\frac{g_{00}}{h}}\, e^{\alpha\beta\gamma} h_{\gamma\nu}. \end{aligned} \tag{3.2}$$

We require that in K' no gravitational field should be present, and under this condition we shall obtain here the components of the material tensor $(\varepsilon')_{lm}^{ik}$. The transformation which brings the metric 3-tensor $h_{\mu\nu}$ of the system K into the unit metric tensor $(h')_{\alpha\beta}$, corresponding to a Galilean space in K', can be written in the form

$$(h')_{\alpha\beta} = \tilde{a}_{\alpha}^{\mu} \tilde{a}_{\beta}^{\nu} h_{\mu\nu}, \quad (h')^{\alpha\beta} = a_{\mu}^{\alpha} a_{\nu}^{\beta} h^{\mu\nu}. \tag{3.3}$$

Expressing the nonvanishing components of the tensor ε_{lm}^{ik} in terms of the corresponding components of $(\varepsilon')_{lm}^{ik}$ with the aid of (3.3) and substituting these expressions into (3.2), it is not difficult to obtain

$$\begin{aligned} (\varepsilon')_{0\delta}^{\mu\nu} &= -\frac{1}{2}\sqrt{g_{00}}\, e_{\delta}^{\mu\nu}, \\ (\varepsilon')_{\gamma\delta}^{0\mu} &= \frac{1}{2}\frac{1}{\sqrt{g_{00}}}\, e_{\gamma\delta}^{\mu}. \end{aligned} \tag{3.4}$$

According to (2.4), (2.5) the relations (3.4) are equivalent to two material equations

$$(D')^{\gamma} = (g_{00})^{-1/2} (E')^{\gamma}, \quad (H')^{\alpha} = (g_{00})^{1/2} (B')^{\alpha}, \tag{3.5}$$

which mean that in the auxiliary problem without the gravitational field we must consider an isotropic homogeneous medium with the permittivity and magnetic permeability $\varepsilon = \mu = (g_{00})^{-1/2}$. In this case the electromagnetic fields in these two physically different systems turn out to be analogous and can be obtained from each other with the aid of formulas of the form (2.9). In other words, a change in the space-time metric in a gravitational field from the point of view of its effect on the electromagnetic fields can be equivalent to the presence of a dielectric medium.[4,18]

4. THREE-DIMENSIONAL FORMULAS FOR THE TRANSFORMATION OF FIELDS AND SOURCES

Although the use of 4-tensor formalism enables us to formulate the comparison method in the simplest and most general form, in the solution of specific problems it turns out to be useful to go over to a 3-coordinate vector description of quantities characterizing the electromagnetic field.

The 3-vector transformation formulas, in particular, are more convenient because they do not depend explicitly on the form of the metric tensor. But in those cases when these formulas will be written in terms of components we shall everywhere for the sake of simplicity assume the metric to be Galilean, making no distinction between covariant and contravariant components. We shall write the linear transformation (2.2) of coordinates and time with the aid of the matrix $\hat{\nu}$ and the numerical factor $\varkappa$ in the following form:

$$\mathbf{r}' = \hat{\nu}(\mathbf{r} - \mathbf{a}t), \quad t' = \varkappa(t - \mathbf{b}\mathbf{r}), \tag{4.1}$$

where $\mathbf{a}$ and $\mathbf{b}$ are arbitrary 3-vectors. Then from (2.9) in the notation of (4.1) (or by a direct substitution of (4.1) into the Maxwell equations) and taking into account the similarity transformations it is not difficult to obtain the rules for the recalculation of the induction vectors $\mathbf{D}$ and $\mathbf{B}$ and of the sources ρ, $\mathbf{j}$ in going from system K to K':

$$\begin{aligned} \mathbf{D}' &= k_D \varkappa \hat{\nu} (\mathbf{D} - \mathbf{a}(\mathbf{b}\mathbf{D}) + c[\mathbf{b} \times \mathbf{H}]), \\ \mathbf{B}' &= k_B \varkappa \hat{\nu} (\mathbf{B} - \mathbf{a}(\mathbf{b}\mathbf{B}) - c[\mathbf{b} \times \mathbf{E}]), \end{aligned} \tag{4.2}$$

$$\mathbf{j}' = k_D \hat{\nu} (\mathbf{j} - \mathbf{a}\rho), \quad \rho' = k_D \varkappa (\rho - \mathbf{b}\mathbf{j}). \tag{4.3}$$

The transformation formulas for the field intensities $\mathbf{E}$ and $\mathbf{H}$ turn out, generally speaking, to be quite awkward, but for the case of further interest to us when the matrix $\hat{\nu}$ is assumed to be diagonal (there is no rotation of axes in the coordinate 3-space) they can be written in a relatively compact form

$$\hat{\nu}\mathbf{E}' = k_B \nu (\mathbf{E} + c^{-1}[\mathbf{a}\times\mathbf{B}]),\ \hat{\nu}\mathbf{H}' = k_D \nu (\mathbf{H} - c^{-1}[\mathbf{a}\times\mathbf{D}]), \qquad (4.4)$$

where $\nu = \mathrm{Det}\,\hat{\nu}$.

The following values of the parameters:

$$\mathbf{a} = c^{-2}\mathbf{b} = \mathbf{V},\ \varkappa = \gamma,\ \hat{\nu} = \begin{pmatrix} \gamma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \qquad (4.5)$$

apparently correspond to the Lorentz transformations (1.1) in the notation of (4.1) and in this case the relations (4.2), (4.4) go over into the Minkowski formulas if we take $k_D = k_B$.

We recall that one of the transformations of the form of (2.2), (4.1) which leave Maxwell equations invariant is the Galilean transformation when the nonvanishing components of the matrix α_k^i in (2.2) are equal to $\alpha_i^i = 1$, $\alpha_1^0 = -a/c$ or, in the notation of (4.1), $b = 0$, $\nu_i^i = \varkappa = 1$. In this case the rules for recalculating fields and sources into the corresponding auxiliary system K' are (for brevity we set $k_D = k_B = 1$):

$$\left.\begin{array}{ll} \mathbf{E}' = \mathbf{E} + c^{-1}[\mathbf{a}\times\mathbf{B}], & \mathbf{D}' = \mathbf{D}, \\ \mathbf{H}' = \mathbf{H} - c^{-1}[\mathbf{a}\times\mathbf{D}], & \mathbf{B}' = \mathbf{B}, \\ \mathbf{j}' = \mathbf{j} - \mathbf{a}\rho, & \rho' = \rho \end{array}\right\} \qquad (4.6)$$

and turn out to be even simpler than the corresponding relativistic relations. In contrast to the latter they do not have any singularities for $a \geqslant c$, and this makes using them equally convenient for arbitrary values of a. If in K the material equations have the form of (1.3), then in K' in accordance with (4.6) we obtain the relations

$$\mathbf{D}' = \varepsilon (\mathbf{E}' - c^{-1}[\mathbf{a}\times\mathbf{B}']),\quad \mathbf{B}' = \mu (\mathbf{H}' + c^{-1}[\mathbf{a}\times\mathbf{D}]), \qquad (4.7)$$

which are also simpler than the Minkowski formulas (1.4) corresponding to the choice of the auxiliary system on the basis of (4.5). As has been already emphasized in the introduction, the case of vacuum is an exception when the Lorentz transformations are priviledged.

Thus, one can say that the transformations of fields and sources obtained on the basis of Galilean formulas can be regarded in principle not only as approximate ones for $\beta = a/c \ll 1$ but also as quite rigorous relations if one has in mind a comparison of media and not a transformation of reference systems. In such a case the induction vectors D and B are invariant so that a difference compared to relativistic formulas of recalculation exists already in the first order in β.

5. SOME EXAMPLES OF THE APPLICATION OF THE METHOD OF COMPARISON

As has been noted already, the use of non-Lorentz transformations and of the comparison method can be convenient for the investigation of fields in media with parameters or sources which vary in space and time in accordance with the law governing a traveling wave of arbitrary profile: $p = p(x - Vt)$, particularly for $V \geqslant c$. As limiting cases this also includes problems with abrupt moving boundaries and point sources.

Suppose, for example, that in the initial system K the material equations have the form of (1.3), i.e., the medium is stationary but the sources move and are functions of the form $j(x - Vt)$, $\rho(x - Vt)$. Setting in (4.1) $a = V \cdot x_0$ (i.e., taking $x' \propto x - Vt$) one can choose the auxiliary system (K') to be stationary (the sources in it will not depend on t'). In contrast to the Lorentz transformations the remaining coefficients in (4.1) can be simultaneously chosen so that the medium in K' will also be stationary and the material equations will preserve the form analogous to (1.3)

$$\mathbf{D}' = \varepsilon'\mathbf{E}',\quad \mathbf{B}' = \mu'\mathbf{H}', \qquad (5.1)$$

where ε', μ' are certain constants. In other words, the anisotropy of the problem due to the relative motion of the medium and the sources can, in contrast to (1.4), be totally "forced" into the transformation formulas for fields and sources (4.2)–(4.4), and this, of course, is simpler for calculations. It is not difficult to verify by the substitution of (4.2) and (4.4) into (5.1) that for this it is necessary and sufficient that the following relation be satisfied

$$\sqrt{\frac{\mu'}{\mu}}\,k_D = \sqrt{\frac{\varepsilon'}{\varepsilon}}\,k_B, \qquad (5.2)$$

which represents the condition of similarity of electromagnetic systems of the kind under consideration, and that the following conditions also hold

$$\begin{array}{l} \mathbf{b} = \varepsilon\mu c^{-2} V \mathbf{x}_0,\quad \nu_1^1 = \varkappa \sqrt{\dfrac{\varepsilon\mu}{\varepsilon'\mu}}, \\ \nu_3^3 = \nu_2^2 = \nu_1^1 \sqrt{1 - \beta^2\varepsilon\mu}\,. \end{array} \qquad (5.3)$$

One can take the values of ε' and μ' in (5.1), for example, to be the same as the initial values ε and μ or corresponding to vacuum ($\varepsilon' = \mu' = 1$). In the latter case in accordance with (5.2), (5.3) the expression replacing (4.1) takes on the form

$$\begin{array}{l} z' = \tilde{\varkappa} z,\quad y' = \tilde{\varkappa} y,\quad x' = \tilde{\varkappa}\,(1 - \beta^2\varepsilon\mu)^{-1/2}\,(x - Vt), \\ t' = \tilde{\varkappa}\,(\varepsilon\mu)^{-1/2}\,(1 - \beta^2\varepsilon\mu)^{-1/2}\,(t - \varepsilon\mu c^{-2} V x). \end{array} \qquad (5.4)$$

The transformations (5.4) remind us of Lorentz transformations and go over into them as $\varepsilon\mu \to 1$, $\tilde{\varkappa} = 1$. Generally speaking, the factor $\tilde{\varkappa}$ and one of the coefficients k_D and k_B remain arbitrary, and they can be chosen, for example, in such a manner as to simplify formulas (4.3) for the recalculation of the sources themselves.

Thus, the fields due to sources moving in a homogeneous medium can be obtained by means of the well-known solutions for stationary sources *in vacuum* by means of a simple recalculation of fields and coordinates in accordance with formulas (4.2), (4.4). We illustrate this on the example of a point charge moving with velocity V along the x axis in a nondispersive dielectric. We take the static solution of the auxiliary problem in the form

$$\mathbf{D}' = \mathbf{E}' = q\,(r')^{-3}\,\mathbf{r}',\quad \mathbf{H}' = \mathbf{B}' = 0. \qquad (5.5)$$

We find the corresponding solution of the initial problem by substituting relations (5.2), (5.3), (5.5) into the formulas for the recalculation of fields (4.2), (4.4). Choosing for the sake of simplifying the result $\varkappa = 1$, $k_D = (\varepsilon\mu)^{-3/2}(1-\beta^2\varepsilon\mu)^{-1}$ we obtain

$$E_x = q\varepsilon^{-1}(x - Vt)R^{-3}, \quad H_x = 0,$$
$$E_y = q\varepsilon^{-1}yR^{-3}, \quad H_y = -\beta qzR^{-3}, \tag{5.6}$$

where $R = [(x - Vt)^2 + (y^2 + z^2)(1 - \beta^2\varepsilon\mu)]^{1/2}$. The remaining components of the fields can be easily obtained from the material equations (1.3) taking into account the cylindrical symmetry of the problem.

Formulas (5.6), naturally, agree with the well-known expressions for fields of uniformly moving charges obtained by the traditional method (cf., for example,[21]). We note, that already from (5.4) we can see the special feature occurring in the Cerenkov case ($\beta^2\varepsilon\mu > 1$), here real x, t correspond to imaginary values of x', t' and conversely. As a result of this in the solution of (5.6) on the surface of the cone $(x - Vt)^2 = (y^2 + z^2)(\beta^2\varepsilon\mu - 1)$ the fields have a divergence associated with Cerenkov radiation.

One can proceed in a similar manner also in investigating fields of oscillators moving in a medium—the use of the transformations (5.4) enables us to express these fields in terms of the well-known solution for a stationary oscillator in vacuum. Here the special features arising in "faster than light" motion are also taken into account in an explicit manner.

We now indicate how one can investigate waves in media with variable parameters within the framework of the class of linear transformations (2.2) considered here (since in a direct application of media with variable ε and μ of transformations of the type of (5.4) the latter are no longer linear). For this one can utilize the fact that the separation of the current induced in the medium into a conduction current **j** and the displacement vector **D** in (1.2) is carried out, generally speaking, in a nonunique manner. Therefore for media with variable parameters (such as density, temperature, etc.) the corresponding part of the polarization current can be included into the quantity **j** leaving, if necessary, ε and μ at values different from unity only for taking into account the homogeneous "background" medium or the effect of retarding systems. Then the material equations (1.3) must be complemented by a relation connecting the current **j** with the electromagnetic field, for example, in the form

$$\mathbf{j} = \sigma\mathbf{E}, \tag{5.7}$$

where the conductivity tensor $\hat{\sigma}$ for dispersive media is a linear operator.[7)]

For media with variable parameters the operator $\hat{\sigma}$ will depend explicitly on r and t. If, in particular, the parameter wave is of the form of a moving layer of constant profile ($\hat{\sigma} = \hat{\sigma}(x - Vt)$), then by the replacement $x' \propto x - Vt$ the problem can again be reduced to a stationary one—to the investigation of reflection and transmission of waves in an auxiliary stationary layer, including the case of $V > c$. In such a case again by a suitable choice of the dependence $t'(t, x)$ one can succeed in simplifying the material relations for the medium in comparison with the relativistic relations. As a result one can directly use for moving layers the methods and the results well known for stationary spatially-inhomogeneous media.[22,23]

Further, since here the coefficients in Eqs. (1.2) do not depend on t', the variables x' and t' are separable, and the problem of finding fields in the form $E' = f'(x')g'(t')$ can be reduced to solving an equation with variable coefficients (but now in terms of ordinary derivatives) involving the function $f'(x')$ the form of which depends on the operator $\hat{\sigma}$. And for the other factor $g'(t')$ an equation with constant coefficients is obtained; its solution can be taken to be harmonic in t' (i.e., proportional to $\exp(i\omega' t')$). Then the procedure of comparison can be conveneiently carried out in a somewhat different form—in a number of specific problems the transformation $t'(x, t)$ can be so chosen that the equation for $f'(x')$ and for the analogous factor $f(x - Vt)$ in the initial problem would be invariant without recalculating the functions being sought (in contrast to (4.2), (4.4)).[11] Physically this means that the comparison system is a layer of the same nature, but stationary. As a result of this when the function $f'(x')$ is known in order to find the desired solution $E(x, t)$ it is sufficient in the expression $E' = f'(x')g'(t')$ to substitute (4.1) in place of x', t'. As an example we consider the case of a moving layer of plasma of variable density $N(x, t) = N(x - Vt)$. The variation of $N(x, t)$ can be, in principle, due both to the motion (drift) of an inhomogeneous plasma, and to ionization–recombination processes (it is in the latter case that $V > c$ is possible and this has been repeatedly noted in the literature—cf., for example,[11,17,24,25]). It should be noted that depending on the mechanism producing a density variation the form of the operator $\hat{\sigma}(r, t)$ and, correspondingly, the behavior of electromagnetic waves turn out to be different in the case of the same dependence $N(\mathbf{r}, t)$ (cf.,[26–28]), and this is often overlooked.

In particular, for high frequency fields in an immobile nonstationary plasma the connection of **j** with **E** can be written with some idealizing assumptions in the form usual for the phenomenological theory of dispersive media

$$\mathbf{j}(t) = \int_0^\infty \sigma(t, \tau)\,\mathbf{E}(t - \tau)\,d\tau, \tag{5.8}$$

where the kernel $\sigma(t, \tau)$ turns out to be equal to[26,27]

$$\sigma(t, \mathbf{r}) = e^2 m^{-1} N(\mathbf{r}, t - \tau)\exp[-(\nu + \delta)\tau];$$

here e, m and $N(\mathbf{r}, t)$ are the charge, mass and the density of electrons which varies in time due to the processes of ionization and recombination, ν is the effective frequency of elastic collisions of electrons with

[7)]Such a relation does not exclude taking into account the effect of the magnetic field of the wave which is essential for moving media since the field **B** can be expressed in terms of **E** in accordance with (1.2).

heavy particles, δ^{-1} is the mean lifetime for free electrons. Generally speaking, taking (5.8) into account one can obtain from the system (1.2) a third order equation for the vector **E** (and consequently, for the factor $f'(x')$). If for the sake of simplicity we neglect the effect of collisions (which is permissible for $N^{-1}|\partial N/\partial t| \gg (\nu+\delta)$) we obtain for the field **E** in a plane wave traveling along the x axis a second order equation of the Klein–Gordon type

$$\frac{\partial^2 E}{\partial x^2}-\frac{\varepsilon}{c^2}\frac{\partial^2 E}{\partial t^2}=\frac{\varepsilon}{c^2}\omega_p^2(x-Vt)E, \tag{5.9}$$

where $\omega_p=(4\pi e^2N/m\varepsilon)^{1/2}$ is the plasma frequency.

Setting in (5.9) $E=f'(x')\exp(i\omega' t')$, where x' and t' are introduced in accordance with (4.1), the problem can be reduced in accordance with the foregoing to the investigation of waves in a stationary (immobile) plasma layer of profile $N'(x')$, similar to the initial one which is situated in a likewise immobile dielectric of permittivity ε'. The specific values of N' and ε' depend on the choice of the coefficients in (4.1). For example, if we take $\nu_1^1=1$, $\varkappa=1/(1-\beta\sqrt{\varepsilon})$, $b=\sqrt{\varepsilon}/c$,

$$N'=(1-\beta^2\varepsilon)N,\quad \varepsilon'=\varepsilon(1-\beta^2\varepsilon)^{-2}; \tag{5.10}$$

but if $\nu_1^1=(1-\beta^2\varepsilon)^{-1/2}$, $b=\varepsilon Vc^{-2}$, then we have

$$N'=N,\quad \varepsilon'=\varepsilon \tag{5.11}$$

etc.

Using the known structure of the field in the auxiliary case it is not difficult by means of recalculating E' and ω' to obtain the amplitude and the frequencies of secondary waves in the initial problem, their group and phase velocities, etc. In particular, the frequencies of the incident (ω_0) and of secondary (ω) waves outside the layer turn out, just as in the case of sharp boundaries,[24] to be connected by the Doppler relations

$$\frac{\omega}{\omega_0}=\frac{1-(V/v_0)}{1-(V/v)}, \tag{5.12}$$

where v_0, v are the phase velocities of the waves. The coefficients of power reflection (R) and transmission (T) of waves in a moving layer can be directly expressed in terms of analogous quantities for the auxiliary layer (R', T'):

$$R=R'\frac{(\omega_0k_r)}{(\omega_rk_0)},\quad T=T'\frac{(k_0'\omega_0k_t)}{(k_t'\omega_tk_0)}, \tag{5.13}$$

where k is the wave number, while the subscripts r, t refer to the reflected and transmitted waves respectively. In particular, if there is no plasma ahead of the ionization front ($v_0=v_r$), then $R=R'$. The condition for total reflection ($T=0$) can here be satisfied even for $T'\neq 0$, if the frequency found in accordance with (5.12) $\omega_t \leqslant \omega_{p\max}$, i.e., $\mathrm{Re}\,k_t=0$.

We note further that the method of comparison enables us also in the general case to obtain relations connecting the total energies (W) and the frequencies of quasimonochromatic wave packets independently of the specific profile of the plasma layer. Indeed, starting from the fact that in this approximation the energy of the field is conserved in the auxiliary system, i.e., $R'+T'=1$, we obtain in the case of an initial layer, moving with a "less than light" velocity ($\beta\sqrt{\varepsilon}<1$),[13]

$$W_0\omega_0=W_r\omega_r+W_t\omega_t. \tag{5.14a}$$

Since in the case of ionization $\partial N/\partial t>0$ and ω_r, $\omega_t>\omega_0$, it follows from (5.14a) that $W_0<W_r+W_t$, i.e., the total energy of the electromagnetic waves is reduced.

In the "greater than light" case ($\beta\sqrt{\varepsilon}>1$) special features appear in the formulas for the recalculation of parameters. For example, in the variant (5.10) the density N' of the auxiliary layer turns out to be negative, which is physically nonrealizable. However, this circumstance does not preclude a formal procedure for seeking a solution and merely means that the dispersion equation for the auxiliary medium has the form $c^2k^2/\varepsilon'=\omega^2+|\omega_p'^2|$.[8] In contrast to the initial problem, when $c^2k^2/\varepsilon=\omega^2-\omega_p^2$, and for $\omega<\omega_p$ the plasma is nontransparent, the quantity k' is real for arbitrary values of ω'. In other words, in the "greater than light" case a perturbation of arbitrarily low frequency is transformed with respect to its spectrum in such a manner that $\omega_{t1}>\omega_p$, and the wave "sneaks" through the plasma. Thus, for $\omega_p\gg\omega_0$ a large coefficient of frequency transformation occurs here as has been repeatedly pointed out in the literature (cf., for example,[12,13,24,25]). We note that now the reflected wave, as such, no longer exists, but we have a second transmitted wave of frequency ω_{t2}, the group velocity of which in the initial problem is in the direction of catching up with the layer. As a result it follows in this case from the equation $R'+T'=1$[13] that

$$W_0\omega_0=W_{t1}\omega_{t1}-W_{t2}\omega_{t2}. \tag{5.14b}$$

In spite of the opposite sign of the last term compared with (5.14a) analysis shows[13] that in this case also we always have $W_{t1}+W_{t2}<W_0$. Physically this is understandable, if we take into account the fact that for any arbitrary velocity of motion of the ionization wave, a part of the electromagnetic energy is expended in imparting translational motion to the newly produced electrons.

In the case of a moving plasma equation (5.9) is no longer valid; here, by the way, it is in principle necessary to take account spatial dispersion. In connection with this it is more convenient to start from the relations of microtheory from which (without taking into account thermal motion, etc.[11,12]) it is possible once again to obtain the Klein–Gordon equation, but this time for vector potential **A** (in this case $\mathbf{E}=-c^{-1}\partial\mathbf{A}/\partial t$, $\mathbf{B}=\mathrm{curl}\,\mathbf{A}$). As a result of this the dispersion law and all the kinematic formulas (for frequencies and propagation

8)In another variant of comparison there are no special features in the formulas of (5.11), but the variables x', t' become imaginary and the dispersion equation for K' is obtained in the same form as in the case that has just been considered.

vectors, phase and group velocities, etc.) remain the same as in the preceding case, but the relations involving amplitudes and energies are significantly altered. Thus, from the condition $R' + T' = 1$ instead of (5.14) we now have[12]

$$\frac{W_0}{\omega_0} = \frac{W_r}{\omega_r} + \frac{W_t}{\omega_t} \quad (\beta\sqrt{\varepsilon} < 1),$$
$$\frac{W_0}{\omega_0} = \frac{W_{t1}}{\omega_{t1}} - \frac{W_{t2}}{\omega_{t2}} \quad (\beta\sqrt{\varepsilon} > 1). \qquad (5.15)$$

Formulas (5.15) in a definite sense generalize the well-known Manley–Rowe[29] relations to the case of an arbitrary aperiodic variation of the parameters of the medium. The first of them, in fact, means that the total number of quanta is conserved in reflection and refraction; the total energy in this case can both increase and decrease depending on the relation between ω_0 and ω_r, ω_t. But if we neglect reflection ($W_r = 0$), then from this we obtain $W/\omega = \text{const}$. For media with smoothly varying parameters the validity of such an adiabatic invariant was established in[30,31] (cf., also[24,28,32]). Thus, when a wave packet "enters" into a denser plasma its frequency and energy increase at the expense of the kinetic energy of the moving plasma. Previously relations of the form (5.15) were obtained for moving sharp boundaries (cf., [24,32]).

And from the second of equations (5.15) it follows that in the case of "faster than light" motion of an inhomogeneous plasma induced production of new quanta occurs, and this can be treated as stimulated Cerenkov radiation from a moving layer in a retarding dielectric medium. It is of interest to note that the dispersion equation for the auxiliary medium in this case coincides with the corresponding equation obtained in[33] for a stationary plasma with inverted population.

Thus, a comparison with an auxiliary stationary layer enables us to carry out an exhaustive investigation of electromagnetic waves in systems with moving plasma layers. An analogous method is also applicable in more complicated cases, for example, within the framework of kinetic theory for a heated moving plasma,[14] when there is, in fact, not one variable parameter $N(x - Vt)$, as above, but a continuous set of parameters—the unperturbed distribution function. In such a case the dispersion equation (and the kinematic relations for the frequencies and the wave numbers) is obtained in a different form; nevertheless, the energy relationships (5.15) remain in force as before, if we neglect losses of a surface nature which are significant only for abrupt boundaries.

6. CONCLUSION

Summarizing we note that in the preceding discussion ~~we~~ were not aiming to carry out a detailed review of specific physical problems and results obtained in the literature with the aid of the method of comparison; one can become acquainted with them utilizing the appended bibliography. The authors wished more to emphasize the principle and the methodological aspects of the problem—to clarify the essence of the procedure of comparison, its relationship to the transformations of STR and GTR and to underline the available advantages in comparison with them. It seems to us that the possibilities of this method for the solution of different problems are far from having been exhausted. We note, first of all, that it can be useful in the investigation not only of electromagnetic fields, but of wave systems of arbitrary nature. Utilizing the invariance of the corresponding dynamic equations with respect to different transformations of the independent variables, here also one can in a number of cases suceed in simplifying the problems significantly. For example, definite results for systems described by Lagrange equations were obtained in[15,16] by such a method. Finally, a comparison of systems is possible which are also related by nonlinear transformations of independent variables. Some problems utilizing the invariance of the equations of electrodynamics under nonlinear transformations of space-time were solved, for example in[11,16,34], but in view of the complexity and the diversity of this problem it is difficult at the present time to advance any general considerations or recommendations concerning the choice of optimal transformations.

The authors are grateful to B. M. Bolotovskiĭ, S. N. Stolyarov and V. A. Ugarov for discussion of the problems raised here and for useful comments.

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Translated by G. Volkoff