Amplitude, phase, frequency—fundamental concepts of oscillation theory

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Usp. Fiz. Nauk **123**, 657–682 (December 1977)

This review is concerned with applying the analytic signal in oscillation theory, where the concept of the analytic signal was hardly ever applied until recently. We treat the mathematical properties of the Hilbert transform and of the analytic signal, which allow one to determine the amplitude, phase, and frequency of any oscillation at any instant of time. For narrow-band oscillations and for broad-band oscillations that arise under slow frequency modulation, this definition agrees with the intuitive meaning of amplitude, phase, and frequency and with the quasistationary approximation, while allowing one to estimate the limits of applicability of the latter. We show that a number of radiotechnical devices (mixers, frequency modulators, detectors, frequency discriminators, etc.) transform the parameters of an oscillation as defined by the analytic signal. We establish the relationship between the adiabatic invariant and the equation of the oscillations for the analytic signal. This relationship allows one to construct a complete theory of the triode oscillator having a cubic characteristic, in which the capacity of the circuit and the transconductance of the tube slowly fluctuate. Here we get a new result in the second approximation, namely: we calculate the influence of the flicker effect on the instantaneous frequency of the oscillator; the corresponding spectral line width is substantial in practice. In conclusion, we treat some paradoxes and supplementary examples that illustrate the technical and physical significance of the introduced concepts.

PACS numbers: 84.30.Ng, 84.30.Qi, 84.20.Es

CONTENTS

---'and I wish you wouldn't keep vanishing so suddenly: you make one quite giddy.'

'All right,' said the Cat; and this time it vanished quite slowly, beginning with the end of the tail, and ending with the grin, which remained some time after the rest of it had gone.

'Well! I've often seen a cat without a grin,' thought Alice; 'but a grin without a cat! It's the most curious thing I ever saw in all my life!'

(*Lewis Carroll, Alice in Wonderland*)

INTRODUCTION

We could also have called this review "The analytic signal in oscillation theory". The analytic signal (AS), which was introduced in 1946 by Gabor, [1,2] permits one to define unambiguously the amplitude, phase, and frequency (APF) of any real time function (random or deterministic), while in the theory of nonmonochromatic wave fields it allows one to define field intensity, coherence functions, etc. In the theory of noise and partly coherent fields, the concept of the analytic signal has acquired full citizen's rights; at least it has become evident that it is easier and simpler to introduce the AS than to do without it. [3-10]

At the same time, APF concepts are commonly used without exact definition in oscillation theory, in radiotechnology, and in other fields that deal with deterministic time processes that don't reduce to strictly harmonic processes. Thus, for example, the phrase "the frequency stability of the generator amounts to 10^{-10} over such and such a period" remains, strictly speaking, undeciphered.

Must one define the amplitude, phase, and frequency in a uniform manner for all cases, and moreover by using the Hilbert transform and the analytic signal, i.e., in a rather complicated way? For narrow-band oscillations, one can introduce APF purely intuitively with uncertainty that lessens as the effective frequency band becomes narrower; one can measure them with the same uncertainty with a detector, phase meter, frequency meter, etc. At the same time, the AS allows one to give an exact definition of the APF that agrees within reasonable limits with their intuitive meaning and practical application. In particular, the convenience of this definition is confirmed by the fact that it satisfies three important requirements.

A. Single-valuedness

Gorelik[11] wrote that the term "sinusoidal oscillation with slowly varying amplitude" is self-contradictory like a "slightly bent straight line," and he noted[12] that one cannot unambiguously discern the amplitude $a(t)$ and the phase $\varphi(t)$ from the observed process:

$$u(t) = a(t) \cos[\omega_0 t + \Phi(t)] = a(t) \cos \varphi(t) . \tag{1}$$

It is not evident how to separate the known function u into the factors a and $\cos\varphi$. This becomes quite evident if we go to the complex notation $w(t) = u(t) + iv(t)$ by supplementing the real oscillation $u(t)$ with an *arbitrary* imaginary part $v(t)$. Then we have

$$w(t) = a(t)\, e^{i\varphi(t)},$$

where the amplitude a, the phase φ, and the instantaneous frequency $\omega = d\varphi/dt = \dot{\varphi}$ are defined by the expressions

$$a(t) = \sqrt{u^2(t) + v^2(t)}, \quad \varphi(t) = \arccos\frac{u(t)}{a(t)} = \arcsin\frac{v(t)}{a(t)}, \tag{2}$$
$$\omega(t) = \frac{\dot{v}(t)\, u(t) - \dot{u}(t)\, v(t)}{a^2(t)}.$$

Hence in the oscillation of (1), the APF are just as arbitrary as the imaginary part is. If we want to give an exact definition of the APF, then we must specify the operator that matches each function $u(t)$ with a function $v(t)$.

Without an unambiguous definition of the APF, certain methods become fruitless that use complex notation and pretend at heightened accuracy. Thus, the response $\tilde{w}(t)$ of a linear circuit to a modulated oscillation $w(t)$ is often calculated by using the quasistationary approximation

$$\tilde{w}(t) = K(i\omega(t))\, w(t), \tag{3}$$

or when the modulation isn't slow enough, by using the correcting asymptotic series[13-15]

$$\tilde{w}(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n K(i\omega)}{d\omega^n} e^{i\omega t} \frac{d^n}{dt^n} [w(t)\, e^{-i\omega t}];$$

Here $K(i\omega)$ is the transfer coefficient of the circuit, while $\omega(t)$ is the instantaneous frequency of the input process (the derivative of the phase), which one must determine before calculating and correcting anything, lest the limits of accuracy be exceeded.

B. Slowness

In many cases (in particular, when one uses the quasistationary approximation, see above), one requires slowness, or smoothness of the functions $a(t)$, $\Phi(t)$, and $\omega(t) = \omega_0 + \dot{\Phi}(t)$. Such a smoothness is obtained in solving many problems, in particular, in nonlinear oscillation theory, by applying the averaging method or related methods. [16-20] However, the APF that one determines here make sense only within the framework of the averaging method, and they have no broader meaning.

Let us take up this point in greater detail. When using the representation of oscillations in the phase plane having the coordinates $(u, -\dot{u}\omega_0)$, one determines the APF from the position of the imaging point at each instant, i.e., from the system of equations

$$u(t) = a(t) \cos[\omega_0 t + \Phi(t)],$$
$$-\frac{\dot{u}(t)}{\omega_0} = a(t) \sin[\omega_0 t + \Phi(t)]. \tag{4}$$

This is equivalent to the common definition (2) for $v(t) = -\dot{u}(t)/\omega_0$. However, the definition (4) implies that the functions $a(t)$ and $\Phi(t)$ contain fast components of frequency $2\omega_0$ for *any* narrow-band oscillation. Hence they are not slow (as compared with $\cos\omega_0 t$ or $\sin\omega_0 t$).

We shall confine the treatment to a simple example. One can naturally set $a = 1 + m \cos\Omega t$ and $\varphi = \omega_0 t$ for an oscillation having slow amplitude modulation (AM):

$$u(t) = (1 + m\cos\Omega t)\cos\omega_0 t, \quad \frac{\Omega}{\omega_0} = \varepsilon \ll 1$$

Yet the definition (4) leads to other expressions: accurate to terms of the order of ε, we get

$$a(t) = 1 + m\cos\Omega t + \frac{\varepsilon}{2} m \sin\Omega t \sin 2\omega_0 t,$$
$$\Phi(t) = \frac{\varepsilon}{2} \frac{m\sin\Omega t}{1 + m\cos\Omega t}(1 + \cos 2\omega_0 t).$$

Thus the amplitude and phase contain fast "vibrational" terms that are not inherent in the original oscillation and which are compensated in the product $u = a\cos\varphi$. Of course, these vibrations drop out upon averaging, and one gets smooth functions of a and Φ. Yet one must account for the vibrational corrections even in the first approximation (with respect to ε), one must find them as an extra step, [21] and often even *redefine* the APF.

A faulty definition of the APF like (4) especially impedes the construction of higher approximations. It turns out that the AS leads to a slow APF that agrees under appropriate conditions with the intuitive treatment, and which allows one easily to solve oscillatory problems (Chaps. 4 and 5).

C. Integral and differential character

Although the APF take on certain values at each instant, they actually are integral in character and they characterize the process $u(t)$ in a certain time interval, more strictly, along the entire time axis for $-\infty<t<\infty$. This situation has been known for a long time: Mandel'shtam[24] pointed out that one can't understand oscillatory phenomena without knowing the entire form of oscillation, the entire time course of the process.

At first glance, we come here to a contradiction with the customary differential (local) representations that many devices and instruments are based on. All servo systems for automatic frequency or phase control, automatic gain control, ordinary amplitude detectors, discriminators, limiters, etc.—all these devices are based on the seemingly indisputable statement that one can measure and alter (tune or limit) the instantaneous frequency and amplitude at each instant without knowing the entire course of the process. On the other hand, attempts to define the APF as purely differential quantities in terms of the values of u and $\dot{u}$ at each instant, e.g., by using Eqs. (4), do not yield satisfactory results. Yet one sometimes gets an evident absurdity—the instantaneous frequency takes on imaginary values, or varies from 0 to ∞ in each period, etc. [25]

This contradiction is overcome by the analytic signal. The integral character of the APF is reflected in the Hilbert transform (Chap. 1), where the integration is performed over the entire time axis. Yet in many cases an *approximate* definition of the APF proves possible, in which the principal contribution to the integral comes from small (but not too small!) regions in the time or in the spectral domain (see Chap. 6). Then these concepts acquire a quasilocal character.

The situation is analogous to the transition from waves to rays in optics. Some radiotechnical devices—frequency converters, certain modulators—resemble an interferometer or spectroscope in the sense that they rely on integral (wave) concepts. Yet others—the cited servo instruments—are analogous to the telescope or microscope, which are based on local (ray) representations. Devices and instruments of the latter group can operate successfully only when the integral concepts become asymptotically degenerate (the waves are converted into rays). Analysis of the pertinent conditions explains certain paradoxrs of oscillation theory, as well as the characteristic features of a number of modern radio systems (see Chap. 8).

We note that one can define a and Φ by a "sliding" time average

$$a(t)\,e^{i\Phi(t)}=2\overline{u(t)\,e^{-i\omega_0 t}} \tag{5}$$

(see Chap. 8 for details). This definition and the AS give similar results for narrow-band oscillations, but the definition (5) becomes non-single-valued and unsuitable as the band broadens (e.g., in frequency modulation), whereas the definition using the AS retains its force and perspicuity (see Chap. 6).

Below we shall barely treat random processes (noise), where as we have noted the AS has long occupied its proper position. Yet in essence, there is no fundamental difference between a regular signal and a concrete realization of noise: a signal can be noise-like, and noise signal-like. In surfeit, this favors the application of the AS not only to noise but also to deterministic signals.

When this article was almost ready, a new edition of Rytov's book[20] was published in which the AS is widely used, though he applies with equal weight the definition (4), the complex notation of oscillations (see Chap. 2 below), and time-averaging (the AS renders it superfluous; see Chaps. 2, 5, and 8 below). Our position is more radical: we consider that the AS gives a universal definition of the APF, while the other definitions that are generally allowable are applicable only as long as they agree with the AS. There are a number of counterexamples (video pulses and oscillations without a clearly marked carrier), in which the notions of the APF themselves seem ill-useful. Yet even in these cases the introduction of the APF via the AS can make sense (Chap. 8).

1. DEFINITION OF THE AMPLITUDE, PHASE, AND FREQUENCY IN TERMS OF THE ANALYTIC SIGNAL

A strictly harmonic oscillation at the frequency ω is given for $-\infty<t<\infty$ by the expressions

$$u(t)=x\cos\omega t-y\sin\omega t=a\cos(\omega t+\Phi)=a\cos\varphi, \tag{1.1}$$

Here x, y, a, Φ, and ω are real constants (without loss of generality we assume that $a>0$ and $\omega>0$). They are interrelated by the equations

$$x=a\cos\Phi,\quad y=a\sin\Phi,\quad a=\sqrt{x^2+y^2},\quad \Phi=\arccos\frac{x}{a}=\arcsin\frac{y}{a}. \tag{1.2}$$

Customarily people call a the amplitude, φ the phase, and $\omega=\dot{\varphi}$ the frequency, and x and y the (quadrature) components of the amplitude. One obtains the complex notation of the oscillation of (1.1):

$$w(t)=ae^{i(\omega t+\Phi)}$$

by supplementing the oscillation $u(t)$ with the imaginary component

$$v(t)=a\sin(\omega t+\Phi),$$

which differs from $u(t)$ by a phase shift of $-\pi/2$.

The Hilbert transform generalizes this rule to arbitrary functions: if $u(t)$ is a superposition of harmonic oscillations, then the function $v(t)$ that is coupled with it by the Hilbert transform is the superposition of the same oscillations phase-shifted by $-\pi/2$. In particular, if

$u(t)$ can be represented by a Fourier integral, then

$$u(t)=\frac{1}{\pi}\int_0^\infty [U_c(\omega)\cos\omega t+U_s(\omega)\sin\omega t]\,d\omega, \tag{1.3}$$

$$v(t)=\frac{1}{\pi}\int_0^\infty [U_c(\omega)\sin\omega t-U_s(\omega)\cos\omega t]\,d\omega. \tag{1.4}$$

In the time domain, this transformation has the form[26]

$$v(t)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(s)}{t-s}\,ds. \tag{1.5}$$

Here the integral is taken in the sense of the principal Cauchy value. The expression (1.5) is applicable both for periodic functions (which are representable by Fourier series) and for random processes (which are representable by Fourier-Stieltjes integrals[20]). We can establish the connection between the formulas (1.4) and (1.5) most easily by using the properties a) and b) (see below).

One can write the relationship (1.5) in the symbolic form

$$v=\mathrm{H}[u],\quad u=-\mathrm{H}[v]. \tag{1.6}$$

Here the second relationship stems from the fact that a double Hilbert transformation shifts all the phases by $-\pi$.

As we see, one can carry out the Hilbert transformation by using an ideal $(-\pi/2)$-phase-shifter (one can do this in real time only approximately; though with arbitrarily high accuracy for a signal that is known over its entire course for $-\infty<t<\infty$).

The functions $u(t)$ and $v(t)$ allow one to create the *analytic signal*. In the case of the Fourier integral of (1.3), it has the form

$$w(t)=u(t)+iv(t)=\frac{1}{\pi}\int_0^\infty U(\omega)\,e^{i\omega t}\,d\omega,\quad U(\omega)=U_c(\omega)-iU_s(\omega). \tag{1.7}$$

We define the amplitude $a(t)$, phase $\varphi(t)$, and instantaneous frequency $\omega(t)$ of the oscillation $u(t)$ according to the formulas of (2).

Later we shall show the fruitfulness of these *definitions* from the mathematical, physical, and technical standpoints. Yet it is useful to bear the following in mind:

1) We do not assert that one must introduce the concepts of the APF for every function $u(t)$; their expediency is determined by the essence of the problem.

2) Upon defining for a given oscillation the amplitude (the envelope) according to (1.3)–(1.7), we treat it as an objective characteristic of the oscillation, which a detector of some given design will reproduce with a certain error. Of course, we could reject this approach and assume that the envelope arises only as a result of detection. Yet then the given signal will have as many envelopes as one pleases – each detector has its own. The abovesaid also holds for the phase and the frequency.

3) The term "analytic signal" is explained by the fact that one can treat the variable t in the integral of (1.7) as being complex, and represent the function $w(t)$ in the form

$$w(t)=\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{u(s)}{t-s}\,ds,\quad t=\xi+i\eta, \tag{1.8}$$

while passing below the point $s=t$. Then $w(t)$ is an analytic function in the upper half-plane $\eta>0$ that declines like $1/t$ or faster as $|t|\to\infty$. Its real and imaginary parts (u and v) are conjugated harmonic functions that are related by the Cauchy-Riemann conditions. The function $w^*(t)=u(t)-iv(t)$ has the same properties in the lower half-plane. All of this shows that the function v, which we match with the function u via the Hilbert transform, is in a certain sense "best-fitted" to u; other fitting methods are cruder (see Chap. 2).

Let us list the fundamental properties of the Hilbert transform (1.5) and of the AS.

A. The superposition principle

The operator H is linear, i.e.,

$$\mathrm{H}\left[\sum_n c_n u_n\right]=\sum_n c_n \mathrm{H}[u_n].$$

Here the c_n are arbitrary numbers and the u_n are arbitrary functions for which $\mathrm{H}[u_n]$ is defined.

B. The principle of harmonic correspondence

If $u(t)$ is the harmonic oscillation of (1.1), then

$$v(t)=x\sin\omega t+y\cos\omega t=a\sin(\omega t+\Phi),$$
$$w(t)=(x+iy)\,e^{i\omega t}=ae^{i(\omega t+\Phi)}.$$

This again implies the relationships (1.2). These relationships stem from the identities

$$\int_{-\infty}^{\infty}\frac{\cos\omega s}{s}\,ds=0,\quad \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin\omega s}{s}\,ds=\operatorname{sgn}\omega=\begin{cases}1 & \omega>0,\\ 0 & \omega=0,\\ -1 & \omega<0.\end{cases}$$

C. Uniformity in time (stationariness)

If we replace $u(t)$ by $u(t-t_0)$ (t_0 = const is an arbitrary lag), then $v(t)$ is replaced by $v(t-t_0)$ and $w(t)$ by $w(t-t_0)$. This means that the Hilbert transform commutes with any transformation that is uniform in time, e.g., differentiation with respect to t:

$$\mathrm{H}\left[\frac{du}{dt}\right]=\frac{d}{dt}\mathrm{H}[u],\quad \mathrm{H}\left[\frac{d^2u}{dt^2}\right]=\frac{d^2}{dt^2}\mathrm{H}[u]\ \text{etc.}$$

D. Uniformity in phase

If we replace all the phases ωt in the spectral expansion of $u(t)$ by $\omega t+\theta$ (θ = const), then we get $u(t,\ \theta)$ in place of $u(t)$ and $w(t,\ \theta)$ in place of $w(t)$, with

$$w(t,\theta)=w(t)\,e^{i\theta},$$

That is, the envelope and the frequency remain invariant while the instantaneous phase receives a (constant) shift of θ.

E. Uniformity in frequency

If we replace all the phases ωt in the spectral expansion by $(\omega+p)t$ (p=const>0), then we get the AS $w(t, p)$, with

$$w(t, p) = w(t)e^{ipt},$$

That is, the envelope is not altered, while the instantaneous frequency is shifted by p. A transformation of this type is carried out by an ideal mixer (Chap. 3).

F. Coincidence of energy spectra

As the expressions (1.3)–(1.4) imply, the functions u and v have the same energy spectrum

$$|U(\omega)|^2 = U_c^2(\omega) + U_s^2(\omega). \qquad \text{(P)}$$

According to Eq. (1.7), the AS is represented by a Fourier integral that extends only to positive frequencies. This is noteworthy since only positive frequencies have physical meaning.

2. THE ANALYTIC SIGNAL AND THE COMPLEX FORM OF OSCILLATIONS

People often represent real oscillations of the type of (1) by complex functions by formally replacing $\cos(\omega_0 t+\Phi)$ by $e^{i}(\omega_0 t+\Phi)$. For example, they match the Gaussian and rectangular radio pulses

$$u(t) = \frac{1}{\sqrt{2\pi}\,T} e^{-t^2/2T^2}\cos\omega_0 t, \quad u(t) = \frac{1}{2T}\Pi(t)\cos\omega_0 t, \qquad (2.1)$$

where $\Pi(t)=1$ when $|t|<T$, and $\Pi(t)=0$ when $|t|>T$, with the complex functions

$$f(t) = \frac{1}{\sqrt{2\pi}\,T} e^{-(t^2/2T^2)+i\omega_0 t}, \quad f(t) = \frac{1}{2T}\Pi(t)e^{i\omega_0 t}. \qquad (2.2)$$

The latter differ from the corresponding analytic signals. Here one has implicitly asserted that the amplitude of the oscillation (2.1) is $|f(t)|$. At first glance this seems more natural than the definition by using (1.7). The "complex signal" $f(t)$ constructed in this way competes with the analytic signal.

The function $f(t)$ is represented by the Fourier integral

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\,d\omega, \qquad (2.3)$$

Here the pulses of (2.1) yield:

$$F(\omega) = e^{-1/2(\omega-\omega_0)^2T^2}, \quad F(\omega) = \frac{\sin(\omega-\omega_0)T}{(\omega-\omega_0)T}, \qquad (2.4)$$

Here the spectrum $U(\omega)$ that corresponds to the AS of (1.7) is related to $F(\omega)$ by the relationship

$$U(\omega) = \frac{1}{2}[F(\omega)+F^*(-\omega)]. \qquad (2.5)$$

The complex notation that employs $f(t)$ has the following defects:

1) We can define $f(t)$ and $F(\omega)$ only under the condition that the original function $u(t)$ is given by the formula $a(t)\cos\varphi(t)$. That is, a separation into amplitude and oscillatory factors has been performed (as we know, this is equivocal). Yet if $u(t)$ is given by a graph, oscillogram, or table, one cannot construct $f(t)$ without additional assumptions.

2) The energy spectrum of the function $\operatorname{Im} f(t)$ with which we supplement $u(t)$ according to Eqs. (2.2)–(2.4) does not agree with the energy spectrum of $u(t)$, since

$$|F(\omega)+F^*(-\omega)|^2 \neq |F(\omega)-F^*(-\omega)|^2.$$

Thus the function $v=\mathrm{H}[u]$ supplements the function u more naturally and generally than the function $\operatorname{Im} f(t)$.

In Chap. 1 we have formulated the properties a)–f) of the analytic signal; now we shall add to them the following:

g) *The distinction of the AS from the function $f(t)$* is given by the inequality

$$|f(t)-w(t)| = \frac{1}{2\pi}\left|\int_{-\infty}^{0}[F(\omega)e^{i\omega t}-F^*(\omega)e^{-i\omega t}]\,d\omega\right| \leqslant \frac{1}{\pi}\int_{-\infty}^{0}|F(\omega)|\,d\omega. \qquad (2.6)$$

If, as in the examples of (2.1), the spectrum $U(\omega)$ is concentrated near the frequency ω_0, then this distinction diminishes with increasing ω_0 or with decreasing band width $\delta\omega \sim 1/T$. However, $f(t)$ and $w(t)$ exactly coincide only when $F(\omega)=0$ when $\omega<0$.

Let us formulate this condition in another way. Owing to the properties a)–b), when $0<\Omega<\omega_0$, we have

$$\mathrm{H}[\cos\Omega t\cos\omega_0 t] = \frac{1}{2}\mathrm{H}[\cos(\omega_0+\Omega)t+\cos(\omega_0-\Omega)t]$$
$$= \frac{1}{2}[\sin(\omega_0+\Omega)t+\sin(\omega_0-\Omega)t] = \cos\Omega t\sin\omega_0 t = \cos\Omega t\,\mathrm{H}[\cos\omega_0 t],$$

That is, we can remove the slowly-varying function $\cos\Omega t$ outside the Hilbert-transform symbol. An obvious generalization of this gives the following property:

h) *The quadrature components $x(t)$ and $y(t)$ can be removed outside the Hilbert-transform symbol* if their spectrum contains no frequencies higher than ω_0, i.e.,

$$u(t) = x(t)\cos\omega_0 t - y(t)\sin\omega_0 t,$$
$$v(t) = x(t)\sin\omega_0 t + y(t)\cos\omega_0 t, \qquad (2.7)$$

if u and v are connected by the relationships (1.5)–(1.6).

We have derived an important generalization of the formulas (1.2): in forming the APF, the slow functions $x(t)$ and $y(t)$ (slow in the sense that their spectra do not overlap ω_0) play the same role as constants; they define

$$a(t) = \sqrt{x^2(t)+y^2(t)}, \quad \Phi(t) = \arccos\frac{x(t)}{a(t)} = \arcsin\frac{y(t)}{a(t)}. \qquad (2.8)$$

Finally, we can easily derive the following property[27] for the product $u(t)=x(t)z(t)$ by generalizing (2.7):

i) *The slow (low-frequency) factor $x(t)$ can be removed outside the Hilbert-transform symbol* if its spectrum does not overlap the spectrum of the high-frequency (fast) factor $z(t)$, i.e.,

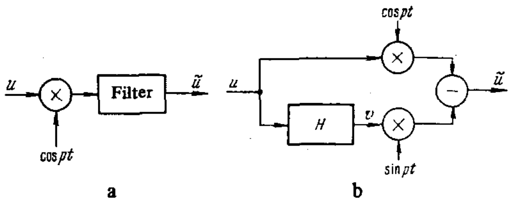

FIG. 1. Two frequency-converter circuits.

$$\mathrm{H}\,[xz] = x\,\mathrm{H}\,[\dot{z}]. \qquad (2.9)$$

This property is analogous to averaging (where also the slow factors are removed). Yet by replacing averaging by Hilbert transformation, we have lent precision to the concept of a slow function and substantially expanded it. For example, one can treat a modulating oscillation of 999 Hz frequency as being "slow" with a 1-kHz carrier. This is useful in solving concrete problems (see Chaps. 4 and 5).

In practice the spectra of the functions $x(t)$ and $y(t)$ can slightly "overflow" the frequency ω_0, and the spectra of the functions $x(t)$ and $z(t)$ in Eq. (2.9) can slightly overlap. In this case we have the inequality

$$\begin{aligned} &|v-\mathrm{H}\,[u]\,|\leqslant\frac{1}{\pi}\int\limits_{\omega_0}^{\infty}[\,|X(\omega)|+|Y(\omega)|\,]\,d\omega,\\ &|x\mathrm{H}\,[z]-\mathrm{H}\,[xz]\,|\leqslant\frac{1}{\pi^2}\int\limits_{0}^{\infty}|X(\omega)|\,d\omega\int\limits_{0}^{\omega}|Z(\nu)|\,d\nu. \end{aligned} \qquad (2.10)$$

This is analogous to the inequality (2.6); here the function v is defined by the second formula of (2.7), while $X(\omega)$, $Y(\omega)$, and $Z(\omega)$ are the spectra of the functions $x(t)$, $y(t)$, and $z(t)$, respectively; see Eq. (2.3).

Certain properties of the AS that seem paradoxical at first glance will be treated below in Chap. 8; one can find calculations of the AS in[28].

3. FREQUENCY CONVERSION AND DETECTION

It turns out that the function of many radiotechnical instruments relies on the AS, although they have been proposed without any connection with the AS and even long before its introduction.

Let us start with frequency conversion.[29] A frequency converter (or mixer) is a linear instrument that consists of a multiplier and a filter (Fig. 1a). In the generally adopted idealization, it shifts the spectrum by the frequency p of the reference oscillation. If we represent the input oscillation by Eq. (1.3), we can write the converted oscillation $\tilde{u}(t)$ by replacing (in upconversion of the frequency) ω by $\omega+p$. Elementary calculations give

$$\tilde{u}\,(t) = u\,(t)\cos pt - v\,(t)\sin pt. \qquad (3.1)$$

Here the Hilbert-conjugated function $v(t)$ that corresponds to Eq. (1.4) arises from the mixer circuit. Equation (3.1) shows that a frequency converter can be made in another way: according to the scheme in Fig. 1b, which contains a Hilbert operator (phase shifter); this variant is used in practice in balanced mixers and in single-band modulators.[5]

In applying a mixer in a superheterodyne receiver or for comparing a frequency to be measured against a standard, we always *assume* that the amplitude of the signal (the envelope) and the variation of the instantaneous frequency are conserved:

$$\text{if } u\,(t) = a\,(t)\cos\varphi\,(t),\text{ then } \tilde{u}\,(t) = a\,(t)\cos[\varphi\,(t)+pt]. \qquad (3.2)$$

Yet if this is true over some range of variation of p, then this implies that the $a\cos\varphi$ and $a\sin\varphi$ that figure here coincide respectively with the u and v in Eq. (3.1). That is, the mixer conserves the properties of the signal that we need only if they are defined in terms of the AS.

The abovesaid implies that if we have applied a mixer in a frequency meter for controlling the stability of a quartz oscillator, we *must* further measure the frequency as measured by the analytic signal, lest further distortions become inevitable. Moreover, the superheterodyne receiver (invented in 1918) satisfactorily reproduces messages because these messages at the transmitting end modulate an amplitude or frequency as defined via the AS, at least to the needed accuracy.

This is confirmed in examining the operation of modulators. In particular, a detailed analysis of the operation of the Armstrong frequency modulator leads to the conclusion[29] that even the first system of frequency telephony (1936) achieved frequency modulation as defined via the AS (see also Chap. 5 below).

One can weaken the preceding conditions by requiring the invariance of the amplitude alone or of the changes in frequency alone, and also by treating the variations of the initial phase[30] of the reference oscillation, rather than of its frequency: in all cases we unavoidably come to the same concepts of the APF.

Let us discuss now the operation of a quadratic detector. The oscillation

$$u\,(t)=\frac{1}{2}\,w\,(t)+\frac{1}{2}\,w^{*}\,(t) \qquad (3.3)$$

after squaring gives

$$u^2\,(t)=\frac{1}{2}\,a^2\,(t)+\frac{1}{4}\,w^2\,(t)+\frac{1}{4}\,w^{*2}\,(t) \qquad (a^2=ww^{*}). \qquad (3.4)$$

Here only the first term passes the filter, i.e., the square of the envelope. People usually think that detection can be carried out only under the condition that the spectrum of $u(t)$ is concentrated near the carrier frequency ω_0 in a band that is narrow in comparison with ω_0 (dotted lines in Fig. 2). Actually the spectrum of $u(t)$ merely must not extend outside the limits $\omega_0/2 < \omega < 3\omega_0/2$. Then the low- and high-frequency parts of the spectrum of $u^2(t)$ will not overlap (Fig. 2, solid lines) and *in principle* one can separate them: separation of the low-frequency part amounts to detection, and separation of the high-frequency part to frequency multiplication.

If the spectra overlap, the detector introduces distor-

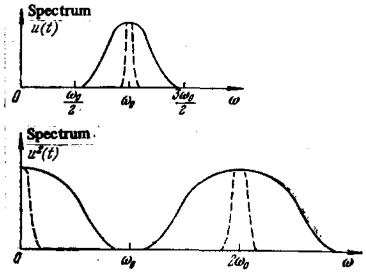

FIG. 2. Illustrating the theory of the quadratic detector.

tions. Hence, upon accounting for property h) of the AS, we have concluded that: a correctly functioning quadratic detector isolates the intensity $I(t)=(1/2)a^2(t)$ as defined in terms of the AS (the same is true of the thermistor and the photomultiplier). In particular, this is the basis of the principle of action of the Brown-Twiss interferometer.[20]

4. THE ADIABATIC INVARIANT AND THE QUASISTATIONARY APPROXIMATION

Let us study an oscillation that obeys the equation

$$\ddot{u}(t)+\Omega^2(t)\,u(t)=0, \tag{4.1}$$

where the function $\Omega^2(t)$ varies slowly (in comparison with $u(t)$; the estimates are given below). For example, this equation describes the motion of a pendulum of variable length, for which, as we know, the amplitude and frequency of the oscillations are related under a slow perturbing action by the condition of adiabatic invariance[18,31]

$$J=a^2(t)\,\omega(t)=\text{const}. \tag{4.2}$$

However, this condition is satisfied if a and ω are defined in terms of the AS, and it is not satisfied under the traditional definition (4). This is a cogent argument in favor of the analytic signal.

In order to convert to the AS, let us apply the Hilbert transform to both parts of (4.1). Upon applying the properties c) and i), we get the same equation for $v=\mathrm{H}[u]$. Therefore the AS obeys the equation

$$\ddot{w}(t)+\Omega^2(t)\,w(t)=0, \tag{4.3}$$

which satisfies the condition (4.2) *exactly*. In order to convince ourselves of this, we need only multiply (4.3) by w^* and take the imaginary part.

We can treat Eq. (4.3) as the equation of motion of a point in the uv plane under the action of a central force (depending on t). Then the condition (4.2) is Kepler's law of areas. Upon seeking a solution of Eq. (4.3) in the form

$$w(t)=\sqrt{\frac{J}{\omega(t)}}\,e^{i\varphi(t)},\quad \varphi(t)=\int\limits_0^t \omega(s)\,ds+\varphi_0, \tag{4.4}$$

we get the following equation for $\omega(t)$:

$$\omega^2+\frac{\ddot{\omega}}{2\omega}-\frac{3\dot{\omega}^2}{4\omega^2}=\Omega^2. \tag{4.5}$$

If we assume that $\omega=\Omega$ (when $\Omega>0$, we do not need the value $\omega<-\Omega$, since the AS corresponds to positive frequencies; see Chap. 1), we get the WKB approximation.[32]

We have made two errors. First, the property i) is satisfied exactly when the spectra of $u(t)$ and $\Omega^2(t)$ do not overlap; this is not true in the general case, but when $\Omega(t)$ is slow the spectra overlap slightly "in the tails," and we can easily estimate this error by using the inequalities (2.10). If we understand slowness in the sense that the function depends on t in terms of the "slow time" $t'=\varepsilon t$ with $\varepsilon\ll 1$ and we assume that the slow functions $\Omega(t')$, $a(t')$, and $\Phi(t')$ possess derivatives up to the mth, inclusive, then the error of the *equation* (4.3) will be of the order of ε^{m+1}. The second error, which involves the replacement of ω by Ω in the *solution* (4.4), is of the order of ε^{m+2} if $\Omega^2=1+\varepsilon^m g(\varepsilon t)$, i.e., if Ω^2 varies slowly and is small (for $m>0$).

We must note that, if we use the definition (4), we get an amplitude and phase for the oscillation of (4.1) that do not agree with the condition (4.2). Only after we have introduced the corrections and redefined the APF (which in essence imply employing the AS) do we arrive at the adiabatic invariant.

We have shown that the AS allows one to go from the real equation (4.1) to the complex equation (4.3). One can show the converse: if we replace Eq. (4.1) by its complex analog (4.3) and obtain the solution $w=u+iv$ with a *slow* $a(t)$ and $\Phi(t)$, then w is the AS. The proof rests on the properties a) and c) (see below), and also on the principle of harmonic correspondence (property b)). When the latter breaks down for the harmonic oscillation u, one gets a rapidly oscillating amplitude and phase. Carson[33] used this approach as applied to frequency modulation as early as 1922. A recent book on nonlinear oscillations[22] also uses the AS implicitly.

Now let us examine the real linear equation

$$L\left(\frac{d}{dt}\right)\tilde{u}=u=a(t)\cos[\omega_0 t+\Phi(t)] \tag{4.6}$$

and its complex analog

$$L\left(\frac{d}{dt}\right)\tilde{w}=w=a(t)\,e^{i[\omega_0 t+\Phi(t)]}, \tag{4.7}$$

which has the solution (3) in the quasistationary approximation, where $K(i\omega)=1/L(i\omega)$. When $L(i\omega)=-\omega^2+i\alpha\omega+\omega_r^2$, this equation describes, in particular, the passage of a FM oscillation through the circuit of a frequency discriminator. In the case $K(i\omega)=1-ke^{-i\omega\tau}$, $0<k<1$, it describes its passage through a Bernstein interferometer;[19,20] when $\Phi(t)$ is slow, the frequency variations are converted into amplitude variations, as is widely used in frequency detectors and meters.

It seems at first glance that the complex equation (4.7) is derived from the original equation (4.6) without the participation of the AS. Actually this is not true. Since the APF that are converted in the discriminator must be understood as objectively *measurable* parameters, one must derive Eq. (4.7) from (4.6) by applying some operator H that relates u and v, and which does not depend

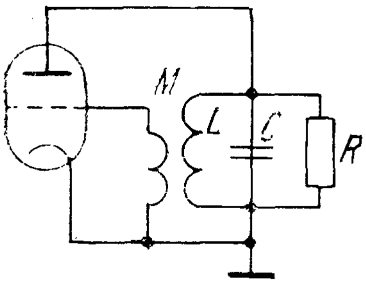

FIG. 3. Triode oscillator.

on the concrete form of the oscillation. Upon comparing the left-hand sides of the equations, we note that the operator is linear and commutative with differentiation (properties a) and c), Chap. 1). That is, it corresponds to a stationary filter having a certain transfer coefficient $H(i\omega)$. On the other hand, upon treating the transformation of the right-hand sides and introducing the quadrature components according to the formulas (2.8), we arrive at the relationship

$$H(i\omega) = -i\,\frac{Z(\omega-\omega_0) - Z(\omega+\omega_0)}{Z(\omega-\omega_0) + Z(\omega+\omega_0)}, \quad Z(\omega) = X(\omega) + iY(\omega). \tag{4.8}$$

Independence of $u(t)$, i.e., of $Z(\omega)$ obtains only if the spectra $Z(\omega-\omega_0)$ and $Z(\omega+\omega_0)$ do not overlap, and then (4.8) gives the Hilbert transform: $H(i\omega) = -i\ \mathrm{sgn}\,\omega$.

Thus the quasistationary approximation (3) is also characterized by two errors: an error in the *equation* (4.7) involving the overlap of the spectra, and an error in its *solution*, which is of the order of ε if a and Φ depend on εt. The latter can be diminished (see the Introduction), but this refinement loses sense if the error arising from overlap of the spectra is large. Moreover, this error is often smaller than the others.

Whenever we apply (under characteristic conditions of slowness and broad-band character) quasistationary treatments for estimating the APF (rather than the oscillation itself) we are *implicitly* employing the AS. A frequency discriminator is a typical quasistationary device. Hence it measures the frequency of the AS.

5. THE VAN DER POL GENERATOR AND ITS FREQUENCY INSTABILITY CAUSED BY THE FLICKER EFFECT

In spite of the importance of the problem and the wealth of literature, the frequency stability of ordinary vacuum-tube oscillators has not been studied with the necessary thoroughness. People have studied only the perturbing factors that allow analysis within the first approximation with respect to ε. These first-order effects include natural fluctuations due to additive noise (thermal and shot noise), amplitude fluctuations due to flicker noise, frequency fluctuations due to random variations in the capacitance of the circuit, and certain others. Yet these effects do not always determine the practical stability. In particular, the natural fluctuations give rise only to an insignificant broadening of the spectral line (of the order of 10^{-15}), while capacitance fluctuations are characteristic only of certain oscillators of poor stability.[19] We shall study below the frequency fluctuations due to the flicker noise of the vacuum tube; they are manifested only in the second approximation, whose construction requires a definition of the APF more accurate than the usual.

Let us examine a Van der Pol generator in which the transconductance S of the tube and the capacitance C of the circuit vary slowly in time (Fig. 3):

$$S(t) = S_0[1+\xi(t)], \quad C(t) = C_0[1+\eta(t)], \quad \xi \ll 1, \quad \eta \ll 1. \tag{5.1}$$

Here small fluctuations in the transconductance arise from flicker noise. For a soft oscillation regime (cubic characteristic of the tube), the oscillation equation is brought into the form

$$\frac{d^2u}{dt^2} + \Omega^2(t)\,u = \varepsilon\frac{d}{dt}\left\{\psi(t)\left[\chi(t)\,u - \frac{1}{3}u^3\right]\right\}. \tag{5.2}$$

Here we have introduced the dimensionless time $\omega_0 t$ and have used the notation

$$\varepsilon = \frac{p}{Q} \ll 1, \quad p = kRS_0 - 1, \quad \Omega^2(t) = \frac{C_0}{C(t)} = 1 - \eta(t),$$
$$\psi(t) = 1 + \xi(t) - 3\eta(t), \quad \chi(t) = 1 + \frac{\xi(t)}{p} + 2\eta(t); \tag{5.3}$$

Here Q is the Q-factor of the circuit, $k = M/L$ is the coupling coefficient, and p is a measure of the limit cycle. For the sake of a certain simplification of the formulas, we have linearized with respect to the small perturbations ξ and η. If perturbations are lacking ($\xi = \eta = 0$, $\Omega = \psi = \chi = 1$), then Eq. (5.2) acquires the simple form

$$\frac{d^2u}{dt^2} + u = \varepsilon\frac{d}{dt}\left(u - \frac{1}{3}u^3\right). \tag{5.4}$$

This is the Van der Pol equation, which has been solved by the most varied methods (see Refs. 16–21). We shall seek a solution of Eq. (5.2) in the form

$$u(t) = u_1(t) + \varepsilon u_3(t) + \varepsilon^2 u_5(t) + \ldots, \tag{5.5}$$
$$u_1(t) = a(t)\cos\varphi(t),$$
$$u_n(t) = x_n(t)\cos[n\varphi(t)] - y_n(t)\sin[n\varphi(t)] \quad (n = 3, 5, \ldots).$$

We assume the functions $a(t)$, $x_n(t)$, and $y_n(t)$ to be slow, as depending on the slow time εt. Hence their derivatives are of the order of ε, while we can assume that their spectra do not overlap the high-frequency spectrum of the oscillations $u(t)$ and $\cos\varphi$ or $\cos n\varphi$ and $\sin n\varphi$ (see Chap. 4). The same conditions are imposed on the functions Ω, ψ, and χ in Eq. (5.2).

We can derive the equation of the first approximation

$$\frac{d^2u}{dt^2} + \Omega^2 u = \varepsilon\frac{d}{dt}\left\{\psi\left[\chi u_1 - \frac{1}{3}u_1^3\right]\right\} = \varepsilon\frac{d}{dt}\left\{\psi\left[\left(\chi - \frac{1}{4}a^2\right)a\cos\varphi - \frac{1}{12}a^3\cos 3\varphi\right]\right\}$$

by substituting the expansion of (5.5) into Eq. (5.2) with account taken of terms of the order of ε. Let us apply the Hilbert transform to it; all the slow functions are taken outside the transform, and as in Chap. 4, we get the following equation for $w = w_1 + \varepsilon w_3$:

$$\frac{d^2w}{dt^2}+\Omega^2 w=\varepsilon\frac{d}{dt}\left(\gamma w_1-\frac{1}{12}\psi w_1^3\right), \tag{5.6}$$

Here we have

$$\gamma=\psi\cdot\left(\chi-\frac{1}{4}a^2\right). \tag{5.7}$$

When a and Φ vary slowly, the spectra of the analytic signals w_1 and w_3 practically do not overlap. Therefore, upon separating the oscillations of different frequencies, we arrive at the system of equations:

$$\begin{aligned}&\frac{d^2w_1}{dt^2}+\Omega^2 w_1=\varepsilon\frac{d}{dt}(\gamma w_1),\\&\frac{d^2w_3}{dt^2}+\Omega^2 w_3=-\frac{1}{12}\frac{d}{dt}(\psi w_1^3).\end{aligned} \tag{5.8}$$

Here the first equation is nonlinear (γ depends on a). We can easily derive from it (by multiplying by w_1^* and taking the imaginary part as in Chap. 4) the equation for the adiabatic invariant $J=a^2\omega$, namely,

$$\dot J=\varepsilon\gamma J, \quad \text{or} \quad \dot J=\varepsilon\psi\left(\chi-\frac{1}{4\omega}J\right)J; \tag{5.9}$$

The frequency ω is determined from the equation

$$\omega^2+\frac{\ddot\omega}{2\omega}-\frac{3\dot\omega^2}{4\omega^2}=\Omega^2-\frac{\varepsilon}{2}\left(\dot\gamma+\frac{\varepsilon}{2}\gamma^2\right). \tag{5.10}$$

The latter equation is analogous to (4.5), with $\omega=\Omega$ within an error of the order of ε^2. This means that in the first approximation the amplitude and frequency modulation (AM and FM) of the triode generator are related in the same way upon slow variation of the resonance frequency of its circuit as in the linear system treated in Chap. 4. When perturbations are lacking, i.e., $\Omega=\psi=\chi=1$, Eq. (5.9) goes over into the abbreviated Van der Pol equation

$$\frac{da^2}{dt}=\varepsilon\left(1-\frac{1}{4}a^2\right)a^2. \tag{5.11}$$

We have arrived at Van der Pol's results in an essentially new way—without averaging. At the same time we have shown that the varying amplitude $a(t)$ as determined from the abbreviated equation, which is usually treated only as an approximation that requires vibrational corrections,[21] is actually the absolute value of the analytic signal $w_1(t)$. As we shall see below, this is true also in the second approximation.

Equation (5.9) generalizes Van der Pol's solution to an oscillator having the variable parameters of (5.2). It defines a quantity that is invariant under adiabatic conditions, but which varies when adiabaticity breaks down. If the perturbations are slow, then

$$J=\text{const},\quad \gamma(t)=0,\quad a^2(t)=4\chi(t). \tag{5.12}$$

The second equation of (5.8) determines the third harmonic; in the given approximation we have

$$w_3=i\frac{\psi w_1^3}{32\Omega}. \tag{5.13}$$

The frequency correction caused by the current fluctuations, i.e., the function $\xi(t)$, arises only in the second approximation. In order to find it, we must keep the terms of the order of ε^2 while substituting the expansion (5.5) into Eq. (5.2): we account for the fifth harmonic on the left-hand side and the third on the right. Upon transforming to the AS and separating the harmonics, we get the following system:

$$\begin{aligned}&\frac{d^2w_1}{dt^2}+\Omega^2 w_1=\varepsilon\frac{d}{dt}\left\{\psi\left[\left(\chi-\frac{1}{4}a^2\right)w_1-\frac{\varepsilon}{4}w_1^{*2}w_3\right]\right\},\\&\frac{d^2w_3}{dt^2}+\Omega^2 w_3=\frac{d}{dt}\left\{\psi\left[\varepsilon\left(\chi-\frac{1}{2}a^2\right)w_3-\frac{1}{12}w_1^3\right]\right\},\\&\frac{d^2w_5}{dt^2}+\Omega^2 w_5=-\frac{1}{4}\frac{d}{dt}(\psi w_1^2w_3).\end{aligned} \tag{5.14}$$

With an error of the order of ε^3, we can restrict the treatment in the first equation of (5.14) to the approximation (5.13) for w_3. Then the first equation acquires the form

$$\frac{d^2w_1}{dt^2}+\Omega^2 w_1=\varepsilon\frac{d}{dt}(\gamma_1 w_1),\quad \gamma_1=\psi\left(\chi-\frac{1}{4}a^2-i\frac{\varepsilon}{128}\frac{a^4}{\Omega}\right). \tag{5.15}$$

If we assume that

$$w_1=W\exp\left(\frac{\varepsilon}{2}\int\gamma_1\,dt\right), \tag{5.16}$$

we get an equation of the form of (4.3) for W in which Ω^2 is replaced by

$$\Omega_1^2=\Omega^2-\frac{\varepsilon}{2}\left(\dot\gamma_1+\frac{\varepsilon}{2}\gamma_1^2\right), \tag{5.17}$$

Here, according to (5.15), the imaginary part of γ_1 introduces a term of the order of ε^3 into the right-hand side of (5.17). Therefore we can assume Ω_1 to be real, upon replacing γ_1 by γ in (5.17), and can take an expression of the type of (4.4) for W.

We must separate the real and imaginary parts of γ_1 in the exponent in the second factor in (5.16), and attribute the latter to a frequency correction. Finally, the amplitude a and the frequency ω of the AS of (5.16) are determined in the second approximation by the relationships:

$$a^2=\frac{|c|^2}{\omega_1}\exp\left(\varepsilon\int\gamma\,dt\right), \tag{5.18}$$

$$\omega=\omega_1-\frac{\varepsilon^2}{256\Omega}\psi a^4, \tag{5.19}$$

Here ω_1 is defined by Eq. (5.10).

Equation (5.18) shows that the second approximation introduces no changes into the amplitude, and Eq. (5.9) continues to hold (with ω replaced by ω_1); the relationships (5.12) are satisfied for an adiabatic regime. In this regime, the dimensionless frequency of the oscillations as defined by Eqs. (5.19) and (5.10) acquires the following form when we account for Eq. (5.12):

$$\omega(t)=1-\frac{\varepsilon^2}{16}\psi(t)\chi^2(t).$$

Here we have set $\eta=0$ and $\Omega=1$, while neglecting modulation of the capacitance. Upon substituting in the values of ε, ψ, and χ from the expressions (5.3), we get the final formulas for the (dimensionless) amplitude and oscillation frequency

$$\begin{aligned}&a(t)=2\left[1+\frac{\xi(t)}{2p}\right],\\&\omega(t)=1-\frac{p^2}{16Q^2}-\mu\xi(t),\quad \mu=\frac{p(p+2)}{16Q^2}.\end{aligned} \tag{5.20}$$

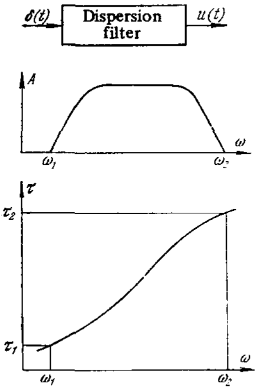

FIG. 4. Passive shaping of an FM signal.

Here $\xi(t)$ is the slow relative perturbation of the tube current caused by the flicker effect, which affects the transconductance S according to the first formula of (5.1).

The expressions (5.20) allow one to make a complete study of the instability of the oscillator when acted on by flicker noise, and to draw the following conclusions:

a) The measure p of the limit cycle has a different effect on the amplitude and the frequency: while decreasing the amplitude fluctuations, an increase in p substantially increases the frequency fluctuations.

b) The mechanism of production of frequency fluctuations consists in the fact that the third harmonic of the oscillation gives rise to a correction to w_1 upon interacting with the first harmonic (the term $w_1^{*2}w_3$ in (5.14)). Just like the third harmonic itself (see Eq. (5.13)), this correction is in quadrature with the original oscillation, and its amplitude fluctuations caused by fluctuations in the transconductance give rise to corresponding phase and frequency fluctuations. The existence of the quadrature correction also gives rise to a static frequency shift (the term $-p^2/16Q^2$ in the second formula of (5.20) Kobzarev[34] derived this static correction as early as 1931).

c) Flicker noise is a slow random process that can usually be considered stationary and normal[19, 20]; according to the formulas of (5.20) it causes amplitude and frequency modulation of oscillations. The frequency modulation determines the width of the spectral line. If we assume an energy spectrum of the process $\xi(t)$ in the form

$$G_\xi(\omega) = \frac{b}{\omega^\lambda} \qquad (b = 10^{-14},\ \lambda = 0.99), \tag{5.21}$$

we get the following expression for the relative width of the spectral line[19]:

$$\Delta\omega = \mu \sqrt{\frac{2b}{1-\lambda}}. \tag{5.22}$$

Here the line shape arising from frequency modulation is close to Gaussian. In particular, if we take $p=100$, then $\Delta\omega \approx 10^{-11}$ for $Q=10^4$, and $\Delta\omega \approx 10^{-9}$ for $Q=10^3$. These values agree with the actual characteristics of stable oscillators. This confirms the substantial role of the discussed effect.

d) The results concerning the influence of the flicker effect on frequency instability of the triode oscillator are apparently new: we have not been able to find them in the literature.[16–23] These results are rather easily derived via the AS, whereas the usual approach leads to excessively cumbersome calculations.

e) The obtained results can be generalized in many directions. Yet perhaps the most remarkable of them is that the complex functions w_1, w_3, ... have proved useful in the nonlinear oscillation problem.

6. PASSIVE AND ACTIVE SHAPING, ASYMPTOTIC PROPERTIES OF BROAD-BAND FM OSCILLATIONS

In radar, in accelerator technology, and in other fields where FM signals of preassigned shape are applied, two different methods are employed of obtaining these signals: passive and active. In the passive method,[35–37] the signal is created at the output of a dispersion line (filter) excited by a delta-function pulse. The amplitude $A(\omega)$ and phase $\alpha(\omega)$ characteristics of the line are selected so as to produce the given spectrum; the oscillation being shaped is determined by these characteristics, and can be represented by the Fourier integral:

$$u(t) = \frac{1}{\pi}\int_{\omega_1}^{\omega_2} A(\omega)\cos[\omega t - \alpha(\omega)]\,d\omega. \tag{6.1}$$

Here we assume that the passband of the line is limited to the interval (ω_1, ω_2); in addition to the phase characteristic $\alpha(\omega)$, it is convenient to treat the group retardation

$$\tau(\omega) = \alpha'(\omega). \tag{6.2}$$

The latter is determined by the derivative of $\alpha(\omega)$, and is usually monotonic in the band (Fig. 4).

Active shaping is closer to traditional transmission technique. It uses a frequency-controlled autogenerator (often a carcinotron). Such oscillators were rejected in the initial stage owing to poor stability and insufficiently exact realization of the frequency-modulation law. Later, however, using appropriate automatic frequency control circuits, active shaping came into use. This has often led to results that are practically unattainable for passive instruments. Figure 5 shows a diagram of a standard active transmitter[38] that shapes (in the millimeter range) a linear FM signal with the specific parameters: pulse duration $T=1$ ms, frequency deviation $\Delta f=1000$ MHz, base $B=T\Delta f=10^6$.

The transmitter functions as follows. An oscillation

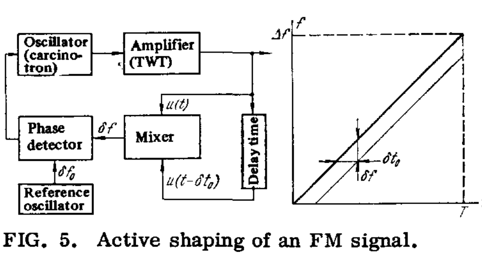

FIG. 5. Active shaping of an FM signal.

from an oscillator having a linear frequency variation is retarded by the time $\delta t_0 = 1\ \mu s$, and it is mixed with the unretarded oscillation. A (constant) frequency difference is formed at the output of the mixer of $\delta f = \delta t_0 / T\Delta f = 1$ MHz. The circuit contains a reference quartz oscillator of frequency $\delta_0 f_0 = 1$ MHz, and the difference frequency is compared with δf_0 in a phase detector. An error signal appears upon phase mismatch that controls the frequency of the main oscillator. The stabilization thus attained in the instantaneous difference frequency (and even in the phase) ensures shaping of the oscillation with strictly linear frequency modulation.

Let us turn our attention to the fundamental distinction between the passive and active methods. The passive method is integral, and the filter takes prior account of the shape of the oscillation over the entire time axis: if we know the characteristics $A(\omega)$ and $\alpha(\omega)$ of the filter, we can say in advance the value that the signal will take at each instant t. Conversely, in the active method the instantaneous frequency is measured and tuned in the short time δt_0, which is determined by the speed of action of the servo system. No element of the circuit "has prior knowledge" of the shape of the oscillation. In other words, the active method uses the adiabatic conception of a slowly varying frequency, while the passive method is based on the spectral (i.e., integral) approach. However, both methods are applied for the same purpose; a comparison of them allows one to establish an important connection between these alternative approaches.

Let us return to the passive method. We can easily construct the AS for the oscillation of (6.1):

$$w(t) = \frac{1}{\pi} \int_{\omega_1}^{\omega_2} A(\omega)\, e^{i[\omega t - \alpha(\omega)]}\, d\omega. \tag{6.3}$$

Upon calculating this integral, we could exactly indicate the amplitude and instantaneous frequency (as defined via the Hilbert transform) at each instant. Yet at each instant t the signal is formed by a superposition of all the spectral components. In order to determine the latter, we must in turn know the entire shape of the oscillation. However, in many cases one can calculate the integral of (6.3) *approximately* by using the asymptotic stationary-phase method. According to this method, the main contribution to (6.3) for each t comes from a small neighborhood of the stationary point ω_s, which is determined by the equation

$$\alpha'(\omega_s) = \tau(\omega_s) = t. \tag{6.4}$$

Here the effective width of this neighborhood is[39]:

$$\delta\omega = (4-8)\sqrt{\frac{1}{|\tau'(\omega_s)|}}, \quad \tau'(\omega_s) = \alpha''(\omega_s). \tag{6.5}$$

If this width is small in comparison with $\omega_2 - \omega_1$ and the neighborhood lies within the interval (ω_1, ω_2), i.e., if

$$\delta\omega \ll \omega_2 - \omega_1, \quad \omega_1 + \delta\omega < \omega_s < \omega_2 - \delta\omega, \tag{6.6}$$

then the integral of (6.3) is approximately equal to

$$w(t) = \sqrt{\frac{2}{\pi\tau'(\omega_s)}}\, A(\omega_s) \exp\left\{ i \left[\omega_s t - \alpha(\omega_s) + \frac{\pi}{4} \right] \right\} \tag{6.7}$$

Hence the instantaneous frequency φ with account taken of (6.4) is $\omega_s(t)$. Thus, *under conditions in which the stationary-phase method is applicable*, the instantaneous frequency equals the frequency of the spectrum that makes the major contribution to the value of w at the given instant. We have arrived at the adiabatic treatment: at each instant the oscillation is characterized by only one frequency—the instantaneous frequency of the AS, and its time variation determines the spectrum of the oscillation. Each instant of time (or more exactly, a small neighborhood δt around it; see below) defines the value of the spectral function $A(\omega)e^{-i\alpha(\omega)}$ at the point $\omega = \omega_s(t)$. This amounts to the traditional conception of a slowly varying frequency on which active shaping is based. However, as is now clear, this treatment is not always applicable, but only when the integral relationship (6.3) leads to the expression (6.7), i.e., when the conditions (6.6) are satisfied.

Under these conditions the band width is $\omega_2 - \omega_1 \approx 2\pi\Delta f$, where Δf is the deviation of the instantaneous frequency, while the difference in group retardation is $|\tau(\omega_2) - \tau(\omega_1)| \approx T$, where T is the pulse duration. Since also we have $\tau'_\omega(\omega_s) \sim T/(\omega_2 - \omega_1)$, we get the following inequality from the first condition of (6.6):

$$B = T\Delta f \gg 5 - 20. \tag{6.8}$$

That is, differential representations (the adiabatic treatment) are applicable to *broad-band* FM oscillations having a large base.

The second condition of (6.6) indicates distortions due to the edge effect when the essential frequency interval $\delta\omega$ exceeds the filter band; this corresponds to the time interval

$$\delta t = \frac{\delta\omega}{|d\omega_s/dt|} = (4-8)\sqrt{\frac{2}{|\ddot{\varphi}(t_s)|}}, \quad \dot{\varphi}(t_s) = \omega_s \tag{6.9}$$

that is taken up by the edge of the pulse. Edge distortions unavoidably appear in active shaping (see Fig. 5), and they restrict its potentialities.

The spectral regions distorted by the edge effect are usually filtered out in a method that narrows the receiver band (weighting treatment). If we allow 10% energy losses, we get the condition $\delta t \leq 0.1T$. Upon accounting for Eq. (6.9), we get the estimate

$$B = T\Delta f \gg 500 - 2000, \tag{6.10}$$

which lends exactness to the condition (6.8). This estimate is confirmed by practice. Thus, the authors of the extensive review[38] of linear FM of pulses shaped by active methods note the advantages of these methods over the passive methods, especially for large bases—of the order of 10^6 and above. Yet, in spite of the substantial differences between the active methods and the differences in ranges and widths of pulse bands, characteristically no signal having a base less than 1000 can be shaped by the active method, in line with our condition (6.10).

Passive shaping dominates for bases measured in tens and hundreds. It does not involve the adiabatic treatment, and as we see, this arises from fundamental rather than technical reasons. If we try to shape such a signal by an active method (at one time one of us (D. V.) participated in such an attempt), we make the same error as in trying to get directed radiation from a parabolic reflector of small dimensions (as compared with the wavelength). That is, we are mechanically applying geometric optics and forgetting that the edge effects here are not small—they fully govern the radiation and render geometric-optics representations inapplicable.

We note also that the difference frequency δf and the retardation time δt_0 chosen in the transmitter in Fig. 5 agree with the estimates of (6.5) and (6.9). This is characteristic also of other active devices of this type.[38]

7. ASYMPTOTIC PROPERTIES OF NARROW-BAND OSCILLATIONS

Above we have been treating *broad-band* oscillations that arise in FM with large bases. The definition of the APF by Eq. (5) is inapplicable to these oscillations, whereas the definition via the AS agrees with the adiabatic treatment and gives its limits of applicability. Examples of narrow-band oscillations are given at the beginning of Chap. 2 (for $\omega_0 T \gg 1$); more generally, if $u(t)$ is defined by the first formula of (2.7) and the quadrature components $x(t)$ and $y(t)$ are slow, i.e., the effective band $\delta\omega$ of their spectral functions $X(\omega)$ and $Y(\omega)$ is small in comparison with the carrier frequency,

$$\delta\omega \ll \omega_0, \tag{7.1}$$

then the AS acquires a simple form

$$w(t) = [x(t) + iy(t)]\, e^{i\omega_0 t}. \tag{7.2}$$

This also agrees with the adiabatic treatment.

Of course, this result involves the properties g)–i) of the AS. Yet it is expedient to interpret it in another way, from the standpoint of approximate calculation of the integrals of (1.7) and (2.3). Just as in the integral of (6.3), these integrals possess a substantial frequency range that adjoins the point ω_0, and which is governed by the effective band $\delta\omega$. The approximate expression (7.2) holds when this band does not exceed the bound $\omega = 0$ in the integral of (1.7). In this treatment the condition (7.1) is quite analogous to (6.6).

Yet we have obtained two qualitatively different conditions under which the integral representations involving the AS degenerate into the ordinary adiabatic treatment of oscillations having slowly varying parameters. The condition (7.1) requires *narrow-band* character with respect to ω_0, as is customary and understandable. Moreover, the condition (6.8) requires a large base, or *broad-band* character, whereby it doesn't involve a carrier frequency in any way.

In this regard we must bear in mind that these conditions are alike in their mathematical essence, and that there is only one condition for asymptotic degeneracy, rather than two different ones. While treating, say, shaping by the scheme of Fig. 4, we assume that the dispersion and damping of the line depend to an equal extent on the frequency. Then it is convenient to treat the integral of (6.3) in the complex plane of ω, and the stationary (saddle) point ω_s will also be generally complex.[40] It happens to lie on the real axis only in the studied limiting cases when only amplitude or only phase variations in the spectrum prevail. However, more general degeneracy conditions (i.e., applicability of the saddle-point method) reduce to the idea that *the spectrum must vary rapidly near* ω_s, while it is not essential whether these variations are amplitude or phase variations.

Narrow-band oscillations amount to an amplitude $A(\omega)$ of the spectrum close to a delta function, while broad-band oscillations amount to a rapidly varying phase $\alpha(\omega)$ caused by the dispersion of the line; in both cases a replacement of the integral by the differential APF is admissible, i.e., an adiabatic treatment.

8. PARADOXES AND COUNTEREXAMPLES

The APF as defined via the AS are objective characteristics of an oscillation that can be measured by an amplitude detector, a frequency discriminator, etc. Yet the principle of operation of these and many other devices is based on the adiabatic conception of slowly varying amplitude and frequency. We have seen that this conception requires that the conditions (6.8) or (7.1) be satisfied; if these conditions are violated the proposed apparatus proves inoperable. We have already encountered this in Chap. 6: Active shaping methods are applicable in practice only with long bases. Let us give some additional examples, while paying major attention to the condition (6.8).

A. The Robinson paradox[41]

Upon starting with the idea of a slowly varying frequency, Robinson proposed eliminating interference from neighboring FM transmitters by using a small deviation Δf that is substantially smaller than the modulating audio frequency F (in 1930). Evidently this assumes that at each instant a region of the spectrum near $\omega(t)$ is being radiated, so that there is no energy outside the limits of the deviation. However, the decrease in the deviation leads to a small frequency-modulation index, $m = \Delta f/F \ll 1$. As we easily note, with tonal modulation this is equal to the base B. Hence the condition (6.8) is violated and the adiabatic conception loses force and becomes inapplicable.

Robinson's error lies in viewing this conception as universal. This is the same error as when one views the conceptions of geometric optics as universal and tries to examine in a microscope an object that is small in comparison with the wavelength.

B. Early attempts at frequency telephony[42]

As early as 1912 attempts were made to modulate the frequency with an arc transmitter (and later with a vacuum-tube transmitter) by switching a condenser micro-

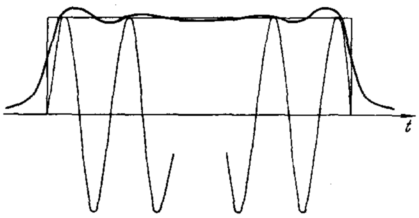

FIG. 6. Envelope of a square-wave radio pulse.

phone into the circuit. The demodulation was effected on the slope of the resonance curve of the receiver. The results were not satisfying—why?

In the then-employed long-wave region the capacitance of the circuit was thousands and tens of thousands of picofarads, while the variable capacitance of the microphone was of the order of 1%. Therefore the deviation did not exceed 1 kHz (with a 100-kHz carrier) and modulation by speech led to a small index $m \lesssim 1$ at which the adiabatic conception is inapplicable.

Here they made the same error as Robinson—the band was not broad. Now we attain such a modulation by employing varicaps or reactive tubes, but at much higher frequency deviations.

The Armstrong modulator (see Chap. 3) also operated with a small index ($m \lesssim 1/2$), but the modulated oscillations were then frequency-multiplied, which increased the FM index by a factor of 500.[42] Hence these oscillations could be received by an ordinary discriminator with an amplitude limiter, i.e., an apparatus based on adiabatic views.

Now we turn to the counterexamples that involve unusual (more exactly, unfamiliar) properties of the AS, and which lead to the relatively widespread opinion that the analytic signal, while convenient and useful in the mathematical theory, does not always make sense from the physical or technical standpoint.[7,8,28,43–45]

C. Violation of causality

Figure 6 shows the envelope of a square-wave radio pulse that has a forerunner and a wake. The perplexing questions arise: How can the envelope, i.e., the consequence, precede the original signal, or cause? What does the envelope enclose when the oscillation hasn't started yet or no longer exists? How can such an envelope correspond to the result of detection, while the signal at the output of a detector does not precede the input agent? How does the envelope manage to behave like the grin of the Cheshire Cat in Wonderland? Of course, from the mathematical standpoint the forerunner and the wake are explained by the analytic character of the function $w(t)$, which allows no jumps. Yet this result is understandable also physically after some reflection.

For narrow-band oscillations, one can introduce the "natural" concept of the envelope and its components by the relationships

$$a^2(t) = \widetilde{2u^2(t)}, \quad x(t) = \widetilde{2u(t)\cos\omega_0 t}, \quad y(t) = -\widetilde{2u(t)\sin\omega_0 t}, \qquad (8.1)$$

Here the wavy overline denotes time-averaging over some effective interval, e.g.,

$$a^2(t) = \frac{1}{T_0}\int_{t-T_0}^{t+T_0} u^2(s)\,ds, \quad \text{or}$$

$$a^2(t) = \frac{2}{\sqrt{2\pi}\,T_0}\int_{-\infty}^{\infty} \exp\left[-\frac{(t-s)^2}{2T_0^2}\right] u^2(s)\,ds. \qquad (8.2)$$

Of course, $a(t)$ will depend on the method and interval of averaging, and many of its useful mathematical properties will be lost, but the main point is that, according to the formulas of (8.2), the envelope *precedes* $u(t)$: the forerunner is formed as the averaging interval "creeps up on" the pulse, and the wake as it "creeps off." Evidently any reasonable definition of the envelope (a physical one that allows one to measure $a(t)$ rather than a mathematical abstraction) leads to a forerunner and a wake of duration $T_0 \gtrsim \pi/\omega_0$.

We note that a quadratic detector operates according to the first formula of (8.1), and synchronous detectors according to the two others; the averaging is performed by a filter that transmits only the low frequencies ($\omega \ll \omega_0$). The result depends, though weakly, on the frequency characteristic of the filter. Moreover the filter yields the functions $a(t)$, $x(t)$, and $y(t)$ with a delay of the order of T_0, which is inversely proportional to the filter band. Under such an averaging there is no forerunner, but the wake is elongated.

The AS ensures the minimal averaging time: we have $T_0 \sim \pi/\omega_0$ for narrow-band oscillations, while for all others the averaging time "tunes itself" in accordance with the structure of the process, as we see from the expression (6.9) for δt. Thus the definition of the APF via the AS is as "differential" as is generally possible.

D. Logarithmic singularity

Figure 7 shows the envelopes of pulses whose durations are comparable with the period $2\pi/\omega_0$ of the carrier frequency. If $u(t)$ possesses jumps, then the envelope goes to infinity at these points. A logarithmic singularity at the points of discontinuity of $u(t)$ is inherent in the Hilbert transform. This also sometimes excites doubt as to the physical significance of the AS.[28,44]

Of course, real signals are defined by continuous functions $u(t)$, but even if we use discontinuous functions

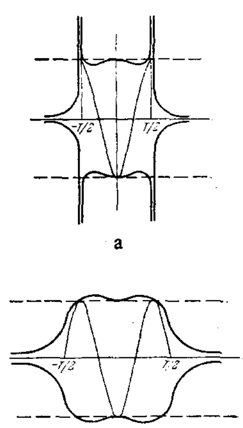

FIG. 7. Envelopes of square-wave radio pulses of short duration.

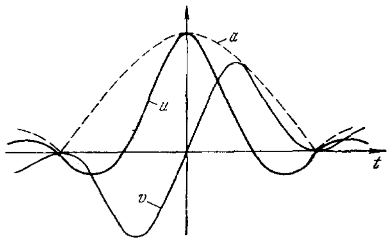

FIG. 8. Envelope of a video pulse having a square-wave spectrum.

while idealizing actual phenomena, then the logarithmic (i.e., very weak) singularity leads to no false conclusions. Yet if the function $u(t)$ is continuous, then there are no singularities.

In the "purely differential" definition of the APF by Eq. (4), the infinity at the points of discontinuity of $u(t)$ turns out to be not logarithmic, but stronger. The AS gives an (inessential) singularity precisely because it ensures the minimal averaging time. Moreover, $a(t)$ and $\Phi(t)$ are usually treated in slow time, so that the overshoots, forerunners, and wakes to be integrated are generally insignificant.

E. Oscillations without a carrier

Figure 7 has already given examples of signals for which the graphic meaning of an envelope is not very clear. One can easily increase the number of such examples. For example, let us examine an oscillation having the square-wave spectrum:[43]

$$U(\omega)=1 \text{ at } |\omega-\omega_0|<\Delta\omega, \quad U(\omega)=0 \text{ at } |\omega-\omega_0|>\Delta\omega. \tag{8.3}$$

When $\omega_0 \gg \Delta\omega$, it has a quasiharmonic nature, and the AS gives a quite natural form of the envelope. Let us displace the spectrum into the low-frequency region by decreasing ω_0. In the limit $\omega_0=\Delta\omega$, we get the broadband spectrum of a video pulse:

$$U(\omega)=1 \text{ at } \omega<2\Delta\omega, \quad U(\omega)=0 \text{ at } \omega>2\Delta\omega. \tag{8.4}$$

Figure 8 shows the envelope of this oscillation. People often assume that it is better not to introduce the envelope for such oscillations at all, since it does not give a pictorial view of the course of the process, and as it varies rapidly, it has only a nominal meaning.[43,45]

However, a linear FM pulse in an intermediate-frequency channel has precisely this type of spectrum after processing in a matched filter: the amplitude spectrum of the signal is a square wave and the phase structure is removed in the filter. Values are quite possible in a channel of, e.g.,[36,37] $f_0=30$ MHz, $\Delta f=25$ MHz (so that $\Delta f \approx f_0$). In this case one *must* introduce the envelope—it defines the result of detection and distance resolution. The downward displacement of the frequency spectrum is performed in the mixer (see Chap. 3), which leaves invariant the envelope as defined via the AS.

A still more complicated oscillation is obtained when the middle part of the spectrum of (8.4) is rejected, e.g., in interference suppression. Narrow bands remain at the edges that approximately correspond to two sinusoids of frequencies ω_1 and ω_2, with $\omega_1 \ll \omega_2$. This is precisely the counterexample that gave the argument for concluding that the envelope (the absolute value of the AS) is devoid of physical content for *broad-band oscillations*.[8] However, one must in practice extract the residual information about the signal also from such envelopes, while in addition, we treated in Chap. 6 oscillations as broad-banded as one pleases, for which the APF and the AS have a distinct and pictorial meaning.

In summing up the results, we can say that the envelopes in many of the counterexamples are unusual, but to declare them unreasonable *a priori* won't do: they are sometimes useful and full of content.

There are also cases (see Chap. 5) in which it is not rational to introduce a *single* AS and the corresponding APF for an oscillation as a whole, while moreover it is convenient to introduce the AS and the APF for a series of nonoverlapping spectral bands (in Chap. 5, these are the neighborhoods of the frequencies ω_0, $3\omega_0$, $5\omega_0$, etc.).

An opinion exists that the AS is suitable for describing modulation of harmonic oscillations, but not those of an arbitrary carrier, e.g., $\sin^2\omega_0 t$. Yet the appropriate generalization was proposed as early as 1958.[2] One can define the amplitude and frequency modulation of an *arbitrary* oscillation $u_0(t)$ as the corresponding variation in the envelope and phase of the AS $w_0(t)=u_0(t)+iv_0(t)$. Here the functions $u_0(t)$ and $v_0(t)$ play the same role as $\cos\omega_0 t$ and $\sin\omega_0 t$ in modulation of harmonic oscillations. In particular, this approach allows one easily to derive and generalize most of the results of Ageev.[46]

CONCLUSION

Above we have recalled the varied applications of the AS in the theory of random processes and fields, in particular, in wave optics[9,10]; in quantum optics the AS retains its significance, since the field is written in the form (3.3). After transformation to operators, one term defines radiation (creation) of photons, while the other defines their absorption (annihilation), since only positive frequencies figure in the one term, and only negative in the other.

With account taken of the abovesaid, this review pursues a rather modest aim: to "pull up" (in the sense of applying the AS) oscillation theory to the level of noise theory, coherence theory, and other fields where the AS has already been applied for a long time and has yielded a number of important results. We have tried to show that the AS allows one better to understand and calculate oscillatory phenomena in the most varied systems, though adopting it requires overcoming a certain psychological block.

The significance of the AS increases to the extent that the theory of oscillations invades new fields and quantitative changes (mainly the expansion of signal bands) grow into qualitative ones; then one must replace the usual views with new ones that are more general, and which reduce to the former only in the appropriate limit. The analytic signal gives us a general definition of the amplitude, phase, and frequency, the fundamental concepts of the theory of oscillations and waves.

The authors thank L. M. Fink for useful discussions.

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Translated by M. V. King