The editorial board of Uspekhi Fizicheskikh Nauk reports with great sorrow the untimely death, on February 27, 1977, in his 56th year, of its long-time member and secretary-in-chief, Candidate of Physical and Mathematical Sciences,

VLADIMIR ALEKSANDROVICH UGAROV

Relativistic strings and dual models of the strong interactions

M. S. Marinov

Institute of Theoretical and Experimental Physics, Moscow
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A review is given of the theory of strong interactions based on the model in which hadrons are represented as one-dimensional elastic relativistic systems ("strings"). The relation between this model and the concepts of quarks and partons is discussed. Basic results pertaining to Veneziano's dual theory, which can be regarded as a consequence of the string model, and modifications of this theory are considered. A detailed account of the classical theory of strings is given. The main emphasis is on those problems which are important in constructing a quantum theory, namely Hamiltonian mechanics and conformal symmetry. The quantization procedure is described and is shown to be self-consistent only in a 26-dimensional space with a special condition on the spectrum of states. A theory of strings with distributed spin is considered. Spin is introduced by means of the formalism of Grassman algebras. Quantization is then possible only in a 10-dimensional space. Interactions of strings take place by virtue of their rupture and recombination. A method of calculating interaction amplitudes is given. Discussions of the Koba-Nielsen representation, the continuum integral, and the two-dimensional conformal group are included in an appendix.

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CONTENTS

1. INTRODUCTION. STRINGS—A MODEL FOR THE DUAL THEORY OF STRONG INTERACTIONS

A. The modern view of hadronic structure

Theoretical ideas about the structure of strongly interacting particles (hadrons) have been greatly enriched during the past few years. It has already been evident for some time that hadrons are neither "elementary" nor "point-like," but our conception of their internal structure has now become clearer and more concrete. Hadrons are pictured as extended objects, apparently consisting of two types of matter: a small number of "fundamental particles"—quarks, which possess quantum numbers such as charge and strangeness, and vector "gluon" fields (from "glue"), which bind the quarks together. This picture is somewhat reminiscent of a system of positive and negative particles (such as electrons and positrons) bound by the electromagnetic field. However, neither free quarks nor strongly interacting "photons" (gluons) have been observed experimentally. This may be due to the unusual properties of the gluon field: unlike photons, gluons interact strongly *with each* *other*. The theory lacks complete clarity, but it can be assumed that, owing to this self-interaction, the gluon field is not dissipated in space like the electric field (rather slowly, falling off according to a power law at large distances from the charges), but is concentrated within narrow tubes which join the sources of the field—the quarks. In this case it is possible, for example, that the energy of a two-body system like the hydrogen atom (or positronium) would increase without limit at large inter-particle distances instead of decreasing. If this is the case, no external force would be capable of dislodging an individual quark from the system. Like quarks, gluons are also "charged," and it is natural to expect that, for the same reason, free gluons cannot be radiated into space.

These considerations stem from three main sources. First, there is the classification of hadrons on the basis of higher symmetries, which led to the concept of quarks. Secondly, there is a new unified theory of the weak and electromagnetic interactions, in which charged vector fields play a major role. Thirdly, there are dual models, one of the most intensively analyzed approaches to the theory of the strong interactions. In fact, dual theories and their physical interpretation constitute the subject of the present review.

The principle on which dual models (henceforth abbreviated DMs) are based—*reggeon-resonance duality*—is a unification of two of the most fruitful approaches to the physics of strong interactions. The first of these approaches makes use of the idea that particles interact through the *exchange* of quanta of some field. For example, the most important part of the nucleon–nucleon interaction in a nucleus is due to the exchange of pions. In scattering at large non-relativistic energies, when the nucleons approach each other more closely, there are exchanges of ρ and ω mesons, which are heavier than pions. At high energies, exchanges of more complex "particle-like" systems—reggeons—play a major role. The second approach involves the idea that collisions of hadrons can give rise to metastable *intermediate states*—resonances (analogous to the compound nuclei formed in nuclear reactions)—which dominate the scattering of hadrons at moderate energies. The principle of duality asserts that these two approaches, which are in general independent, are not "complementary," but have a common dynamical character.

It has recently become apparent that this idea, which originated as an elegant mathematical construction, can be associated with a physical picture which is surprisingly easy to understand intuitively. All particles, both stable and short-lived, are represented as stationary states of a *one-dimensional material system*, usually known as a *relativistic string*. It has been proved that the quantum theory of interacting strings leads to the same predictions about the nature of hadronic collision processes as those obtained from formal considerations in the dual theory. Although DMs have many attractive features, they suffer from certain serious defects which prevent them from being adopted as a realistic theory of hadrons. Their interpretation in the language of strings has made it possible to greatly simplify the formalism and to achieve a better understanding of the reason for the difficulties inherent in this approach. Many theorists working in this field hope that it will still be possible to modify the dual scheme in such a way that it will be able to serve as a theory of the strong interactions. From this point of view, the theory of strings is of definite heuristic value.

B. General principles and hadronic phenomenology

For over a quarter of a century, the theory of the interaction of electrons and photons—quantum electrodynamics—has been the model of a consistent theory of elementary particles. It is the only example of a local relativistic quantum field theory in which the probability of any physical process due to the electromagnetic interaction can in principle be based on the principle of least action, with a Lagrangian for the interacting fields which is constructed in analogy with the classical theory. The method of calculation is perturbation theory—an expansion in powers of the electron charge. Despite incredible efforts, it has not been possible to construct a theory of the strong interactions on the basis of this model. The reason for this lack of success is not only that the "charge" which determines the strength of the hadronic interaction (the coupling constant) must be large, thus rendering perturbation theory inapplicable, nor even the fact that the renormalization technique and the summation of the perturbation series for large coupling constants are of doubtful validity. The principal difficulty is that it has not been possible to formulate a basic framework for the theory and to determine the fundamental fields and their interaction Lagrangian. In electrodynamics, "matter" consists of structureless "atoms"—point-like electrons—and a massless photon "field" which is of long range and which has therefore been well studied by macroscopic methods. These circumstances, which have made it possible to construct the classical and quantum theories of electromagnetism, do not apply to the physics of hadrons, so that we can hardly hope to find a local Lagrangian formalism in this case.

It is usually assumed that every process involving particle interactions is described by some particular probability amplitude. These amplitudes are related to the matrix elements of the S-matrix, a unitary operator in the space of states characterized by a set of free particles having definite momenta. Each amplitude is a function of the kinematic variables of the process, the momenta and spins of the colliding particles and of the particles produced as a result of their interaction. Even if it is not possible to construct a complete theory which could be used to calculate the amplitudes, we can at least attempt to find certain general properties of the amplitudes as elements of an overall S-matrix. The "S-matrix approach" to the theory of hadrons is to formulate general laws for constructing amplitudes and to analyze their applications to specific processes.

The generally accepted principles of the theory are as follows:

I. *The homogeneity and isotropy of four-dimensional space-time.* This condition implies that an amplitude is non-zero only when the total energy and momentum are conserved. A consequence of relativistic invariance is that the amplitudes are functions of kinematic invariants—masses and scalar products of the momenta. When allowance is made for the spin variables, the amplitude become matrices with definite transformation properties in going from one coordinate system to another.

II. *The conservation of probability.* The sum of the probabilities for all possible outcomes of any particular physical process is equal to unity. This requirement can be satisfied by taking the S-matrix to be a unitary operator. The unitarity condition leads to a system of quadratic integral equations for the (complex) interaction amplitudes.

III. *Causality.* A signal cannot propagate in space (either in a vacuum or in a medium) faster than light in a vacuum. If the space-time formulation is translated into one involving the energies and momenta of the particles, this natural requirement implies that the amplitudes must possess certain analytic properties as functions of complex variables. In particular, an amplitude can have no singularities in the energy, apart from those due to the intermediate states which can occur in the process in question. The nature of the singularities which do occur is determined by the unitarity condition.

IV. *Crossing symmetry (or simply "crossing").* It is easy to see that if a particle is described by a field which is local in space-time, then the same field also describes its antiparticle. Let $A(p)$ be the amplitude for some process involving the absorption of a particle a with 4-momentum p. Then, by virtue of the locality condition, $A(-p)$ describes a process in which the antiparticle $\bar{a}$ is emitted. On the other hand, by the principle III (analyticity), the functions $A(p)$ and $A(-p)$ can be related to one another by analytic continuation. Thus, for example, the amplitudes for processes such as $\pi^+p\to\pi^+p$, $\pi^-p\to\pi^-p$, and $p\bar{p}\to\pi^+\pi^-$ are described by a single analytic function, evaluated in different regions of its variables.

The principles I–IV hold in local Lagrangian field theory to arbitrary order of the expansion in the coupling constant. However, the corresponding equations arising from the unitarity condition, which are quadratic in the amplitudes, are naturally not satisfied identically, but only in an approximation in which terms of higher order are neglected.

Apart from perturbation theory, no concrete example of a set of interaction amplitudes satisfying the general principles has hitherto been considered. DMs provide an interesting example of this type, although they satisfy the unitarity condition only in an approximation in which all intermediate states other than single-particle states are neglected. Ideas about the nature of the strong interactions based on the interpretation of extensive experimental data suggest that this approximation is reasonable.

The most important qualitative features of the strong interactions include the following two circumstances. First, hadronic collisions have a large probability of exciting a whole spectrum of short-lived compound systems, or "resonances." Scattering of particles at low and intermediate energies depends mainly on resonant intermediate states. Experiments have shown that there exist resonances with rather high masses, exceeding the mass of the nucleon by a factor 2–3, and that these resonances have relatively large lifetimes (small "widths"). Secondly, many characteristics of interaction processes at high energies are well described in terms of reggeons, which in a sense constitute an "analytic continuation" of the resonances in the crossed channel. Thus pole singularities in the interaction amplitudes play a major role: at low energies we have poles describing resonances, whereas at high energies we have poles in the complex angular-momentum plane—reggeons.

C. Dual theories: their advantages and disadvantages

Attempts to construct a theory which provides a good description of both high and low energies led to the concept of *reggeon–resonance duality.* The hypothesis of duality is the assumption that the interaction amplitudes have only pole singularities and satisfy the principles I–IV. This assumption naturally requires an infinite number of poles; in other words, the model requires an infinite spectrum of resonances. The theory is constructed in such a way that at high energies the superposition of a large number of resonances effectively leads to reggeon exchanges, while at low energies the summation of many reggeon poles gives a resonant behavior in the energy. An explicit example of such an amplitude for the simplest process of the pion–pion interaction was first given by Veneziano in 1968 (the dual theory for the interaction of scalar particles is generally known as the Veneziano model). The Veneziano amplitude is expressed in terms of the Euler Γ-function (see Sec. 2 and Sec. A of the Appendix). It has been found that the hypothesis of duality is equivalent to the imposition of very strong constraints on the form of the amplitudes, so that the structure of the entire theory is practically unique. The model predicts the entire spectrum of resonances. The qualitative predictions of the model are, on the whole, consistent with experiment.

However, the model is too primitive and has the properties of unitarity and analyticity only in the limit of a crude "single-particle" form. Only the contribution of single-particle intermediate states is taken into account in the unitarity condition, and the only singularities of the amplitudes are sequences of poles in the complex planes of the invariant variables. There are no cuts due to multi-particle intermediate states. The resonant states are assumed to have infinitesimally small widths, with a spectrum that is equally spaced and highly degenerate. Moreover, it is impossible to construct a theory which incorporates a reggeon having the quantum numbers of the vacuum (the Pomeranchuk pole), which plays a major role in elastic scattering at high energies. However, it is possible that the simplest dual amplitudes should be regarded as a first approximation, or a set of "Born terms" which, when iterated, would ultimately lead to a consistent theory. Although this procedure is far from simple, the hope of the enthusiasts of this approach is that the non-trivial character of the first approximation and the richness of its properties will guarantee that the method converges rapidly. In fact, cer-

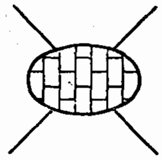

FIG. 1. A Feynman "net" diagram. The virtual momenta around each contour are assumed to be small.

tain qualitative features of the strong interactions, such as the "integral" properties of multi-particle processes, are already given correctly by the first approximation. However, it should be pointed out that, even from this point of view, the dual theory cannot be regarded as complete in principle. The theory is self-consistent (contains no negative probabilities) only in a space with an unphysical number of dimensions (26 or 10, instead of 4). In addition, the states of lowest mass are very different from the observed particles. Certain variants of the theory contain particles with imaginary mass—"tachyons."

The literature on DMs, even the review literature, is very extensive.[1)] Veneziano[1] gave a lucid formulation of the basic principles underlying the construction of DMs and their physical motivation. Kaĭdalov[2] also provided an introduction to the physics of DMs. Applications of DMs to specific physical processes and comparisons of the theoretical predictions with experimental data have been reviewed by Levin[3a] and by Jenkovszky and Shelest.[3b] A review of Sivers and Yellin[4] contains an analysis of the properties of the interaction amplitudes in various DMs and their possible modifications. The mathematical formalism has been discussed in detail by Alessandrini *et al.*[5] (the operator method) and by Schwarz[6] (further development of the operator method, the Virasoro algebra, the Neveu–Schwarz model, the Shapiro–Virasoro model, etc.). Gervais and Sakita[7] gave an account of applications of functional integration to various DMs. Rebbi[8] reviewed the approach to DMs based on the concept of a relativistic string. Discussions of DMs which include the theory of strings have also been published by Mandelstam[9] and Scherk.[10] A phenomenological approach to DMs can be found in a paper by Phillips and Roy.[11] A concise and lucid review of the subject was given by Olive[12] in his report at the 1974 London Conference.

D. The microscopic picture: partons and quarks

After the formulation of simple and elegant expressions for dual amplitudes, attempts were made to understand DMs from the standpoint of local quantum field theory and to establish a space-time description of interactions which possess dual properties. In particular, it was shown[13] that an approximate summation of a certain class of Feynman diagrams (Fig. 1) leads to Veneziano amplitudes. In other words, dual amplitudes arise when particles are exchanged in the form of complex systems consisting of a large number of strongly interacting particles. This led to the view[14] that DMs are closely related to parton dynamics (for a discussion of partons, see, e.g., Feynman's paper[15]).

Harari[16a] and Rosner[16b] had already pointed out that reggeon–resonance duality can be expressed in a natural way in the language of quarks. It can be regarded as established that there are neither resonances nor Regge poles with "exotic" quantum numbers (such as mesons with isospin 2 or hyperons with strangeness +1) and that all particles belong to the simplest representations of the group $SU(3)$: the singlet, the octet, or (for baryons) the decuplet. This observation can also be understood as follows: all the meson states (resonances and Regge poles) are constructed from quark-antiquark pairs ($q\bar{q}$), while the baryon states are constructed from three quarks (qqq). Of course, from the point of view of quantum field theory, a particle is also necessarily associated with an indefinite number of "virtual pairs" $q\bar{q}$, but these pairs are coupled in such a way that they do not alter the total quantum numbers of the system. It can be assumed that the same types of intermediate states ($q\bar{q}$ or qqq) play the major role in all strong interactions of particles. (In addition, elastic scattering is dominated by diffraction, which is due to the exchange of the Pomeranchuk pole; this component must be taken into account separately.) This hypothesis can be formulated in terms of simple diagrams such as those shown in Fig. 2, known as Harari–Rosner diagrams, which by definition contain no virtual quark loops or intersections of quark lines, i.e., they are "planar." Independently of the details of quark dynamics and the accuracy of $SU(3)$ symmetry, this picture leads to a number of qualitative predictions, which are generally in agreement with experiment. On the other hand, the Harari–Rosner diagrams provide an intuitively obvious representation of duality if each channel contains only single-particle "allowed" ($q\bar{q}$ or qqq) intermediate states.

The parton and quark approaches are consistent with one another if it is assumed that the lines around the edge of the Feynman diagram in Fig. 1 represent the motion of quarks, while the internal lines represent the virtual particles ("gluons") which bind the quarks together. A particularly clear picture emerges if the gluons are represented as quark-antiquark pairs and the concepts of "quarks" and "partons" are identified. A

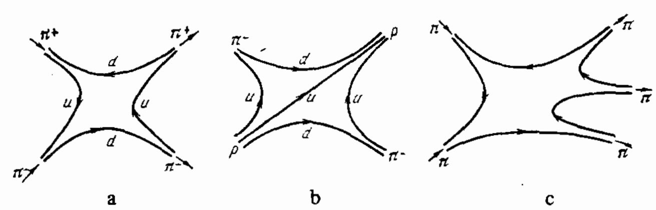

FIG. 2. Scattering of particles as a process involving quark interactions. Harari–Rosner diagrams: a) $\pi^+\pi^-\to\pi^+\pi^-$, with duality of the s- and t-channels; b) $\pi^-p\to p\pi^-$, with duality of the s- and u-channels and an important contribution to backward scattering; c) $\pi\pi\to\pi\pi\pi$, an example of an inelastic process (u and d in the diagrams label the "proton" and "neutron" quarks).

[1)]The cited literature includes those papers which, in the opinion of the author, are most lucidly written and which may be particularly useful for gaining an understanding of the subject. An exhaustive bibliography which scrupulously reflects the priorities of all investigations of dual models is hardly appropriate in the present review.

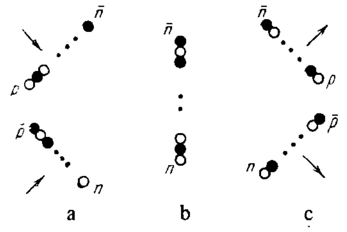

FIG. 3. Meson scattering. a) Quark chains collide and join at their extremities; b) the intermediate state —a single chain; c) rupture of the chain and emission of mesons.

meson is represented as a chain ($q\bar{q}q\bar{q}\cdots q$) in which only "neighboring" (in momentum space) pairs $q\bar{q}$ interact. Each quark is "bivalent," so that only the quarks at the extremities are responsible for the interaction with other particles. The scheme of $\pi^+\pi^-$ scattering is shown in Fig. 3. A chain can be in an excited state (a resonance). A change in the length of a chain under the action of an external force leads to a change in the number of particles, while the average density of mass per unit length is determined by the local dynamics and does not change. This density, which has the dimensions $m^2c/\hbar$, is a fundamental constant of the theory; as we shall show later, it is related to the slope α' of a Regge trajectory.

When all the virtual momenta are small in comparison with the masses, the diagram of Fig. 1 leads to a dual amplitude in the limit of an infinite number of virtual particles. The production of a $q\bar{q}$ pair with a large relative momentum must be interpreted as a rupture of the chain. There exist arguments that this process has a relatively small probability. This fact can be regarded as an argument in favor of DMs, since it explains why intermediate states containing a single resonance are dominant.

E. The relativistic string as a model of a hadron

The planar character of the dual diagrams indicates that the field which mediates the interaction between quarks is concentrated for some reason within a narrow region of space near the line joining the quarks. On the other hand, it is natural to interpret the equal-spacing character of the spectrum associated with DMs as the result of an excitation of a large number of harmonic oscillators with multiple frequencies. A classical system of this type is an elastic string. This was the origin of the idea that particles are quantum states of a one-dimensional continuous system.[17–19] These considerations lead to the problem of the classical and quantum descriptions of a relativistic one-dimensional structure—a "string."

The classical Lagrangian of a relativistic string in the context of DMs was first written down by Nambu[20] and was subsequently analyzed by several Japanese physicists[21–23]. A method of quantization was proposed by Goddard *et al.*[24] In addition to the model which describes the interaction of scalar particles in terms of generalized Veneziano amplitudes, there exist other DMs, in particular the Shapiro–Virasoro model (see[25,26]), which possesses the property of "non-planar" duality, and the "fermion" models of Neveu and Schwarz[27] and Ramond.[28] The Shapiro–Virasoro model arises naturally in considering closed strings.[29] The "fermion" models are obtained from strings with distributed spin. The idea of constructing a chain of spin-$\frac{1}{2}$ partons was first proposed by Aharonov *et al.*[30] A consistent theory of strings with spin was developed by Iwasaki and Kikkawa.[31] The next step was a theory of the interaction of strings, which was developed by Mandelstam, who used the method of functional integration for both ordinary strings[32] and strings with spin.[33]

Thus the basic results which have hitherto been obtained from DMs are reproduced by the theory of relativistic strings. This means that it may be possible to gain a better understanding of the physical significance of the hypothesis which lead to DMs and to avoid their inherent difficulties. We note that the formal identification of DMs with the theory of strings was not totally unexpected, since it has long been known[34] that it is possible to translate DMs into the language of quantum field theory with one spatial dimension, and the introduction of strings merely made it possible to identify this dimension with a line in ordinary space.[35]

The mechanical model which leads to the dual theory and which is an idealization of a quark-parton chain should not be called a "string," which we normally take to be of fixed length, but a "spring." We shall show that this object is similar to the "spring" that forms the American toy known as "slinky," which is used in Crawford's textbook[36] to illustrate wave phenomena. This is a helix of thin elastic wire consisting of a large number of looops. In its equilibrium state the stationary spring is compressed into a length of several centimeters, but it can be stretched to a length of several meters without irreversible changes. Owing to the absence of transverse elasticity and the large range of elastic extensions, the reactions of the spring to external forces are surprising and amusing; in particular, the name "slinky" originates from the fact that it easily and quickly avoids obstacles, i.e., it "slinks." When the spring is stretched to a length L, there is a tension $T=\varkappa(L-l)$, where l is its initial length and $\varkappa$ is its coefficient of rigidity. For sufficiently large deformations $L\gg l$, the force is proportional to the length, and harmonic oscillations of large amplitude can occur. As an idealization for small l we can take such a spring as a model of a material point with an internal structure which is excited by external forces. To apply this model to elementary particles, we must also assume that the wire is not only infinitesimally thin, but is also infinitesimally light, so that the "proper mass" of the spring is equal to zero. Of course, it is then necessary to work within the framework of the theory of relativity. Extensions lead to an elastic energy, i.e., a distributed mass. However, we shall continue to employ the conventional term "string" in what follows.

The simplest non-trivial state of a free string (in the absence of an external field) is shown in Fig. 4. The extremities of the string are not stretched, so that they have zero mass and move with the velocity of light, and the elasticity produces a centripetal force. The state of such a string is completely determined by its length. In this connection, there is an important relation between the mass and the angular momentum: $J=\alpha' M^2$

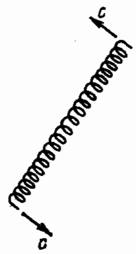

FIG. 4. The simplest non-trivial state of a string—the leading Regge trajectory.

(where α' is a constant of the theory). After quantization, J (as well as M and the length of the string) takes only discrete values; in addition, there is a new constant α_0 such that $J = \alpha' M^2 + \alpha_0 \hbar$. This is the obvious interpretation of the family of particles contained on the leading Regge trajectory. More complex states are described in Chap. 3.

The fact that this model involves a linear Regge trajectory can be understood by means of the following qualitative argument. The total energy Mc^2 of the string consists of the elastic potential energy and the kinetic energy of the internal motion. It is natural to assume that these two terms are of the same order of magnitude (as is usually the case in stationary systems) and that the elastic energy is proportional to the length, i.e., $M \sim L$. On the other hand, $J \sim PL$, where P is the effective momentum of the rotational motion. Since the extremities of the string move with the velocity of light, we have $P \sim Mc$ and hence $J \sim ML \sim M^2$. In the quantum case, the energy must also include a contribution $\hbar n\nu/2$ from the zero-point oscillations, where n is the effective number of oscillators and ν is the characteristic frequency. Since the oscillation propagates along the string with the velocity of light, we have $\nu \sim c/L$ and $M \sim L + \beta L^{-1}$, where β is some constant. If the second term here is smaller than the first, then $J \sim ML \sim M^2 + (\hbar\alpha_0/\alpha')$, where $\alpha_0 \sim \beta$.

Interactions of particles and resonances are represented as ruptures or recombinations of strings. Decays of resonances are described as follows: a string of length L_0 breaks into two strings of length L_1 and L_2, which in turn break, leading finally to the production of a certain number of compressed strings having no mass and moving with the velocity of light. A rupture involves a partial contraction and a transformation of elastic energy into kinetic energy of motion. This picture corresponds to the idea of a cascade mechanism of resonance decay. For example, the A_2 meson (of mass 1340 MeV/c^2) decays mainly according to the scheme $A_2 \to \rho\pi \to (\pi\pi)\pi$. It is somewhat more difficult to imagine the inverse process—the recombination of strings. When two strings collide, their extremities become linked and they exist in the form of a single string until a rupture occurs, possibly at a different position. This is the description of a binary process—the scattering of two particles (or resonances) (see Figs. 2 and 3). Of course, it is difficult to imagine the contact interaction representing the collision of two material points (the extremities of the strings) in classical mechanics. However, this is perfectly natural in the quantum theory; it is sufficient to recall the example of νe scattering in Fermi's theory. By virtue of the uncertainty relation, a beam of particles having a definite momentum is described by a wave function which extends over a large region of space. There is therefore a large probability amplitude for the encounter of two particles at a single point. This argument holds in any local theory and is perfectly applicable to the interaction between the extremities of the strings. It should be noted, incidentally, that a rupture, like any spontaneous process, cannot be described within the framework of classical mechanics.

It is extremely significant that the foregoing intuitive arguments, together with the contemporary formalism of quantum theory, have made it possible to calculate the interaction amplitudes and to obtain the equations of the dual theory (this result is due to Mandelstam). Of course, the pole character of the interaction amplitude is obvious from the outset, since the intermediate state here is a string with an equally spaced discrete spectrum. However, the crossing symmetry of the amplitude is non-trivial. It is also significant that strings reproduce the entire dual theory, together with corrections for multi-particle intermediate states (virtual resonance loops). The contribution from a two-resonance intermediate state in a scattering process has a particularly simple interpretation: the string breaks, the fragments recombine, and a final rupture then occurs (see Chap. 2 for further details).

Despite its apparent simplicity, the theory of strings is by no means trivial. The presence of an infinite number of degrees of freedom leads to difficulties in quantization. Moreover, despite the formulation of several new variants, nobody has yet been able to find a classical model for a non-local one-dimensional system which can be quantized in a consistent manner. Thus the fundamental difficulties which had previously been found to be inherent in dual models acquired an explicit physical interpretation.

Apart from its possible application to the physics of strong interactions, the relativistic string is of great interest from a purely theoretical point of view. This is a beautiful example of a non-local relativistic system which is constructed as a natural generalization of the mechanics of a material point. The difficulties in quantizing the theory are specific to quantum field theory, the simplest two-dimensional variant of which is the mechanics of a string, and are due to the typical peculiarities of the contemporary theory, such as the non-Abelian gauge group and the Schwinger terms in current algebra (in this connection, see Chap. 4).

F. Content of the review

In Chap. 2 we give the basic results concerning Veneziano's dual model and its modifications. On the whole, this section contains no proofs, but merely describes the qualitative features of the theory. Chapter 3 contains the classical theory of relativistic strings. We concentrate on problems which are important for the construction of a quantum theory—the canonical formalism and conformal symmetry. In Chap. 4 we describe the canonical quantization procedure and show that the quantum theory is self-consistent only with special conditions on the character of the spectrum and in a 26-dimensional space. Chapter 5 is devoted to the classical and quantum theories of strings with distributed spin. To describe the spin degrees of freedom in the classical theory, it is necessary to introduce anticommuting classical variables, i.e., mechanics with a Grassman algebra. An account is given of the theory of a classical particle with spin. Strings with spin are constructed as a natural generalization. The quantization procedure is self-consistent in this case only in a 10-dimensional space. It is shown in Sec. 6 how integrals over trajectories can be used to construct quantum amplitudes for the interactions of strings and to reproduce DMs. Some technically complicated problems are relegated to the appendix.

Several interesting problems are beyond the scope of the present review. One such problem is why the field energy is concentrated within a narrow, almost one-dimensional, region of space, in other words the problem of the "microstructure" of strings. It has been suggested that such a situation might occur in field theories which admit spontaneous symmetry breaking.[37] The appearance of structures such as strings is related to problems like "quark confinement" and infrared divergences in Yang–Mills theories of vector mesons. A discussion of these ideas can be found in[38–42].

Spontaneous symmetry breaking of dual models connected with a degeneracy of the resonances in the mass and spin may significantly alter current ideas and render the theory more realistic. This approach to duality and its quark interpretation has been developed by Volkov *et al.*[43,44] and by Bardakci and Halpern.[45,46]

In dual theories such as the Veneziano model, which can be interpreted in terms of interacting strings, hadronic resonances have zero widths in the first approximation. In the present review, we do not consider modifications of the dual theory which dispense with the zero-width approximation. In such models, the initial amplitudes already have the branch points required by the unitarity condition. An example of such a theory which has been analyzed in detail is the construction of dual amplitudes with Mandelstam analyticity.[47] By introducing finite widths in the dual amplitudes, interesting predictions are obtained for the observed resonances.[48]

Dual theories with infinitesimally narrow resonances cannot yet be taken to be a realistic description of the observed particles: their advantages are the intuitively clear space-time picture of the interactions which they provide and their mathematical elegance.

2. BASIC PROPERTIES OF DUAL MODELS[2)]

A. The construction of dual amplitudes

Consider the elastic scattering of identical scalar particles. In the simplest DM, proposed by Veneziano,[49] the scattering amplitude is represented in the form

$$A(s,t,u)=g^2[V(s,t)+V(t,u)+V(u,s)],$$
$$V(s,t)=\frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))}\equiv \mathrm{B}(-\alpha(s),\ -\alpha(t)), \tag{2.1}$$

where $\alpha(z)\equiv\alpha_0+\alpha' z$; g, α_0, and α' are constants, s, t, and u are the Mandelstam variables, and Γ and B are the well-known Euler functions. In the case of scattering of non-identical particles, the coefficients of V for the various channels and the parameters α_0 for the functions $\alpha(s)$, $\alpha(t)$, and $\alpha(u)$ may be different. However, the "slope parameter" α' must be universal. The function $V(s,t)$ has simple poles at $\alpha(s)=0,1,2,\ldots$, with residues which depend polynomially on t. The amplitude takes a particularly symmetric form if we impose the further condition

$$a\equiv\alpha(s)+\alpha(t)+\alpha(u)=3\alpha_0+4\alpha'\mu^2=-1, \tag{2.2}$$

where μ is the mass of the colliding particles. In this case, the amplitude (2.1) can be written in the form

$$A(s,t,u)=\frac{4}{\pi}g^2\prod_{z=s,t,u}\left[\cos\left(\frac{1}{2}\pi\alpha(z)\right)\Gamma(-\alpha(z))\right]. \tag{2.3}$$

At first sight, the condition (2.2) seems unrealistic. In fact, if we require that the first pole of the scattering amplitude corresponds to the external particle, i.e., $\alpha(\mu^2)=0$, then (2.2) implies that $\alpha_0=1$ and $\mu^2=-1/\alpha'$. Of course, the occurrence of a particle with imaginary mass, a "tachyon," is a major defect of the model. However, as we shall see later, the condition $\alpha_0=1$ is very attractive and is even necessary in this variant of the DM.

The amplitude (2.1) is a solution of the problem of finding a symmetric function of two variables which possesses the following properties (which together define the property of duality): a) there are no singularities in either of the variables, apart from poles on the real semi-axis; b) the residue at each pole in one of the variables is a polynomial in the other variable. It was shown by Coon[50] that Eq. (2.1) gives the only elementary solution of this problem[3)] if we reject a model with logarithmic pole trajectories, which has certain defects. The requirement of duality has the following consequences: a) an equal-spacing rule in the spectrum, i.e., there are poles at the points $s=M_k^2\equiv(k-\alpha_0)/\alpha'$ with $k=0,1,\ldots$; b) a linear relation between the square of the mass and the spin, since the residue $R_k(t)$ at the k-th pole in the variable s is a polynomial in t of degree k; c) degeneracy in the masses of the states with different angular momenta, since $R_k(t)$ does not reduce to $P_k(\cos\theta)$, where $\cos\theta=1+2t(M_k^2-4\mu^2)^{-1}$; d) a power-law "Regge" asymptotic behavior as $|s|\to\infty$ with $\arg s>0$, i.e., $V\sim\exp[\alpha(t)\ln s]$.

Multi-particle dual amplitudes can be constructed on the basis of a requirement of "planar duality" which generalizes the foregoing properties of the 4-particle amplitude, i.e., meromorphy and polynomial behavior of the residues. More precisely, let $V_N(p_1,\ldots,p_N)$ be an invariant function of the momenta of N particles with $p_1+\cdots+p_N=0$ (Fig. 5), this function being symmetric with respect to cyclic permutations of its arguments.

[2)]The reader who is familiar with DMs or who is not interested in the application of the theory of relativistic strings to the strong interactions may omit this section.

[3)]The general solution has the form of a sum of terms of the type (2.1) with the substitutions $\alpha(s)\to\alpha(s)-m$ and $\alpha(t)\to\alpha(t)-n$, where m and n are positive integers (see Eq. (A.5) in the Appendix).

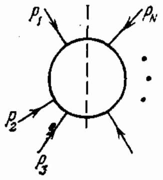

FIG. 5. An N-particle interaction amplitude. The cut of the diagram corresponds to the interaction channel.

We can assume that V_N depends only on the invariant variables of the form $s_{mn}=(p_{m+1}+\cdots+p_n)^2=(p_{n+1}+\cdots+p_m)^2$. Each of these variables corresponds to a particular channel—a reaction in which $(n-m)$ particles are transformed into $(N-n+m)$ particles. Planar duality can be formulated in terms of the following requirements: a) V_N has no singularities, apart from poles in the variables s_{mn}:

$$V_N=\sum_k (s_{mn}-M_k^2)^{-1} R_k^{(mn)}; \tag{2.4}$$

b) the residues at these poles have no singularities in the "transverse" variables $s_{m'n'}$, where $m'<m$ and $n'<n$ (or $m'>n$ and $n'>m$). Functions which possess these properties have been determined (see, e.g., the work of Goebel and Sakita[51]), and a manifestly symmetric integral representation was constructed for these functions[52] (see also Sec. A of the Appendix). However, it was necessary to verify that this purely analytic approach leads to amplitudes which are consistent with unitarity (which was naturally expressed in the single-particle form assumed in DMs). This was shown to be the case by Fubini and Veneziano,[53] who also established new important properties of the particle spectrum.

B. The spectrum of single-particle states

The poles of the amplitudes are interpreted as resonances. We must therefore obviously assume that all the amplitudes have poles at the same positions, i.e.,

$$M_k^2=\frac{k-\alpha_0}{\alpha'}, \tag{2.5}$$

as in the simplest case (2.1). Now unitarity implies that the residue at a pole must have the form of a product of transition matrix elements or, if there is degeneracy, a sum of such products:

$$R_k^{(mn)}=\sum_{l=0}^{k}\sum_{\nu} T^{(l,\nu)}_{\mu_1\ldots\mu_l}(p_{m+1},\ldots,p_n)\,T^{(l,\nu)}_{\mu_1\ldots\mu_l}(p_{n+1},\ldots,p_m); \tag{2.6}$$

here $T^{(l,\nu)}$ is an irreducible tensor[4] of rank l, i.e., l is the angular momentum of a resonance and ν is an index associated with the additional degeneracy. The tensors $T^{(l,\nu)}$ have the same duality properties as functions of the momenta as for the amplitude as a whole; in particular, they have pole singularities in their "internal" invariant variables. It is very significant that the expansion (2.6) holds for any residue of any amplitude, where $T^{(l,\nu)}$ is a universal system of tensor functions. The first proof of this fact[53] was very cumbersome. It was possible to simplify it greatly[54] by introducing an operator formalism. However, the factorization of the residues (Eq. (2.6)) is obtained only at the expense of a large degeneracy: the number of terms in the sum rises rapidly with increasing k.

This circumstance is easily understood. For a given value of l, the irreducible tensors $T^{(l,\nu)}_{\mu_1\ldots\mu_l}(p_1,\ldots,p_j)$ differ from one another essentially by the way in which their indices $\mu_1,\ldots,\mu_l$ are distributed among their vector arguments $p_1,\ldots,p_j$. (Since we are considering amplitudes with an arbitrary number of particles, we can always assume that $j>l$). Crudely speaking, the number of different tensors $T^{(l,\nu)}$ (for fixed l) is therefore equal to the number of representations of the number l as a sum of integers, $P(l)$. The solution of this classical problem in the theory of numbers has long been known; in particular, for large l we have[55] $P(l)\sim cl^{-1}\exp(\gamma\sqrt{l})$, where c and γ are certain constants. Thus, for each value of the mass M_k, there exist a certain number of resonances, $D(k)$, and this number rises rapidly with increasing k (and mass): $D(k)\sim\exp(\gamma\sqrt{k})$.

Nevertheless, this problem has not been completely solved. From the point of view of unitarity, the residues for elastic transitions must be positive, i.e., the operators $\hat{R}_k$ with matrix elements (2.6) must be positive definite. In other words, there must not be any "ghosts"—states with negative norm—among the resonances. However, owing to the pseudo-Euclidean metric, the tensor contraction (2.6) contains negative terms associated with the energy components of the momenta. These terms give no ghosts only if they are cancelled by contributions from the spatial components of the momenta. An analysis has shown (see the review by Schwarz[6] for further details) that ghosts can be universally eliminated only under the condition $\alpha_0=1$. This is a highly undesirable restriction, since it makes it impossible to identify the external scalar particles with the basic state contained on the trajectory $\alpha(s)(k=0)$. The theory also contains a massless particle of spin 1 (a strongly interacting "photon"). At $M^2=1/\alpha'$ there is a pair of degenerate states—particles with spins 2 and 0. At $M^2=2/\alpha'$ we have a state with spin 3 and two states with spin 1 and opposite parities, etc. The odd poles are cancelled in the fully symmetrized amplitude (2.1). However, odd-spin resonances occur in the multi-particle amplitudes.

The foregoing picture is quite unlike the observed resonance spectrum. Nevertheless, we can hardly dispense with the restriction $\alpha_0=1$ in this approach. The formal reason for this is that the dual amplitudes with $\alpha_0=1$ possess a strong additional symmetry associated with the Lie algebra found by Virasoro[56] (see Sec. C of the Appendix), which makes it possible to eliminate the time components.

These shortcomings of the simplest model described above were one of the reasons why more complex DMs have been constructed; however, the latter have also not yet overcome the difficulties which are inherent in this approach.

[4]A symmetric tensor is said to be irreducible if its contraction with respect to any pair of indices is equal to zero.

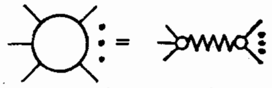

FIG. 6. An interaction amplitude as a sum over resonances (the pole approximation).

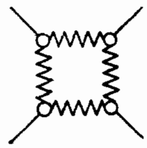

FIG. 8. A single-loop diagram of the dual theory. The wavy lines represent infinite sequences of resonances.

C. Higher orders and various modifications of the dual theory

The simplest dual amplitudes of the form (2.1) can serve as a description of the contribution from single-particle intermediate states (Fig. 6). Iterations of the pole amplitudes should in principle take into account more complex intermediate states (Fig. 7). It was shown by Kikkawa *et al.* [57,58] see also the review[5]) that it is possible to construct a consistent theory of the iteration of dual amplitudes in analogy with perturbation theory by representing the interaction amplitudes as series whose terms correspond to diagrams similar to Feynman diagrams (Fig. 8).

Incidentally, we note that these series reduce to precisely the perturbation series of ordinary quantum field theory for $g^2 \equiv \alpha'\lambda^2 \to 0$, where λ is fixed. In fact, it is easy to see (see Sec. A of the Appendix) that

$$\lim_{\alpha'\to 0} g^2 V(s, t) = -\lambda^2 [(s-M^2)^{-1} + (t-M^2)^{-1}], \tag{2.7}$$

where $M^2 = -\alpha_0/\alpha'$, i.e., the Veneziano amplitude reduces in this limit to the Born amplitude in the theory of a scalar field with the interaction $\lambda\varphi^3$. It can be shown (see[59-61]) that if $\alpha_0 = 1$, the limit gives massless scalar electrodynamics, while for $\alpha_0 = 2$ (the Virasoro model) it gives a theory similar to gravitation. A clear indication of the possibility of such a limit is the fact that when $\alpha' \to 0$ the distance between the poles tends to infinity, leaving only the lowest state of the entire family of resonances.

Complex (non-pole) dual resonance diagrams can also be interpreted in terms of ordinary Feynman diagrams (Fig. 9), as has been pointed out by Fairlie and Nielsen[62] (see also[34,63] in this connection). Thus the expansion in the number of resonance loops corresponds to isolating classes of Feynman diagrams characterized by a given number of contours which carry a large virtual momentum. In the language of quarks and gluons, a resonance loop corresponds to virtual production of a quark-antiquark pair (Fig. 10).

We can in principle calculate dual resonance diagrams with any number of loops, but this calculation is by no means as simple as in the case of Feynman diagrams. However, an analysis shows that the expansion obtained in this way contains no small parameter and that each new term radically alters the amplitude. In particular, the single-loop approximation (see Fig. 8) already leads to complicated singularities[64] and leaves no trace of the simple dual picture. Nevertheless, it is remarkable that the single-loop approximation can be used to obtain an amplitude which has no singularities other than simple poles on the new ("loop") linear trajectory

$$\alpha_L(s) = 2\alpha_0 + \frac{1}{2}\alpha' s, \tag{2.8}$$

provided that the dimensionality of space is 26 (see[65]). The resulting amplitude is then completely symmetric and can be reduced to the form proposed previously by Virasoro.[25] The Virasoro amplitude

$$W(s, t, u) = g_W^2 \prod_{z=s,\, t,\, u} \frac{\Gamma(-(1/2)\,\alpha(z))}{\Gamma(-(1/2)\,a - (1/2)\,\alpha(z))}, \tag{2.9}$$

where $a = \alpha(s) + \alpha(t) + \alpha(u)$, has poles in each of the three channels, i.e., it possesses the property of "non-planar" duality. If $a = -1$, the amplitude W reduces to the form (2.1) (or (2.3)), provided that $g_W^2 = g^2/\sqrt{\pi}$. However, it is more appropriate for the amplitude (2.9) to adopt the condition $a = -2$, which is in agreement with Eq. (2.8) with $\alpha_0 = 1$ if $\mu^2 = -4/\alpha'$ for the external particles. With a trajectory of the form

$$\alpha(s) = 2 + \frac{1}{2}\alpha' s$$

the Virasoro model gives no ghost states (like the Veneziano model with $\alpha_0 = 1$), and a symmetric and completely dual integral representation can be written for the multi-particle interaction amplitude (see Sec. A of the Appendix). It was pointed out by Shapiro[26] that the amplitude (2.9) can be obtained from a Feynman diagram spanning a closed surface, with small virtual momenta (as in the diagram of Fig. 1).

A crude interpretation of these results is as follows. The trajectory which appears in the Veneziano amplitude (2.1) corresponds to the physical mesons (such as the ρ meson). The resonances which appear in the Virasoro amplitude (2.9) are two-particle bound states of "ρ-meson" resonances and lie on the vacuum trajectory, which has double the intercept on the vertical axis and half the slope (according to Eq. (2.8)). This picture was an attractive one, since it has been "established" experimentally that the vacuum trajectory has $\alpha_0^{(P)} \approx 1$ and $\alpha'_P \approx 0.5$ GeV^{-2}, while the ρ-meson trajectory has $\alpha_0^{(\rho)} \approx 0.5$ and $\alpha'_\rho \approx 1$ GeV^{-2}. However, we must remember that a consistent theory is obtained only when $\alpha_0^{(\rho)} = \alpha_0^{(P)}/2 = 1$ and in a 26-dimensional space!

The fact that some of the defects of the simplest DMs due to Veneziano (or Virasoro) are avoided in a 26-di-

FIG. 7. A two-resonance intermediate state.

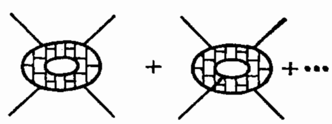

FIG. 9. Feynman diagrams for the single-loop dual diagram.

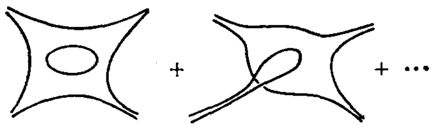

FIG. 10. Quark interpretation of the diagram in Fig. 8.

mensional space suggests that the theory should include some additional degrees of freedom which might assume the function of some of the spatial components. This heuristic argument turns out to be quite correct: by classifying the states according to new quantum numbers, it is possible to reduce the critical dimensionality of the space. The most direct (but by no means trivial) method of doing this is to introduce an internal symmetry with respect to the group $SU(N)$; this is the Bardakci–Halpern model[66] (see also[67] and the review by Schwarz[6]). This procedure reduces the critical dimensionality to $d_{cr} = 26 - N$, but the model is clearly not a realistic one.

A more interesting generalization of the DM involves the introduction of additional fermion degrees of freedom. In particular, this makes it possible to construct a DM for the interaction of fermions (the Ramond model[28]). From the point of view of the quark picture, this approach for the interactions of bosons (the Neveu–Schwarz model[27]) can be interpreted simply as a method of allowing for the spin $\frac{1}{2}$ of the quarks. One finds two families of resonances produced by a quark–antiquark pair in states of total spin 0 and 1 ("pion" and "ρ-meson" trajectories). The amplitudes for the meson–meson interaction which are obtained in this way have the same structure as the Veneziano amplitude (2.1). In particular, for $\pi\pi$ scattering we have

$$A_{\pi\pi} = g^2 \frac{\Gamma(1-\alpha_\rho(s))\,\Gamma(1-\alpha_\rho(t))}{\Gamma(1-\alpha_\rho(s)-\alpha_\rho(t))}. \tag{2.10}$$

However, the spectrum of states is different from that described in Sec. 2B. For example, the lowest states on the trajectories have $\mu_\pi^2 = -1/2\alpha'$ and $\mu_\rho^2 = 0$ (instead of $\mu^2 = -1/\alpha'$, as in the Veneziano model). It is also possible to construct a variant of the Neveu–Schwarz model which satisfies non-planar duality.[68] With a certain modification[69–71] of the Neveu–Schwarz theory (the introduction of an additional fermion variable), it has even been possible to eliminate the states with $\mu^2 < 0$ ("tachyons"), although this model is far from reality. The critical dimensionality for DMs with spin is 10 (instead of 26). A simple interpretation of the Neveu–Schwarz and Ramond models involves the idea of a "string" with distributed spin, which is described in Chap. 5.

In Table I we show the main qualitative characteristics of dual models (tachyons are absent in the Neveu–Schwarz model only if an appropriate modification is made[69–71]).

The mathematics required for the description of DMs is very formal and rather complicated. The introduction of relativistic strings has made these models much easier to understand and more accessible to physical intuition, and in certain cases it has even simplified the calculations.

3. RELATIVISTIC STRINGS. THE CLASSICAL THEORY

A. Kinematics and the variational principle

The motion of a point in the theory of relativity is described by a line in four-dimensional space-time: $x_\mu = x_\mu(\tau)$, where τ is a parameter which varies along the line. The classical action for a free material point is proportional to the arc length of this line[72]:

$$\mathcal{S} = -m \int_{\tau_i}^{\tau_f} \sqrt{\dot{x}^2}\, d\tau, \tag{3.1}$$

where m is the mass of the point and $\dot{x}_\mu = dx_\mu/d\tau$ is the

TABLE I.

Model	Leading trajectory $\alpha(s)$	Spin at $M=0$	Tachyon, $\alpha'\mu^2$	Critical dimensionality d	Limit $\alpha' \to 0$	Properties of string
Veneziano	$1+\alpha' s$	1	-1	26	$\lambda\varphi^3$ scalar theory	Free string
Virasoro	$2+1/2\alpha' s$	2	-4	26	Quantized gravitation	Closed string
Ramond	$1/2+\alpha' s$	1/2	None	10	Massless electrodynamics	String with distributed spin. Quarks at the ends
Neveu–Schwarz	(pf): $1+\alpha' s$ $(\pi\omega)$: $\alpha' s$	1 0		10	Scalar electrodynamics	String with distributed spin. Quark and antiquark at the ends
Addition of an internal symmetry	As in the original dual model			$d-N$	Yang–Mills theory	No simple interpretation

tangent vector, which lies inside the light cone, $\dot{x}^2 > 0$. The action $\mathscr{S}$ is invariant with respect to Lorentz transformations and the choice of the parameter τ. It is usual to use one of two possible parametrizations: 1) τ is the "laboratory" time, $x_0 = \tau$, and $\dot{x}^2 = 1 - v^2$; 2) τ is the "proper" time and $\dot{x}^2 = 1$.

Consider a one-dimensional material system of finite length, whose points are characterized by an "internal coordinate" σ in the range $0 \leqslant \sigma \leqslant \sigma_0$. The motion of such a system is described by a two-dimensional surface $x_\mu(\tau, \sigma)$ in Minkowski space. Systems for which the action is invariant with respect to the choice of the parameters τ and σ are naturally of special interest. This property holds for an action[5)] which is proportional to the area of the region on the surface bounded by the lines $\tau = \tau_i$ and $\tau = \tau_f$:

$$\mathscr{S} = -A \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_0} d\sigma \sqrt{(\dot{x}x')^2 - \dot{x}^2 x'^2}; \tag{3.2}$$

here A is a constant of dimensionality m^2, $\dot{x}_\mu = dx_\mu/d\tau$, and $x'_\mu = dx_\mu/d\sigma$. The expression under the square root sign in (3.2) is positive if the surface is time-like, i.e., if at each point of this surface there exists a tangent vector contained within the light cone.[6)] We also take $\dot{x}^2 > 0$, so that τ can be regarded as "intrinsic time." In principle, we can also consider two- and three-dimensional systems[7)] (see, e.g., [80]).

The action (3.2) is obviously invariant with respect to changes of variables of the form

$$\tau \to \tilde{\tau} = f(\tau, \sigma), \quad \sigma \to \tilde{\sigma} = h(\tau, \sigma), \tag{3.3}$$

where f and h are arbitrary differentiable functions. It is important only that the Jacobian of the transformation (3.3) is everywhere non-zero and that the boundary of the surface has its original form, i.e., $h(\tau, 0) = 0$ and $h(\tau, \sigma_0) = \tilde{\sigma}_0 = \text{const}$.

It is convenient to define an orthonormal coordinate system on the surface by the invariant conditions

$$(\dot{x}x') = 0, \quad \dot{x}^2 = -x'^2, \tag{3.4a}$$

or, equivalently,

$$(\dot{x} \pm x')^2 = 0. \tag{3.4b}$$

An important property of a two-dimensional pseudo-Euclidean surface is that the conditions (3.4a) and (3.4b) do not completely fix the system of parameters, but they merely distinguish certain systems—in fact, a very wide class of systems. It is easy to see that these conditions, as well as the boundary of the surface, are invariant with respect to transformations of the form

$$\begin{aligned} \tilde{\tau} &= \tau_1 + \tau + g(\tau + \sigma) + g(\tau - \sigma), \\ \tilde{\sigma} &= \sigma + g(\tau + \sigma) - g(\tau - \sigma), \end{aligned} \tag{3.5}$$

where τ_1 is a constant and $g(u)$ is an arbitrary periodic function $g(u) = g(u + 2\sigma_0)$ such that $\partial(\tilde{\tau}, \tilde{\sigma})/\partial(\tau, \sigma) \neq 0$, i.e.,

$$2\frac{dg}{du} + 1 \neq 0 \tag{3.6}$$

for any u. Equation (3.5) describes a conformal transformation (see Sec. C of the Appendix).

The simplest and most obvious parametrization is the "laboratory" parametrization

$$x_0 = \tau, \quad \dot{x}_0 = 1, \quad \dot{x}_i = v_i, \quad x'_0 = 0, \quad x'_i = w_i; \tag{3.7a}$$

here $\mathbf{v}$ is the velocity of the point, $\mathbf{w}$ is a vector tangent of the curve $\mathbf{x}(x_0, \sigma)$ with $|\mathbf{w}| = dl/d\sigma$, and dl is the element of length along this curve. It follows from the conditions (3.4) that

$$(\mathbf{vw}) = 0, \quad \mathbf{w}^2 = 1 - \mathbf{v}^2. \tag{3.7b}$$

Thus the velocity of any point is always directed along the normal to the instantaneous position of the curve. The proper length L_0 of the curve and its "laboratory" length L (which takes into account the Lorentz contraction) are defined by the equations

$$\begin{aligned} L_0(\tau) &= \int dl = \int_0^{\sigma_0} \sqrt{1 - \mathbf{v}^2}\, d\sigma, \\ L(\tau) &= \int \sqrt{1 - \mathbf{v}^2}\, dl = \int_0^{\sigma_0} (1 - \mathbf{v}^2)\, d\sigma. \end{aligned} \tag{3.8}$$

The length L is in general not conserved in the process of motion. The action (3.2) has the form $\mathscr{S} = -\int L(\tau) d\tau$, so that the principle of minimum $\mathscr{S}$ can be regarded as a combination of Hamilton's dynamical principle for points of the string and the static condition of the minimum length L. In spite of the established terminology, it would therefore be more correct to think of the system in question not as a string, but as a thin and elastic spring. It is essential, however, that the system is relativistic. In the simplest case, the spring has zero length and zero mass and moves with the velocity of light. Non-trivial solutions are described in the following subsection.

B. Equations of motion and conservation laws

Let us determine the variation of the action (3.2) for an arbitrary variation $\delta x_\mu(\tau, \sigma)$ (of course, $\delta x_\mu(\tau_i)$

5) Lagrangians of the form (3.2) were considered by Dirac[73] and by Barbashov and Chernikov[74] before the advent of dual models and the interest in relativistic strings which they created. In addition to the work which we have already mentioned,[20-24] the system described by the action (3.2) has been studied by Chang, Mansouri, and Nambu,[75-77] Konisi,[78] and Barbashov and Chernikov.[79]

6) An arbitrary vector tangent to the surface has the form $y_\mu = a\dot{x}_\mu + bx'_\mu$. A part of the surface lies inside the light cone if there exist two different zero vectors, $y^2 = 0$. This implies that $(\dot{x}x')^2 - \dot{x}^2 x'^2 > 0$.

7) In this case, $x_\mu = x_\mu(\tau, \sigma_1, \sigma_2)$ or $x_\mu = x_\mu(\tau, \sigma_1, \sigma_2, \sigma_3)$, and the action has the form

$$\mathscr{S} = A \int d\tau \int \Pi\, d\sigma\, F,$$

where $F^2 = \det \| (x^{(\alpha)} x^{(\beta)}) \|$ with $\alpha, \beta = 0, 1, 2, 3$, and $x_\mu^{(\alpha)} = dx_\mu/d\sigma_\alpha$ with $\sigma_0 \equiv \tau$.

$= \delta x_\mu(\tau_f) = 0$):

$$\delta\mathcal{S} = \int d\tau \Big[\int_0^{\sigma_0} d\sigma \Big(\frac{\partial}{\partial\tau}\frac{\partial\mathcal{L}}{\partial\dot{x}_\mu} + \frac{\partial}{\partial\sigma}\frac{\partial\mathcal{L}}{\partial x'_\mu} \Big) \delta x_\mu - \Big(\frac{\partial\mathcal{L}}{\partial x'_\mu}\delta x_\mu \Big)_{\sigma=0}^{\sigma_0} \Big], \tag{3.9}$$

where

$$\mathcal{L} = -A\sqrt{(\dot{x}x')^2 - \dot{x}^2 x'^2}.$$

Let

$$p^\mu(\tau,\sigma) = \frac{\partial\mathcal{L}}{\partial\dot{x}_\mu}, \quad \pi^\mu(\tau,\sigma) = \frac{\partial\mathcal{L}}{\partial x'_\mu}. \tag{3.10}$$

The equations of motion corresponding to the variational principle $\delta\mathcal{S} = 0$ take the form

$$\frac{\partial p^\mu}{\partial\tau} + \frac{\partial\pi^\mu}{\partial\sigma} = 0. \tag{3.11}$$

The boundary conditions for a free string[8] (with arbitrary variations δx_μ at the end-points), which also follow from (3.9), have the form

$$\pi^\mu(\tau, 0) = \pi^\mu(\tau, \sigma_0) = 0. \tag{3.12}$$

Note that the explicit form of $\mathcal{L}$ implies that

$$p^2 = -A^2 x'^2, \quad \pi^2 = -A^2\dot{x}^2, \tag{3.13}$$

and, in particular, that $\dot{x}^2 = 0$ if $\sigma = 0$ or σ_0. In other words, the ends of the string move with the velocity of light. This is a perfectly natural result, since the system is characterized by zero density of rest mass—the mass y of the system as a whole is due to the internal motion and tension.

The invariance of the action leads to various conservation laws. Using Eq. (3.9) with $\delta x_\mu = \text{const}$, we obtain the conservation law for the total momentum, which is given by

$$P^\mu = \int_\Gamma p_\Gamma^\mu d\gamma, \tag{3.14}$$

where Γ is an arbitrary curve intersecting the surface and $p_\Gamma^\mu d\gamma \equiv p^\mu d\sigma - \pi^\mu d\tau$. In particular, if Γ is a line $\tau = \text{const}$, then $p_\Gamma^\mu d\gamma \equiv p^\mu d\sigma$. Of course, the conservation of the momentum (3.14) can also be regarded as a trivial consequence of Eq. (3.11), which has the form of a conservation law for the flux of the two-dimensional vector (p, π) on the surface (τ, σ). Taking $\delta x_\mu = \omega_{\mu\nu}x^\nu$, Eq. (3.9) leads to a conservation law for the total angular momentum

$$J^{\mu\nu} = \int_\Gamma (x^\mu p_\Gamma^\nu - x^\nu p_\Gamma^\mu)\, d\gamma. \tag{3.15}$$

We shall now adopt an orthogonal parametrization (τ, σ), i.e., we add the supplementary conditions (3.4) to the Lagrangian. The Lagrangian can then be represented in the manifestly conformally invariant form

$$\mathcal{L} = \frac{1}{2}A(\dot{x}^2 - x'^2), \tag{3.16}$$

while the equations of motion and boundary conditions take the particularly simple form[9]

$$\ddot{x}_\mu - x''_\mu = 0, \tag{3.17a}$$

$$x'_\mu(\tau, 0) = x'_\mu(\tau, \sigma_0) = 0. \tag{3.17b}$$

In the laboratory parametrization (3.7), Eq. (3.17a) can be rewritten in the form of Newton's second law for an element of the string:

$$\frac{\partial}{\partial\tau}\Big(\rho\frac{\mathbf{v}}{\sqrt{1-\mathbf{v}^2}}\Big) = \frac{\partial}{\partial\sigma}(T\mathbf{n}), \tag{3.18}$$

where $\rho = A\sqrt{1-\mathbf{v}^2}$ is the linear density of mass, $T = A\sqrt{1-\mathbf{v}^2}$ is the tension at a given point, and $\mathbf{n}$ is a unit tangent vector. Thus points of the string at which the tension is equal to zero move with the velocity of light, since the density of mass at these points is also equal to zero.

The general solution of the problem (3.17) has the form

$$x_\mu(\tau, \sigma) = r_\mu\tau + f_\mu(\tau+\sigma) + f_\mu(\tau-\sigma), \tag{3.19}$$

where r_μ is a constant vector and $f_\mu(u)$ is a differential vector function satisfying the identities

$$f_\mu(u) = f_\mu(u + 2\sigma_0), \tag{3.20a}$$

$$\Big(r_\mu + 2\frac{df_\mu}{du}\Big)^2 = 0. \tag{3.20b}$$

The first identity follows from the boundary condition (3.17b), while the second follows from (3.4b). The initial data determine the function $f_\mu(u)$ on the interval $[-\sigma_0, \sigma_0]$:

$$f_\mu(u) = \frac{1}{2}\Big[x_\mu(0, |u|) + \varepsilon(u)\int_0^{|u|}\dot{x}(0,\sigma)\,d\sigma - r_\mu u\Big], \qquad \varepsilon(u) \equiv \frac{u}{|u|}. \tag{3.21}$$

The vector r_μ is related to the total momentum of the system by the equation

$$r_\mu = \sigma_0^{-1}\int_0^{\sigma_0}\dot{x}_\mu(0,\sigma)\,d\sigma = (A\sigma_0)^{-1}P_\mu. \tag{3.22}$$

We note that in the laboratory parametrization (3.7) we have $f_0(u) \equiv 0$ and r_μ is a unit vector with components $(1, 0, 0, 0)$, so that the parameter σ_0 in this case is related to the mass by the simple equation

$$M = A\sigma_0. \tag{3.23}$$

Let us consider an interesting particular case. Suppose that the initial conditions have the form

[8]See Sec. 3D for the case of a closed string.

[9]Let us verify that the conditions (3.4) are consistent with the equations of motion. By Eq. (3.17a), the scalar functions $\varphi_\pm = (\dot{x} \pm x')^2$ satisfy the condition $\dot{\varphi}_\pm = \pm\varphi'_\pm$. Hence if $\varphi_\pm = 0$ for all σ at the initial instant $\tau = \tau_i$, then $\varphi_\pm \equiv 0$ for all τ and σ.

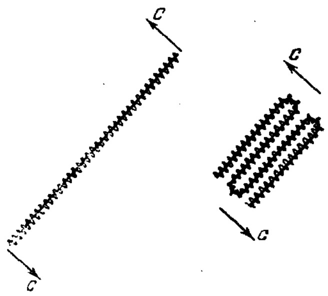

FIG. 11. A folded rotating string—the basis solution (3.26). a) $N=1$; b) $N=4$.

$$x_\mu(0,\sigma)=\omega_N^{-1}W_\mu\cos\omega_N\sigma,\quad \dot{x}_\mu(0,\sigma)\,r_\mu+V_\mu\cos\omega_N\sigma,\quad \omega_N=\frac{N\pi}{\sigma_0}\quad (N=1,2,\ldots), \tag{3.24}$$

Then it follows from (3.4) that

$$(rV)=(rW)=(VW)=0,\quad r^2=-V^2=-W^2. \tag{3.25a}$$

Let us choose the center-of-mass system, in which $\mathbf{P}=0$ and $P_0=M$ is the total mass, and introduce units of measurement on the surface (τ,σ) which lead to the "laboratory" parametrization (3.7). Then

$$r_0=\frac{M}{A\sigma_0}=1,\quad \mathbf{r}=0,\quad V_0=W_0=\mathbf{VW}=0, \tag{3.25b}$$

where $\mathbf{V}$ and $\mathbf{W}$ are mutually orthogonal unit vectors. The solution of (3.19) takes the form

$$x_0(\tau,\sigma)=\tau,\quad \mathbf{x}(\tau,\sigma)=\omega_N^{-1}\cos\omega_N\sigma\,[\mathbf{W}\cos\omega_N\tau+\mathbf{V}\sin\omega_N\tau]. \tag{3.26}$$

It is easy to see that these equations describe a string folded N times in the form of a straight-line segment and rotating in the plane $(\mathbf{V},\mathbf{W})$ about its midpoint with an angular velocity such that its end-points move with the velocity of light (see Fig. 11). Let us calculate the angular momentum J of the system, using Eq. (3.15). This is a vector normal to the plane $(\mathbf{V},\mathbf{W})$ and of length

$$J=(2\pi AN)^{-1}M^2. \tag{3.27}$$

Thus we obtain the linear dependence between the angular momentum and the square of the mass which is characteristic of dual models (in the classical limit $J\gg 1$ or $\alpha_0\to 0$). It is now obvious that the fundamental constant A is related to the slope of the Regge trajectory by the equation

$$A=(2\pi\alpha')^{-1}. \tag{3.28}$$

The state with $N=1$ corresponds to the leading trajectory while for $N>1$ we obtain some of the resonances on the "daughter" trajectories. We recall that the quantity A is equal to the tension of the string at the point with $v=0$ (see (3.18)). Taking $\alpha'=m_p^{-2}$ and transforming the usual units, we obtain $A=m_pc^2/\lambda_p=1.6\times10^{-3}/1.3\times10^{-13}$ erg/cm ≈ 13 m (here $\lambda_p=2\pi\hbar/m_pc$ is the Compton wavelength of the proton). Noting that $v^2=\cos^2\omega_N\sigma$ for the solution (3.26), we find from (3.8) that the "laboratory" and "property" lengths of the string are given by $L=\sigma_0/2=M/2A$ and $L_0=2\sigma_0/\pi$; the length of the string is proportional to its mass. The length of the string is independent of the time for the solution (3.26).

C. The Hamiltonian formalism

To carry out the canonical quantization, we must determine the Hamiltonian corresponding to the proper time τ, rewrite the equations of motion in Hamilton's form, and introduce the algebra of Poisson brackets. Owing to the invariance with respect to the general transformations (3.3), the original Lagrangian (3.9) describes a system which is degenerate in Dirac's sense,[81] since the equation

$$p^\mu(\tau,\sigma)=\frac{\partial\mathscr{L}}{\partial\dot{x}_\mu} \tag{3.29}$$

cannot be solved for $\dot{x}_\mu$. It is easy to see that this equation leads to the conditions[10)]

$$\varphi_\pm(x,p)\equiv(p\pm Ax')^2=0. \tag{3.30}$$

According to Dirac's theory, the Hamiltonian density in this case has the form

$$\mathscr{H}(x,p)=\dot{x}_\mu\frac{\partial\mathscr{L}}{\partial\dot{x}_\mu}-\mathscr{L}+v_+\varphi_++v_-\varphi_-, \tag{3.31}$$

where $v_\pm(\sigma)$ are arbitrary functions, whose choice fixes the "gauge." The first two terms cancel one another. Putting $v_+=v_-=-1/4A$, we have

$$H=\int_0^{\sigma_0}\mathscr{H}(x,p)\,d\sigma=-\frac{1}{2}\int_0^{\sigma_0}\left(\frac{p^2}{A}+Ax'^2\right)d\sigma. \tag{3.32}$$

The Poisson brackets for the canonical variables have the usual form[11)]

$$\{p_\mu(\sigma_1),\,x_\nu(\sigma_2)\}_{\text{P.B}}=-g_{\mu\nu}\,\delta(\sigma_1-\sigma_2). \tag{3.33}$$

The equations of motion

$$\dot{x}_\mu=\frac{\delta H}{\delta p^\mu}=-\frac{p_\mu}{A},\quad \dot{p}_\mu=-\frac{\delta H}{\delta x^\mu}=-Ax''_\mu \tag{3.34}$$

are equivalent to (3.17a), so that our gauge agrees with the choice of the orthonormal coordinate system on the surface given by the conditions (3.4).

We now replace σ by an independent variable θ in the range $-\sigma_0\leqslant\theta\leqslant\sigma_0$ and transform to the new dynamical variables $y_\mu(\tau,\theta)$ given by

$$y_\mu(\tau,\theta)=\frac{1}{2}\{g_{\mu\nu}p^\nu(\tau,\sigma)\pm Ax'_\mu(\tau,\sigma)\}\quad\text{for}\quad\theta=\pm\sigma. \tag{3.35}$$

We impose the boundary conditions $y_\mu(\tau,+0)=y_\mu(\tau,-0)$ and $y_\mu(\tau,\sigma_0)=y_\mu(\tau,-\sigma_0)$, which are equivalent to (3.17b). In these variables, the Poisson brackets take the form

[10)]Similarly, in the case of a material point with the action (3.1), invariance with respect to the choice of the parameter τ leads to the condition $p^2=m^2$.

[11)]We employ the metric $g_{00}=1$ and $g_{kk}=-1$ for $k=1, 2, 3$. The Poisson brackets are defined as in the books[81,84].

$$\{y_\mu(\theta_1),\ y_\nu(\theta_2)\}_{\text{P.B}} = \frac{1}{2} A g_{\mu\nu} \delta'(\theta_1 - \theta_2), \tag{3.36}$$

with the condition

$$y^2(\theta) = 0, \tag{3.37}$$

and the Hamiltonian and total momentum are given by

$$\mathscr{H} = -\frac{1}{A}\int_{-\sigma_0}^{\sigma_0} y^2(\theta)\,d\theta, \tag{3.38}$$

$$P_\mu = \int_{-\sigma_0}^{\sigma_0} y_\mu(\theta)\,d\theta. \tag{3.39}$$

It is natural to interpret $y_\mu(\theta)$ as a generalized momentum density. "Neighboring" points interact, leading to a non-canonical form for the Poisson brackets; thus, owing to the "contact" character of the interaction, the Hamiltonian has a quasi-free form. We note that the Hamiltonian for a point particle obtained from the action (3.1) under the condition $\dot{x}^2 = 1$ (where τ is the proper time) is $H = -(p^2 - m^2)/2m$ (see Appendix A of Feynman's paper[82] and the work of Casalbuoni *et al.* [83]). The condition (3.37) indicates that the string consists of "massless" matter. We recall that two vectors $y_\mu(\sigma)$ and $y_\mu(-\sigma)$ are specified at each point of the string, one of which can be interpreted as a parton momentum and the other as an antiparton momentum.

Note that the Hamiltonian (3.38) determines the evolution of the dynamical variables as a function of the parameter τ and is not identical with the energy P_0 (Eq. (3.39), which determines the time development of the system from the point of view of an external observer.

The equations of motion for the Hamiltonian (3.38)

$$\frac{\partial y_\mu}{\partial \tau} = \frac{\partial y_\mu}{\partial \theta} \tag{3.40}$$

with the boundary condition $y_\mu(\tau, \sigma_0) = y_\mu(\tau, -\sigma)$ have the simple solution

$$\begin{aligned} y_\mu(\tau,\ \theta) &= Y_\mu(\tau + \theta), \\ Y_\mu(u) &\equiv y_\mu(0,\ u) = Y_\mu(u + 2\sigma_0). \end{aligned} \tag{3.41}$$

It will be useful for what follows to make a further change of variables. Let us expand $y_\mu(\theta)$ in a Fourier series:

$$\begin{aligned} y_\mu(\tau,\ \theta) &= \frac{1}{2\sigma_0}\Big[P_\mu + \sqrt{\pi A}\sum_{m\neq 0} a^m{}_\mu(\tau)\,e^{i\omega_m\theta}\Big], \\ \omega_m &= \frac{\pi m}{\sigma_0}, \quad a_\mu^{-m} = (a_\mu^m)^*; \end{aligned} \tag{3.42}$$

the term with $m = 0$ is written separately here, in accordance with (3.39). The equations of motion in the variables a_μ^m and their solutions have the form

$$\dot{a}_\mu^m = i\omega_m a_\mu^m, \tag{3.43}$$

$$a_\mu^m(\tau) = \alpha_\mu^m e^{i\omega_m \tau}. \tag{3.44}$$

If we regard the string as an oscillating system with an infinite number of degrees of freedom, the quantities a_μ^m are analogous to the "normal coordinates" (see, e.g., [84]). Of course, the solution (3.42) and (3.44) can be obtained directly from Eq. (3.19) by expanding the function $f_\mu(u)$ in a Fourier series. The particular solution (3.26) corresponds to the initial condition $\alpha_\mu^m = 0$ with $m \neq N$, i.e., an excitation of only the N-th oscillator.

The Poisson brackets for the normal coordinates, obtained from (3.36), are

$$\{a_\mu^m,\ a_\nu^n\}_{\text{P.B}} = i m g_{\mu\nu}\delta(m+n), \tag{3.45}$$

where $\delta(k) = 0$ for $k \neq 0$ and $\delta(0) = 1$. A specific feature of this system is the presence of constraints. Expanding the left-hand side of Eq. (3.37) in a Fourier series, we obtain

$$L_0 \equiv -\frac{\sigma_0 H}{\pi} = (2\pi A)^{-1}P^2 + \sum_{n>0}(a^n a^{-n}) = 0, \tag{3.46a}$$

$$L_m = L^*_{-m} = (\pi A)^{-1/2}(Pa^m) + \frac{1}{2}\sum_{n\neq 0,\, m}(a^n a^{m-n}) = 0. \tag{3.46b}$$

The quantities $L_m(\tau)$ obviously satisfy equations of the form (3.43), so that $L_m(\tau) = \Lambda_m e^{i\omega_m\tau}$, where Λ_m are constants, and the conditions $L_m = 0$ are consistent with the equations of motion. Using (3.45), we find

$$\{a_\mu^m,\ L_n\}_{\text{P.B}} = i m a_\mu^{m+n}, \tag{3.47a}$$

$$\{L_m,\ L_n\}_{\text{P.B}} = i(m-n)L_{m+n}. \tag{3.47b}$$

Since the Poisson brackets for the quantities L_m are expressed linearly in terms of these same quantities (the L_m form a Lie algebra), the conditions (3.46) define constraints of the first kind (in Dirac's terminology[81]), and the L_m are generators of a symmetry group. It is easy to see that this is the group of conformal transformations (3.5) (for further details, see Sec. C of the Appendix).

The real and imaginary parts of the variables a_μ^m are related to the Fourier expansions of the momenta and coordinates of points of the string:

$$\begin{aligned} p_\mu(\tau,\ \sigma) &= \sigma_0^{-1}\Big[P_\mu + (\pi A)^{1/2}\sum_{m\neq 0} a_\mu^m(\tau)\cos\omega_m\sigma\Big], \\ x_\mu(\tau,\ \sigma) &= X_\mu(\tau) + (\pi A)^{-1/2}\sum_{m\neq 0}(im)^{-1}a_\mu^m(\tau)\cos\omega_m\sigma. \end{aligned} \tag{3.48}$$

Here X_μ is the coordinate of the center of mass; it follows from the equations of motion that $X_\mu(\tau) = q_\mu + \tau P_\mu / A\sigma_0$, where q_μ is a constant vector. Using (3.15), we can evaluate the total angular momentum in the new variables:

$$J_{\mu\nu} = (X_\mu P_\nu - X_\nu P_\mu) + i\sum_{m>0} m^{-1}(a_\mu^m a_\nu^{-m} - a_\nu^m a_\mu^{-m}). \tag{3.49}$$

The first term in this expression is the orbital angular momentum of the string as a whole, while the second term is the total spin associated with its internal motion. According to Eq. (3.46a), the mass of the string is given by

$$M^2 = -2\pi A\sum_{m>0}(a^m a^{-m}). \tag{3.50}$$

D. Closed strings

The theory described in the preceding subsections can easily be extended to a system represented by a closed curve in space. Such a system is of interest in connection with the Shapiro–Virasoro model. We shall assume that the parameter σ in the case of a closed string varies within the range $[-\sigma_0, \sigma_0]$, so that $x_\mu(-\sigma_0) = x_\mu(\sigma_0)$. The equations of motion (3.11) remain valid. We must take $\delta x_\mu(\sigma_0) = \delta x_\mu(-\sigma_0)$ in varying the action; the extremal condition then leads to the condition $\pi_\mu(\sigma_0) = \pi_\mu(-\sigma_0)$ instead of (3.12). Thus, in the orthogonal coordinate system (3.4) we have the equation (3.17a) with the boundary conditions

$$x_\mu(\tau, \sigma_0) = x_\mu(\tau, -\sigma_0), \quad x'_\mu(\tau, \sigma_0) = x'_\mu(\tau, -\sigma_0). \tag{3.51}$$

Instead of (3.19), the general solution takes the form

$$x_\mu(\tau, \sigma) = r_\mu \tau + f_\mu^{(+)}(\tau+\sigma) + f_\mu^{(-)}(\tau-\sigma), \tag{3.52}$$

where

$$f_\mu^{(\pm)}(u) = f_\mu^{(\pm)}(u + 2\sigma_0), \tag{3.53a}$$

$$\left(r_\mu + 2\frac{df_\mu^{(\pm)}}{du}\right)^2 = 0. \tag{3.53b}$$

The functions $f_\mu^{(\pm)}(u)$ on the interval $[-\sigma_0, \sigma_0]$ and the vector r_μ are determined by the initial conditions:

$$\begin{aligned} r_\mu &= (2\sigma_0)^{-1} \int_{-\sigma_0}^{\sigma_0} \dot{x}_\mu(0, \sigma)\, d\sigma, \\ f_\mu^{(\pm)}(\pm\sigma) &= \frac{1}{2}\left\{x_\mu(0, \sigma) \pm \int_0^\sigma [\dot{x}_\mu(0, \sigma') - r_\mu]\, d\sigma'\right\}. \end{aligned} \tag{3.54}$$

In analogy with (3.35), we can introduce new variables in phase space:

$$y_\mu^{(\pm)}(\sigma) = \frac{1}{2}[p_\mu(\sigma) \pm Ax'(\sigma)], \quad -\sigma_0 \leqslant \sigma \leqslant \sigma_0. \tag{3.55}$$

The general solution (3.52) can then be written in the form

$$\begin{aligned} y_\mu^{(\pm)}(\tau, \sigma) &= (2\sigma_0)^{-1}\left[P_\mu^{(\pm)} + (\pi A)^{1/2} \sum_{m\neq 0} a_\mu^{(\pm)m}(\tau)\, e^{\pm i\omega_m \sigma}\right], \\ a_\mu^{(\pm)m}(\tau) &= [a_\mu^{(\pm)-m}(\tau)]^* = \alpha_\mu^{(\pm)m} e^{i\omega_m \tau}. \end{aligned} \tag{3.56}$$

The general momentum is $P_\mu = P_\mu^{(+)} + P_\mu^{(-)}$. Thus we have two independent sets of normal coordinates $a^{(+)}$ and $a^{(-)}$, and accordingly two systems of constraints of the form (3.46), $L_m^{(+)}$ and $L_m^{(-)}$. The total Hamiltonian has the form $H = -\pi(L_0^{(+)} + L_0^{(-)})/2\sigma_0$.

A closed string possesses all the solutions which exist for an open string. These solutions are obtained by "folding" the string and requiring that the bends have the velocity of light (the tension at the bends is equal to zero). This corresponds to initial conditions for which $f_\mu^{(+)}(u) \equiv f_\mu^{(-)}(u)$. In particular, we obtain the "linear trajectories" described by Eq. (3.27), although we now have $N = 2$ for the leading trajectory (Fig. 12). In effect, this leads to half the slope, in complete agreement with

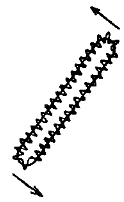

FIG. 12. The simplest non-trivial motion of a closed string.

the ideas discussed in Sec. 2C (see Eq. (2.8)).

There are also solutions which are specific to a closed string (details have been given by Barbashov and Chernikov[79]). For example, the string can have the form of a circle at any instant of time. In this case,

$$\mathbf{x}(\tau, \sigma) = R \sin\frac{\pi\tau}{\sigma_0}\cdot \mathbf{n}(\sigma), \quad \mathbf{v}(\tau, \sigma) = \cos\frac{\pi\tau}{\sigma_0}\cdot \mathbf{n}(\sigma), \tag{3.57}$$

where $\mathbf{n}(\sigma)$ is a unit vector with components $(\cos\pi\sigma/\sigma_0, \sin\pi\sigma/\sigma_0, 0)$. The maximum velocity of points of the string is equal to the velocity of light c, the maximum radius of the circle is $R = \sigma_0/\pi$, and the mass is $M = 2\sigma_0 A$. The string pulsates with a frequency $\nu = 1/2\sigma_0 = (2\pi\alpha' M)^{-1}$ between zero radius and the maximum radius R. It is easy to find even more complicated planar motions. Suppose that the string has the shape of a certain planar curve at some instant of time. Then the velocity, which lies in the same plane, is uniquely determined in direction and magnitude by the conditions (3.7). Having found the velocity, it is easy to determine the solution at an arbitrary instant of time.

4. RELATIVISTIC STRINGS. THE QUANTUM THEORY

A. Canonical quantization

The transition to the quantum theory is effected by means of the usual prescription[104] of replacing the Poisson brackets for the canonical variables by the commutators: $-i\hbar\{\cdots\}_{\mathrm{P.B}} \to [\cdots]$. A complete set of variables for the string consists of the vectors X_μ, P_μ, and a_μ^m (see (3.48)), so that the covariant quantization conditions have the form

$$[\hat{P}_\mu, \hat{X}_\nu] = i\hbar g_{\mu\nu}, \tag{4.1a}$$

$$[a_\mu^m, a_\nu^n] = \hbar m g_{\mu\nu}\delta(m+n). \tag{4.1b}$$

Thus $\hat{c}_k = a_k^m/\sqrt{m}$ ($k = 1, 2, 3$ and $m > 0$) is the creation operator and $\hat{c}_k^+ = a_k^{-m}/\sqrt{m}$ is the annihilation operator for the m-th oscillator. The time components have the opposite sign on the right-hand side, leading to states with a negative norm ("ghosts"). This is a consequence of the vector character of the dynamical variable, and a similar situation is well known in the theory of the quantization of the electromagnetic field. In quantum electrodynamics, the difficulty is eliminated by gauge invariance; the physically admissible states are restricted by a condition which corresponds to the choice of the Lorentz gauge in the classical theory (see, e.g., the book of Akhiezer and Berestetskiĭ[85]):

$$\frac{\partial A_\mu^-}{\partial x_\mu}|\Phi\rangle = 0; \tag{4.2}$$

here A_μ^- is the part of the electromagnetic field operator $A_\mu(x)$ which contains the annihilation operators of the photon, and $|\Phi\rangle$ is a physical state vector. Similarly, owing to the presence of the group of conformal transformations (3.5), "ghosts" are eliminated when the string is quantized.

Let us begin by writing down the generators of the symmetry in normal operator form:

$$L_m=\frac{\sigma_0}{\pi A}\int_{-\sigma_0}^{\sigma_0}:y^2(\theta):e^{-i\omega_m\theta}\,d\theta, \tag{4.3}$$

$$L_0=(2\pi A)^{-1}P^2+\sum_{m>0}(a^m a^{-m}), \tag{4.4a}$$

$$L_m=(L_{-m})^+=(\pi A)^{-1/2}(Pa^m)+\sum_{n>0}(a^{m+n}a^{-n})+\frac{1}{2}\sum_{n=1}^{m-1}a^{m-n}a^n,\quad m>0. \tag{4.4b}$$

The operators in the sum that determines L_m commute, and we have merely regrouped the terms. As to L_0, the transition to the quantum expression is not unique: the operator L_0 is determined only to within a c-number term, a fact which must be taken into account in what follows. Using (4.1), we obtain[12)]

$$[L_m,\ L_n]=(m-n)\,L_{m+n}+\frac{1}{12}dm\,(m^2-1)\,\delta(m+n), \tag{4.5}$$

where d is the dimensionality of space-time (we are considering the d-dimensional case because the theory was inconsistent for $d=4$). Quantization leads to an extra term in this commutator in comparison with (3.47b). This effect is characteristic of systems with an infinite number of degrees of freedom (see the discussion in[86]). Such "extra" terms in the quantum theory are known as Schwinger terms (see, e.g., the book[87]). It is easy to derive Eq. (4.5) by calculating the vacuum average

$$\langle 0|L_mL_{-m}|0\rangle=\left\langle 0\left|\frac{1}{4}\sum_{n=1}^{m-1}(a^{m-n}a^n)\sum_{l=1}^{m-1}(a^{-l}a^{l-m})\right|0\right\rangle =\frac{1}{2}g_{\mu\nu}g^{\mu\nu}\sum_{n=1}^{m-1}n\,(m-n). \tag{4.6}$$

Thus the operators L_m no longer form a closed Lie algebra; this is the formal reason for the inconsistency of the theory.

As usual, quantization transforms the constraints into conditions on the admissible vectors in Hilbert space describing the physical states:

$$L_0\,|\Phi\rangle=-\alpha_0\,|\Phi\rangle, \tag{4.7a}$$

$$L_{-m}\,|\Phi\rangle=0,\qquad m>0; \tag{4.7b}$$

here α_0, a new constant of the theory, is a number connected with the ambiguity in the choice of the operator L_0. As in electrodynamics, the physical states are annihilated only by the "annihilation" operators L_{-m}, and the condition (4.7b) for all m would contradict the commutation relations. However, since $L_m=(L_{-m})^+$, the matrix elements involving physical states vanish for any operator L_m. We have a reasonable theory if the subspace of physical states is closed and contains no states with a negative norm.

A complete basis in Hilbert space can be constructed in terms of the creation operators

$$|\Psi_N(P)\rangle=\prod_{n=1}^{N}a_{\mu n}^{m_n}|0,\,P\rangle,\qquad m_n>0, \tag{4.8}$$

where $|0,P\rangle$ describes a non-excited state of the string with momentum P_μ. These vectors include, in particular, vectors with a negative norm, due to the action of the operators a_0^m. The physical states $|\Phi\rangle$ are certain linear combinations of the vectors (4.8) satisfying the conditions (4.7). It can be proved that the vectors $|\Phi\rangle$ include no states with a negative norm if $d\leq 26$ and that the subspace is closed if $d=26$ and $\alpha_0=1$. This proof is very complicated and will not be reproduced here (see, e.g., the review of Scherk[10]). We consider now a simpler approach developed by Goddard *et al.*,[24] which makes it possible to see that all the physical states have a positive norm.

B. Non-covariant quantization and "transverse" physical states

The construction of a Hamiltonian formalism and the quantization of systems with constraints in phase space usually pose a dilemma: one can either postulate the Poisson brackets and commutators in the usual form to eliminate some of the variables, thus "distorting" phase space and replacing the Poisson brackets by Dirac brackets. (The quantization of systems with constraints was first considered by Dirac, who gave a review of this work in his book.[81] Dirac's ideas were further developed by Faddeev[88] and by Hanson, Regge, and Teitelboim.[89]) The simplest example is the case of a relativistic point particle. The first approach is to postulate the commutator (4.1a) for the coordinate and momentum, imposing the condition $(P^2-M^2)|\Phi\rangle=0$ on the states (this leads to the invariant Klein–Gordan equation). The second approach is to postulate the commutators for the space components of $\mathbf{P}$ and $\mathbf{X}$ and to express P_0 in terms of $\mathbf{P}$ in the form $P_0=(\mathbf{P}^2+M^2)^{1/2}$. In this case, the states are arbitrary functions of $\mathbf{P}$, and we have a commutator $[P_0,\mathbf{X}]\neq 0$. The invariant quantization of a string was discussed in the preceding subsection; we found that it leads to difficulties in constructing the physical states. Let us consider the second approach.

We take the x_1-axis along the vector $\mathbf{P}$. We shall make use of the gauge invariance (3.5) and identify the parameter τ with the coordinate x_+:

$$x_+(\tau,\sigma)\equiv\frac{x_0+x_1}{\sqrt{2}}=\tau. \tag{4.9}$$

This parametrization is analogous to the "laboratory" parametrization (3.7a), but it makes use of a variable on the light cone instead of the usual time variable. The role of the energy is now played by the variable $P_-=(P_0-P_1)/\sqrt{2}$, which must be expressed in terms of the remaining momentum components[13)]:

[12)]We employ a system of units in which $\hbar=1$ in what follows.

[13)]We employ the following notation: for any pair of vectors b_μ and c_μ, the scalar product is given by

$$(bc)=b_+c_-+b_-c_+-(b_\perp c_\perp),\ \text{where}\ (b_\perp c_\perp)=\sum_{j=2}^{d-1}b_jc_j.$$

$$P_-=\frac{P_\perp^2+M^2}{P_+}=\frac{M^2}{P_+}. \tag{4.10}$$

The conditions (3.46) and (4.9) enable us to explicitly eliminate the longitudinal components of the vectors a_μ^m (this is the advantage of the choice (4.9)):

$$a_+^m=0, \tag{4.11a}$$

$$a_-^m=\sqrt{\pi A}\frac{1}{P_+}\Big[L_m+\frac{1}{2}\sum_{n\neq 0,\, m}(a_\perp^n a_\perp^{n-m})\Big] \tag{4.11b}$$

for all $m\neq 0$. The transverse components of the vectors $\mathbf{a}^m$ are independent, and it is only these components which must be used to construct the physical states:

$$|\Phi_N(P)\rangle=\prod_{n=1}^{N} a_{j_n}^{m_n}|0, P_+\rangle, \tag{4.12}$$

where $j_n=2,\ldots,d-1$. Thus we have eliminated the "time" oscillators with negative norm, together with one of the space dimensions.

To calculate the mass spectrum, it is sufficient to make use of the condition (4.7a):

$$M^2|\Phi_N\rangle=2\pi A\,(-\alpha_0+\sum_{m>0} mN_m)\,|\Phi_N\rangle, \tag{4.13}$$

where N_m are the occupation numbers, which are the eigenvalues of the operator $(a_\perp^m a_\perp^{-m})/m$, satisfying $\sum N_m = N$. Thus we have obtained the equally spaced and highly degenerate spectrum described in Sec. 2. The state (4.12) is described by a tensor of rank N in the "transverse" space. By decomposing this tensor into irreducible components, we obtain the various spins from 0 to N. If only the first oscillator is excited, i.e., if $N_1 = N$ and $N_m=0$ for $m>1$, then the classical motion is as shown in Fig. 4, where the x_1-axis is perpendicular to the plane of rotation. Calculating the spin of this state by means of Eq. (3.49), we obtain an equation for the leading trajectory in the form

$$J = N_1 = \alpha_0 + \alpha' M^2, \tag{4.14}$$

where $\alpha'=(2\pi A)^{-1}$, in accordance with the classical argument given in Sec. 3B.

Since the construction was not relativistically invariant, we must verify that the result is covariant. In other words, a Lorentz transformation must not take a vector out of the space of physical states; this condition requires that the operators $_{\mu\nu}$ constructed from Eq. (3.49) satisfy the ordinary commutation relations. It was easy to verify that this condition was satisfied for the Poisson brackets in the classical case. In the quantum case, there appear Schwinger terms, which spoil the algebra. To see this, we must find the commutator $[\ _{i-},\ _{j-}]$, which should be equal to zero. However, it can be shown that

$$[\mathscr{Y}_{i-}, \mathscr{Y}_{j-}] = 2(\pi A P_+^2)^{-1}\sum_{m>0}\Big[m\Big(1-\frac{d-2}{24}\Big)+m^{-1}\Big(\frac{d-2}{24}-\alpha_0\Big)\Big](a_j^m a_i^{-m}-a_i^m a_i^{-m}). \tag{4.15}$$

This calculation makes use of Eq. (4.11b), and a nonzero result is obtained for the same reason as in the case of the second term in the commutator (4.5). Thus the theory is consistent only if

$$\alpha_0 = 1, \quad d = 26. \tag{4.16}$$

The requirement $\alpha_0 = 1$ has a simple interpretation. A vector particle corresponding to $N=N_1=1$ has an invariant description in terms of a transverse wave function only if its mass is equal to zero. The condition $d=26$ does not have such a simple interpretation. It can be formulated as the requirement that the actual number of degrees of freedom of the system is $4! = 24$. Dirac pointed out in his book[81] that there may exist systems with constraints for which it is not possible to construct a quantum theory. A string is an example of such a system.

Actually, if we are considering an individual string, we can forego the 26-dimensional space, provided that we are not concerned with the excitations of the oscillators associated with "extra" 22 transverse coordinates. From the formal point of view, the result (4.16) is unsatisfactory because it is not always possible to transform to the "conical" time (4.9) (this observation is due to Patrascioiu[90]). To make the change of variables, we require nondegeneracy of the Jacobian (3.6), which in this case takes the form

$$r_+ + 2f_+'(u)\neq 0 \tag{4.17}$$

for all u; here r_μ and $f_\mu(u)$ determine the solution according to Eq. (3.19). On the other hand, by virtue of the condition (3.20b), the inequality (4.17) can hold only if the velocity of the extremity of the string is never in the "forward" direction, i.e., parallel to $\mathbf{P}$. The class of such states is not defined in an invariant manner, and it is not surprising that the generators of the Lorentz group do not form a closed algebra. Several authors[90–93] have attempted to find a quantization procedure which does not involve variables on the cone. However, the conclusion that the conditions (4.16) are necessary appears to remain valid.[94] A simple field-theoretic interpretation of the critical dimensionality of space-time was proposed by Brink and Nielson.[95] These authors calculated the energy of the zero-point quantum fluctuations and showed that renormalization leads to a reasonable result only if the conditions (4.16) hold.

Nevertheless, even if it were possible to quantize a free string, we would still require a self-consistent description of the interaction. It was shown by Mandelstam[32] that this again leads to the requirement that $d=26$.

5. STRINGS WITH SPIN

Since it has not been possible to construct a consistent quantum theory of a relativistic string, the question arises as to whether it is possible to modify the theory in order to make it self-consistent and more realistic. The most interesting possibility would be to introduce spin variables in addition to the coordinates. This is also motivated by the fact that an ordinary string cannot serve as a model for a particle with half-integral spin.

The introduction of spin is a particularly obvious step from the point of view of the parton picture: it is natural to assume that the fundamental particles have spin $\frac{1}{2}$ (this idea is due to Aharonov *et al.* [30]). Dual models with spin variables have been constructed, [27, 28] and Gervais and Sakita[96] showed that these models are related to the introduction of fermion fields in two-dimensional space-time. This work, together with other independent approaches, [97, 98] served as a basis for a new theoretical trend—the study of symmetry groups with anticommuting parameters, or "supersymmetries" (see the review by Ogievetskiĭ and Mezincescu, [99] which contains an extensive bibliography).

A quantum theory with anticommutators can be constructed by quantizing generalized classical mechanics in the case of a Grassman algebra with anticommuting generators. Such a generalization of mechanics is described in[100]. A quantum action principle for the anticommuting canonical variables was first considered by Schwinger. [101] The idea of constructing a Grassman analog of mechanics was also clearly formulated by Martin. [102] In the next subsection, we give an account of the classical theory of a relativistic particle with spin, as considered in detail in[100]. After doing this, it is perfectly natural to introduce strings with spin.

A. Classical dynamics of a particle with spin

In the non-relativistic theory, we shall describe the spin by a three-dimensional vector ξ with anticommuting components:

$$\xi_k\xi_l + \xi_l\xi_k = 0; \quad k, l = 1, 2, 3 \tag{5.1}$$

(in particular, $\xi_k^2 = 0$). The phase space of a particle with spin is constructed by supplementing the ordinary six-dimensional space $(\mathbf{q}, \mathbf{p})$ with a three-dimensional space (ξ). In other words, the dynamical variables, i.e., functions in phase space, are elements of a Grassman algebra G_3 with the three generators ξ_k. Although the elements of the Grassman algebra are not functions in the ordinary sense, it is possible to construct for them analogs of the concepts of ordinary analysis, such as differentiation and integration [14]. Suppose that the total action is an even real element of the algebra G_3. Then it can be written in the form

$$\mathscr{S} = \int_{t_i}^{t_f} \left(\mathbf{p}\dot{\mathbf{q}} + \frac{i}{2}\xi\dot{\xi} - \mathscr{H}(\mathbf{p}, \mathbf{q}, \xi)\right) dt, \tag{5.2}$$

where $\mathscr{H}$ is the Hamiltonian function. Any element of G_3 can be written as a polynomial in ξ of degree no greater than 3. Since the Hamiltonian H is also an even element of G_3 (it commutes with any ξ_k), it depends linearly on the spin angular-momentum vector

$$S_k = -\frac{i}{2} e_{klm}\xi_l\xi_m. \tag{5.3}$$

[14] Precise definitions and many results can be found in Berezin's book. [103]

The most general local Hamiltonian has the form

$$\mathscr{H}(\mathbf{p}, \mathbf{q}, \xi) = \frac{p^2}{2m} + V_0(\mathbf{q}) + \mathbf{LS}V_1(\mathbf{q}) + \mathbf{SB}(\mathbf{q}), \tag{5.4}$$

where $L = \mathbf{q}\times\mathbf{p}$ is the orbital angular momentum, $V_0(\mathbf{q})$ and $V_1(\mathbf{q})$ are potential functions, and $\mathbf{B}(\mathbf{q})$ is a vector field. The third term in the Hamiltonian describes the spin-orbit interaction, while the last term describes the interaction with the external magnetic field.

The action (5.2) leads to the following expression for the Poisson brackets:

$$\{f, g\}_{\text{P.B}} = \frac{\partial f}{\partial \mathbf{p}}\frac{\partial g}{\partial \mathbf{q}} - \frac{\partial f}{\partial \mathbf{q}}\frac{\partial g}{\partial \mathbf{p}} + i\frac{f\overleftarrow{\partial}}{\partial\xi}\frac{\overrightarrow{\partial}g}{\partial\xi}, \tag{5.5}$$

where $\overleftarrow{\partial}$ and $\overrightarrow{\partial}$ denote the right and left derivatives, respectively (see the book[103]). In particular,

$$\{\xi_k, \xi_l\}_{\text{P.B}} = i\delta_{kl}. \tag{5.6}$$

The dynamics is determined by the "Heisenberg" equations

$$\dot{f} = \{\mathscr{H}, f\}_{\text{P.B}} \tag{5.7}$$

for any function $f(\mathbf{p}, \mathbf{q}, \xi)$. In particular, the equations of motion which follow from the variational principle for the action (5.2) have the form

$$\dot{\mathbf{p}} = -\frac{\partial\mathscr{H}}{\partial\mathbf{q}}, \quad \dot{\mathbf{q}} = \frac{\partial\mathscr{H}}{\partial\mathbf{p}}, \quad \dot{\xi} = i\frac{\mathscr{H}\overleftarrow{\partial}}{\partial\xi}. \tag{5.8}$$

The quantization procedure is to replace the fundamental bracket (5.6) by an anticommutator divided by $-i\hbar$:

$$[\hat{\xi}_k, \hat{\xi}_l]_+ = \hbar\delta_{kl}. \tag{5.9}$$

Introducing the notation $\hat{\xi}_k = \sqrt{\hbar/2}\hat{\sigma}_k$, we find that the operators $\hat{\sigma}_k$ satisfy the relations $[\hat{\sigma}_k, \hat{\sigma}_l] = 2\delta_{kl}$, which are realized for the Pauli matrices. The spin operator (5.3) then takes the usual form

$$\hat{S}_k = -\frac{i}{2} e_{klm}\hat{\xi}_l\hat{\xi}_m = \frac{1}{2}\hbar\hat{\sigma}_k. \tag{5.10}$$

Thus the quantization procedure reproduces the usual theory of a spin-$\frac{1}{2}$ particle.

The action for a free relativistic particle can be written in the form

$$\mathscr{S} = \int_{\tau_i}^{\tau_f} \left\{-mz + \frac{i}{2}[-(\xi\dot{\xi}) + (u\xi)(u\dot{\xi})]\right\} d\tau, \tag{5.11}$$

where $z = (\dot{x}^2)^{1/2}$, $u_\mu = \dot{x}_\mu/z$, and ξ_μ is a four-dimensional vector with anticommuting components. The action is constructed in such a way that in the rest system it is independent of ξ_0 and agrees with (5.2). The matrix of the bilinear form in ξ and $\dot{\xi}$ in (5.11) is degenerate, so that an equation of motion cannot be written for the longitudinal component of the vector ξ. To completely define the dynamics, we require a constraint, which can be written in an invariant form by introducing a fifth coordinate in spin space:

$$(u\xi) - \xi_5 = 0. \tag{5.12}$$

Following Dirac,[81] we incorporate the constraint in the action by means of an undetermined multiplier:

$$\mathcal{S} = \int_{\tau_i}^{\tau_f} \left\{ -mz + \frac{i}{2} [-(\xi\dot{\xi}) + \xi_5\dot{\xi}_5 + m((u\xi) - \xi_5)\lambda] \right\} d\tau, \tag{5.13}$$

where λ anticommutes with all the ξ. The canonical momentum

$$p^\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}_\mu} = -mu^\mu + \frac{i}{2} m (\xi^\mu - (u\xi) u^\mu) \lambda \tag{5.14}$$

satisfies the constraint equations

$$p^2 - m^2 = 0, \ (p\xi) - m\xi_5 = 0. \tag{5.15}$$

The total action for a free particle in Hamilton's form can be written

$$\mathcal{S} = \int_{\tau_i}^{\tau_f} \left\{ -(p\dot{x}) + (p^2 - m^2) v - \frac{i}{2} [(\xi\dot{\xi}) - \xi_5\dot{\xi}_5] + \frac{i}{2} [(p\xi) - m\xi_5] \lambda \right\} d\tau, \tag{5.16}$$

where v and λ are arbitrary functions.

The equations (5.15) are consequences of the symmetry of the Lagrangian with respect to two groups of transformations. The first symmetry, "gauge" symmetry, is due to the freedom in choosing the parameter τ (as in the case of a spinless particle). The infinitesimal transformations of the second kind have the form

$$\begin{gathered} \xi_\mu \to \bar{\xi}_\mu = \xi_\mu + u_\mu \eta, \quad \xi_5 \to \bar{\xi}_5 = \xi_5 + \eta, \\ x_\mu \to \tilde{x}_\mu = x_\mu + \frac{i\xi_\mu \eta}{m}, \end{gathered} \tag{5.17}$$

where $\eta(\tau)$ is an anticommuting "parameter," which has an arbitrary dependence on τ but satisfies $\eta(\tau_i) = \eta(\tau_f) = 0$. Following the terminology introduced in[96], we shall call the transformation (5.17) a "supergauge" transformation. Here λ can be chosen arbitrarily and fixes the dependence of ξ_5 on τ. Note that the momentum (5.14) is not proportional to the velocity $\dot{x}_\mu$, as in the ordinary case, but also contains spin terms. This phenomenon was considered by Schrödinger in the quantum theory (for discussions, see the books by Dirac[104] (Sec. 69) and Bethe[105]) and is known as the "Zitterbewegung" of the electron.

Quantization leads to the relations

$$[\hat{\xi}_\mu, \hat{\xi}_\nu]_+ = -\hbar g_{\mu\nu}, \quad [\hat{\xi}_5, \hat{\xi}_5]_+ = \hbar, \quad [\xi_\mu, \xi_5]_+ = 0, \tag{5.18}$$

which can be represented by the algebra of the Dirac matrices:

$$\hat{\xi}_\mu = \sqrt{\frac{\hbar}{2}} \gamma_5 \gamma_\mu, \quad \hat{\xi}_5 = \sqrt{\frac{\hbar}{2}} \gamma_5. \tag{5.19}$$

The constraints reduce to the following conditions on the state:

$$(p^2 - m^2)\psi = 0, \ [(p\gamma) - m]\psi = 0. \tag{5.20}$$

Thus, we have derived the Dirac equation from the classical action (5.11).

In the case of a massless particle ($m = 0$), it is natural to rewrite the action (5.16) in the form

$$\mathcal{S} = \int_{\tau_i}^{\tau_f} \left[-(p\dot{x}) - \frac{i}{2} (\xi\dot{\xi}) + \frac{i}{2} \xi_5\dot{\xi}_5 + vp^2 + \frac{i}{2} (p\xi) \lambda \right] d\tau. \tag{5.21}$$

The constraints then have the form $p^2 = 0$ and $(p\xi) = 0$, and $\xi_5 = 0$, i.e., the theory is "γ_5-invariant." We note that there is no Lagrangian action principle for a massless particle.

B. The classical theory of strings with spin

To describe a string with spin, we introduce not only the generalized momenta $y_\mu(\tau, \theta)$ of points of the string (see Sec. 3C), but also anticommuting spin variables $\xi_\mu(\tau, \theta)$. In other words, at every point of the string we specify a pair of vectors $\xi_\mu(\tau, \sigma)$ and $\xi_\mu(\tau, -\sigma)$, one of which describes the spin of a parton and the other the spin of an antiparton. We postulate canonical Poisson brackets analogous to (5.6),

$$\{\xi_\mu(\theta_1), \ \xi_\nu(\theta_2)\}_{\mathrm{P.B}} = -ig_{\mu\nu}\delta(\theta_1 - \theta_2). \tag{5.22}$$

The constraint equations which generalize (3.37) and (5.15) can be written in the form

$$(y\xi) = 0, \quad y^2 + \frac{i}{2} A\xi\xi' = 0. \tag{5.23}$$

The first of these equations enables us to eliminate the longitudinal component of the vector ξ; the second equation is determined by the closure condition of the Lie albegra for the constraint equation (see Dirac's book[81]). To derive this result, we use (3.36) and (5.22) to calculate the Poisson brackets for the functionals

$$\Phi(\varphi) = \int_{-\sigma_0}^{\sigma_0} (y\xi)\varphi(\theta)\, d\theta, \quad F(f) = \int_{-\sigma_0}^{\sigma_0} \left(y^2 + \frac{i}{2} A\xi\xi'\right) f(\theta)\, d\theta, \tag{5.24}$$

where $\varphi(\theta)$ and $f(\theta)$ are differentiable functions satisfying $\varphi(-\sigma_0) = \varphi(\sigma_0)$ and $f(-\sigma_0) = f(\sigma_0)$. We have

$$\begin{gathered} \{\Phi(\varphi_1), \Phi(\varphi_2)\}_{\mathrm{P.B}} = -iF(\varphi_1\varphi_2), \\ \{\Phi(\varphi), F(f)\}_{\mathrm{P.B}} = A\Phi\left(\frac{1}{2}\varphi f' - \varphi' f\right), \\ \{F(f_1), F(f_2)\}_{\mathrm{P.B}} = AF(f_1 f_2' - f_1' f_2). \end{gathered} \tag{5.25}$$

The constraint equations (5.23) are equivalent to the conditions $\Phi \equiv 0$ and $F \equiv 0$ for any φ and f. The presence of a closed system of constraints leads to a symmetry of the system with respect to a group of transformations. The infinitesimal transformations of this group can be determined by considering the functionals

$$B_\mu(b) = \int_{-\sigma_0}^{\sigma_0} y_\mu b(\theta)\, d\theta, \quad \Omega_\mu(\omega) = \int \xi_\mu \omega(\theta)\, d\theta, \tag{5.26}$$

$$\begin{gathered} \{F(f), B_\mu(b)\}_{\mathrm{P.B}} = AB_\mu(fb'), \\ \{F(f), \Omega_\mu(\omega)\}_{\mathrm{P.B}} = A\Omega_\mu\left(f\omega' + \frac{1}{2} f'\omega\right), \end{gathered} \tag{5.27}$$

$$\begin{gathered} \{\Phi(\varphi), B_\mu(\omega)\}_{\mathrm{P.B}} = -\frac{1}{2} A\Omega_\mu(\varphi b'), \\ \{\Phi(\varphi), \Omega_\mu(\omega)\}_{\mathrm{P.B}} = iB_\mu(\varphi\omega). \end{gathered} \tag{5.28}$$

It follows from (5.28) that $\Phi(\varphi)$ generates the supergauge transformations (cf. Eq. (5.17))

$$\delta\xi_\mu = y_\mu\eta, \quad \delta y_\mu = \frac{i}{2} A (\xi'_\mu\eta + \xi_\mu\eta'), \tag{5.29}$$

where $\eta = \varepsilon\varphi(\theta)$ is a function which anticommutes with ξ_μ, and ε is an infinitesimal parameter of the supergroup. It is also easy to see that the functional $F(f)$ generates a transformation of the form

$$y_\mu(\theta) \to \beta y_\mu(\tilde\theta), \quad \xi_\mu(\theta) \to \beta^{1/2}\xi_\mu(\tilde\theta), \tag{5.30}$$

where for small f we have $\tilde\theta = \theta - Af(\theta)$ and $\beta = 1 - Af'$. For finite f, the function $\tilde\theta = g(\theta)$ is determined by the equation $h(g) = h(\theta) + 1$, where $h' = [-Af(\theta)]^{-1}$; we then have $\beta = f(\tilde\theta)/f(\theta) = g'$ (see also Sec. C of the Appendix).

For a string with spin which is symmetric with respect to these transformations, the Hamiltonian action can be written in the form

$$\mathcal{F} = \int_{\tau_i}^{\tau_f} d\tau \int_{-\sigma_0}^{\sigma_0} d\theta \left(x\dot y + \frac{y^2}{A} + \frac{i}{2}\xi\xi' - \frac{i}{2}\xi\dot\xi \right). \tag{5.31}$$

This form corresponds to introducing the constraints (5.23) with the coefficients 0 and $-A^{-1}$, respectively (as in (3.38)). The variational principle leads to the equations of motion

$$\frac{\partial y_\mu}{\partial\tau} = \frac{\partial y_\mu}{\partial\theta}, \quad \frac{\partial\xi_\mu}{\partial\tau} = \frac{\partial\xi_\mu}{\partial\theta} \tag{5.32}$$

and the boundary conditions

$$y_\mu(\sigma_0) = y_\mu(-\sigma_0), \quad \xi\delta\xi\,|_{\sigma=\sigma_0} = \xi\delta\xi\,|_{\sigma=-\sigma_0}, \tag{5.33}$$

where $\delta\xi_\mu$ is the variation of ξ_μ. The solution for $y_\mu(\tau,\sigma)$ is again of the form (3.41), while for the spin variable we have[15)]

$$\xi_\mu(\tau, \theta) = \zeta_\mu(\tau + \theta), \quad \zeta_\mu(u) \equiv \xi_\mu(0, u),$$
$$\zeta_\mu(u + 2\sigma_0) = \varepsilon\zeta_\mu(u), \qquad \varepsilon = \pm 1. \tag{5.34}$$

The last result follows from the boundary condition, and $\varepsilon = +1$ describes a string which behaves as a whole like a fermion (the Ramond model), while $\varepsilon = -1$ describes a boson string (the Neveu–Schwarz model). These solutions are invariant with respect to the conformal transformations (3.5). The constraints (5.23) are also invariant if the pair of functions $y_\mu(\tau, \pm\sigma) = y^\pm_\mu$ transforms like a two-dimensional conformal vector, while the pair $\xi_\mu(\tau, \pm\sigma) = \xi^\pm_\mu$ transforms like a two-dimensional spinor on the surface (τ, σ) (see also[7]), in accordance with (5.30).

15)The form of the Hamiltonian in (5.31) shows that the solution of the equations of motion can be regarded as a result of the transformation (5.30) with $f = -\tau/A$. In this case, $\tilde\theta = \theta + \tau$ and $\beta \equiv 1$.

We can go over to the normal variables by taking the Fourier transform[16)]:

$$\xi_\mu(\tau, \theta) = (2\sigma_0)^{-1/2} \sum_r b^r_\mu(\tau)\, e^{i\omega_r\theta}, \quad (b^r_\mu)^* = b^{-r}_\mu, \tag{5.35}$$

$$\{b^r_\mu, b^s_\nu\}_{\text{P.B}} = -ig_{\mu\nu}\delta(r+s). \tag{5.36}$$

The solution of the equations of motion has the form

$$b^r_\mu(\tau) = \beta^r_\mu e^{i\omega_r\tau}. \tag{5.37}$$

Rewriting the constraints (5.23) in terms of the normal variables (3.42) and (5.35), we obtain

$$G_r = \sqrt{\frac{2\sigma_0}{\pi A}} \int_{-\sigma_0}^{\sigma_0} (y\xi) \exp(-i\omega_r\theta)\, d\theta$$
$$= (\pi A)^{-1/2}(Pb^r) + \sum_{n\neq 0} (a^n b^{r-n}) = 0, \tag{5.38}$$

$$L_m = \frac{\sigma_0}{\pi A} \int_{-\sigma_0}^{\sigma_0} \left(y^2 + \frac{i}{2} A\xi\xi' \right) \exp(-i\omega_m\theta)\, d\theta,$$
$$L_0 = (2\pi A)^{-1} P^2 + \sum_{n>0} (a^n a^{-n}) - i \sum_{r>0} r\, (b^r b^{-r}) = 0, \tag{5.39}$$
$$L_m = (\pi A)^{-1/2}(Pa^m) + \frac{1}{2} \sum_{n\neq 0,\, m} (a^n a^{m-n}) - \frac{i}{2} \sum_r \left(r - \frac{m}{2} \right) (b^r b^{m-r}) = 0,$$
$$m \neq 0.$$

Using (5.25), we can calculate the Poisson brackets:

$$\{G_r, G_s\}_{\text{P.B}} = -2iL_{r+s}, \quad \{G_r, L_m\}_{\text{P.B}}$$
$$= i\left(r - \frac{1}{2} m \right) G_{r+m}, \quad \{L_m, L_n\}_{\text{P.B}} = i(m-n) L_{m+n}. \tag{5.40}$$

The mass and total angular momentum are also readily expressed in terms of these variables:

$$M^2 = -2\pi A\, [\sum_{n>0} (a^n a^{-n}) - i \sum_{r>0} r\, (b^r b^{-r})], \tag{5.41}$$

$$J_{\mu\nu} = \int_{-\sigma_0}^{\sigma_0} (x_\mu y_\nu - x_\nu y_\mu - i\xi_\mu\xi_\nu)\, d\theta$$
$$= X_\mu P_\nu - X_\nu P_\mu + i \sum_{n>0} n^{-1} (a^n_\mu a^{-n}_\nu - a^n_\nu a^{-n}_\mu) - i \sum_{r\geq 0} b^r_\mu b^{-r}_\nu. \tag{5.42}$$

(Cf. Eqs. (3.49) and (3.50).)

C. Quantization

In quantizing the string with spin, we must supplement the commutators (4.1) with the Fermi commutation relations for the operators $\hat b^r_\mu$ (we put $\hbar = 1$):

$$[\hat b^r_\mu, \hat b^s_\nu]_+ = -g_{\mu\nu}\delta(r+s). \tag{5.43}$$

The operators $\hat a$ and $\hat b$ always commute. As in the case of a spinless string, we take the normal ordering of the factors in the sums in the operator $\hat L_0$. The algebra of the symmetry operators, Eq. (5.40), is modified in the relations containing $\hat L_0$ (cf. (4.5)):

$$[\hat G_r, \hat G_s]_+ = -2\hat L_{r+s} + \frac{1}{2} d \left(r^2 - \frac{1}{8}(1-\varepsilon) \right) \delta(r+s),$$
$$[\hat G_r, \hat L_m]_- = \left(r - \frac{1}{2} m \right) \hat G_{r+m}, \tag{5.44}$$
$$[\hat L_m, \hat L_n]_- = (m-n)\hat L_{m+n} + \frac{1}{8} dm \left(m^2 - \frac{1}{2}(1-\varepsilon) \right) \delta(m+n).$$

16)In this subsection, r, $s = 0, \pm 1, \pm 2, \ldots$ for $\varepsilon = 1$ (a fermion string) and r, $s = \pm 1/2, \pm 3/2, \ldots$ for $\varepsilon = -1$ (a boson string); m and n are integers and $\omega_k = k\pi/\sigma_0$.

The state vectors are distinguished by the occupation numbers for the boson and fermion oscillators. In the fermion case ($\varepsilon=+1$), there is also a vector operator $\hat{b}^0_\mu$, which is represented by the matrices $\gamma_5\gamma_\mu/\sqrt{2}$ and which has the interpretation of the average spin vector of the string. The lowest state in this case has spin $\frac{1}{2}$ and mass 0, and the constraint $G_0=0$ reduces to the Dirac equation $(P\gamma)\psi=0$. As in the spinless case, the leading trajectory has the form $J=\alpha_0+\alpha' M^2$. The constant α_0 is related to the constraint equation $L_0|\Phi\rangle = -(\alpha_0-\frac{1}{2})|\Phi\rangle$, in analogy with (4.7a). In the case of a fermion string, the theory is self-consistent only if $\alpha_0 = \frac{1}{2}$, so that $\mu=0$ (where μ is the mass of the lowest state on the leading trajectory). This can be seen simply from the fact that $\hat{L}_0=-\hat{G}_0^2$ and $G_0|\Phi\rangle=0$. In the meson case, we also have the condition $\alpha_0=\frac{1}{2}$, and the lowest state (the analog of the pion) is a tachyon with $\mu^2=-\frac{1}{2}\alpha'$. The next trajectory corresponds to an excitation of a spin oscillation with the minimum frequency, $r=\frac{1}{2}$ (a spin wave of length $4\sigma_0$). The equation of this trajectory has the form $J=\alpha_0+\frac{1}{2}+\alpha' M^2$. The lowest state on this trajectory (the ρ meson) is a massless vector particle. We note that the model predicts the correct relation $\alpha_0^{(\rho)} - \alpha_0^{(\pi)} = \frac{1}{2}$.

Since the Lagrangian is an even function of ξ, the theory of a string with spin possesses a symmetry with respect to the inversion $\xi\to-\xi$, which Neveu and Schwarz[27] expressed in terms of g-parity. This symmetry manifests itself in the fact that the parity of the number of spin excitations is conserved. In other words, we have a conservation law for the operator $\hat{C}=(-1)^{\hat{F}}$, where $\hat{F}=-\sum_{r>0}(\hat{b}^r\hat{b}^{-r})$. The transition $\rho\to 2\pi$ is literally forbidden in this model. The picture becomes more realistic if we introduce the isospin I; we then have a conservation law for the operator $\hat{G}=(-1)^{F+I}$, and the transition $\pi\to 2\rho$ instead of $\rho\to 2\pi$ is forbidden. A more detailed discussion can be found in Mandelstam's review.[9]

As in Sec. 4B, states satisfying the constraint equations can be constructed by means of variables on the light cone. The supergauge transformation can be used to impose the additional condition

$$b^r_+=0, \quad r\neq 0, \tag{5.45}$$

and b^r_- can be expressed in terms of the transverse components in Eq. (5.38). By performing calculations similar to those which led to (4.15), it can be seen that the theory is self-consistent only if

$$\alpha_0=\frac{1}{2}, \quad d=8+2=10. \tag{5.46}$$

The interpretation of this result is that for each orbital degree of freedom there are two spin degrees of freedom, and the required number of transverse oscillations is given by the rule $8+2\times 8=24$.

6. INTERACTIONS

We have so far been considering free strings. In going over to a theory describing the motion of a string under the action of forces, we must bear in mind that, if quantization is to be possible, it is essential to have a theory which is invariant with respect to changes of the variables (τ,σ) (Eq. (3.3) and (3.5)). This symmetry must not be violated when an interaction is introduced, otherwise we are again faced with the problem of "ghosts." A consequence of this restriction is that only the extreme points are "valent."

A. A string with an external field

The effect of an external electromagnetic field with intensity $F^{\mu\nu}(x)$ on a string can be described in a conformally invariant manner by adding the following term to the action (see the work of Ademollo *et al.* [29]):

$$\begin{aligned}\mathcal{S}_{\text{e.-m}} &= \int d\tau \int d\sigma \mathcal{L}_{\text{e.-m}},\\ \mathcal{L}_{\text{e.-m}} &= -eF^{\mu\nu}(x)\dot{x}_\mu x'_\nu = -e\,(\mathbf{E}\mathbf{w}+\mathbf{w}\,[\mathbf{v}\times\mathbf{H}])\,,\end{aligned} \tag{6.1}$$

where e is the charge, $\mathbf{v}$ is the velocity, and $\mathbf{w}$ is the tangent vector (3.7). Equation (6.1) describes the energy of an electric dipole in an external field $(\mathbf{E},\mathbf{H})$ and is consistent with the idea of a string as a parton-antiparton chain. In this picture, however, the material of the string is neutral on the average. In fact, by substituting $F^{\mu\nu}=\partial A^\nu/\partial x_\mu - \partial A^\mu/\partial x_\nu$ and integrating with respect to σ, we can transform $\mathcal{S}_{\text{e.-m}}$ to the form

$$\mathcal{S}_{\text{e.-m}} = e\int_{\tau_i}^{\tau_f} d\tau\,[(\dot{x}A)|_{\sigma=\sigma_0}-(\dot{x}A)|_{\sigma=0}]. \tag{6.2}$$

This action corresponds to a neutral string with charges $\pm e$ at the extremities. Additional arbitrary charges can also be placed at either extremity. We note that this picture predicts a gyromagnetic ratio equal to 1 for resonances on the leading trajectory. The magnetic moment due to a charge e_0 at the extremity of a rotating string (Fig. 4) is given by

$$\varkappa = \frac{1}{4}e_0L_0 = \frac{e_0}{M}J, \tag{6.3}$$

where L_0 is the proper length of the string.

This simple model is clearly in conflict with the known electromagnetic structure of hadrons. Owing to the need to preserve conformal symmetry, no reasonable procedure has yet been proposed for introducing the electromagnetic (or weak) current in "string matter"; this is one of the generally recognized defects of dual models. (In this connection, see the papers of Nambu[75] and Willemsen.[106])

B. The interaction of strings

As we pointed out in Sec. 3A, the physical picture of interactions between strings is very simple: strings can divide or recombine at their extremities. However, this intuitive idea leads to non-trivial mathematics. The complexity of the theoretical description of interactions is due to the fact that the free motion has a simple form in the normal coordinates, while the interaction is localized in x-space. Certain additional complications re-

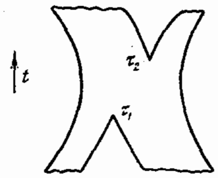

FIG. 13. The surface in Minkowski space representing the scattering of two strings.

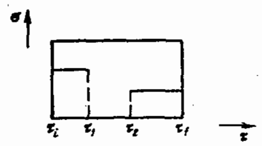

FIG. 15. Mapping of the surface of Fig. 13 into the complex plane.

sult from the presence of conditions due to conformal invariance.

The mathematical model for the rupture of a string is formulated as follows in the classical theory. Suppose that the function $f(\sigma)$ is specified on the interval $[A, C]$ by a Fourier series with coefficients a^m and on the interval $[C, B]$ by a Fourier series with coefficients b^m. Then the coefficients c^m of the Fourier series for the interval $[A, B]$ can be expressed linearly in terms of $\{a\}$ and $\{b\}$. To apply this model to the string (see, e.g., the paper of Rebbi[107]), we must match the coordinate systems (τ, σ) on three interacting strings. It is most convenient to use the "conical" gauge (4.9). The parameter σ_0 is then not proportional to the mass, as in the "laboratory" gauge (3.23), but is proportional to the "conical momentum" P_+, and the law of conservation of momentum leads to the conservation of lengths along the σ-axis: for the decay $c \to a+b$, we obtain $\sigma_0^{(c)} = \sigma_0^{(a)} + \sigma_0^{(b)}$. In addition, the constraints (4.11), which eliminate the longitudinal and "time" oscillations, are automatically taken into account. However, this approach again entails the problem of relativistic invariance. Mandelstam[108] showed that the model is self-consistent in the quantum case only under the condition (4.16).

The general principle for constructing interaction amplitudes can be formulated in a natural way in the language of Feynman's continuum integral. The S-matrix is represented as an integral of $\exp(i\mathscr{S}/\hbar)$ over all possible two-dimensional surfaces in Minkowski space, where $\mathscr{S}$ is the classical action (3.2). The topology of the surfaces determines the initial, intermediate, and final states. For example, for the pole amplitude for (inelastic) scattering of two resonances (see Fig. 3), the surfaces have the form shown in Fig. 13. The integration is carried out over the coordinates of all the points of the surface and over the instants of "time" at which the recombination (τ_1) and rupture (τ_2) occur. To illustrate this formalism, let us consider the usual theory of scalar fields $\varphi(x)$ and $\Phi(x)$ with masses m and M and an interaction Lagrangian $\lambda\varphi^2\Phi$. The simplest pole amplitude for the scattering of φ particles corresponds to classical trajectories of the form shown in Fig. 14. The integral contains only the action for free particles, and it is easy to evaluate the integrals with respect to $x_\mu(\tau)$. For $\tau_i \to -\infty$ and $\tau_f \to +\infty$, the result depends only on $\tau_2 - \tau_1$. The integration with respect to τ_1 and τ_2 leads to a propagator $(P^2 - M^2)^{-1}$ (for details on this approach, see[83]). From this point of view, the Feynman diagrams determine the topology of the world-lines in x-space which contribute to the continuum integral for the scattering amplitude. The action for a string, as in the case of a particle, has the form of a quadratic functional (see (3.16) or (3.32)), and this enables us to calculate the continuum integral. Mandelstam[32] carried out this calculation by transforming to the variable $\tilde{\tau} = i\tau$ (the "Wick rotation") and reducing the problem to an integration over functions of the complex variable $\rho = \tilde{\tau} + i\sigma$ (an analog of the conformal variable $\tau+\sigma$) in a strip with cuts (Fig. 15). Applying the Christoffel–Schwarz transformation, this region is mapped into the unit circle and the calculation of the continuum integral is reduced to a Neumann problem (see Sec. B of the Appendix). For the N-particle amplitude, the integration with respect to $\tau_1, \ldots, \tau_{N-2}$ (the instants of rupture and recombination) leads to the Koba–Nielson representation (see Sec. A of the Appendix). Owing to the presence of constraints, one must either apply Faddeev's method,[88] preserving the explicit invariance (as was done by Gervais and Sakita[35]), or integrate only with respect to the transverse coordinates. The second procedure is simpler and has been successfully carried through,[32] but the invariance of the result again requires the condition (4.16).

As in ordinary field theory, functional integration is not the only method of defining the S-matrix. The interaction can be described by considering the evolution of a quantum string under the action of the classical field produced by another string. This approach was developed by Ademollo *et al.*[29] (see also the review by Rebbi[8]). Its drawback is that it lacks explicit duality. Several authors[109–111] have constructed non-local field theories for quantized strings which also lead to dual amplitudes. Independently of the result, the procedure for constructing the amplitudes leads to simple rules analogous to the Feynman rules. Incidentally, it should be pointed out that, in contrast with the case of Schwinger–Feynman electrodynamics, these rules were already known before string theory was developed (see, e.g., the reviews[4,5]). Suppose that we are interested in the no-loop ("Born") N-particle amplitude. Let us represent this amplitude by a Harari–Rosner quark diagram (such as that shown in Fig. 2) and construct (in an arbitrary manner) a "tree" diagram, as in Fig. 16. The vertices at which particles are emitted and the internal

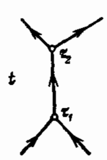

FIG. 14. World-lines of scattering point particles in the pole approximation.

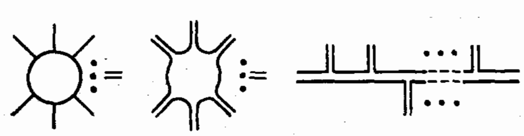

FIG. 16. Feynman rules for the dual amplitude.

resonance lines correspond to operators which are expressed in terms of the normal variables a_μ^m:

$$\hat{\Gamma}_{u,d}(p) = ig(\alpha')^{-1/2} : \exp[ip\hat{x}^{(u,d)}]:, \quad \hat{D} = -i\alpha'(\hat{L}_0 - \alpha_0)^{-1}, \quad (6.4)$$

where p_μ is the momentum of the emitted particle, $\alpha' = (2\pi A)^{-1}$, and g is the coupling constant. The index u is used for a particle which is emitted "upward" in the diagram of Fig. 16, the index d is used for a particle which is emitted "downward," $\hat{x}_\mu^{(u)} = \hat{x}_\mu|_{(\sigma=0)}$, $\hat{x}_\mu^{(d)} = x_\mu|_{(\sigma=\sigma_0)}$, the operator $\hat{x}_\mu$ has the form (3.48), and the operator $\hat{L}_0$ is defined in (4.4a). Clearly, $\hat{D}$ has poles at $P^2 = M_k^2$, where M_k are the resonance masses (2.5). The amplitude corresponding to Fig. 16 has the form

$$V_N(p_1, \ldots, p_N) = \langle 0, p_1 | \hat{\Gamma}_u(p_2)\hat{D}\hat{\Gamma}_u(p_3)\hat{D}\hat{\Gamma}_d(p_4) \ldots \hat{D}\hat{\Gamma}_u(p_{N-1}) | 0, p_N \rangle. \quad (6.5)$$

This matrix element can be calculated by the method of "coherent states." To construct more complicated diagrams, we also introduce an operator for resonance emission (the "triple-reggeon vertex") and a "twisting" operator $\hat{T}$ such that $\hat{T}\hat{\Gamma}_d\hat{T} = \hat{\Gamma}_d$ and $\hat{T}^2 = 1$. Each internal line can be "twisted," in which case it is represented by the operator $\hat{D}\hat{T}$.

Interactions of strings with spin have also been considered; this was done by Mandelstam[33] using the continuum integral, and by Kaku[112] using non-local quantum field theory. Mandelstam's method appears to be more suitable for the problem. In particular, it was Mandelstam who first calculated dual amplitudes for fermion–antifermion scattering in Ramond's model. This problem has not been fully solved using the operator approach (see[113, 114]).

When they interact, open strings can combine into closed strings, which are represented by tubular surfaces in Minkowski space (Fig. 17) and have the interpretation of resonances contained on the Pomeranchuk trajectory and its daughter trajectories ("pomerons"). The theory of resonance–pomeron interactions has been considered in[9, 29, 115]. The dual theory of interacting pomerons in lowest order (the no-loop approximation; Fig. 18) is the Shapiro–Virasoro model.

7. CONCLUSIONS

The theory of interacting strings which has been developed in recent years is an important stage in theoretical physics. This is the first model of extended relativistic objects which has a firm basis in the form of a complete classical theory. Intuitive ideas are very attractive, but the physical world is too complex; in attempting to describe it, we are forced to digress from the simple "mechanical" picture. The first step, which is not yet totally divorced from geometry, is the introduction of spin. In the same way, it is possible to describe an internal symmetry, for example by introducing an anticommuting field with the properties of an isovector on the string. To construct the most interesting example of a dual theory, the Bardakci–Halpern model,[66] we must introduce a Fermi spinor field with the properties of the quark representation $SU(N)$. Whether it is worth retaining the name "string" after introducing such complications is a question of terminology.

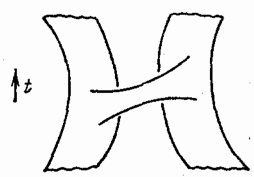

FIG. 17. The surface representing the scattering of two resonances with exchange of the Pomeranchuk pole.

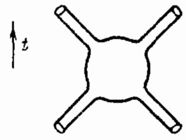

FIG. 18. Pomeron-pomeron scattering —the Virasoro amplitude.

New "string-like" objects can be devised by abandoning the space-time picture. In particular, a model which permits a consistent quantization in any number of dimensions has been found.[116] However, the value of this model is greatly lessened by an arbitrariness in the energy-momentum operator. But the main drawback of this theory is that, in the absence of a clear geometric picture, there is no means of constructing a casual interaction.

From a formal point of view, a string is an example of a system with an infinite number of internal degrees of freedom, for which the generators of the Poincaré group (energy, momentum, and angular momentum) are represented as integrals over the internal variables (Eqs. (3.14), (3.15), and (3.39)). In the classical theory, these observable quantities form a Lie algebra which is correct from a geometric point of view. Thus a string is a non-trivial example of a realization of the general program of Dirac[117] for constructing a relativistic system of dynamics. Nobody has yet succeeded in finding a satisfactory variant of the model. It will probaby be necessary to consider a more complicated geometry, different internal spaces, and new types of fields.

We conclude by recalling the necessary elements of particle physics which are absent in the contemporary theory of strings: 1) a realistic spectrum of states with no "tachyons" and with the correct multiplet structure; 2) a consistent quantization procedure in a four-dimensional space; 3) a convergent "perturbation theory" in the number of loops and renormalization procedure; 4) a model for the baryons as states constructed from triplets of quarks (and a set of pairs?); 5) electromagnetic and weak currents.

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APPENDICES

A. INTEGRAL REPRESENTATIONS OF DUAL AMPLITUDES

1. The Veneziano amplitude

The B-function in the expression (2.1) for the Veneziano amplitude can be represented in the form

$$V(s,t)=\int_0^1 x^{-\alpha(s)-1}(1-x)^{-\alpha(t)-1}\,dx. \tag{A.1}$$

This integral is defined for $\alpha(s)<0$ and $\alpha(t)<0$. It can be analytically continued into the physical region by rewriting it in the form of a contour integral in the complex plane (see, e.g., the book[118]). Expanding the second factor under the integral sign in a series in x, we obtain

$$V(s,t)=\sum_{n=0}^{\infty}\frac{G_n(1+\alpha(t))}{n!\,[n-\alpha(s)]}, \tag{A.2}$$

where $G_n(a)\equiv\Gamma(a+n)/\Gamma(a)$ is the Pochhammer polynomial.[116] The asymptotic form for $|\alpha(s)|\to\infty$ and $\arg s\neq 0$ can be found by making the substitution $x=e^{-v}$. Putting $\alpha(s)=\alpha_0+\alpha' s<0$ and $|s|\to\infty$, we obtain

$$V(s,t)=\int_0^{\infty} e^{v\alpha(s)}(1-e^{-v})^{-\alpha(t)-1}\,dv\approx\int_0^{\infty} e^{v\alpha(s)}v^{-\alpha(t)-1}\,dv$$
$$=\Gamma(-\alpha(t))\,(-\alpha' s)^{\alpha(t)}. \tag{A.3}$$

Thus $V(s,t)$ has only pole singularities (with polynomial residues) and a "Regge" asymptotic form, i.e., it possesses the property of "duality."

The model can be generalized by introducing a "weight" function $\rho(x)$ with no singularities for $0\leqslant x\leqslant 1$ under the integral sign. If this function is represented in the form of a series

$$\rho(x)=\sum_{m,n=0} P_{m,n}(s,t)\,x^m(1-x)^n, \tag{A.4}$$

where $P_{m,n}(s,t)$ is a polynomial in the variables s and t of degrees m and n, respectively, then the amplitude takes the form

$$\tilde V(s,t)=\sum_{m,n=0} P_{m,n}(s,t)\,B(m-\alpha(s),\,n-\alpha(t)). \tag{A.5}$$

This does not alter the character of the spectrum or the asymptotic behavior. This limit as $\alpha'\to 0$ (Eq. (2.7)) can be evaluated directly from (A.1) by dividing the range of integration into the two parts $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$ and making an expansion in the parameter α'.

2. The Koba-Nielsen representation[52]

The integrand in (A.1) has power singularities for $x\to 1$, ∞, and 0, which give rise to poles in the amplitude in the variables t, u, and s, respectively. Let us make the change of variable

$$x=\frac{(y-y_3)(y_2-y_1)}{(y-y_2)(y_3-y_1)}, \tag{A.6}$$

where $y_3<y_1<y_2$ are some fixed points on the y-axis. The variable x is equal to the cross-ratio of the four points (y_1,y_2,y_3,y) and is therefore invariant with respect to projective (bilinear) transformations[17)] of the y-axis. We shall show that the substitution (A.6) is convenient for the generalization to multi-particle amplitudes, particularly in establishing the property of crossing symmetry. Writing $y_4=y$ and $\beta=\alpha_0-2(\alpha_0+\alpha'\mu^2)$ and making use of the relations

$$\alpha(s)=2\alpha' p_1p_2-\beta,\quad \alpha(s)+\alpha(t)+\alpha(u)=\alpha_0-2\beta,$$
$$1-x=\frac{(y-y_1)(y_3-y_2)}{(y-y_2)(y_3-y_1)},\quad \frac{dx}{x(1-x)}=\frac{(y_3-y_1)\,dy}{(y-y_1)(y-y_3)},$$

we obtain

$$V=C\int d\mu_4(y)\prod_{n>m}(y_n-y_m)^{-2\alpha'(p_mp_n)}, \tag{A.7}$$

where the element of integration has the form

$$d\mu_4(y)=\prod_{n=1}^{4}(y_n-y_{n-1})^{\beta-1}\,[(y_3-y_1)(y_4-y_2)]^{\alpha_0-\beta}\,dy_4,$$
$$C=(y_2-y_1)(y_3-y_2)(y_3-y_1),\quad y_0\equiv y_4,$$

and the range of integration is $y_3<y_4<y_1$. The integral (A.7) is invariant with respect to cyclic permutation of the indices 1, 2, 3, 4. The integral representation (A.7) can be generalized in a natural way to an arbitrary number of particles N. The element $d\mu_N(y)$ simplifies considerably if we adopt a "self-consistency" condition: the external particles lie on the same trajectory $\alpha(p^2)$ as the resonances. In this case, $\alpha(\mu^2)=0$ and $\beta=\alpha_0$. An equivalent but even more symmetric representation is obtained by transforming the line $-\infty<y<\infty$ into the unit circle $z=e^{i\theta}$ with $-\pi<\theta\leqslant\pi$ and applying the complex bilinear transformation

$$y=-i\frac{z-1}{z+1}=\operatorname{tg}\frac{\theta}{2} \tag{A.8}$$

(Fig. 19). We note that if $\beta=\alpha_0$, the element of integration is invariant with respect to bilinear transformations. Thus the amplitude for the N-particle depicted in Fig. 16 has the representation

$$V_N(p_1,\ldots,p_N)=C\int\prod_{m\neq n}|z_n-z_m|^{-\alpha'(p_np_m)}\,d\mu_N(z),$$
$$d\mu_N(z)=\prod_{n=1}^{N}|z_n-z_{n-1}|^{\alpha_0-1}\prod_{n=4}^{N}d\theta_n, \tag{A.9}$$

where $C=|(z_1-z_2)(z_2-z_3)(z_3-z_1)|$ and the integration is over the region $\theta_2\leqslant\theta_3\leqslant\cdots\leqslant\theta_N\leqslant\theta_1$. The amplitude constructed in this way has the property of "planar duality," as suggested by the region of integration and the expression for the element of integration. We note that a bilinear transformation which leaves the unit circle invariant can be used to map the points (z_1,z_2,z_3) into any other triplet of points without changing the amplitude (it is important here that $\sum_{n=1}^{N}p_n=0$). This implies that

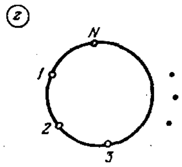

FIG. 19. The Koba–Nielsen variables.

[17)] The relation between the special functions and the group of projective transformations of the line is discussed in Chap. VII of Vilenkin's book.[119]

the function V_N is symmetric with respect to all cyclic permutations of its arguments.[18]

The integral (A. 9) becomes particularly symmetric when $\alpha_0 = 1$ (the Virasoro condition[56]). In this case, the element of integration is invariant with respect to *all* conformal transformations of the z-plane and has the representation

$$d\mu_N(z) = \text{const} \cdot \prod_{n=4}^{N} d\sigma_n, \tag{A. 10}$$

where $z_n = Z(\sigma_n)$ are points belonging to a closed curve Z in the z-plane which can be obtained by applying a certain conformal transformation to the unit circle, and σ is a variable which is equal to the arc length on the curve Z. In this case, the element of integration is also independent of the order of the indices, so that the total N-particle amplitude is given by the integral (A. 9) with $-\pi \leq \theta_n \leq \pi$ for any n.

Analyses of certain group properties of the function V_N can be found in[120–122].

3. The Shapiro–Virasoro amplitures

In the preceding subsection, we gave an integral representation of an amplitude which has the property of "planar" duality. Shapiro[26] found an analogous representation for a fully dual amplitude. In the case when $N = 4$ and $\alpha(s) + \alpha(t) + \alpha(u) = -2$, his result has the form

$$W(s, t, u) = c \int \prod_{n>m} |y_n - y_m|^{-2\alpha'(p_n p_m)} |(y_2 - y_1)(y_3 - y_2)(y_3 - y_1)|^2 d^2 y_4, \tag{A. 11}$$

where $c = g_w^2/\pi$, the points y_n are in the *complex plane*, and the integration is over all y_4. We shall evaluate this integral and show that it leads to Eq. (2. 10) when $\alpha_0 = 2$. If $\sum p_n = 0$, the integral (A. 11) is invariant with respect to bilinear transformations and is independent of the choice of the points y_1, y_2, and y_3. Let $y_1 = 1$, $y_2 = \infty$, and $y_3 = 0$; then

$$W(s, t, u) = c \int |y|^{-2\alpha'(p_1 p_2)} |1 - y|^{-2\alpha'(p_2 p_3)} d^2 y. \tag{A. 12}$$

The integral is symmetric with respect to the substitutions $y \to (1 - y)$ and $y \to y^{-1}$, so that $W(s, t, u)$ is a symmetric function of its arguments. To evaluate the integral, we make use of the identity

$$\Gamma(\nu)\, a^{-\nu} = \int_0^\infty t^{\nu-1} e^{-at}\, dt. \tag{A. 13}$$

Using this representation for the powers of the variables $a_0 = |y|^2$ and $a_1 = |1 - y|^2$, we can transform the integral (A. 12) to a Gaussian form in y. The remaining double integral with respect to (t_0, t_1) can be evaluated by making the substitutions $t_0 = tz^{-1}$ and $t_1 = t(1 - z)^{-1}$.

The generalization of the integral (A. 11) to the case of an arbitrary number of particles N is just as obvious as the Koba–Nielsen representation. By mapping the complex y-plane onto a sphere by means of a stereographic projection (analogous to (A. 8)), we can transform (A. 11) to an explicitly dual form.

By considering an integral over an n-dimensional sphere, Brower and Goddard[123] found dual amplitudes with $\alpha_0 = n$, of which (A. 9) and (A. 11) are particular cases with $n = 1$ and 2. By analytically continuing this integral in n, Kudryavtsev[124] found an integral representation for any α_0.

B. THE CONTINUUM INTEGRAL IN THE DUAL THEORY

The interaction amplitudes in dual models can be represented as continuum integrals. This representation has a universal character, since it provides a unified form which can be used to determine the contributions of diagrams with any number of loops in various models, including fermion models. This approach has been applied by Sakita *et al.* [7, 13b, 34, 35]

Let us first calculate the Gaussian continuum integral (Appendix C in Feynman's paper[125]):

$$I[g] = \frac{\int D[f] \exp\left[-\frac{1}{2}(f, Af) + i(g, f)\right]}{\int D[f] \exp\left[-\frac{1}{2}(f, Af)\right]}, \tag{B. 1}$$

where f and g are vectors in some Hilbert space $\mathcal{H}$, A is a linear Hermitian operator in $\mathcal{H}$, and $D[f]$ is the element of integration. In our case, $\mathcal{H}$ is the set of square-integrable functions and A is a differential operator. Evaluating $I[g]$ by taking the limit for multiple integrals, we obtain

$$I[g] = \exp\left[-\frac{1}{2}(g, Bg)\right], \quad B = A^{-1}. \tag{B. 2}$$

Let f be functions in a region D of the plane (ξ, η) bounded by a curve Γ, and let g be a function on Γ such that

$$(f, Af) = -\int_D d\xi\, d\eta \left[\left(\frac{\partial f}{\partial \xi}\right)^2 + \left(\frac{\partial f}{\partial \eta}\right)^2\right]$$
$$= \int_D f \Delta f\, d\xi\, d\eta - \int_\Gamma f \frac{\partial f}{\partial n} d\gamma, \quad (g, f) = \int_\Gamma g(\gamma) f(\gamma)\, d\gamma, \tag{B. 3}$$

where $\Delta = \partial^2/\partial\xi^2 + \partial^2/\partial\eta^2$, $\partial/\partial n$ is the normal derivative, and γ is the arc length on the curve Γ. The kernel of the operator which is inverse to A satisfies the equations ($\zeta_1 = \xi$, $\zeta_2 = \eta$)

$$\Delta_\zeta B(\zeta, \zeta') = \delta^{(2)}(\zeta, \zeta'), \quad \left.\frac{\partial B}{\partial n}\right|_{\zeta, \zeta' \to \Gamma} = -\delta(\gamma - \gamma'). \tag{B. 4}$$

Thus $B(\zeta, \zeta')$ is the Green's function of the Neumann problem for Laplace's equation in the region D (see Chaps. 6 and 7 of the book[126]).

If D is the unit circle, then

$$B(\zeta, \zeta') = -\frac{1}{2\pi} \ln\left(|z - z'| \left|z - \frac{1}{z'^*}\right|\right), \tag{B. 5}$$

where $z = \xi + i\eta$ is the complex coordinate in the plane. On the circle $|z| = 1$ we have $1/z = z^*$, so that

[18]When all the variables $(p_n p_m)$ are equal, the integral (A. 9) reduces to a product of Γ-functions (see[128,129]).

$$(g, Bg) = -\frac{1}{\pi}\int_{-\pi}^{\pi} g(\gamma_1) g(\gamma_2) \ln|e^{i\gamma_1} - e^{i\gamma_2}| d\gamma_1 d\gamma_2. \tag{B.6}$$

Taking the functions f and g to be four-dimensional vectors and putting

$$g = g^{\mu}(\gamma) = \sqrt{2\pi\alpha'} \sum_{n=1}^{N} p_n^{\mu} \delta(\gamma - \theta_n), \tag{B.7}$$

we obtain the representation (B.1) for the integrand in (A.9). (We must also eliminate self-energy terms $\sim p_n^2[\delta(\gamma-\theta_n)]^2$ here; see[7] for further details.)

The quantum theory of the interaction of strings leads naturally to a continuum integral (see Sec. 6) in which $f^{\mu} = x^{\mu}$ is the coordinate of a point of the string. The strip with cuts (see Fig. 15) is transformed into a half-plane by means of the substitution

$$\rho \equiv \tau + i\sigma = \sum_{n=1}^{N} \lambda_n \ln(y - y_n), \quad \sum \lambda_n = 0, \tag{B.8}$$

where $\lambda_n = P_+^{(n)}$ and the lengths of the strings are $\sim |\lambda_n|$. The half-plane $\mathrm{Re}\, y \geq 0$ is then transformed into the unit circle by means of the mapping (A.8). Multi-loop diagrams are described by the representation (B.1)–(B.3) with multiply connected regions D. To introduce fermion degrees of freedom, we must add a two-dimensional fermion field to the "Lagrangian" (B.3) (see, e.g., (C.9)). The integral (B.1) can also arise from the calculation of the "net" diagrams shown in Fig. 1 (from[13b, 19]). To obtain this integral, the propagators for $p^2 \ll m^2$ must be represented in the form $(p^2 - m^2)^{-1} \approx -m^2[1 + (p^2/m^2)] \approx -m^2 \exp(p^2/m^2)$. The limit of an infinite number of contours leads to the Gaussian continuum integral.

C. THE TWO-DIMENSIONAL CONFORMAL GROUP AND THE SUPERGROUP

The conformal group (CG) is the set of all continuous transformations of space for which the interval between two infinitesimally spaced points is multiplied by a positive function (for a discussion, see, e.g., Rosen's paper[127]). These transformations preserve the angles formed by pairs of lines having a common point. For a d-dimensional space, the elements of the CG depend on $(d+1)(d+2)/2$ parameters. The CG for $d=2$ is infinite-dimensional and is isomorphic to the direct product $\Gamma \times \Gamma$, where Γ is the group of all continuous transformations of a single variable.

Let us first consider the structure of the group Γ. Suppose that ζ is a coordinate on the line with $-\infty < \zeta < \infty$ and $\varphi(\zeta)$ is an infinitely differentiable function. The change of variable

$$\zeta \to \tilde{\zeta} = g(\zeta), \tag{C.1}$$

where $g(\zeta)$ is a smooth monotonic function, can be represented in the operator form

$$\varphi(\zeta) \to \tilde{\varphi}(\zeta) \equiv \varphi(\tilde{\zeta}) = \hat{T}\varphi(\zeta), \quad \hat{T} = \exp\left[f(\zeta)\frac{d}{d\zeta}\right], \tag{C.2}$$

where the functions $f(\zeta)$ and $g(\zeta)$ are related by the equation

$$\frac{f(g)}{f(\zeta)} = \frac{dg}{d\zeta}. \tag{C.3}$$

The Laurent expansion of the function $f(\zeta)$ gives the system of generators of the group Γ:

$$\begin{gathered} f(\zeta) = \sum_{k=-\infty}^{\infty} a_k \zeta^k, \quad \hat{T} = \exp\Big(-\sum_{m=-\infty}^{\infty} a_{m+1} \hat{L}_m\Big), \\ \hat{L}_m = -\zeta^{m+1} d/d\zeta, \quad [\hat{L}_m, \hat{L}_n]_- = (m-n) L_{m+n}. \end{gathered} \tag{C.4}$$

Thus the Lie algebra of the group Γ is isomorphic to the Virasoro algebra (cf. (3.47b)). The operators $\hat{L}_{+1}$, $\hat{L}_0$, and $\hat{L}_{-1}$ form a closed subalgebra which is isomorphic to the Lie algebra of the group $SU(1,1) \sim SL(2,R)$ (for the definitions, see Vilenkin's book[119]):

$$[L_{+1}, L_{-1}]_- = 2L_0, \quad [L_{\pm 1}, L_0]_- = \pm L_{\pm 1}. \tag{C.5}$$

This triplet of operators generates the group of projective (bilinear) transformations of the line. In this case, the function $f(\zeta)$ is a quadratic trinomial, and the substitution (C.1) can be represented in the form

$$\frac{\tilde{\zeta} - c_1}{\tilde{\zeta} - c_2} = \exp[\alpha(c_1 - c_2)] \frac{\zeta - c_1}{\zeta - c_2}, \tag{C.6}$$

if $f(\zeta) = \alpha(\zeta - c_1)(\zeta - c_2)$.

Consider now the CG for a pseudo-Euclidean plane with coordinates (τ, σ). Introducing the variables $\zeta_\pm = \tau \pm \sigma$, the element of length in the plane can be represented in the form $ds^2 \equiv d\tau^2 - d\sigma^2 = d\zeta_+ d\zeta_-$. Independent substitutions of the form (C.1) for the variables $\zeta_\pm$ lead to conformal transformations of the plane. For these transformations, the neighborhood of each point of the plane is stretched by a scale factor μ and undergoes a hyperbolic rotation through an angle ω:

$$\mu = \sqrt{g'_+ g'_-}, \quad \omega = \frac{1}{2} \ln \frac{g'_+}{g'_-}, \tag{C.7}$$

where $g'_\pm = d\tilde{\zeta}_\pm / d\zeta_\pm$. Following Gervais and Sakita,[7] we shall specify an irreducible conformal field by a pair of indices $\varkappa$ and j, and we write the transformation law in the form

$$\varphi_{\varkappa, j}(\zeta) \to \tilde{\varphi}_{\varkappa, j}(\zeta) = \mu^{\varkappa} e^{-j\omega} \varphi_{\varkappa, j}(\tilde{\zeta}) \equiv (g'_+)^{(\varkappa - j)/2} (g'_-)^{(\varkappa + j)/2} \varphi_{\varkappa, j}(\tilde{\zeta}). \tag{C.8}$$

A pair of fields with $\varkappa = -\frac{1}{2}$ and $j = \pm\frac{1}{2}$ is called a conformal spinor, and a pair of fields with $\varkappa = -1$ and $j = \pm 1$ is called a conformal vector. In particular, $\partial_\pm \varphi_{0,0}$ is a conformal vector (here $\partial_\pm \equiv \partial/\partial\zeta_\pm$).

The two-dimensional field theory with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\int d\tau\, d\sigma\, [\partial_+\varphi\, \partial_-\varphi + i\psi_+ \partial_-\psi_+ + i\psi_- \partial_+\psi_-], \tag{C.9}$$

where φ is a scalar field and $\psi_\pm$ is an anticommuting conformal spinor, is invariant with respect to the CG. This theory also possesses an additional symmetry. Let $\eta_\pm = \eta_\pm(\zeta_\pm)$ be an anticommuting field with $\varkappa = +\frac{1}{2}$ and $j = \mp\frac{1}{2}$ (the conjugate conformal spinor; $\psi_+\eta_+ + \psi_-\eta_-$ is an invariant of the CG). The Lagrangian (C. 9) remains unchanged under the transformations

$$\delta\psi_{\pm} = \partial_{\pm}\varphi\eta_{\pm},\ \ \delta\varphi = i\,(\psi_{+}\eta_{+} + \psi_{-}\eta_{-}). \tag{C.10}$$

This is the simplest example of a supersymmetry (for further details, see the review[99]). Conformally invariant two-dimensional theories were considered by Itabashi.[130]

The conformal transformations which leave a line in the plane fixed form a subgroup $\Gamma \subset \Gamma \times \Gamma$ (for example, for the line $\sigma = 0$ we have $g_{+}(\zeta) \equiv g_{-}(\zeta)$). The requirement that a pair of parallel lines is fixed leads to a contraction onto a subgroup for whose elements the function $g(\zeta)$ is periodic. A relativistic string with spin is an example of a system with the Lagrangian (C. 9) but somewhat complicated by the presence of an additional vector index, which is invariant with respect to transformations which leave the lines $\sigma = 0$ and $\sigma = \sigma_0$ fixed.

D. COMMENTS ON THE BIBLIOGRAPHY

Within the volume of the present paper, we have not been able to touch upon a number of new lines of development of the basic theme. Moreover, it is not clear at the present time which of the theoretical results that have been obtained will be important for an understanding of particle physics and which are merely formal exercises. In order to assist the reader who proposes to work actively in this field to find the papers on the subject in which he is interested, we have appended a supplementary list of references to this review. This list also contains some interesting papers on the basic theme of the review which were not cited in the main text. As a guide to both lists of references, we give here a subject index to the literature:

1) Reviews: [1–12, 42, 44, 47, 206, 239, 267, 276, 293, 298].

2) The classical and quantum theory of relativistic strings: [8–10, 17–24, 73–80, 89–95, 201, 208, 227, 228, 279, 280, 283, 286].

3) The closed string and the Virasoro–Shapiro model: [25, 26, 29, 31, 65, 68, 115, 249, 252, 253, 258, 271, 282, 295].

4) The string with spin and supersymmetry: [27, 28, 33, 68–71, 96–100, 112, 207, 221, 226, 253, 256, 258, 259, 273, 283, 295, 296, 298].

5) Internal symmetries: [6, 66, 87, 204, 205, 207, 218, 231, 256, 268, 273, 274].

6) Specific models of baryons and mesons: [17, 92, 212–214, 220, 222, 232, 235, 237, 244, 246–248, 251, 254, 264, 265, 279].

7) Interactions of strings. Strings in external fields: [9, 29, 32, 33, 35, 75, 94, 106–115, 201, 209, 213, 217, 243, 245, 249, 252, 258, 263, 265, 270, 275, 278, 284, 285, 295].

8) Partons, quarks, and Feynman "net" diagrams: [13–19, 30, 213, 250, 266, 268, 273, 297].

9) Functional integration in dual models: [7, 9, 32, 33–35, 240, 262, 263, 266, 285].

10) Operator algebra: [5, 6, 25, 27, 28, 54, 56, 69–71, 113, 114, 210, 214, 221, 223, 242, 257, 287].

11) Miscellaneous variants of dual models: [11, 47–53, 92, 123, 124, 203, 206, 210, 236, 238, 240, 257, 264, 270, 274, 275, 297].

12) Integral representations of dual amplitudes: [26, 51, 52, 62–65, 120–124, 128, 129, 236, 261, 262].

13) Higher orders ("loop diagrams"): [4, 5, 57–59, 204, 209, 269, 277].

14) Spontaneous symmetry breaking: [43–46, 202, 204, 250, 278].

15) Modifications of string theory: distributed mass, quarks at the extremities, etc.: [116, 208, 212, 215, 218, 219, 225, 229–232, 244, 246, 251, 255, 259, 265, 268, 280, 284, 288, 291, 292].

16) Field models of string production: gauge fields, vortices, monopoles, etc.: [37–42, 211, 216–219, 224, 233, 234, 239, 241, 248, 255, 260, 268, 281, 282, 286, 289–292, 294].

17) The limit $\alpha' \to 0$ and the relation to local theories: [59–61, 271, 272, 294].

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