

# The renormalization-group method in the theory of phase transitions

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The renormalization group for the statistical mechanics of a wave field is described. The hypothesis of scaling of correlations near an order-disorder phase-transition point is written in renormalization-group terms. The renormalization-group equations near a fixed point are investigated. The scaling dimensions (critical indices) of the principal quantities are found in the  $\epsilon$ -approximation. The stability of many-component systems and the question of asymptotic symmetry are investigated. The renormalization method is applied to the study of the dynamics of systems near a phase-transition point.

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## INTRODUCTION

The foundations of the general theory of second-order phase transitions were laid in the well-known papers of Landau (cf. <sup>[1,2]</sup>). It is now generally acknowledged that a second-order transition is none other than the spontaneous breaking of the symmetry of the system. L. D. Landau introduced a quantitative characteristic of the symmetry breaking—the order parameter  $\varphi$ . A simple and universal theory (the self-consistent field theory), in which fluctuations were assumed to be negligibly small, made it possible to describe a number of phenomena in superconductors and ferroelectrics with extreme elegance and good agreement with experiment.

At the same time, a number of facts did not fit the self-consistent field theory. The exact solution of the two-dimensional Ising model, propounded by Onsager in 1944, displayed a logarithmic singularity in the specific heat instead of the finite jump predicted by self-consistent field theory. Experiments to measure the specific heat  $C_p$  of helium near the  $\lambda$ -point<sup>[3]</sup> and the specific heat  $C_v$  near the critical liquid-vapor point of argon<sup>[4]</sup>

revealed a growth of the specific heat approximately describable by the logarithmic law:  $C \sim A \ln |T - T_c| + C_{reg}$ . These effects, and many other deviations from the Landau theory, must obviously be ascribed to strongly developed fluctuations of the order parameter. Near the transition point the size  $r_c$  of these fluctuations (the correlation length) becomes very large. In each region with linear dimensions  $\sim r_c$  the resultant order parameter  $\Phi_c$  (the analog of the magnetic moment of a ferromagnet) increases with increase of the size of the region, although not in proportion to the volume. A substance near a transition point can be visualized as an aggregate of macroscopic regions (cells) interacting with each other. Change of temperature reduces to a change in the sizes of the cells. If all lengths are measured in units of  $r_c$ , and the resultant order parameter in units of  $\Phi_c$ , a change of temperature does not lead to any change in the thermodynamic quantities or correlation functions.

In this way, a scaling hypothesis was formulated in papers by the authors of this review<sup>[5]</sup> and by Kada-

noff.<sup>[6]</sup> In the framework of this hypothesis, which is universal for all second-order phase transitions, the behavior of the thermodynamic quantities and  $\nu_c$  as functions of  $T - T_c$ ,  $p - p_c$  and external fields, and also of the correlators as functions of the distances and thermodynamic variables, is expressed by power laws. The power exponents are called critical exponents. The scaling hypothesis made it possible to express a large number of critical exponents in terms of any two. The state of the theory in 1967 was described in a review by one of the authors.<sup>[7]</sup> There is now a multiplicity of experimental confirmations of the scaling hypothesis (cf. <sup>[2,8]</sup>). However, the calculation of the critical exponents remained an impenetrable problem up to 1972. New ideas were proposed in papers by Wilson<sup>[9]</sup> (the renormalization-group method) and by Wilson and Fisher<sup>[10]</sup> (the  $\epsilon$ -expansion). These ideas bring little that is new, perhaps, to our physical understanding of phase transitions, but they have turned out to be a powerful tool of mathematical analysis and have found application not only in the theory of phase transitions but also in a series of related problems, such as the Kondo effect, percolation theory, the theory of long-chain molecules, and, finally, quantum field theory.

In the present review an account is given of the renormalization-group method and its application to the theory of phase transitions. The static critical indices are calculated in the  $\epsilon$ -approximation. The question of the dynamical exponents is examined in detail. Here we are concerned, first and foremost, with the dependence of the kinetic coefficients on  $T - T_c$  and with the wave-vector dependence of the relaxation time  $t_r$  of the fluctuations:  $t_r \sim \nu^{-z}$ . The quantity  $z$  is called the dynamical critical exponent. Is this exponent independent, or is it determined by the static indices? The answer to this question is not unique; it depends on the set of conserved quantities that characterize the system. In this part of the review we have followed the papers of Halperin, Hohenberg and Ma.<sup>[11]</sup>

Another physical group of questions considered in the review is that concerned with instabilities and first-order phase transitions induced by fluctuations. It is found that, even in extremely simple many-component systems, sufficiently strong anisotropy or interaction with a weakly fluctuating quantity can lead to instability and a first-order phase transition. Here the fluctuations play a role similar to that of phonons in superconductors: the exchange of "fluctuons" leads to an effective attraction between other quasi-particles and, in a number of cases, to instability. The theory predicts, with certain provisos, the character of the phase diagram and of the order parameter that arises as a result of the transition.

To a greater degree than the previous review<sup>[7]</sup> or the book by the authors,<sup>[2]</sup> the present review is devoted to an account of the mathematical apparatus. Apart from the reasons indicated above, this is connected with the fact that we want to show the theory in action and make it possible for those wishing to study independently to use the theory. We have examined the question of the limits of applicability of the theory in detail, since

papers in which authors attempt to apply the Gell-Mann-Low renormalization-group technique in situations in which it is in principle inapplicable have recently begun to proliferate.

Another feature of our review is the small amount of experimental data. This is due, principally, to the fact that in all cases the real indices are close to those of the Landau theory. The difference between the exponents for one-, two- and three-component order parameters is not great. This similarity of the exponents has the consequence that the experiments become disproportionately difficult, and their results are not too convincing. The theory also leads to close values of the exponents, but, unfortunately, in a number of cases the theoretical difference between the indices lies beyond the bounds of experimental capabilities (e.g., the change in the exponents as a result of the magnetic-dipole interaction of the spins in a ferromagnet<sup>[12]</sup>). As regards the instabilities and first-order phase transitions, there are still very few experiments on these.

There are already rather many reviews of the renormalization group, amongst which we mention those by Wilson and Kogut<sup>[13]</sup> and Fisher.<sup>[14]</sup> In the former the fundamental problems of the theory and methods of calculating the dimensions are examined in detail. In the latter the exponents are found for a number of more complicated models, which describe the properties of different magnets. The review offered below differs from these both in method of exposition and in choice of material, particularly as regards applications. In particular, the theory of instabilities in many-component systems, and the problems of the kinetics, are described in a connected form for, apparently, the first time.

## 1. THE PROBABILITY AND HAMILTONIAN OF FLUCTUATIONS. A LITTLE MORE HISTORY

In the Landau theory an order parameter  $\varphi$ , which can be a many-component quantity, is introduced. Its meaning and the number of its components are determined by the physics of the phase transition being studied. For example, in the case of a ferromagnetic Curie point the order parameter is the magnetic moment, calculated per site or per unit volume, in an antiferromagnetic transition it is the difference in the moments of the two sublattices, in the transition to the superfluid state it is the complex wavefunction of the condensate, and at a liquid-vapor critical point  $\varphi$  is equal to the difference between the mean density and the critical density. The order parameter  $\varphi$  is a classical quantity, defined as an average over a volume containing a large number of particles. It is legitimate, however, to consider its fluctuations, i.e., to regard it as dependent on the coordinates  $x$  of a point in the system under consideration. If we fix  $\varphi(x)$  at each point of the system, then, by carrying out the summation over the remaining degrees of freedom in the partition function  $Z$ , we can find the free energy  $F$  (for fixed volume) or the thermodynamic potential  $\Phi$  (for fixed pressure) as a functional of  $\varphi(x)$ . In the following we shall not specify under which conditions the system is being considered, and

instead of  $F/T$  and  $\Phi/T$  ( $T$  is the temperature) we shall use the symbol  $\mathcal{H}$  and the conventional term "Hamiltonian." Landau postulated that the Hamiltonian can be expanded in an integro-power series in powers of  $\varphi(x)$  and its derivatives. Confining ourselves to the lowest powers of  $\varphi$  and  $\nabla\varphi$ , we arrive at the Landau Hamiltonian  $\mathcal{H}_L$ :

$$\mathcal{H}_L = \int \left[ \frac{1}{2} (\nabla\varphi)^2 + \frac{1}{2} \tau_0 \varphi^2 + g_0 \varphi^4 - h\varphi \right] dx, \quad (1.1)$$

where the parameters  $\tau_0$ ,  $g_0$  and  $h$  are functions of the temperature, pressure and external fields. The well-known argument of Landau shows (cf. <sup>[11]</sup>) that the quantity  $\tau_0$  should change sign, going to zero at the transition point (or, for small  $g_0$ , almost at the transition point). It is natural to call  $\tau \approx (T - T_c)/T_c$  the dimensionless temperature. The quantity  $g_0$  does not vary so strongly—in the following it will be assumed to be constant and will be called the interaction constant. We shall call the quantity  $h$  the external field. The probability  $W[\varphi]$  of a given realization  $\varphi(x)$  of the ordering field is equal to

$$W[\varphi] = Z^{-1} \exp(-\mathcal{H}[\varphi]), \quad (1.2)$$

where  $Z$  is the statistical integral.

In the self-consistent field approximation (Landau theory) the fluctuations of  $\varphi$  are assumed to be negligibly small, and the equilibrium properties of the system are determined by the condition that  $\mathcal{H}_L$  is an extremum with respect to  $\varphi$ . The condition for applicability of the self-consistent field approximation was found by Levanyuk<sup>[64a]</sup> and Ginzburg<sup>[64b]</sup> and has the form (cf., e.g., <sup>[2]</sup>)

$$\frac{g_0^2 T_c^2}{\tau} \ll 1. \quad (1.3)$$

We shall call the quantity  $\mathbf{Gi} = g_0^2 T_c^2 / \tau$  the Ginzburg number. It is a characteristic of the substance and does not depend on the temperature. At the phase-transition point the Ginzburg criterion is not fulfilled and the self-consistent field approximation becomes inapplicable. When  $\mathbf{Gi} \gg 1$  the Landau theory has no domain of applicability. It is necessary to consider the statistical mechanics of the wave field  $\varphi(x)$  with Hamiltonian  $\mathcal{H}_L$ . The field has a macroscopic meaning—it is smoothed (in its Fourier expansion the short-wavelength harmonics are absent). This approach to the problem of a phase transition in the region in which strongly interacting long-wavelength fluctuations of the ordering arise was suggested by L. D. Landau at the end of the 1950's.

In a paper by the authors<sup>[15]</sup> it was shown that the problem of the fluctuations at a phase transition can be reduced systematically to the problem of a wave field  $\varphi(x)$  in three-dimensional space. It was found to be possible to represent correlation functions in the form of series in which each term is expressed entirely in terms of the same correlation functions. If one assumes that the correlation functions are homogeneous functions of their arguments (i.e., of the coordinates and correlation length), one can choose the power exponents in such a way that all terms of the series have

the same dimension. It then remains to ensure that certain algebraic equations for the constants of the theory are fulfilled. Thus, the assumption that the correlators have a power-law form and that there are relations between the scaling powers in the correlators (the scaling hypothesis) was formulated for the first time in<sup>[15]</sup>. However, specific values for the powers (scaling dimensions) were obtained in<sup>[15]</sup> at the cost of additional assumptions. As we now understand, such a solution of the field-theory equations does in fact exist. However, the equations themselves are nonlinear and have an infinite set of power-law solutions, and the solution found in<sup>[15]</sup> does not satisfy certain additional physical requirements (locality, conformal invariance). These conditions, formulated in papers by A. A. Migdal and A. M. Polyakov (cf. <sup>[16,17]</sup>), give the possibility in principle of finding the scaling dimensions. However, the calculational difficulties have prevented us from going beyond statements of principle and the calculation of certain general properties.

In the interval between the papers<sup>[15]</sup> and <sup>[16,17]</sup>, papers by Widom<sup>[18]</sup> and Domb and Hunter,<sup>[19]</sup> in which homogeneity properties of thermodynamic quantities were postulated, and papers by the authors<sup>[5]</sup> and Kadanoff,<sup>[6]</sup> in which the homogeneity of the thermodynamic quantities was proved on the basis of the hypothesis of scaling of fluctuations, appeared almost simultaneously. The subsequent development of the scaling hypothesis made it possible to describe a wide range of experimental results (cf. <sup>[21]</sup>).

A new push was given to the theory of phase transitions by Wilson,<sup>[9]</sup> who applied the Gell-Mann-Low renormalization-group method<sup>[20]</sup> familiar in quantum field theory. From a fundamental point of view this method is yet another formulation of the scaling hypothesis. However, this new formulation turned out to be extremely fruitful. Using the method it turned out to be possible to calculate the critical exponents approximately as functions of the number of components of the order parameter, the dimensionality of space, and the form of the interaction, to calculate the equation of state, and to analyze the stability conditions for different systems. As a rule, the supplementary idea of proximity to four-dimensional space—the so-called  $\epsilon$ -expansion<sup>[10]</sup>—was used in these investigations. Methods of direct computer calculation of critical indices by the renormalization-group method without the use of the  $\epsilon$ -expansion have recently been found.<sup>[21]</sup> The direct methods are good in that they permit us, in principle, to improve the accuracy of the calculations without limit.

In the following we shall need certain general information and definitions from the fluctuation theory of phase transitions. Below we give an account of these without justifications, which can be found in the book.<sup>[2]</sup>

## 2. DIMENSIONS, THE ALGEBRA OF FLUCTUATING QUANTITIES, CORRELATORS

At the transition point the characteristic size of a fluctuation—the correlation length—becomes infinite. When the length scale is changed by a factor  $\lambda$  the physi-

cal quantities such as the order parameter, energy density  $\varepsilon(x)$ , temperature  $\tau$ , field  $h$ , etc., are multiplied by a certain power of  $\lambda$ . By definition, a quantity  $A(x)$  has scaling dimension  $\Delta_A$  if the scale transformation

$$\begin{aligned} x &\rightarrow x' = \lambda x, \\ A &\rightarrow A'(x') = \lambda^{-\Delta_A} A(x) \end{aligned} \quad (2.1)$$

does not change the statistical properties of this quantity. The assumption that fluctuating fields  $A(x)$  have this property is the scaling hypothesis for the quantities  $A$ , and the different formulations of the hypothesis are associated with the possibility of writing the requirement of invariance of the statistical properties in different ways. Physical considerations determine certain relations for the scaling dimensions  $\Delta_A$ . For example, the dimension of the length  $x$  is  $\Delta_x = -1$ , and the dimension of a particular extensive quantity—the singular part  $\Phi_{\text{sing}}$  of the density of the thermodynamic potential (Hamiltonian)—is  $\Delta_\Phi = d$ , where  $d$  is the number of spatial dimensions of the system. The sum of the dimensions of thermodynamically conjugate quantities is equal to  $d$ , e.g.,

$$\Delta_\tau + \Delta_h = d. \quad (2.2)$$

An important role is played by the quantity  $\varepsilon(x)$  conjugate to the temperature  $\tau$ . It may be called the entropy density or (more commonly) the energy density. According to the general rules,

$$\Delta_\varepsilon + \Delta_\tau = d. \quad (2.3)$$

The character of the singularities of different quantities near the transition point ( $\tau=0$ ,  $h=0$ ) is determined from scaling-dimension considerations. For example, for  $\tau \rightarrow 0$ ,  $h \rightarrow 0$  the thermodynamic potential  $\Phi(\tau, h)$  has the form<sup>[5]</sup>

$$\Phi(\tau, h) = \Phi_{\text{reg}} + V\tau^{2-\alpha} f(k). \quad (2.4)$$

In formula (2.4)  $\Phi_{\text{reg}}$  is the part of  $\Phi$  that is regular at the transition point and  $f(k)$  is a certain function of the scale-invariant  $k$ :

$$k = h\tau^{-\Delta_h/\Delta_\tau}. \quad (2.5)$$

The power exponent  $2 - \alpha$  in formula (2.4) is expressed in terms of the dimension  $\Delta_\tau$ :

$$2 - \alpha = \frac{d}{\Delta_\tau}. \quad (2.6)$$

We also write an important formula for the irreducible correlation functions  $K(x_1 \dots x_n) = \langle \varphi(x_1) \dots \varphi(x_n) \rangle$  at a phase-transition point<sup>[23]</sup>:

$$K(\lambda x_1 \dots \lambda x_n) = \lambda^{-n\Delta_\varphi} K(x_1 \dots x_n). \quad (2.7)$$

Homogeneity relations of the type (2.7) specify in the language of correlators the requirement that the statistical properties of the system be unchanged under the transformation (2.1), and are one of the formulations of the scaling hypothesis.

Physical quantities, generally speaking, do not possess a definite scaling dimension but can be represented as a superposition of quantities  $A_i(x)$  possessing a definite dimension, or of the "fields"  $h_i$  thermodynamically conjugate to them. This important postulate (the hypothesis of an algebra of fluctuating quantities) was formulated by Polyakov and Kadanoff<sup>[23]</sup>:

$$\begin{aligned} \varphi(x) &= \sum_{i=1}^{\infty} a_i A_i(x), \\ h &= \sum_{i=1}^{\infty} \alpha_i h_i. \end{aligned} \quad (2.8)$$

The coefficients  $a_i$  and  $\alpha_i$  and similar coefficients are not universal. The lowest dimension  $\Delta_i$  for which  $a_i \neq 0$  in the series (2.8) will be called the dimension of the physical quantity (e.g.,  $\varphi$ ). It is this dimension that is manifested in correlations at large distances. There are, however, situations in which subsequent terms of the series (2.8) are also important. A number of physical applications of the algebra of fluctuating quantities can be found in the book.<sup>[2]</sup>

An important model, which permits an exact calculation, is the free-field model. The field of the optical phonons in a solid in the complete absence of anharmonic terms provides a simple physical picture of this model. The free-field Hamiltonian is a particular case of the Landau Hamiltonian  $\mathcal{H}_L$ , with  $g_0=0$ :

$$\mathcal{H}_L = \frac{1}{2} \int [(\nabla\varphi)^2 + \tau\varphi^2] dx. \quad (2.9)$$

For a free field, averages of the form

$$K(x_1 x_2 \dots x_n) = \langle \varphi(x_1) \varphi(x_2) \dots \varphi(x_n) \rangle$$

are calculated by Wick's theorem

$$K(x_1 x_2 \dots x_n) = \sum \Pi K(x_i - x_j). \quad (2.10)$$

In formula (2.10) the summation runs over all possible partitions of the arguments  $x_1, x_2, \dots, x_n$  into pairs. The quantity  $K(x_i - x_j)$  is the pair correlator

$$K(x - y) = \langle \varphi(x) \varphi(y) \rangle. \quad (2.11)$$

Calculations give for  $K(x)$

$$K(x) = 2T (2\pi r_c r)^{1-(d/2)} K_{(d/2)-1} \left( \frac{r}{r_c} \right), \quad (2.12)$$

where  $r = |x|$ ,  $r_c = \tau^{-1/2}$ , and  $K_\nu(x)$  is a Macdonald function of order  $\nu$ . For  $r \ll r_c$

$$K(x) \sim r^{2-d} \quad (2.13)$$

and for  $r \gg r_c$  we have  $K(x) \sim \exp(-r/r_c)$ . In the two-dimensional case ( $d=2$ )  $K(x) \sim \ln r$ . Of importance in characterizing the interaction in the system are the irreducible correlators  $G(x_1 x_2 \dots x_n) = \langle \langle \varphi(x_1) \varphi(x_2) \dots \varphi(x_n) \rangle \rangle$ , which are obtained from  $K(x_1 \dots x_n)$  by subtracting all possible products of averages of groups of factors  $\varphi(x_i)$ , e.g.,

$$\begin{aligned}
K(x, y) &= G(x, y) + \langle \varphi(x) \rangle \langle \varphi(y) \rangle, \\
K(x, y, z) &= G(x, y, z) + G(x, y) \langle \varphi(z) \rangle \\
&\quad + G(y, z) \langle \varphi(x) \rangle + G(z, x) \langle \varphi(y) \rangle.
\end{aligned}
\tag{2.14}$$

For a free field with Hamiltonian (2.9) all the irreducible correlators except the pair correlator  $G(x, y) = K(x, y)$  are equal to zero; in particular,  $\langle \varphi(x) \rangle = 0$ . The replacement of two factors  $\varphi(x)$  and  $\varphi(y)$  by their average  $K(x, y)$  is called pairing. Wick's theorem (2.10) states that the average of a product of free fields is a sum over all possible pairings. In the following we shall need the concept of the normal product:  $\varphi(x_1)\varphi(x_2)\dots\varphi(x_n)$ : of free fields. The normal product is obtained from an ordinary product by subtracting all possible pairings; e. g.,

$$\begin{aligned}
:\varphi(x)\varphi(y): &= \varphi(x)\varphi(y) - K(x, y), \\
:\varphi(x)\varphi(y)\varphi(z): &= \varphi(x)\varphi(y)\varphi(z) - K(x, y)\varphi(z) \\
&\quad - K(y, z)\varphi(x) - K(z, x)\varphi(y).
\end{aligned}
\tag{2.15}$$

The normal product  $:\varphi(x_1)\dots\varphi(x_n):$  with  $x_1 = x_2 = \dots = x_n = x$  is the quantity  $:\varphi^n(x):$ , which has the properties of a power of a fluctuating field. The quantities

$$A_n(x) = :\varphi^n(x): \tag{2.16}$$

form a complete set in terms of which any function of the field  $\varphi(x)$  can be expanded. For  $\tau = 0$  the quantities  $A_n(x)$  satisfy the scaling properties (1.1) with scaling dimensions  $\Delta_n^0$ :

$$\Delta_n^0 = n\Delta_\varphi^0, \quad \Delta_\varphi^0 = \frac{d-2}{2}. \tag{2.17}$$

A more detailed account of the properties of free fields is given in the book.<sup>[2]</sup> For an interacting field the irreducible correlators  $G(x_1 \dots x_n)$  do not vanish for  $n > 2$ , and formulas (2.10)–(2.17) are not fulfilled. The magnitude of the anomalous dimension

$$\eta \equiv 2(\Delta_\varphi - \Delta_\varphi^0) \tag{2.18}$$

has been found to be small for three-dimensional systems ( $d = 3$ ):  $\eta \lesssim 1/20$ . As was known from investigations in quantum field theory (Landau, Abrikosov, and Khalatnikov, and Sudakov<sup>[24]</sup>), for  $d = 4$  there is a solution with an interaction that vanishes logarithmically at large distances. For an arbitrary  $n$ -component field  $\varphi$  with interaction  $g_0(\varphi^2)^2$  the same result was obtained by Larkin and Khmel'nitskiĭ.<sup>[25]</sup> Wilson and Fisher<sup>[10]</sup> considered the problem in a space of  $4 - \varepsilon$  dimensions. In this case the difference from the free-field theory is small for  $\varepsilon \ll 1$  and a calculation by perturbation theory (the  $\varepsilon$ -expansion) is possible. For small  $\varepsilon$  the interaction at large distances is small, and, therefore, the index  $\eta$  is small. The fact that the experimental value of  $\eta$  is small gives us grounds to expect that the expansion in powers of  $\varepsilon$  is still sufficiently good when  $\varepsilon = 1$ . To a certain degree, the results of the calculations have justified these hopes.

Undoubtedly, each of the methods—the field-theoretical and the renormalization-group—has its merits and its deficiencies. The renormalization-group method has

turned out to be the more constructive and the more convenient for calculations. True, by invoking the idea of the  $\varepsilon$ -expansion, Tsuneto and Abrahams<sup>[26]</sup> have calculated indices in the framework of the field-theoretical approach. But we cannot yet see any possibility of improving the accuracy of calculations that use field-theory series.

We proceed to our account of the renormalization-group method.

### 3. THE SEMIGROUP OF RENORMALIZATIONS

We shall consider a fluctuating field  $\varphi(x)$ . This field is smoothed. In the Fourier expansion

$$\varphi(x) = \int \varphi_q e^{iqx} \frac{d^d q}{(2\pi)^d} \tag{3.1}$$

the quantity  $\varphi_q$  is equal to zero for  $q \geq q_0$ . We write the probability  $W$  of the realization  $\varphi(x)$  of this field in the form

$$W = A_0 \exp[-\mathcal{E}(\varphi, q_0)], \tag{3.2}$$

where the functional  $\mathcal{E}(\varphi, q_0)$  plays the role of the Hamiltonian of the fluctuating field. The Hamiltonian  $\mathcal{E}(\varphi, q_0)$  depends on the cutoff parameter  $q_0$ . In order to study this dependence we shall sum the probability (3.2) over all possible values of  $\varphi_q$  for  $q$  lying in the interval  $\lambda q < q < q_0$  ( $0 < \lambda < 1$ ). We represent the result in the form

$$W_\lambda = A_\lambda \exp[-\mathcal{E}(\varphi, \lambda q_0)] = A_0 \int \exp[-\mathcal{E}(\varphi, q_0)] \prod_{q=\lambda q_0}^{q_0} d\varphi_q. \tag{3.3}$$

If  $\mathcal{E}(\varphi, q_0)$  is known, the definition (3.3) enables us to find  $\mathcal{E}(\varphi, \lambda q_0)$  for all  $\lambda$ . We shall call the transformation from  $\mathcal{E}(q)$  to  $\mathcal{E}(\lambda q)$  a smoothing transformation, and denote it by  $S(\lambda)$ :

$$\mathcal{E}(\varphi, \lambda q) = S(\lambda) \mathcal{E}(\varphi, q). \tag{3.4}$$

The smoothing operation is nonlinear. Obviously, successive application of smoothing operations gives

$$S(\lambda_2) S(\lambda_1) = S(\lambda_1 \lambda_2).$$

The set of operations  $S(\lambda)$  forms what in mathematics is called a semigroup.

We also carry out a scale transformation of the quantities  $q$ :

$$q \rightarrow q' = \lambda^{-1} q$$

and, simultaneously, a transformation that changes the normalization of the fields:

$$\varphi_q \rightarrow \varphi'_q = Z(\lambda) \varphi_{\lambda q}. \tag{3.5}$$

We denote the combination of these two transformations by  $D(\lambda)$ . The transformation  $D$  also changes the Hamiltonian  $\mathcal{E}$ . This change reduces to the replacement of the wave numbers  $q \rightarrow q'$ , and to a change of all lengths by a factor  $\lambda$  and of all fields by a factor  $Z$ :

$$D(\lambda) \mathcal{H}(\varphi_q, k) = \mathcal{H}(Z^{-1}\varphi_{\lambda^{-1}q}, \lambda^{-1}k) = \mathcal{H}'(\varphi'_q, \lambda^{-1}k). \quad (3.6)$$

We shall call the successive application of the operations  $S(\lambda)$  and  $D(\lambda)$  a renormalization operation:

$$R(\lambda) = D(\lambda) S(\lambda). \quad (3.7)$$

The smoothing operation  $S(\lambda)$  decreases the region in which the field  $\varphi_q$  is defined, from the interval  $0 < q < q_0$  to  $0 < q < \lambda q_0$ . The scale transformation  $D(\lambda)$  expands the reduced interval to its previous size. Thus, the transformed Hamiltonian

$$R(\lambda) \mathcal{H}(\varphi_q, q_0) = \mathcal{H}'(\varphi'_q, q_0) \quad (3.8)$$

describes the fluctuations of a new field  $\varphi'_q$  in an unchanged range of scales. The set of operations  $R(\lambda)$  also forms a semigroup ( $\mathbf{R}$ ). For our purposes the difference between a semigroup and a group is unimportant. Therefore, following established tradition, we shall call  $\mathbf{R}$  the renormalization group.

Starting from a certain initial Hamiltonian  $H(\varphi, q_0)$  and applying the operations  $R(\lambda_1)$ ,  $R(\lambda_2)$ , ... repeatedly to it, we obtain a sequence of Hamiltonians  $H'_1, H'_2, \dots$ . The question of the limit properties of this sequence naturally arises.

If the system is not at a critical point, then, by successively applying the renormalization operation, we reach the fairly large length-scales (small  $\lambda q_0$ ) that lie in the domain of the thermodynamic theory of fluctuations. In this domain the distribution of the thermodynamic quantities is Gaussian. The Hamiltonian  $\mathcal{H}_k$  has the form (cf. [1, 2])

$$\mathcal{H}_k = \frac{1}{2} \int_{q < k} (\chi^{-1} + cq^2) |\varphi_q|^2 \frac{dq}{(2\pi)^d}. \quad (3.9)$$

The quantity  $\chi$  is the susceptibility of the system:  $\chi = \partial \langle \varphi \rangle / \partial h$ , where  $h$  is the field conjugate to  $\varphi$ . We first carry out the smoothing operation  $S(\lambda)$  on the Hamiltonian (3.9). Since  $\varphi_q$  with different values of  $q$  do not interact, the smoothing operation in this case reduces to discarding terms with  $q$  in the interval from  $\lambda k$  to  $k$  in (3.9):

$$S(\lambda) \mathcal{H}_k = \frac{1}{2} \int_{q < \lambda k} (\chi^{-1} + cq^2) |\varphi_q|^2 \frac{dq}{(2\pi)^d}. \quad (3.10)$$

We now carry out the scale transformation  $D(\lambda)$  on (3.10):

$$R(\lambda) \mathcal{H}_k = D(\lambda) S(\lambda) \mathcal{H}_k = \frac{\lambda^{dZ-2}}{2} \int_{q < k} (\chi^{-1} + c^2 \lambda^2 q^2) |\varphi_q|^2 \frac{dq}{(2\pi)^d}. \quad (3.11)$$

We draw attention to the fact that the renormalization transformation does not affect the thermodynamic variables (e.g.,  $\tau$  and  $h$  or  $\chi$  and  $c$ ). It differs, therefore, from the scale transformations (2.1). In the case of the scale transformation, the thermodynamic quantities change in such a way that the Hamiltonian  $\mathcal{H}$  remains invariant:

$$\chi \rightarrow \chi' = \chi \lambda^{-dZ^2}, \quad c \rightarrow c' = c \lambda^{d+2Z-2}.$$

We return to formula (3.11). As  $\lambda \rightarrow 0$  the transformed Hamiltonian tends to a finite limit

$$\mathcal{H}^* = \frac{1}{2} \int_{q < k} \chi^{-1} |\varphi_q|^2 \frac{dq}{(2\pi)^d}, \quad (3.12)$$

if  $Z = \lambda^{d/2}$ . For a degenerate system below the transition point, the quantity  $\chi^{-1}$  goes to zero for  $h = 0$ . [2] In this case the finite limit of  $\mathcal{H}_k$  is reached with the condition  $Z = \lambda^{(d+2)/2}$ .

At the critical point the domain of thermodynamic fluctuations is absent. Therefore, the question of the limiting behavior cannot be solved so simply. The hypothesis that there exists a limiting Hamiltonian  $\mathcal{H}^*$ , describing a nongaussian probability distribution, is the scaling hypothesis.

It is convenient to seek the limit  $\mathcal{H}^*$  by considering an infinitesimal renormalization transformation  $R(1 - \zeta)$  ( $\zeta$  is an infinitesimal quantity):

$$R(1 - \zeta) \mathcal{H}(\varphi, \lambda q_0) = \mathcal{H} + \zeta f\{\mathcal{H}\}. \quad (3.13)$$

Equation (3.13) can be written in the form

$$\frac{\partial \mathcal{H}}{\partial \ln \lambda} = -f\{\mathcal{H}\}. \quad (3.14)$$

The renormalization procedure  $R(1 - \zeta)$  depends only on the size of  $\zeta$ . In Eq. (3.13) the quantity  $\lambda$  enters only through the dependence of the Hamiltonian  $\mathcal{H}(\lambda)$ . Therefore, the functional  $f\{\mathcal{H}\}$  that has appeared in the differential equation (3.14) does not depend on  $\lambda$ . It is convenient to change to the variable  $\xi = -\ln \lambda$ , in view of which we introduce the new notation

$$\mathcal{H}(\varphi, \lambda q_0) \equiv \mathcal{H}(\varphi, \xi).$$

In the new notation Eq. (3.14) takes the form

$$\frac{\partial \mathcal{H}}{\partial \xi} = f\{\mathcal{H}\}. \quad (3.15)$$

The simplicity of Eq. (3.15) is only apparent. In reality it is a system of nonlinear equations for the coefficients determining the Hamiltonian  $\mathcal{H}$ . We shall confine ourselves to the class of Hamiltonians  $\mathcal{H}$  which can be represented in the form of series in powers of the quantity  $\varphi$  and of its derivatives. The coefficients in these series obey Eqs. (3.15). The simplest form of limiting behavior of the solutions of Eq. (3.15) is associated with the possible vanishing of the functional  $f\{\mathcal{H}\}$ . A "root" of the equation

$$f\{\mathcal{H}^*\} = 0 \quad (3.16)$$

is a fixed point of the renormalization transformation  $R(\lambda)$ .

For nonlinear systems other types of limiting behavior, e.g., limit cycles (cf. [2, 7]), are also known. We do not see any reason to investigate such limiting behaviors at the present time; we shall concentrate on the study of fixed points.

The general technique for applying renormalization

transformations in the theory of phase transitions is as follows. The starting point is the Landau Hamiltonian

$$\mathcal{H}_L = \frac{1}{2} \int [\tau_0 \varphi^2 + (\nabla \varphi)^2 + g_0 \varphi^4] dx. \quad (3.17)$$

The parameters  $\tau_0$  and  $g_0$  are assumed to be regular functions of the thermodynamic variables—temperature, pressure, etc. For a certain special choice  $\tau_0^*$ ,  $g_0^*$  of these parameters, successive application of the renormalization transformations leads to a nongaussian limiting Hamiltonian  $\mathcal{H}^*$ . It is physically obvious that this point is unstable. For small deviations of  $\tau_0$  and  $g_0$  from their critical values,  $\mathcal{H}(\xi)$  will be found to be rather close to  $\mathcal{H}^*$  for a certain value of  $\xi$  but will tend to the Gaussian fixed point on further increase of  $\xi$ . Renormalization-group equations were found by Gell-Mann and Low<sup>[20]</sup> for the case of quantum electrodynamics. The renormalization group in quantum field theory was investigated in detail by Bogolyubov and Shirkov.<sup>[28]</sup> The application of renormalization-group ideas to the theory of phase transitions is due to Wilson,<sup>[9]</sup> who developed approximate methods of calculation. The idea that a nongaussian fixed point is possible was expressed by Gell-Mann and Low.<sup>[20]</sup> A general analysis of the renormalization-group equations is given in papers by Wegner<sup>[29]</sup> and Wilson and Kogut.<sup>[13]</sup>

#### 4. PROPERTIES OF SOLUTIONS NEAR A FIXED POINT

It was shown in the preceding section that the probability distribution at a phase-transition point is described by a limiting Hamiltonian  $\mathcal{H}^*$ , which is a fixed point of the nonlinear transformation  $R$ . We shall examine the properties of the solutions of Eqs. (3.15) near the point  $\mathcal{H}^*$ .

As before, we consider a set of Hamiltonians  $\mathcal{H}(\varphi)$  that can be represented in the form of series in powers of the quantity  $\varphi$  and of its derivatives:

$$\mathcal{H}(\varphi) = \sum_n \int \dots \int d\mathbf{q}_1 \dots d\mathbf{q}_n g_n(\mathbf{q}_1, \dots, \mathbf{q}_n) \delta \left( \sum_{i=1}^n \mathbf{q}_i \right) \varphi_{\mathbf{q}_1} \dots \varphi_{\mathbf{q}_n}. \quad (4.1)$$

The Hamiltonian  $\mathcal{H}$  is completely defined by the set of quantities  $g_n$  and is a vector (a point)  $\mathbf{g}$  in the infinite-dimensional space of the coefficients  $g_n$ . On change of the normalization of the fields ( $\varphi \rightarrow \varphi' = Z\varphi$ ) the Hamiltonian (4.1) changes—a transformation of the quantities  $g_n$  occurs:  $g_n \rightarrow g'_n = Z^{-n} g_n$ . Therefore, Hamiltonians corresponding to the vectors  $g_n(Z) = Z^{-n} g_n$  must be regarded as coinciding. In order to remove this arbitrariness it is sufficient to require that  $g_n = 1$  for some value of  $n$ . We shall assume that the coefficient of  $\frac{1}{2} \int k^2 \varphi_{\mathbf{k}} \varphi_{-\mathbf{k}} d\mathbf{k}$  in (4.1) is equal to unity. With this normalization the quantities  $g_n(\xi)$  are called the invariant charges.

The equations (3.15) are equations of motion in the space of the  $g_n$ . The quantity  $\xi$  plays the role of the time. The functional that appears in (3.15) is, in fact, a vector function of the point  $\mathbf{g}$ . We shall denote this function by the same letter  $\mathbf{f}(\mathbf{g})$ . In this notation Eqs. (3.15) take the form

$$\frac{d\mathbf{g}}{d\xi} = \mathbf{f}(\mathbf{g}). \quad (4.2)$$

Let  $\mathcal{H}$  be close to  $\mathcal{H}^*$ . This means<sup>1)</sup> that the corresponding point is close to the point  $\mathbf{g}^*$ :

$$\mathbf{g} = \mathbf{g}^* + \boldsymbol{\gamma}. \quad (4.3)$$

The small vector  $\boldsymbol{\gamma}$  obeys the system of linear equations

$$\frac{d\boldsymbol{\gamma}}{d\xi} = K\boldsymbol{\gamma}, \quad (4.4)$$

where the linear operator  $K$  is related to  $\mathbf{f}(\mathbf{g})$  by

$$K = \left. \frac{\partial \mathbf{f}}{\partial \mathbf{g}} \right|_{\mathbf{g}=\mathbf{g}^*}. \quad (4.5)$$

We shall denote the eigenvalues of the operator  $K$  by  $\Delta_i$  and the corresponding eigenvectors by  $\boldsymbol{\gamma}_i$ :

$$K\boldsymbol{\gamma}_i = \Delta_i \boldsymbol{\gamma}_i. \quad (4.6)$$

The general solution of Eq. (4.4) has the form

$$\boldsymbol{\gamma} = \sum_i h_i \boldsymbol{\gamma}_i, \quad h_i = h_{i0} e^{\Delta_i \xi}, \quad (4.7)$$

where the  $h_{i0}$  are arbitrary constants.<sup>2)</sup> We ignore the exotic possibility, which arises for complex values of  $\Delta_i$ , of an oscillatory dependence of the coefficients of the Hamiltonian on the degree of closeness to the critical point—we shall assume that all the  $\Delta_i$  are real.

We note that, in place of Eq. (4.4) for the eigenvectors, we can write analogous equations for the conjugate fields:

$$\frac{dh_i}{d\xi} = \Delta_i h_i. \quad (4.8)$$

Equations (4.8), like Eqs. (4.4), are exact to within terms quadratic in  $h$ .

We have arrived at the scale transformation of the fields  $h_i$  (cf. Sec. 2). Again we emphasize the close connection and the difference between the renormalization-group transformations and scale transformations of the fields  $h_i$ . The former change the Hamiltonian and the latter compensate these changes in such a way that the Hamiltonian remains unchanged. We can assume that under the renormalizations the fields  $h_i$  transform according to the law  $h_i = h_{i0} e^{-\Delta_i \xi}$ .

Those vectors  $\boldsymbol{\gamma}_i$  whose eigenvalues are positive are of special importance. As  $\xi$  increases, deviations associated with these vectors grow and, in the end, lead the vector  $\mathbf{g}$  away from the fixed point. Thus, the quantities  $h_i$  play the role of external fields. The eigenvalues  $\Delta_i$  are the scaling dimensions of these fields. The corresponding vectors  $\boldsymbol{\gamma}_i$  define the fluctuating quantities  $A_i$ . We draw attention to the direct correspondence

<sup>1)</sup>The renormalization-group equations near  $\mathcal{H}^*$  were considered in<sup>[29]</sup> by Wegner.

<sup>2)</sup>We have assumed that the operator  $K$  can be brought to diagonal form. In the general case it can be represented in a Jordan form. In this case, in addition to the exponentials  $e^{\Delta_i \xi}$ , terms of the form  $\xi^k e^{\Delta_i \xi}$  appear in the solutions.



between the theory described here and the hypothesis of an algebra of fluctuating quantities (cf. the Introduction). Namely, according to the algebra hypothesis, the quantity  $\gamma_i$  in the coordinate representation can be represented as the integral of  $A_i(\mathbf{x})$ :

$$\gamma_i = \int A_i(\mathbf{x}) d\mathbf{x}.$$

If  $\Delta_i > 0$ , switching on the field  $h_i$  takes the system away from the critical point. This property is possessed by the magnetic field (in a ferromagnet) and by the temperature. The thermodynamic quantities conjugate to these are called strongly fluctuating quantities. The dimensions  $\Delta_{A_i}$  of these quantities are smaller than the spatial dimensionality  $d$ .

Fields  $h_i$  with dimensions  $\Delta_i < 0$  are conjugate to weakly fluctuating quantities. If the initial values  $h_{i0}$  are such that all the  $h_{i0}$  pertaining to strongly fluctuating quantities are equal to zero, the solution  $\gamma$  tends to zero as  $\xi \rightarrow +\infty$ .

It is obvious that the positive eigenvalues  $\Delta_i$  have an upper bound. Otherwise, the solution (4.7) becomes meaningless. This implies that, amongst the strongly fluctuating quantities, one fluctuates more strongly than the others. Usually, this quantity is associated with the order parameter.

We shall examine Eqs. (4.8) with inclusion of terms quadratic in  $h_i$ :

$$\frac{dh_i}{d\xi} = \Delta_i h_i - \sum_{k,l} a_{ikl} h_k h_l.$$

We introduce the refined quantity  $h_i^{(1)}$ , related to  $h_i$  and the other fields by

$$h_i^{(1)} = h_i + \sum_{k,l} \frac{a_{ikl} h_k h_l}{\Delta_i - \Delta_k - \Delta_l}.$$

Then, in the equation for  $h_i^{(1)}$ , the quadratic terms disappear. It is obvious that further refinement of the quantities  $h_i$  will give the possibility of getting rid of the cubic terms, and so on. Elimination of the nonlinear terms in the renormalization-group equations is impossible only in the case of a "resonance," when  $\Delta_i = \Delta_k + \Delta_l$ . The simplest resonance situation arises in the case when one of the fields  $h_0$  has eigenvalue zero. In this case, of course, of the quadratic terms only the resonance terms need (of necessity) be kept. Taking this fact into account, we write out the renormalization-group equations for this case:

$$\frac{dh_0}{d\xi} = -b h_0^2, \quad (4.9)$$

$$\frac{dh_i}{d\xi} = \Delta_i h_i + a_i h_i h_0. \quad (4.10)$$

The solution of this system of equations has the form

$$h_0 = \frac{h_{00}}{1 + b h_{00} \xi}, \quad (4.11)$$

$$h_i = h_{i0} e^{\Delta_i \xi} \left( \frac{h_0}{h_{00}} \right)^{a_i/b}. \quad (4.12)$$

Thus, if  $b \neq 0$  the dependence of the fields  $h_i$  and  $h_0$  on

the scale becomes more complicated; logarithmic corrections appear. If, however, we ignore these slowly varying corrections, the fundamental dimensions  $\Delta_i$  remain unchanged. For  $b > 0$  the solution (4.11), (4.12) is valid in the entire region and describes the approach to the fixed point. In the case  $b < 0$  the fixed point is unstable: it can be seen from Eq. (4.9) that in this case small deviations grow. However, Eqs. (4.9) and (4.10) and their solutions (4.11) and (4.12), which were derived only under the assumption that  $h_0$  is small, remain correct right up to values of  $\xi$  close to the pole  $\xi_0 = -(b h_{00})^{-1}$  of the function  $h_0(\xi)$ . Namely, it is required that  $(\xi - \xi_0)^{-1} b^{-1}$  be a large quantity. For  $b = 0$  it follows from Eqs. (4.9) and (4.10) that

$$h_0 = h_{00} = \text{const}, \quad (4.13)$$

$$h_i = h_{i0} e^{\tilde{\Delta}_i \xi}, \quad \tilde{\Delta}_i = \Delta_i + a_i h_{00}. \quad (4.14)$$

In this approximation the scaling dimensions  $\tilde{\Delta}_i$  depend continuously on the quantity  $h_0$ .

We can now make precise the condition for the indices to vary continuously. For such a dependence it is necessary and sufficient that  $h_0$  not depend on  $\xi$  in any (power) order.

## 5. RENORMALIZABLE AND ALMOST RENORMALIZABLE HAMILTONIANS

We imagine that the point  $\mathbf{g}$  is moving along one of the paths that lead to the fixed point  $\mathbf{g}^*$ . This means that it is possible to determine corrections to the vectors  $\gamma_i$  with negative dimensions  $\Delta_i$  such that, under the renormalization, in the equations for  $h_i$  terms with positive dimensions do not appear in any order in the small deviations. In the case when one of the eigenvalues is equal to zero, as can be seen from (4.12), all the  $h_i$  ( $i \neq 0$ ) become small if the condition  $\xi \gg \ln h_{i0}/\Delta_i$  is fulfilled. On the other hand, as follows from (4.11),  $h_0$  changes substantially over the range of values  $\xi \sim h_{00}^{-1}$ . For sufficiently small  $h_{00}$  there exists a range of variation of  $\xi$ :

$$\Delta_i^{-1} \ll \delta \xi \ll h_{00}^{-1}, \quad (5.1)$$

in which we can neglect all the fields except  $h_0$  and assume that the change of  $h_0$  is quite small. Here  $\Delta_i$  is the greatest negative dimension. In this case, in the equations for  $h_0$  and  $h_i$  it is legitimate to keep terms of any power order in  $h_0$ . Therefore, in the case under consideration the system of equations of motion (3.15) takes the form<sup>3)</sup>

$$\frac{dh_0}{d\xi} = f(h_0), \quad (5.2)$$

$$\frac{d \ln h_i}{d\xi} = \Delta_i(h_0). \quad (5.3)$$

These equations have already been written out, up to the quadratic terms, in Sec. 4 ((4.9), (4.10)). In deriving (5.2) and (5.3) we neglected terms of order  $h_i \sim e^{\Delta_i \xi}$  for  $\xi \sim h_0^{-1}$ . In other words, Eqs. (5.2) and (5.3) are valid

<sup>3)</sup>Equations of this form were obtained by Gell-Mann and Low in quantum electrodynamics.<sup>[20]</sup>



with relative accuracy  $e^{-\Delta_1/h_0}$ . The restriction arising from (5.1) on the value of  $h_0$ :

$$h_0 \ll \Delta_1^{-1} \sim 1,$$

does not mean, generally speaking, that we can confine ourselves to the first term in Eqs. (5.2) and (5.3). The function  $f(x)$  could, for example, have an additional zero at a small value  $x = x_0$ . Then an additional fixed point appears, with  $h_0 = x_0$ , and  $\Delta_i = \Delta_i(x_0)$ . Equations (5.3) are also true for fields with positive dimensions. In this case we need take into account only the resonance terms in any order in  $h_0$ , since the others are small or are eliminated in the refinement of the quantity  $h_i$ .

Hamiltonians for which one or several eigenvalues vanish are customarily called renormalizable.<sup>4)</sup> In the case when  $m$  vectors correspond to eigenvalue zero, the behavior of the corresponding fields is described by a system of coupled equations, of the form

$$\frac{dh_i}{d\xi} = f_i(h_1, \dots, h_m) \quad (i = 1, 2, \dots, m). \quad (5.4)$$

For the remaining fields  $h_k$  ( $k = m + 1, m + 2, \dots$ ) the renormalization-group equations take the form

$$\frac{d \ln h_k}{d\xi} = \Delta_k(h_1, \dots, h_m). \quad (5.5)$$

In the simplest case, when in the expansion of the functions  $f_i$  we can confine ourselves to terms quadratic in the quantities  $h_1, \dots, h_m$ , Eqs. (5.4) become homogeneous. Therefore, they can be reduced to a system of  $m - 1$  equations for the ratios  $h_i/h_1$ . An investigation of systems of this kind is given in Chap. 9.

The ideas used in the study of renormalizable Hamiltonians also turn out to be useful in the case when one or several eigenvalues  $\Delta_i$  are small compared with unity (the case of an almost renormalizable Hamiltonian). In this case, just as in renormalizable theories, the renormalization-group equations near the fixed point are divided into the two groups (5.4) and (5.5). The fields  $h_1, \dots, h_m$  with small dimensions appear in the first group, and all the other fields in the second. The characteristic length  $\xi_0$  over which the fields  $h_1, \dots, h_m$  vary is equal to  $\Delta^{-1}$  in order of magnitude, where  $\Delta$  is the smallest of the eigenvalues. In deriving Eqs. (5.4) and (5.5) (or (5.2) and (5.3)) we neglected terms of order  $e^{\Delta_k \xi}$ , where the  $\Delta_k$  are the eigenvalues that are not small. This means that the equations (5.4) and (5.5) for a renormalizable Hamiltonian are valid with exponential relative accuracy  $e^{-|\Delta_k/\Delta|}$  in the case under consideration.

We shall consider in more detail an almost renormalizable Hamiltonian with one small eigenvalue  $\Delta_0$ . In this case, in Eq. (5.2) it is necessary to keep not only the terms linear in  $h_0$  but also terms of higher order. In the simplest case we confine ourselves to the linear

<sup>4)</sup>Sometimes, Hamiltonians for which the number of nonpositive eigenvalues is finite are called renormalizable. Inasmuch as we consider only such Hamiltonians in the following, for us the definition adopted above is more convenient.

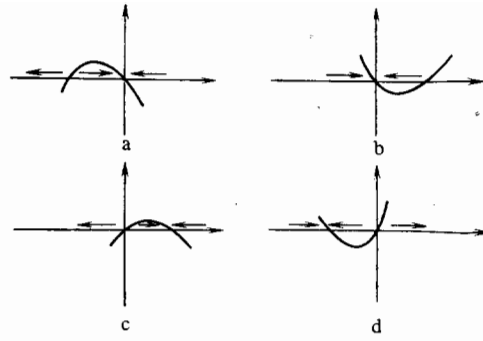


FIG. 1.

and quadratic terms:

$$\frac{dh_0}{d\xi} = \Delta_0 h_0 - b h_0^2. \quad (5.6)$$

Graphs of the right-hand side of Eq. (5.6) in different possible situations are shown in Fig. 1. In all cases a new fixed point  $h_0^* = \Delta_0/b$  arises. In cases c) and d), as can be seen from the figure, it is this fixed point which is the stable one. In the other cases the stable fixed point is the point  $h^* = 0$ . In the cases c) and d) the dimensions  $\Delta_i$  are slightly changed relative to their values at the original fixed point. By means of Eqs. (4.10) it is not difficult to relate the new values to the old values and  $h_0^*$ :

$$\tilde{\Delta}_i = \Delta_i + a \frac{\Delta_0}{b}, \quad \tilde{\Delta}_0 = -\Delta_0. \quad (5.7)$$

Where necessary it is possible to refine the values of  $h^*$  and  $\tilde{\Delta}_i$  by using the subsequent terms of the expansions of Eqs. (5.6) and (4.10) in powers of  $h$ .

## 6. VICINITY OF THE GAUSSIAN FIXED POINT

The free-field Hamiltonian

$$\mathcal{H}_0 = \frac{1}{2} \int q^2 \varphi_{\mathbf{q}} \varphi_{-\mathbf{q}} \frac{d\mathbf{q}}{(2\pi)^d} = \frac{1}{2} \int (\nabla \varphi)^2 dx \quad (6.1)$$

is a fixed point of the renormalization-group transformations for any spatial dimensionality  $d$ . We shall study now how almost free Hamiltonians behave under renormalization transformations. An extra term of the form

$$\mathcal{H}_\tau = \frac{1}{2} \tau \int \varphi^2 dx = \frac{1}{2} \tau \int |\varphi_{\mathbf{q}}|^2 \frac{d\mathbf{q}}{(2\pi)^d} \quad (6.2)$$

leaves the Hamiltonian Gaussian, but, as already pointed out, takes it away from the fixed point (6.1). We shall find the dimension of  $\tau$  in this case. The smoothing does not change the Gaussian Hamiltonian  $\mathcal{H}_0 + \mathcal{H}_\tau$ . In order to preserve the form of (6.1) under a scale transformation  $\mathbf{q} = \lambda \mathbf{q}'$  it is necessary to perform also a transformation of the field:  $\varphi_{\mathbf{q}} = Z^{-1} \varphi'_{\mathbf{q}'}$ , with  $Z = \lambda^{(d+2)/2}$ . The quantity  $\tau$  then transforms according to the law

$$\tau \rightarrow \tau' = \lambda^{-2} \tau = e^{2\xi} \tau. \quad (6.3)$$

Thus, the positive eigenvalue  $\Delta_\tau^0 = 2$  corresponds to the field  $\tau$ . We shall calculate the dimension of the field  $h$  associated with a perturbation of the Hamiltonian (6.1) of the form

$$\mathcal{H} = -h \int \varphi(x) dx. \quad (6.4)$$

Analogous calculations give  $\Delta_h^0 = (d+2)/2$  for the dimension of  $h$ . The role of the eigenvectors  $\gamma_h$  is played by the quantities  $\int : \varphi^k(\mathbf{x}) : d\mathbf{x}$ ; here  $: \varphi^k(\mathbf{x}) :$  signifies the normal product  $: \varphi(\mathbf{x}_1) \varphi(\mathbf{x}_2) \cdots \varphi(\mathbf{x}_k) :$  for  $\mathbf{x}_1 = \mathbf{x}_2 = \cdots = \mathbf{x}_k = \mathbf{x}$ . The normal product was defined in Sec. 2 (cf. formulas (2.15), (2.16)). (For more detail, cf., e.g., [2].) The fields  $h_k$  corresponding to these have dimensions

$$\Delta_k = d - \frac{k(d-2)}{2}. \quad (6.5)$$

In particular, for  $k=1$  we obtain the dimension  $\Delta_h^0 = \Delta_1^0$ . The condition for the existence of an operator  $h_k$  with dimension equal to zero has the form

$$d = \frac{2k}{k-2} = 2 + \frac{4}{k-2}, \quad (6.6)$$

where  $k$  is any integer greater than 2.

Integer values of  $d$  are obtained only for  $k=3, 4, 6$ . The corresponding values of  $d$  are  $d=6, 4, 3$ . In three-dimensional space the quantity  $h_6$  has dimension zero. In this case the quantities  $h_1, h_2, h_3, h_4$  and  $h_5$  have positive dimensions. Even if, by virtue of the symmetry of the system,  $h_3$  and  $h_5$  vanish automatically together with  $h_1 = h$ , the condition  $h_1 = h_2 = h_4 = 0$  can be fulfilled only when the number of thermodynamic variables is not less than three. In particular, this condition cannot be fulfilled for a one-component system. In a two-component system such a point is called a tricritical point.

For  $d=4$  the quantity  $h_4$  has dimension zero and in this case the quantities  $h_1, h_2$  and  $h_3$  possess positive dimensions. In the general case, by a simple transformation  $\varphi \rightarrow \varphi + \text{const}$  we can make either  $h_1$  or  $h_3$  go to zero. Therefore, the fixed point is specified by two equations, e.g.,  $h_1 = 0, h_2 = 0$ . For a one-component system this corresponds to a critical point.

Formally we can consider a space with dimensionality  $d = 4 - \varepsilon$  close to 4 ( $\varepsilon \ll 1$ ). In this case the dimension  $\Delta_4$  of the quantity  $h_4$  is no longer equal to zero, but is small:  $\Delta_4^{(0)} = \varepsilon$ . Therefore, to within quantities of order  $e^{-\text{const}/\varepsilon}$ , we can use the equations (5.2), (5.3) of the renormalizable theory.

Thus, we shall consider a Landau Hamiltonian  $\mathcal{H}_L$  in a four-dimensional or almost four-dimensional space. When  $h_1 = h = 0, h_2 = \tau = 0$  and  $h_4 = g = 0$  this Hamiltonian is transformed into a Gaussian Hamiltonian, which is a fixed point of the renormalization transformations. In order to find the behavior of  $\mathcal{H}_L$  for small  $g \neq 0$  we calculate the coefficients of the renormalization-group equations (5.2), (5.3). This calculation is elementary but fairly cumbersome; therefore, we shall explain the routine of the calculations. To apply Eqs. (5.2) and (5.3) it is necessary to consider a finite renormalization-group transformation, with  $\delta\xi \gg 1$  in order to satisfy the requirement (5.1). Here we make use of perturbation theory in the quantity  $g_0 = h_{00} \ll 1$ . We represent the expression for  $g(\xi, g_0)$  in the form of a series in powers of  $g_0$ . We then differentiate this series with respect to  $\xi$ . We thereby find the function  $f(g)$  in the form

of a series in the quantity  $g_0$ . In lowest order we retain in  $g$  only terms linear and quadratic in  $g_0$ . We obtain

$$f(g) = \varepsilon g_0 - b g_0^2. \quad (6.7)$$

For  $g_0 \rightarrow 0$ , for any  $\xi$ , the quantities  $g$  and  $g_0$  coincide; therefore, in the same approximation we can replace  $g_0$  by  $g$  in  $f(g)$ , so that

$$f(g) = \varepsilon g - b g^2. \quad (6.8)$$

This replacement is legitimate because, as we have proved, the function  $f(g)$  depends only on  $g$ . Then, in the next approximation, it is necessary to find  $g(\xi, g_0)$  to terms of order  $g_0^3$ . This makes it possible to find the function  $f(g)$  to third order in  $g$  by first eliminating the terms of third order in  $g_0$  in  $f(g) = \varepsilon g_0 - b g_0^2 + c g_0^3$ , and so on. The nontrivial fact that the dependence on  $\xi$  and  $g_0$  disappears as a result of successive application of this procedure is guaranteed by the general theory.

The postulate of the existence of a nongaussian fixed point close to the Gaussian one was first enunciated by A. A. Migdal. [30]

## 7. PERTURBATION THEORY AND GRAPHS

We proceed to concrete calculations. We subject the Landau Hamiltonian  $\mathcal{H}_L$  to the smoothing operation  $S(\lambda)$  with  $\lambda \ll 1$ . According to the definition of the smoothing operation  $S(\lambda)$  we must calculate the quantity

$$\exp(-S(\lambda) \mathcal{H}_L) = \text{const} \cdot \int e^{-\mathcal{H}_L} \prod_{\lambda q_0 < q < q_0} d\varphi_q. \quad (7.1)$$

We introduce notation for the operation of smoothing of an arbitrary functional  $A$  with the free-field Hamiltonian  $\mathcal{H}_0$ :

$$(A)_S = \int e^{-\mathcal{H}_0} A \prod_{\lambda q_0 < q < q_0} d\varphi_q / \int e^{-\mathcal{H}_0} \prod_{\lambda q_0 < q < q_0} d\varphi_q. \quad (7.2)$$

In this notation we obtain

$$\exp(-S(\lambda) \mathcal{H}_L) = \exp(-S(\lambda) \mathcal{H}_0) (e^{-\mathcal{H}_{\text{int}}})_S, \quad (7.3)$$

where  $\mathcal{H}_{\text{int}}$  is the interaction Hamiltonian:

$$\mathcal{H}_{\text{int}} = g_0 \int \varphi^4(x) dx. \quad (7.4)$$

We expand the exponential  $e^{-\mathcal{H}_{\text{int}}}$  in a series in powers of  $g_0$ . The general term of the series has the form

$$(-1)^n \frac{g_0^n}{n!} \left\langle \left( \int \varphi^4(x) dx \right)^n \right\rangle_S. \quad (7.5)$$

To calculate the expression (7.5) we represent  $\varphi(\mathbf{x})$  in the form of a sum:

$$\varphi = \varphi_0(\mathbf{x}) + \varphi_1(\mathbf{x}), \quad (7.6)$$

where  $\varphi_0(\mathbf{x})$  is the slow part of the function  $\varphi(\mathbf{x})$ , containing Fourier harmonics  $\varphi_q$  with wave vectors  $q \leq \lambda q_0$ ; the rapidly varying part  $\varphi_1(\mathbf{x})$  contains harmonics with wave vectors  $q$  lying in the interval  $(\lambda q_0, q_0)$ . The

smoothing operation does not affect  $\varphi_0$ , which plays the role of a fixed external field. In respect to  $\varphi_1$  the smoothing is equivalent to ordinary Gibbsian averaging with the free-field Hamiltonian. Therefore, terms containing different pairings of the field  $\varphi_1$  appear in (5.5). It is convenient to carry out concrete calculations in the Fourier representation, in which the pairings have a very simple appearance:

$$\overline{\varphi_{1\mathbf{q}_1}\varphi_{1\mathbf{q}_2}} = \langle \varphi_{1\mathbf{q}_1}\varphi_{1\mathbf{q}_2} \rangle = \begin{cases} \frac{\delta(\mathbf{q}_1 + \mathbf{q}_2)}{q^2} & (\lambda q_0 < q < q_0), \\ 0 & (q \leq \lambda q_0). \end{cases} \quad (7.7)$$

We shall denote a pairing by a line linking the quantities  $\varphi$  being paired (cf. Sec. 2). For example, in first order in  $g_0$  we obtain

$$\begin{aligned} & -g_0 \int \langle \varphi_{\mathbf{q}_1}\varphi_{\mathbf{q}_2}\varphi_{\mathbf{q}_3}\varphi_{\mathbf{q}_4} \rangle \delta(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3 + \mathbf{q}_4) \frac{d\mathbf{q}_1}{(2\pi)^d} \dots \frac{d\mathbf{q}_4}{(2\pi)^d} \\ &= -g_0 \int_{q_i \leq \lambda q_0} \varphi_{0\mathbf{q}_1}\varphi_{0\mathbf{q}_2}\varphi_{0\mathbf{q}_3}\varphi_{0\mathbf{q}_4} \delta(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3 + \mathbf{q}_4) \frac{d\mathbf{q}_1}{(2\pi)^d} \dots \frac{d\mathbf{q}_4}{(2\pi)^d} \\ &\quad - 6g_0 \int_{q \leq \lambda q_0} \varphi_{0\mathbf{q}}\varphi_{-\mathbf{q}} \frac{d\mathbf{q}}{(2\pi)^d} \int_{q_0 > q' > \lambda q_0} \overline{\varphi_{1\mathbf{q}}\varphi_{1-\mathbf{q}'}} \frac{d\mathbf{q}'}{(2\pi)^d} \\ &\quad - 3g_0 \int_{q_0 > q, q' > \lambda q_0} \overline{\varphi_{1\mathbf{q}}\varphi_{1-\mathbf{q}}}\overline{\varphi_{1\mathbf{q}'}\varphi_{1-\mathbf{q}'}} \frac{d\mathbf{q}}{(2\pi)^d} \frac{d\mathbf{q}'}{(2\pi)^d}. \end{aligned} \quad (7.8)$$

The calculation is conveniently represented in diagrammatic form. With each  $g_0$  we associate a point (vertex), from which four lines representing the quantities  $\varphi_{\mathbf{q}}$  emerge. The pairing of the quantities  $\varphi_{\mathbf{q}_1}$  and  $\varphi_{\mathbf{q}_2}$  is depicted by a line linking together the  $\varphi_{\mathbf{q}_1}$  and  $\varphi_{\mathbf{q}_2}$  lines. The integration is performed over all wave vectors  $\mathbf{q}$ . The sum of the wave vectors of the lines entering each vertex is equal to zero. We recall that the integration over the momenta of the pairing lines is performed in the limits  $\lambda q_0 < q < q_0$ . The lines that remain unpaired correspond to  $\varphi_0$  and, consequently, have wave vectors  $q \leq \lambda q_0$ . The graphs corresponding to the three terms of the expression (7.8) are

$$\begin{array}{c} \text{X} + \text{O} + \text{O} \end{array} \quad (7.9)$$

Each graph occurs with a factor equal to the number of combinatoric ways of realizing it.

Thus, according to (7.3), in first order in  $g_0$  we find

$$S(\lambda) \mathcal{F}_L = \mathcal{F}_L + \delta \mathcal{F}, \quad (7.10)$$

where  $\delta \mathcal{F}$  is represented by the sum of graphs

$$\text{O} + \text{O} \quad (7.11)$$

We have made use of the fact that  $\ln(1+x) \approx x$  if  $x \ll 1$ . The first term in (7.11) corresponds to the change of  $\tau$ :

$$\tau \rightarrow \tau_R = \tau_0 + \delta\tau, \quad \delta\tau = 6g_0 \int \overline{\varphi_{1\mathbf{q}}\varphi_{1-\mathbf{q}}} \frac{d\mathbf{q}}{(2\pi)^d}. \quad (7.12)$$

We shall choose  $\tau_0$  in such a way that the renormalized  $\tau_R$  vanishes. The subsequent terms of the expansion in  $g_0$  introduce changes of higher order of smallness in  $\tau_R$ . Putting  $\tau_R = 0$  we find the value  $\tau_0^*$  corresponding to the fixed point. One can convince oneself

that, for any spatial dimensionality  $d > 2$ , the integrals over  $q$  that determine the corrections to  $\tau_0$  formally diverge as  $q_0 \rightarrow \infty$ . This means that the principal contribution to them is made by values  $q \sim q_0$ , and so  $\tau_0^*$  is indeed a constant quantity, independent of  $l$ . We now assume that  $\tau_0 = \tau_0^*$  and forget, for a while, about the perturbations associated with nonzero  $\tau_R$ .

The second term in (7.11) does not depend on  $\varphi_0$  and changes the Hamiltonian by an unimportant constant. In the following we shall not take graphs of this kind into account.

All that remains is to keep track of graphs with four external lines. In second order of perturbation theory in  $g_0$  the only graph that it is necessary to take into account has the form



The corresponding contribution to  $g$  is of the form

$$-36g_0^2 \int_{\lambda q_0 < q < q_0} \frac{d^d q}{(2\pi)^d q^4}. \quad (7.13)$$

We are interested in the case when the wave vectors of the external lines are much smaller than those of the internal lines ( $\lambda \ll 1$ ). Therefore, we have put the external  $\mathbf{q}$  values equal to zero. The integral in (7.13) is easily calculated:

$$\int_{\lambda q_0 < q < q_0} \frac{d^d q}{(2\pi)^d q^4} = \frac{S_d}{(2\pi)^d} \int_{\lambda q_0}^{q_0} q^{d-5} dq = K_d \frac{q_0^{-\epsilon} (\lambda^{-\epsilon} - 1)}{\epsilon}, \quad K_d = \frac{S_d}{(2\pi)^d}, \quad (7.14)$$

where  $S_d$  is the surface area of a sphere of unit radius in  $d$ -dimensional space:

$$S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \quad (7.15)$$

Assuming  $\epsilon$  to be a small quantity, in the lowest approximation we put  $\epsilon = 0$  ( $d = 4$ ) in (7.14) and (7.15). In this approximation,

$$g = g_0 + \frac{9}{2\pi^2} g_0^2 \ln \lambda = g_0^2 - \frac{9}{2\pi^2} g_0^2 \xi. \quad (7.16)$$

Differentiating with respect to  $\xi$ , we find

$$\frac{dg}{d\xi} = -\frac{9}{2\pi^2} g_0^2. \quad (7.17)$$

We now replace  $g_0$  by  $g$  in (7.17). We thereby obtain, according to the general ideology (cf. Sec. 6), the renormalization-group equation in four-dimensional space, exact to small  $g^2$ :

$$\frac{dg}{d\xi} = -\frac{9}{2\pi^2} g^2 = -36K_4 g^2. \quad (7.18)$$

In a space of  $4 - \epsilon$  dimensions, as was shown in the preceding section, this equation must be replaced by Eq. (6.7) with a known coefficient  $b$ :

$$\frac{dg}{d\xi} = \epsilon g - 36K_4 g^2. \quad (7.19)$$

Equation (7.19) has a stable fixed point

$$K_d g^* = \frac{\varepsilon}{36}, \quad (7.20)$$

close to the Gaussian fixed point. At this point, according to (5.7), the dimension of  $g$  in first order in  $\varepsilon$  is equal to  $-\varepsilon$ . The quantity  $g^*$  at the nongaussian fixed point has turned out to be a small quantity of order  $\varepsilon$ . Therefore, the expansion in powers of  $g$  is at the same time an expansion in powers of  $\varepsilon$ .

We shall find the dimension of the quantity  $h_2 = \tau$  near the stable fixed point. For this it is necessary to write out the quantity  $\tau$  to first order in  $\varepsilon$ . Graphically, the field  $\tau$  can be represented by a point from which two lines, corresponding to the quantities  $\varphi_q$  and  $\varphi_{-q}$ , emerge. The first-order correction to  $\tau$  is depicted by the graph



This correction corresponds to the second term in Eq. (4.10).

A simple calculation using the rules formulated above gives

$$\tau = \tau_0 (1 - g_0 \cdot 12K_d \xi). \quad (7.21)$$

Differentiating  $\ln \tau$  with respect to  $\xi$ , we find

$$\frac{d \ln \tau}{d \xi} = \frac{d \ln \tau_0}{d \xi} - 12K_d g_0. \quad (7.22)$$

The dimension  $\Delta_\tau^0 = d \ln \tau_0 / d \xi$  of the quantity  $\tau_0$  at the Gaussian fixed point is known:  $\Delta_\tau^0 = 2$  (cf. (6.3)). In the renormalization-group equations exact to quantities of first order in  $g$  it is necessary to replace  $g_0$  by  $g$ :

$$\frac{d \ln \tau}{d \xi} = 2 - 12K_d g. \quad (7.23)$$

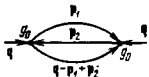
Finally, putting  $g = g^*$  we find the dimension of the quantity  $\tau$  at the stable fixed point:

$$\Delta_\tau = \Delta_\tau^0 - \frac{5}{2\pi^2} g^* = 2 - \frac{\varepsilon}{3}. \quad (7.24)$$

A systematic  $\varepsilon$ -expansion procedure was proposed by Wilson.<sup>[31]</sup>

## 8. THE SECOND $\varepsilon$ -APPROXIMATION

In the first approximation in  $g$  (or  $\varepsilon$ ) the coefficient of  $q^2 \varphi_q \varphi_{-q}$  is not changed after the smoothing  $S(\lambda)$ . In the second approximation, however, changes already arise. In second order in  $g_0$  the change in the coefficients of  $\varphi_{0q} \varphi_{0-q}$  is described by a graph of the form



When  $q_0 \rightarrow \infty$  the integral diverges in the region of large  $q$ . For finite  $q_0$  and for  $q \ll q_0$  the quantity  $\Sigma_2(q)$  is es-

entially the constant  $\Sigma_2(0)$  that renormalizes the quantity  $\tau_0$ . Therefore, we need to calculate not  $\Sigma_2(q)$  itself, but the difference

$$\Sigma_2(q) - \Sigma_2(0),$$

which is proportional to  $q^2$  for small  $q$ . The calculation looks simplest in the  $x$ -representation:

$$\Sigma_2(q) - \Sigma_2(0) = -48g_0^2 \int dx G^3(x) (e^{iqx} - 1), \quad (8.1)$$

where

$$G(x) = \int \frac{e^{ipx}}{p^2} \frac{d^d p}{(2\pi)^d}. \quad (8.2)$$

The coefficient  $48 = 4 \cdot 4 \cdot 6/2$  in front of the integral is half the number of combinatoric ways of realizing the necessary pairings.

For small  $q$  we can expand the exponential  $e^{iqx}$  in (8.1) in a series and keep only the quadratic terms:

$$\Sigma_2(q) - \Sigma_2(0) \approx \frac{48q^2 g_0^2}{2d} \int dx G^3(x) x^2. \quad (8.3)$$

In the lowest  $\varepsilon$ -approximation it is sufficient to put  $\varepsilon = 0$  ( $d = 4$ ) in the integrals (8.2) and (8.3). Then

$$G(x) = \frac{1}{4\pi^2 x^2}. \quad (8.4)$$

The integral (8.3) diverges logarithmically. Therefore,

$$\Sigma_2(q) - \Sigma_2(0) = \frac{48}{2 \cdot 4} q^2 S_4 \frac{g_0^2}{64\pi^6} \xi. \quad (8.5)$$

The changed coefficient of  $q^2 \varphi_q \varphi_{-q}$  is equal to

$$\frac{1}{2} \left( 1 + \frac{3g_0^2}{8\pi^4} \xi \right). \quad (8.6)$$

In order that the coefficient of  $q^2 \varphi_q \varphi_{-q}$  remain unchanged, we must carry out a dilatation operation and renormalize the field [ $Z_\lambda \equiv Z(\lambda)^{-1}$  (cf. (3.5) transl. note)]:

$$q \rightarrow \lambda^{-1} q, \quad \varphi_q = Z_\lambda \varphi'_{\lambda^{-1} q}, \quad (8.7)$$

where

$$Z_\lambda = \lambda^{-(d+2)/2} \left( 1 - \frac{3g_0^2}{16\pi^4} \xi \right). \quad (8.8)$$

The quantity  $Z_\lambda$  is associated with the scale change of the field  $h$  conjugate to  $\varphi$ . Namely, the renormalization transformation  $R(\lambda)$  as applied to  $h$  has the form

$$R(\lambda) h = Z_\lambda h. \quad (8.9)$$

According to the general theory, we can write the renormalization-group equation as

$$\frac{d \ln h}{d \xi} = f_h(g). \quad (8.10)$$

We see that there is no linear term in the expansion of  $f_h(g)$  in powers of  $g$ . According to (8.8),

$$\frac{d \ln h}{d \xi} = \Delta_h^0 - 12(K_4 g)^2, \quad (8.11)$$

where  $\Delta_h^0 = (d+2)/2$ . The change in the dimension of  $h$  is given by the second term in (8.11) with  $K_4 g = K_4 g^* = \epsilon/36$ :

$$\Delta_h = \Delta_h^0 - \frac{\epsilon^2}{108}. \quad (8.12)$$

The dimension  $\Delta_\psi$  is equal to

$$\Delta_\psi = d - \Delta_h = \Delta_\psi^0 + \frac{\epsilon^2}{108}. \quad (8.13)$$

In this approximation the anomalous-dimension index  $\eta = 2(\Delta_\psi - \Delta_\psi^0) = \epsilon^2/54$ . We shall calculate the dimension of  $\tau$  in the second approximation in  $\epsilon$ . For this, in the smoothing it is sufficient to take into account the graphs<sup>5)</sup>

$$S(l) \tau_0 = \begin{array}{c} \diagdown \\ \diagup \end{array} + \begin{array}{c} \diagdown \\ \diagup \\ 12 \end{array} + \begin{array}{c} \diagdown \\ \diagup \\ 12 \end{array} + \begin{array}{c} \diagdown \\ \diagup \\ 12 \end{array} + \begin{array}{c} \diagdown \\ \diagup \\ 288 \end{array} \quad (8.14)$$

The numbers under the graphs indicate the numbers of combinatorial ways of doing the pairings. Carrying out next the dilatation operation, we obtain finally

$$\tau = R(\lambda) \tau_0 = \lambda^{-d} Z_\lambda^2 (1 - 12I g_0 + 144I^2 g_0^2 + 288J g_0^2) \tau_0, \quad (8.15)$$

where  $Z_\lambda$  is determined by formula (8.8), and

$$I = \int_{\lambda < q < 1} \frac{d^2 q}{(2\pi)^d q^2}, \quad J = \int_{\lambda < p < 1, |p+q| < 1} \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{1}{p^2 q^2 (p+q)^2}. \quad (8.16)$$

From (6.15) we find

$$\frac{d \ln \tau}{d \xi} = 2 - 12 \frac{dJ}{d \xi} (g_0 - 36g_0^2 I) + g_0^2 \left[ 288 \frac{d}{d \xi} \left( J - \frac{1}{2} I^2 \right) - 24K_4^2 \right]. \quad (8.17)$$

The combination  $g_0 - 36g_0^2 I$  is none other than the vertex  $g$  in the second approximation. Thus, to order  $g^2$  the renormalization-group equation for  $\ln \tau$  has the following appearance:

$$\frac{d \ln \tau}{d \xi} = 2 - 12K_d g + \left[ 288 \frac{d}{d \xi} \left( J - \frac{1}{2} I^2 \right) - 24K_4^2 \right] g^2. \quad (8.18)$$

The calculation of the integrals in  $J - \frac{1}{2} I^2$  is elementary. The principal contributions, proportional to  $\xi^2$ , cancel in the combination  $J - \frac{1}{2} I^2$ :

$$J - \frac{1}{2} I^2 \approx \frac{1}{2} K_4^2 \xi^2. \quad (8.19)$$

Substituting (8.19) into (8.18), we find

<sup>5)</sup> Here and in the following we do not consider graphs of the form



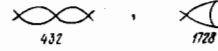
since they are cancelled by graphs of the form



where the cross denotes the corresponding correction to  $\tau_0$ .

$$\frac{d \ln \tau}{d \xi} = 2 - 12K_d g + 120K_4^2 g^2. \quad (8.20)$$

It is necessary to find the quantity  $K_d g^*$  to order  $\epsilon^2$  and substitute it into (8.20). For this it is necessary to write out the renormalization-group equation for  $g$  to order  $g^3$ . In the smoothing it is necessary to add the third-order graphs



and take into account the change of  $Z_\lambda$  in second order in  $g_0$ . The calculations, which are similar to those given above, then give

$$\frac{dg}{d \xi} = \epsilon g - 36K_d g^2 + 816K_4^2 g^3. \quad (8.21)$$

To second order in  $\epsilon$  we find

$$K_d g^* = \frac{\epsilon}{36} + \frac{17\epsilon^2}{972}. \quad (8.22)$$

Finally, substituting  $g = g^*$  into (8.20), we obtain

$$\Delta_\tau = 2 - \frac{\epsilon}{3} - \frac{19\epsilon^2}{162}. \quad (8.23)$$

With the same accuracy we find the exponent  $\nu$ :

$$\nu = \frac{1}{3\tau} = \frac{1}{2} + \frac{\epsilon}{12} + \frac{7\epsilon^2}{162}. \quad (8.24)$$

We leave it to the reader as an exercise to obtain the exponents for a degenerate  $n$ -component system in the second  $\epsilon$ -approximation.

The renormalization-group method makes it possible to find not only the exponents but also the equation of state near the phase-transition point. The calculations performed by Avdeeva and Migdal<sup>[32]</sup> and Brézin, Wallace and Wilson<sup>[33]</sup> have confirmed, to second order in  $\epsilon$ , the correctness of the phenomenological equation of state proposed by Migdal.<sup>[34]</sup> The equation of state of a many-component system has a number of distinctive features, considered in<sup>[35,36]</sup>.

## 9. ASYMPTOTIC SYMMETRY

According to the Landau theory, a phase transition can be described as a spontaneous breaking of symmetry. A system which above the transition point possessed a symmetry group  $G$  has, below this point, a lower symmetry whose corresponding group  $G_1$  is a subgroup of  $G$ . If the Ginzburg number  $G_i$  is small and there is a region in which the Landau theory is applicable, the conclusions of the theory concerning the possible symmetry selection rules in a second-order phase transition<sup>[1]</sup> remain valid. Indeed, all that is required for the derivation of these rules is that the thermodynamic potential in the form proposed by Landau have the same form above and below the phase-transition point. Moreover, it is obvious that the Landau selection rules also remain valid up to a certain critical value, of order unity, of the number  $G_i$ . The question of whether the Landau selection rules are replaced by

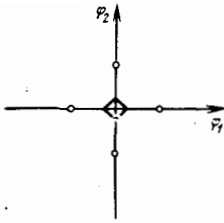


FIG. 2.

others at a certain finite  $G$  remains open.

Fluctuations lead to an interesting phenomenon: the symmetry at the phase-transition point can turn out to be higher than the symmetry  $G$  of the initial phase. We call this phenomenon asymptotic symmetry. We emphasize that we are not talking about the groups of scale and conformal transformations, which also characterize additional symmetry appearing at the transition point. We are talking about additional rotational symmetry elements that do not appear in the group  $G$ . Moreover, the asymptotic-symmetry group can turn out to be continuous even in the case when the initial symmetry is discrete. The phenomenon of asymptotic symmetry arises only in the case when there are several fields (charges) of the same dimension in the system.

We shall consider the simplest example, first investigated by Wilson and Fisher<sup>[10]</sup>—two coupled scalar fields  $\varphi_1$  and  $\varphi_2$ , describable by the Hamiltonian

$$\mathcal{H} = \int dx \left[ \frac{1}{2} \tau (\varphi_1^2 + \varphi_2^2) + \frac{1}{2} (\nabla \varphi_1)^2 + \frac{1}{2} (\nabla \varphi_2)^2 + g_1 (\varphi_1^4 + \varphi_2^4) + g_2 \varphi_1^2 \varphi_2^2 \right]. \quad (9.1)$$

The model of two coupled scalar fields with the Hamiltonian (9.1) is not purely academic. We can point to at least two applications of it to structural transitions in crystals. Let one of the elements of the initial group  $G$  be a fourth-order axis  $C_4$  (tetragonal symmetry). In the transition, atoms situated on the  $C_4$  axis are displaced to nonsymmetric positions. Such phase transitions of the displacement type have been well studied in ferroelectrics. It is obvious that there exist four equivalent displacements of the position of an atom. They differ from one another by a rotation  $C_4$  (Fig. 2). Then  $\varphi_1$  describes the displacement along axis 1 and  $\varphi_2$  describes that along axis 2.

The Hamiltonian (9.1) describes the behavior of the two-component field of the displacements in the case when, in addition to the  $C_4$  axis, a plane containing the  $C_4$  axis appears in the group (the group  $C_{4v}$ ). Another example that leads to the Hamiltonian (9.1) is the ordering of a two-component alloy that has a body-centered cubic lattice in the disordered state (Sec. 139 of<sup>[11]</sup>). If a superstructure with  $T_d$  symmetry with reciprocal-lattice vectors  $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$  and  $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$  is formed, the change in the density of the components is described by the functions  $\varphi_1 = \cos \pi x \cdot \cos \pi y \cdot \cos \pi z$  and  $\varphi_2 = \sin \pi x \cdot \sin \pi y \cdot \sin \pi z$ . From these we can construct two independent fourth-order invariants,  $\varphi_1^4 + \varphi_2^4$  and  $\varphi_1^2 \varphi_2^2$ .

The Hamiltonian (6.1) is invariant under the transformation

$$\varphi_{1,2} \rightarrow \frac{\varphi_1 \pm \varphi_2}{\sqrt{2}}, \quad g_1 \rightarrow \frac{g_1}{2} + \frac{g_2}{4}, \quad g_2 \rightarrow 3g_1 - \frac{g_2}{2}. \quad (9.2)$$

We shall denote the ratio  $g_2/g_1$  by  $y$ . The transformation (9.2) leads to a change of  $y$ :

$$y \rightarrow \frac{6-y}{1+(y/2)}. \quad (9.3)$$

The fixed point of the transformation (7.3) is  $y=2$ . In this case the fourth-order terms in the Hamiltonian (9.1) can be written in the form  $(\varphi_1^2 + \varphi_2^2)^2$  and, consequently, the Hamiltonian (9.1) becomes invariant under rotations in the  $(\varphi_1, \varphi_2)$  plane. For other values of  $y$  this symmetry is absent. It appears, however, in the region of strongly developed fluctuations. We shall preface our investigation of this question by a brief analysis of the phase diagram of the system under consideration, from the point of view of the Landau theory. A simple analysis shows that in the  $(\tau, y)$  plane the region  $\tau > 0, y > -2$  corresponds to the symmetric phase (phase I), in the region  $\tau < 0, -2 < y < 2$  a phase  $\varphi_1 = \pm \varphi_2 \neq 0$  is realized (phase II), and in the region  $\tau < 0, y > 2$  a phase  $\varphi_1 \neq 0, \varphi_2 = 0$  or  $\varphi_1 = 0, \varphi_2 \neq 0$  is realized (phase III). The transformation (9.2), (9.3) carries phase II into phase III. In particular, this transformation carries the interval  $-2 < y < 2$  into  $2 < y < \infty$ . The line  $y=2$  is a line of first-order phase transitions between phases II and III. The region in which the Hamiltonian (9.1) is positive-definite, corresponding to the limits of thermodynamic stability, is determined by the inequalities

$$g_1 > 0, \quad y > -2. \quad (9.4)$$

We turn now to the region of strongly developed fluctuations. As before, we shall use the  $\varepsilon$ -expansion method of Wilson and Fisher, with the intention of applying it to ordinary three-dimensional systems. However, the asymptotic-symmetry effect of interest to us already appears in the zeroth approximation in  $\varepsilon$ . Therefore, we start from an analysis of the situation in four-dimensional space.

The renormalization-group equations for the model under consideration have, up to quadratic terms, the form

$$\frac{dg_1}{d\xi} = -36g_1^2 - g_2^2, \quad \frac{dg_2}{d\xi} = -24g_1g_2 - 8g_2^2. \quad (9.5)$$

We change to the variables  $y = g_2/g_1$  and  $t = -\ln g_1$ . Since the derivative  $dg_1/d\xi$  is everywhere negative,  $t$  is a monotonically increasing function of  $\xi$ . The renormalization-group equation in these variables has the form

$$\frac{dy}{dt} = \frac{y(y-2)(y-6)}{y^2+36}. \quad (9.6)$$

Equation (9.6) has three fixed points:  $y=0, y=2$  and  $y=6$ . We sketch the "phase line" of this equation (Fig. 3). The signs of the derivative, i.e., the directions of motion along the straight line in the different regions, are shown by arrows. The points 0 and 6 are repulsive centers and the point 2 an attractive center. This means

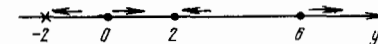


FIG. 3.

that if, by virtue of the initial conditions,  $0 < y < 6$ ,  $y$  tends to 2 in the asymptotic region of large distances. The asymptotic rotation group  $O_2$ , in the  $(\varphi_1, \varphi_2)$  plane, arises. We note that, if the initial value of  $y$  lies in the region  $-2 < y < 0$  or  $y > 6$ , then, by virtue of Eqs. (9.6), the thermodynamic-stability conditions (9.4) are violated in the asymptotic region. Therefore, we should expect that, before this, a first-order phase transition will occur. The next section is devoted to this effect.

The point  $y=0$  corresponds to two noninteracting scalar fields  $\varphi_1$  and  $\varphi_2$ , and the point  $y=6$  corresponds to the noninteracting fields  $(\varphi_1 \pm \varphi_2)/\sqrt{2}$ .

We shall show that the asymptotic symmetry does not disappear in any order in  $\epsilon$ . The renormalization-group equations for small  $\epsilon$  have the form

$$\frac{dg_1}{d\epsilon} = f_1(g_1, g_2), \quad \frac{dg_2}{d\epsilon} = f_2(g_1, g_2). \quad (9.7)$$

Amongst the fixed points of Eqs. (9.7) there should undoubtedly be a point, corresponding to  $O_2$  symmetry, at which  $g_2 = 2g_1$ . Indeed, if the initial Hamiltonian is invariant under  $O_2$ , the symmetry at the transition point cannot be lower. The only question that can arise is whether the symmetric point is stable. But for small  $\epsilon$  the equation for  $y$  that arises from (9.7) must have the form

$$\frac{dy}{d\epsilon} = \frac{y(y-2)(y-6)}{y^2+36} + (y-2)O(\epsilon). \quad (9.8)$$

Therefore, the sign of the derivative near the symmetric point  $y=2$  does not change for small  $\epsilon$ .

The points  $y=2$  and  $y=6$  remain fixed points of Eqs. (9.8) in any approximation in  $\epsilon$ , since they correspond to the absence of interaction between the scalar fields. Therefore, the arrangement of the arrows on the "phase diagram" remains the same as in Fig. 3 in any approximation in  $\epsilon$ . Of course, this in no way implies that everything should remain unchanged when  $\epsilon=1$ . However, the "phase line" can only change discontinuously, at a finite value of  $\epsilon$ . The successful calculation of the indices by means of the  $\epsilon$ -approximation gives rise to the hope that this will happen when  $\epsilon > 1$ .

In the above we followed the paper<sup>[37]</sup> by Lyuksyutov and Pokrovskii.

We shall consider the somewhat more complicated example of fields describable by the Hamiltonian

$$\mathcal{H} = \int dx \left[ \frac{1}{2} (\tau_1 \varphi_1^2 + \tau_2 \varphi_2^2) + \frac{1}{2} (\nabla \varphi_1)^2 + \frac{1}{2} (\nabla \varphi_2)^2 + g_1 (\varphi_1^2)^2 + g_2 (\varphi_2^2)^2 + |g_{12} \varphi_1^2 \varphi_2^2| \right], \quad (9.9)$$

in which we shall assume that  $\varphi_1$  is an  $m$ -component field and  $\varphi_2$  is an  $n$ -component field. Such a situation arises in the case when two phase transitions of a different nature can occur in the system, e.g., ferromagnetic and antiferromagnetic, magnetic and structural, or a transition to the superconducting state and a magnetic transition. If the phase-transition lines  $\tau_1=0$  and  $\tau_2=0$  intersect, in principle the asymptotic symmetry  $O_{m+n}$

( $g_{12} = 2g_1 = 2g_2$ ) can arise at the point of intersection. As in the preceding case, such a point is always a fixed point of the renormalization-group equations. But it is far from always the case that it is stable. One can convince oneself of this by studying the renormalization-group equations in four-dimensional space:

$$\begin{aligned} \frac{dg_1}{d\epsilon} &= -4(m+8)g_1^2 - ng_{12}^2, & \frac{dg_2}{d\epsilon} &= -4(n+8)g_2^2 - mg_{12}^2, \\ \frac{dg_{12}}{d\epsilon} &= -4g_{12}[(m+2)g_1 + (n+2)g_2] - 8g_{12}^2. \end{aligned} \quad (9.10)$$

We shall linearize the equations for the two ratios  $x = g_2/g_1$  and  $y = g_{12}/g_1$  near the symmetric point. An elementary analysis shows that this point is stable only if  $m+n < 4$ . The case  $m+n=4$  is special; one of the eigenvalues vanishes. Therefore, a more careful analysis in the second  $\epsilon$ -approximation is necessary. This shows that the point with  $O_{m+n}$  symmetry is already unstable when  $m+n=4$ . The highest possible asymptotic symmetry is  $O_3$ .

In what physical effects is asymptotic symmetry manifested? The simplest effect is that the critical indices are determined not by the symmetry of the original system but by the asymptotic symmetry. The critical indices  $\alpha$  of the specific heat differ the most sharply. Unfortunately, the accuracy of the determination of the index  $\alpha$ , especially in solids, is not great.

More interesting is the behavior of correlators of the type  $\langle\langle \varphi_\alpha(\mathbf{x}) \varphi_\beta(\mathbf{x}') \rangle\rangle$  near the transition point; in a number of cases this behavior can be established from critical-scattering data. For example, in the case of intersection of curves of second-order phase transitions the correlators  $\langle\langle \varphi_1(\mathbf{x}) \varphi_1(\mathbf{x}') \rangle\rangle$  and  $\langle\langle \varphi_2(\mathbf{x}) \varphi_2(\mathbf{x}') \rangle\rangle$  turn out to be equal. The example considered here has been analyzed in papers by Lyuksyutov, Pokrovskii and Khmel'nitskii.<sup>[38]</sup> A less general situation ( $m=1$ ) was considered earlier by Nelson, Kosterlitz and Fisher.<sup>[39]</sup> Different aspects of the phenomenon of asymptotic symmetry are treated in papers by Fisher and Aharony (see the review<sup>[41]</sup>) and in the paper<sup>[65]</sup> by Brézin, le Guillou and Zinn-Justin.

Of course, phase transitions without an increase of symmetry are possible. An example of such a transition that is interesting from a physical point of view is given by Mukamel<sup>[40]</sup>; this is a structural transition in crystals with tetragonal symmetry (e.g.,  $\text{NbO}_2$ ). It is described by a four-component order parameter, the role of which is played by the density components with wave vectors  $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ ,  $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ ,  $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$  and  $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ . In this case the fourth-order invariants are

$$I_1 = \varphi_1^4 + \varphi_2^4 + \varphi_3^4 + \varphi_4^4, \quad I_2 = \sum_{i \neq k} \varphi_i^2 \varphi_k^2, \quad I_3 = \varphi_1^2 \varphi_2^2 + \varphi_3^2 \varphi_4^2.$$

The critical indices differ from those in the symmetric case ( $\nu = \frac{1}{2} + \epsilon/8 + 7\epsilon^2/96$ ,  $\eta = \epsilon^2/48$ ).

## 10. INSTABILITY AND FIRST-ORDER PHASE TRANSITIONS

What happens in a system of two coupled scalar fields if the initial value  $y = y_0$  lies in the interval  $(-2, 0)$  or



(6, 0)?<sup>[37]</sup> The invariance under the transformation (9.3) enables us to confine ourselves to one of the intervals, e. g.,  $(-2, 0)$ . We again start from the case of four-dimensional space. The renormalization-group equations (9.5), (9.6) also remain valid for finite  $\tau > 0$ , but  $\xi$  in this case must be understood to be the smaller of the two quantities  $-\ln\lambda$  and  $-\frac{1}{2}\ln\tau$ . It has been shown that for  $-2 < y_0 < 0$  the function  $y(\xi)$  decreases monotonically and reaches the stability boundary  $y = -2$  at a certain finite value of  $\xi$ . In the weak-coupling case under consideration the stability boundary  $\xi(y_0)$  differs little from the phase-equilibrium curve. At  $y = -2$  the Hamiltonian (9.1) becomes unstable and in the Hamiltonian it is necessary to take into account anharmonic terms of higher than fourth order in the condensate.<sup>[2]</sup> Corresponding to the  $n$ -th order anharmonicity is a vertex part  $T_n$  that cannot be cut in two through a single line. We are interested only in those graphs in which all the external momenta are equal to zero. The graphs containing the smallest number of integrations for a given number of vertices give the largest contribution. This property is possessed by the ring diagrams

$$\text{X} + \text{II}$$

Summation of these graphs is made easier by the consideration that for  $y_0 < 0$  the symmetric phase I and the phase II ( $\varphi_1 = \pm \varphi_2 \neq 0$ ) can be in equilibrium. Changing to the variables  $\psi_{1,2} = (\varphi_1 \pm \varphi_2)/\sqrt{2}$ , we find that all the external lines correspond to one of the fields  $\psi$ , say  $\psi_1$ , and the internal lines to the other ( $\psi_2$ ). The summation reduces in essence to calculating a sum of terms of a geometric progression and then integrating.<sup>[6]</sup> Discarding the gradient terms and substituting the renormalized values  $\tau$ ,  $g_1$ , we find the thermodynamic potential of the condensate:

$$\Phi = 2\tau\varphi^2 + (2+y)g_1\varphi^4 + \frac{g_1^2}{32\pi^2}\varphi^4 \ln \frac{4g_1\varphi^2}{\tau}, \quad (10.1)$$

where  $\varphi = \varphi_1 = \pm \varphi_2$ . In deriving (10.1) we assumed that the inequality  $4g_1\varphi^2 \gg \tau$  is fulfilled. The equilibrium value of  $\varphi$  is found from the condition that  $\Phi$  is a minimum. On the equilibrium curve,  $\Phi = 0$ . The equations  $\Phi = 0$  and  $\partial\Phi/\partial\varphi = 0$  determine the equilibrium values  $\varphi(\xi)$  and  $y_0(\xi)$  and, consequently, the curve in the  $(y_0, \tau)$  plane. The equation of this curve for small  $y_0$  has the form

$$\tau \sim \exp\left(-\frac{2}{|y_0|^3}\right). \quad (10.2)$$

The equilibrium curve has an essential singularity at the point  $y_0 = 0$  at which it merges with the line of second-order phase transitions. We also write out the formula for the quantity  $\varphi^2$  on the equilibrium curve:

$$\varphi^2 \sim \tau |y_0|^{-6} \sim |y_0|^{-6} e^{-2/|y_0|^3}. \quad (10.3)$$

The results obtained are easily extended to a space of

<sup>6)</sup>For the analogous summation for the case of scalar electrodynamics, see<sup>[41]</sup>.

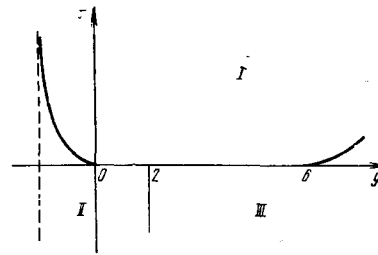


FIG. 4.

$4-\epsilon$  dimensions. For this, to first order in  $\epsilon$  it is sufficient to replace the logarithmic integrals by power integrals, i. e., to assume that

$$\xi = \min\left(\frac{\lambda^{-\epsilon}-1}{\epsilon}, \frac{\tau^{-\epsilon/2}-1}{\epsilon}\right). \quad (10.4)$$

The equation of the equilibrium curve takes the form

$$\tau = \left(1 + \frac{\epsilon}{|y_0|^3}\right)^{-2/\epsilon}, \quad (10.5)$$

and the value of the order parameter on the equilibrium curve is given by

$$\varphi^2 \sim \tau^{(1/2)+(\epsilon/4)} |y_0|^{-3-(3/4)\epsilon}. \quad (10.6)$$

The diagram of state in the  $(y_0, \tau)$  plane is shown schematically in Fig. 4. On it there are three points at which curves of first-order and second-order phase transitions meet. We emphasize that the appearance of two of the three lines of first-order phase transitions is associated entirely with fluctuations. In the Landau theory these lines are absent. For example, the Landau Hamiltonian is completely stable in the region  $-2 < y < 0$ , and only the developed fluctuations lead to the instability of the symmetric state.

The physical reason for the instability can be best understood using the example of two interacting fields with different transition points (the Hamiltonian (9.9)).<sup>[38,42]</sup> Suppose that the system is close to one of the transition points. This means that, e. g., the field  $\varphi_1$  is already strongly fluctuating. If we eliminate the field  $\varphi_1$  a new Hamiltonian  $\mathcal{H}_2$  for the field  $\varphi_2$  arises. We shall show that in this Hamiltonian the coefficient of  $\varphi_2^4$ , which describes the self-interaction of the field  $\varphi_2$ , is negative and of large modulus. In fact, by definition,

$$\mathcal{H}_2 = -\ln \int e^{-\mathcal{H}} \prod_q d\varphi_{1q}. \quad (10.7)$$

To second order in  $g_{12}$ , we find

$$\mathcal{H}_2 = \int \left[ \frac{1}{2}(\tau_2\varphi_2^2 + (\nabla\varphi_2)^2) + g_2\varphi_2^4 \right] dx - g_{12}^2 \int \langle\langle \varphi_1^2(x)\varphi_1^2(y) \rangle\rangle \varphi_2^2(x)\varphi_2^2(y) dx dy. \quad (10.8)$$

The averaging symbol here refers only to the field  $\varphi_1$ . We now neglect fluctuations of the field  $\varphi_2$ , i. e., we assume it to be independent of the coordinates. Then the interaction is described by two terms:

$$g_2\varphi_2^4 - g_{12}^2\varphi_2^4 \int \langle\langle \epsilon_1(x)\epsilon_1(0) \rangle\rangle dx, \quad (10.9)$$

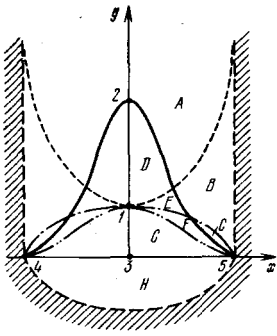
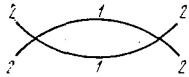


FIG. 5.

where  $\varepsilon_1(\mathbf{x}) = \varphi_1^2(\mathbf{x})$ . The second term is proportional to the specific heat of the field  $\varphi_1$  and is negative. If the specific heat diverges ( $\alpha > 0$ ), it will play the principal role. The graph corresponding to the second term of (10.8) has the form



Exchange of fluctuations in second order of perturbation theory leads to attraction. The lines 1 can pertain to an arbitrary fluctuating field, e. g., to acoustic phonons.<sup>[43]</sup>

It is necessary also to take into account anharmonicities of higher orders in  $\varphi_2$ , since the coefficients of these increase as the transition curve  $\tau_1 = 0$  is approached. This is easily done, if the fluctuations of the field  $\varphi_2$  can be neglected. In this case the interaction of the fields  $\varphi_1$  and  $\varphi_2$  leads to the simple replacement  $\tau_1 \rightarrow \tau_1 + g_{12}\varphi_2^2$ . The singular part of the thermodynamic potential in this case is proportional to the quantity  $(\tau_1 + g_{12}\varphi_2^2)^{2-\alpha}$ . Elimination of the field  $\varphi_1$  in this case gives the following thermodynamic potential in the variables  $\varphi_2$ :

$$\Phi = -A(\tau_1 + g_{12}\varphi_2^2)^{2-\alpha} + \frac{1}{2}\tau_2\varphi_2^2 + g_2\varphi_2^4. \quad (10.10)$$

The constant  $A$  is positive and equal in order of magnitude to  $g_1^{-1+2\alpha}$ . The order of magnitude of  $A$  is determined by matching the thermodynamic potential calculated by the Landau theory to the singular part  $\Phi_{sing} = A\tau^{2-\alpha}$  at the boundary of the region of applicability of the Landau theory ( $\tau \propto \mathbf{G}i \propto g_1^2$ ). In particular, the previous result (10.9) is easily obtained from (10.10) by expanding the singular part in powers of  $\varphi_2^2$ . The value of  $\varphi_2$  can be found from the condition

$$\frac{\partial \Phi}{\partial \varphi_2} = 2\varphi_2[-(2-\alpha)Ag_{12}(\tau_1 + g_{12}\varphi_2^2)^{1-\alpha} + \tau_2 + 2g_2\varphi_2^2] = 0. \quad (10.11)$$

Equation (10.11) has a nonzero solution  $\varphi_2 \neq 0$  ( $\tau_1 = 0$ ) provided that  $g_{12}^{2-\alpha} > g_2/A \propto g_1^{1-2\alpha}g_2$ . Neglecting the small quantity  $\alpha$ , we obtain  $g_{12}^2 > g_1g_2$ . Thus, near the curve of the second-order phase transition with respect to  $\varphi_1$  there can appear a region in which  $\varphi_2$  condenses out.

At the intersection point  $\tau_1 = \tau_2 = 0$  of the phase-transition lines both fields fluctuate strongly. An analysis of the real three-dimensional situation is possible only in

the  $\varepsilon$ -approximation. We introduce the ratios  $x = (g_1 - g_2)/(g_1 + g_2)$  and  $y = g_{12}/(g_1 + g_2)$ . The topological structure of the phase plane of the renormalization-group equations in the four-dimensional case is shown in Fig. 5. For all values of  $m$  and  $n$  there are five fixed points. One of them (1) is an attractive center and the others are repulsive centers or saddle points. In the region bounded by the separatrices (2-4), (2-5) and (4-5) points move toward the attractive center. Outside this region, points move away to infinity as the length-scale increases ( $\xi \rightarrow \infty$ ). The stability boundary ( $x^2 + y^2 = 1$ ,  $x = \pm 1$ ) is denoted by a dashed line. The attractive center corresponds either to the asymptotic symmetry  $O_{n+m}$  ( $n+m < 4$ ) or to the initial symmetry  $O_n \times O_m$ . Depending on which part of the phase plane the initial values of  $g_1$ ,  $g_2$  and  $g_{12}$  lie in, different diagrams of state are possible. Some of them are given in Fig. 6. For a detailed analysis, see the paper by Lyuksyutov, Pokrovskii and Khmel'nitskii.<sup>[38]</sup> Evidently, the general tendency is that an increase in the number of independent fields with dimension zero leads to the disappearance of stability. For example, in the physically interesting case when the number of fields is equal to four, we can construct three fourth-order cubic invariants:

$$I_1 = \sum_{i=1}^4 \varphi_i^4, \quad I_2 = \sum_{i \pm k}^4 \varphi_i^2 \varphi_k^2, \quad I_3 = \varphi_1 \varphi_2 \varphi_3 \varphi_4. \quad (10.12)$$

In this case, as can be illustrated without difficulty by analyzing the renormalization-group equations, none of the fixed points is stable (Brazovskii and Dzyaloshinskii<sup>[44]</sup>). Therefore, we should expect that in this case all transitions are first-order transitions (with, of course, the caveat that the  $\varepsilon$ -expansion may possibly be inapplicable for  $\varepsilon = 1$ ). Physically, a model with a four-component order parameter corresponds to a magnetic phase transition in a crystal with a face-centered cubic lattice. The role of the parameters  $\varphi_1$  and  $\varphi_2$  is played by the projections along the diagonals of the cube of the spins situated at the vertices of the cube and at the centers of its faces. As an example of a substance in which a first-order magnetic phase transition is observed we cite MnO.<sup>[45]</sup>

Apparently, an analysis of instability in superconductors and liquid crystals was first given in the paper<sup>[46]</sup> by Halperin, Lubensky and Ma. Applications in the theory of liquid crystals and in field theory have been analyzed in papers by Vigman, Larkin and Filev.<sup>[47,48]</sup>

More-complicated magnetic systems have been ana-

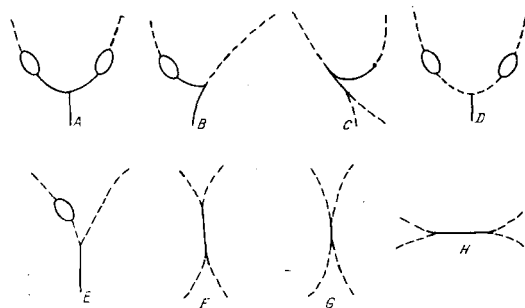


FIG. 6.

lyzed in papers by Mukameļ, Bak and Krinsky<sup>[49]</sup> and Brazovskii, Dzyaloshinskii and Kukhareno.<sup>[50]</sup>

## 11. DYNAMICS NEAR A PHASE-TRANSITION POINT

Together with the correlation length the characteristic time of the relaxation of fluctuations also grows. This leads to a number of distinctive features in dynamical phenomena: anomalous behavior of the kinetic coefficients associated with the strongly fluctuating quantities (cf. <sup>[51]</sup>), narrowing of the Rayleigh line in light scattering near a critical mixing point,<sup>[52]</sup> and anomalies in the absorption of sound and other oscillations. The most important question for the theory is the dependence of the relaxation time (frequency) of the fluctuations on the dimensionless temperature.

Two theoretical approaches to the dynamics near a phase transition have been developed. One of these considers the interaction of the different hydrodynamic motions (sounds, thermal conduction, diffusion) with the fluctuational degrees of freedom and with each other (Kawasaki,<sup>[53]</sup> Kadanoff and Swift,<sup>[54]</sup> Polyakov<sup>[55]</sup>). The other (Halperin and Hohenberg<sup>[56]</sup>) is based on simple phenomenological arguments. It is assumed that the fluctuational motions become hydrodynamic at a length of the order of the correlation length. Since the spectrum of the oscillatory motions (sounds, spin waves) is associated only with the thermodynamic quantities, with such assumptions the relaxation frequency of the fluctuations is determined entirely by the static indices. For example, in the case of superfluid helium it is reasonable to identify the relaxation frequency  $\omega_r$  with the second-sound frequency  $\omega = u_2 q$  for  $q \sim r_c^{-1}$  ( $u_2$  is the velocity of second sound). This gives

$$\omega_r \sim u_2 r_c^{-1} \propto \left(\frac{\rho_2}{\rho}\right)^{1/2} r_c^{-1} \propto r_c^{-3/2} C^{-1} \propto \tau^{-1}.$$

Above the transition point the hydrodynamic motion associated with the fluctuations of the superfluid component is thermal conduction. The dispersion law  $\omega = i(\lambda/C)q^2$ , where  $\lambda$  is the thermal-conductivity coefficient, is characteristic for this. From dimensionality arguments we find  $\lambda \propto r_c^{1/2} \propto \tau^{-1/3}$ , which is in good agreement with experiment.

On the other hand, an exact microscopic analysis (Polyakov<sup>[55]</sup>) shows that there are no reasons, *a priori*, for fluctuational motions to become hydrodynamic when  $q \sim r_c^{-1}$ . Therefore, the question of whether the dependence of  $\omega_r$  on  $\tau$  is determined entirely by the static indices is important for the theory. Usually, the dynamical index  $z$  is introduced:  $\omega_r \propto r_c^{-z}$ . To calculate the index  $z$  it would be necessary to start from the exact dynamical equations for the system. But such a program is not realistic. It is clear that the interaction of the fluctuational degree of freedom with the others deprives us of the possibility of treating the fluctuations as a closed dynamical system. For the statics this was unimportant, since by eliminating superfluous degrees of freedom we could always reduce the original problem to the universal problem of a fluctuating field. In the case of the dynamics there is no such universality. It is pos-

sible to consider the purely relaxational motions of the order parameter, but, in principle, vibrational motions (second-sound, soft optical modes in crystals) are also possible. In order to exclude this possibility we stipulate that we are considering a system above the transition point. It is natural to start from the simplest kinetic equation of the type

$$\frac{1}{\Gamma_0} \frac{\partial \varphi}{\partial t} = - \frac{\delta \mathcal{H}}{\delta \varphi}. \quad (11.1)$$

Near the equilibrium value  $\varphi_0$  the derivative  $\delta \mathcal{H} / \delta \varphi$  is small. In the linear approximation we can represent Eq. (11.1) in the form

$$\frac{1}{\Gamma_0} \frac{\partial \varphi}{\partial t} = - \frac{\delta^2 \mathcal{H}}{\delta \varphi^2} \Big|_{\varphi_0} \delta \varphi = - \frac{1}{\chi} \delta \varphi, \quad (11.2)$$

where  $\chi$  is the static susceptibility. If we assume that  $\Gamma_0$  is regular at the transition point, the scaling dimension  $\omega_r$  coincides with the dimension of  $\chi^{-1}$ :

$$z \equiv \Delta_\omega = -\Delta_\chi = 2 - \eta. \quad (11.3)$$

Let the energy relaxation be described by an analogous equation:

$$\frac{1}{\lambda_0} \frac{\partial e}{\partial t} = - \frac{\delta \mathcal{H}}{\delta e}. \quad (11.4)$$

To this process we must ascribe another frequency  $\omega_\epsilon \sim \lambda_0 (\delta^2 \mathcal{H} / \delta e^2) \sim \lambda_0 / C$ , with the dimension of  $C^{-1}$ , i. e.,  $\Delta_\omega = \alpha / \nu$ .

If the order parameter is a conserved quantity, such as, e. g., the magnetic moment of a ferromagnet, then, in a uniform departure from equilibrium, relaxation does not occur. The simplest isotropic generalization of the kinetic equation (11.1) that possesses this property has the form

$$\frac{1}{\Gamma_0} \frac{\partial \varphi}{\partial t} = - \Delta \frac{\delta \mathcal{H}}{\delta \varphi}; \quad (11.5)$$

here  $\Delta$  is the Laplacian. Arguments completely analogous to the preceding ones show that in this case  $\Delta_\omega = 4 - \eta$ . If the energy of the fluctuations can be regarded as a conserved quantity, then  $\Delta_\omega = 2 + (\alpha/\nu)$ . Naturally, the relaxation in the system is determined by the slowest process. For example, if the order parameter is not conserved while the energy is conserved and  $\alpha > 0$ , then  $\Delta_\omega^\epsilon > \varphi_\omega^\epsilon$ . In this case the slowest process is the energy (temperature) relaxation.

But how legitimate is it to assume that the kinetic coefficients  $\Gamma_0$  and  $\lambda_0$  are nonsingular quantities? The answer to this question is given in the following sections.

## 12. CRITICAL DYNAMICS FROM THE VIEWPOINT OF THE RENORMALIZATION GROUP

The renormalization-group transformations can also be applied to the equations of the dynamics of critical fluctuations. In the simplest case these equations have the form (cf. Sec. 11)

$$\frac{1}{\Gamma_0} \frac{\partial \varphi}{\partial t} = - \frac{\delta \mathcal{H}}{\delta \varphi} + \hbar. \quad (12.1)$$

Here, to keep the notation the same, we have denoted the thermodynamic potential by the symbol  $\mathcal{H}$ . In the general case this equation has the form

$$L \{ \varphi(\mathbf{x}, t) \} = h. \quad (12.2)$$

The field  $\varphi(\mathbf{x}, t)$  is already smoothed—it contains only Fourier harmonics  $\varphi_q(t)$  with wave vectors  $q$  smaller than  $q_0$ . We represent  $\varphi(\mathbf{x}, t)$  in the form of a sum

$$\varphi = \varphi_0 + \varphi_1, \quad (12.3)$$

where the “slowly” varying field  $\varphi_0$  contains harmonics with wave vectors  $q \leq \lambda q_0$  and the “rapidly” varying field  $\varphi_1$  contains harmonics  $\varphi_q$  in the range  $\lambda q_0 < q < q_0$ . In principle, it is possible to solve the equations for the field  $\varphi_1$  while assuming  $\varphi_0$  to be a fixed quantity. The result depends on the initial conditions for the field  $\varphi_1$ . We shall assume that at the initial moment of time the field  $\varphi_1$  was a random, Gibbs-distributed quantity. We substitute the solution  $\varphi_1$  into the equation for  $\varphi_0$ . Then the equation for  $\varphi_0$  depends parametrically on the initial conditions for  $\varphi_1$ . We average over these initial conditions. The equation obtained can be written in the form

$$L' \{ \varphi_0(\mathbf{x}, t) \} = S(\lambda) L \{ \varphi(\mathbf{x}, t) \} = S(\lambda) h. \quad (12.4)$$

Thus, we have defined a smoothing operation on the non-linear operator  $L$ . We now carry out the scale transformation  $\mathbf{x} \rightarrow \lambda \mathbf{x}$  ( $\mathbf{q} \rightarrow \lambda^{-1} \mathbf{q}$ ),  $\varphi_q \rightarrow Z(\lambda) \varphi_q$ .

At the same time, the Hamiltonian  $\mathcal{H}$  is also smoothed, by the method indicated in Sec. 3. We now carry out the dilatation  $D(\lambda)$  (cf. Sec. 3) in such a way that, as before, the coefficient of  $q^2 \varphi_q \varphi_{-q}$  remains equal to  $\frac{1}{2}$ . The choice of the renormalization factor  $Z(\lambda)$  is thereby uniquely fixed. There remains only the possibility of multiplying both sides of the dynamical equation by an arbitrary factor. We choose  $Z'$  in such a way that the renormalized field  $h' = R(\lambda)/h$  appears in the right-hand side of the renormalized Eq. (12.2). Thus,  $Z' = Z(\lambda)$ . Now all the transformations have become unique, and the kinetic coefficients  $\Gamma_i$  have acquired the properties of invariant charges. At the same time, the Hamiltonian  $\mathcal{H}$  has been changed in the way indicated in Sec. 3. We recall that, in the next renormalization step, the Hamiltonian  $R(\lambda)\mathcal{H}$  determines the initial conditions, or the way of averaging over  $\varphi_1$ .

We turn to the differential formulation. Under an infinitesimal change of  $\lambda = e^{-\xi}$  the renormalized operator  $L$  differs from the initial operator by a quantity proportional to  $\delta\xi$ . This time, however, the derivative  $\partial L / \partial \xi$  depends not only on  $L$  but also on  $\mathcal{H}$ , which determines the initial conditions. Therefore, in place of one equation there arises a system of renormalization-group equations:

$$\begin{aligned} \frac{dL}{d\xi} &= F(L, \mathcal{H}), \\ \frac{d\mathcal{H}}{d\xi} &= f(\mathcal{H}). \end{aligned} \quad (12.5)$$

The second equation of the system (12.5) is purely static. Its “roots”  $\mathcal{H}^*$  define the fixed points of the renormaliza-

tion group. It is obvious that the fixed point  $L^*, \mathcal{H}^*$  (the dynamical fixed point) of the system (12.5) is automatically a fixed point for the static case. Generally speaking, the converse statement is not true. We shall assume that the  $\mathcal{H}^*$  corresponding to the critical point is simultaneously a component of the dynamical fixed point  $L^*, \mathcal{H}^*$ . This is the dynamic-scaling hypothesis.

Let  $L$  and  $\mathcal{H}$  be close to the fixed point  $L^*, \mathcal{H}^*$ :

$$L = L^* + \delta L, \quad \mathcal{H} = \mathcal{H}^* + \delta \mathcal{H}. \quad (12.6)$$

The deviations  $\delta L$  and  $\delta \mathcal{H}$  obey the linear equations

$$\frac{d\delta L}{d\xi} = \hat{M} \delta L + \hat{N} \delta \mathcal{H}, \quad \frac{d\delta \mathcal{H}}{d\xi} = \hat{K} \delta \mathcal{H}, \quad (12.7)$$

where  $\hat{M}$ ,  $\hat{N}$  and  $\hat{K}$  are derivatives of the functionals  $F$  and  $f$ . As in the static case, the system of equations (12.7) determines the dimensions of the various quantities near the critical point. We put  $\delta \mathcal{H} = 0$  ( $\mathcal{H} = \mathcal{H}^*$ ). The remaining equation

$$\frac{d\delta L}{d\xi} = \hat{M} \delta L \quad (12.8)$$

determines the dynamical dimensions.

### 13. DYNAMICS IN THE VICINITY OF THE GAUSSIAN FIXED POINT

If one of the eigenvalues of the matrix  $\hat{K}$  is small, then, repeating almost word-for-word the arguments that led to Eqs. (5.2) and (5.3), we obtain equations of the same form in the case of the dynamics, too. In the dynamical case it is necessary to include in the fields  $h_i$  the coefficient  $\Gamma_i$  of the different time derivatives of the quantity  $\varphi$ . The equations for these have the form

$$\frac{d \ln \Gamma_i}{d\xi} = F_i(h_0), \quad (13.1)$$

where  $h_0$  is a quantity with small dimension.

In a space of  $4 - \epsilon$  dimensions a quantity having small dimension is the invariant charge  $g$ —the coefficient of  $\int \varphi^4 d\mathbf{x}$  in the Hamiltonian. The problem then consists in ascertaining the form of the functions  $F_i$ . As in the static case, we obtain these functions by the method of perturbation theory. Here it is necessary to proceed to a specific problem. We shall consider three different cases.

#### A. The order parameter and energy are not conserved

In this case (cf. also Chap. 6 of [2]), the dynamical equation has the form

$$\frac{1}{\Gamma_0} \frac{\partial \varphi}{\partial t} = -(\tau_0 \varphi - \Delta \varphi + 4g_0 \varphi^3) + h. \quad (13.2)$$

We shall solve Eq. (13.2) by the method of iteration in the small quantity  $g_0$ . In the lowest approximation in  $g_0$ ,  $\varphi_1$  does not depend on  $\varphi_0$  and obeys the linear equation

$$\frac{1}{\Gamma_0} \frac{\partial \varphi_1}{\partial t} = -(\tau_0 \varphi_1 - \Delta \varphi_1) + h_1; \quad (13.3)$$

here  $h_1$  are the extraneous random forces, normalized by the condition (cf. <sup>[11]</sup>)

$$\langle h_1(x, t) h_1(x', t') \rangle = 2\Gamma_0 \delta(x - x') \delta(t - t'). \quad (13.4)$$

We transform to the Fourier representation  $\varphi_{1q\omega}$  with respect to the coordinates and time. The fluctuation-dissipation theorem (cf. <sup>[11]</sup>) establishes a relation between the average values  $\langle \varphi_{q\omega} \varphi_{-q-\omega} \rangle$  and the linear-response function  $G_0(q, \omega)$ :

$$G(q, \omega) = \langle \varphi_{1q\omega} \varphi_{1-q-\omega} \rangle = \frac{2T}{\omega} \text{Im} G_0(q, \omega), \quad (13.5)$$

where

$$G_0(q, \omega) = \left( \frac{i\omega}{\Gamma_0} + \tau_0 + q^2 \right)^{-1}. \quad (13.6)$$

The normalization coefficient in (13.5) and (13.6) is chosen in such a way that the equal-time correlator  $\langle \varphi_{1q} \varphi_{1-q} \rangle$  be equal to  $T/q^2$  for  $\tau = 0$ . The equation for the "slow" function  $\varphi_0$  (cf. Sec. 12) has the form

$$\frac{1}{\Gamma_0} \frac{\partial \varphi_0}{\partial t} + \tau_0 \varphi_0 - \Delta \varphi_0 = G_0^{-1} \varphi_0 = -12g_0 \varphi_1^2 \varphi_0 + h_0. \quad (13.7)$$

Iterating (13.7) to second order in  $g_0$ , we find

$$\varphi_0 = G_0 h_0 - 12g_0 G_0 \varphi_1^2 G_0 h_0 + 144g_0^2 G_0 \varphi_1^2 G_0 \varphi_1^2 G_0 h_0. \quad (13.8)$$

Averaging over  $\varphi_1$ , we obtain

$$\varphi_0 = \hat{G} h_0, \quad (13.9)$$

where

$$\hat{G} = G_0 - 12g_0 G_0 \langle \varphi_1^2 \rangle G_0 + 144g_0^2 G_0 \langle \varphi_1^2 G_0 \varphi_1^2 \rangle_{q, \omega}. \quad (13.10)$$

Changing to the  $q, \omega$ -representation, we transform (13.10) to the form

$$G^{-1}(q, \omega) = G_0^{-1}(q, \omega) + 12g_0 \langle \varphi_1^2 \rangle - 144g_0^2 \langle \varphi_1^2 G_0 \varphi_1^2 \rangle_{q, \omega}. \quad (13.11)$$

The second term in the right-hand side of (13.11) leads to the already-known static renormalization of  $\tau_0$ . The last term leads not only to a static renormalization of  $\tau_0$  and of the coefficient of  $q^2 \varphi_{q\omega}$ , but also to a change of the coefficient of  $\omega \varphi_{q\omega}$ :

$$\omega \delta \left( \frac{1}{\Gamma_0} \right) = -144g_0^2 [ \langle \varphi_1^2 G_0 \varphi_1^2 \rangle_{\omega, \omega} - \langle \varphi_1^2 G_0 \varphi_1^2 \rangle_{\omega, 0} ]. \quad (13.12)$$

For the concrete calculation of (13.12) it is necessary to find the quantity

$$\begin{aligned} \Sigma_2(q, \omega) &= 144g_0^2 \langle \varphi_1^2 G_0 \varphi_1^2 \rangle_{q\omega} \\ &= 144g_0^2 \cdot 2 \int G(p_1, \omega_1) G(p_2, \omega_2) G(p_1 + p_2 + q, \omega_1 + \omega_2 + \omega) \\ &\quad \times \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \frac{d\omega_1}{2\pi} \frac{d\omega_2}{2\pi} \end{aligned} \quad (13.13)$$

for  $q=0$  and subtract  $\Sigma_2(0, 0)$  from it. After the subtraction, keeping the term linear in  $\omega$  we find

$$\delta \left( \frac{1}{\Gamma_0} \right) = \frac{1}{\Gamma_0} g_0^2 S_4^2 \cdot 144 \ln \frac{4}{3}. \quad (13.14)$$

Next, in accordance with the general prescription, we carry out the dilatation  $\varphi_q = Z(L)^{-1} \varphi'_q$ , where  $Z(L) = L^{(d+2-\eta)/2}$ . We find, for  $g = g^*$ ,

$$\frac{d \ln \Gamma}{d \ln L} = 2 - \eta + \frac{1}{9} \varepsilon^2 \ln \frac{4}{3} = 2 + c\eta, \quad (13.15)$$

where  $c = 6 \ln(4/3) - 1$ . In this case there appears a new dynamical exponent, unrelated to the static exponents.

### B. The order parameter is conserved and the energy is not conserved

In this case the dynamical equation has the form (11.5). The corresponding response function  $G_0$  for  $\tau = 0$  has the form

$$G_0 = \frac{1}{(i\omega/\Gamma_0 q^2) + q^2}.$$

It is necessary to study the quantity

$$\frac{i\omega}{q^2} \delta \left( \frac{1}{\Gamma_0} \right) = \Sigma_2(\omega, q) - \Sigma_2(0, 0),$$

where  $\Sigma_2$  is determined by Eq. (13.13). Investigation shows that when  $\omega$  and  $q$  tend to zero this quantity tends to a well-defined limit. Therefore,  $\delta(1/\Gamma_0) = 0$ . The kinetic coefficient  $\Gamma_0$  is not renormalized and remains finite at the transition point. In this case, as shown in the preceding section,  $\Delta_\omega = 4 - \eta$ ,

The same result is obtained in the case when both the order parameter and the energy are conserved.

### C. The order parameter is not conserved and the energy is conserved

Here a convenient model is one with two independent field  $\varphi$  and  $\varepsilon$ , obeying the equations

$$\begin{aligned} \frac{\partial \varphi}{\partial t} &= -\Gamma_0 \frac{\delta H}{\delta \varphi} + h(x, t), \\ \frac{\partial \varepsilon}{\partial t} &= -\lambda_0 \Delta \frac{\delta H}{\delta \varepsilon} + \tau(x, t), \end{aligned} \quad (13.16)$$

where  $h$  and  $\tau$  are extraneous random fields and

$$\mathcal{H} = \int \left[ \frac{1}{2} \tau_0 \varphi^2 + \frac{1}{2} (\nabla \varphi)^2 + g_0 \varphi^4 + \gamma_0 \varepsilon \varphi^2 + \frac{1}{2} C_0 \varepsilon^2 \right] dx. \quad (13.17)$$

In the static case the integration over the field  $\varepsilon$  can be carried out exactly and leads to a Landau Hamiltonian with a slightly changed constant  $g_0 \rightarrow g_0 - (C_0 \gamma_0^2 / 2)$ . Therefore, it is obvious immediately that the quantity  $\varepsilon$  has the same dimension as  $\varphi^2$ . Why, then, was it necessary to introduce two independent quantities into the theory? In the real system, describable by the true Hamiltonian, hydrodynamic modes arise; one of these is thermal conduction. Whatever the bare interaction constant  $g_0$ , the coupling between the hydrodynamic mode and the fluctuational degrees of freedom turns out to be not small, and, moreover, possesses singularities at  $k=0$ . Therefore, a direct calculation in the framework of the formalism of statistical mechanics comes up against difficulties that have not yet been overcome. At the same time, there exists the hope, supported by the

experimental facts, that the coupling will turn out to be weak at the phase-transition point. The introduction of the variable  $\varepsilon$  enables us to start from a small region about the Gaussian fixed point and gives a phenomenological way of describing the singular part (corresponding to the hydrodynamic mode) of the interaction.

We proceed to the calculations. We shall assume  $\varphi$  to be an  $n$ -component field. As in the static case, the calculations become easier if we use a diagrammatic method. We shall represent a pairing  $\langle \varphi_1 \varphi_1 \rangle$  by a solid ordinary line, a pairing  $\langle \varepsilon \varepsilon \rangle$  by a solid wavy line, the response function  $G_0$  of the field  $\varphi$  (in the linear approximation) by a dashed ordinary line, and the response function  $D_0$  by a dashed wavy line.

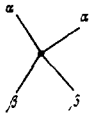
The response of the field  $\varepsilon$  in the linear approximation has the form

$$D_0(k, \omega) = \left( \frac{i\omega}{\lambda_0 k^2} + C_0^{-1} \right)^{-1}. \quad (13.18)$$

The response of the field  $\varphi$  in the same approximation is

$$G_{\alpha\beta}(k, \omega) = \frac{\delta_{\alpha\beta}}{(i\omega/\Gamma_0) + \tau_0 + k^2}. \quad (13.19)$$

With the quantity  $g_0$  we associate a point at which two pairs of ordinary lines, corresponding to  $\varphi_\alpha$  and  $\varphi_\beta$ , meet:



The quantity  $\gamma_0$  is represented by a point at which one wavy line ( $\varepsilon$ ) and two ordinary lines ( $\varphi$ ) meet. The change in the coefficient  $\lambda$  after the smoothing is determined by the graph of the form



to which corresponds the integral

$$-\gamma_0^2 \int \frac{d^4 p}{(2\pi)^4} \frac{d\omega'}{2\pi} \frac{1}{(i\omega'/\Gamma) + p^2} \cdot 2T_c \operatorname{Im} \frac{1}{[i(\omega - \omega')/\Gamma] + (k - p)^2}. \quad (13.20)$$

The integral (13.20) for  $k \rightarrow 0$  does not contain terms of the form  $i\omega/k^2$ . Therefore, the coefficient  $\lambda_0$  is not changed in the smoothing. In the scale dilatation it is multiplied by  $L^2$ .

We turn to the smoothing of the quantity  $\Gamma$ . In lowest order it is determined by the two graphs



which, in the sum, represent the integral

$$\Sigma = \Sigma_1 + \Sigma_2 = -\gamma^2 C \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2} \frac{[\Gamma + (i/c)] p^2}{i\omega + [\Gamma + (i/c)] p^2}. \quad (13.21)$$

We have already seen that the quantity  $\gamma^2 C$  has the same order (i. e.,  $\varepsilon$ ) as  $g$ . Therefore, the graph



which made the principal contribution in case a), can be disregarded. We have everywhere written  $\gamma$ ,  $C$ ,  $\Gamma$  and  $\lambda$  in place of  $\gamma_0$ ,  $C_0$ ,  $\Gamma_0$  and  $\lambda_0$ , since they coincide in lowest order of perturbation theory.

After the subtraction  $\Sigma(\omega) - \Sigma(0)$ , the subsequent calculation of the integral, and the scale transformation (as in the free-field case), we find

$$-\frac{d \ln \Gamma}{d\xi} = \frac{C\gamma^2 K_4}{1+\mu} + 2, \quad (13.22)$$

where

$$\mu = \frac{\lambda}{\Gamma C}. \quad (13.23)$$

It is necessary to calculate the values of the quantities  $C\gamma^2 K_4$  and  $\mu$  at the fixed point. The first problem is a purely static one. The smoothing of  $C$  in the first  $\varepsilon$ -approximation is determined by the graph

$$\text{Graph} = -\frac{1}{2} n \gamma^2 \int \frac{d^4 p}{p^4} = -\frac{1}{2} n \gamma^2 K_4 \xi. \quad (13.24)$$

The subsequent scale transformation and differentiation with respect to  $\xi$  give

$$\frac{d \ln C}{d\xi} = -\frac{n}{2} \gamma^2 C K_4. \quad (13.25)$$

According to the general theory,  $C$  is none other than the specific heat, the dimension of which is equal to  $\Delta_C = \alpha/\nu$ . Hence,

$$\frac{n}{2} \gamma^2 C^* K_4 = \frac{\alpha}{\nu}. \quad (13.26)$$

Next, from (13.23) and (13.22) we find

$$\frac{d \ln \mu}{d\xi} = -\frac{d \ln \Gamma}{d\xi} - \frac{d \ln C}{d\xi} + \frac{d \ln \lambda}{d\xi} = -\frac{C\gamma^2 K_4}{1+\mu} + \frac{\alpha}{\nu}. \quad (13.27)$$

Equation (13.27) has three fixed points:  $\mu^* = 0$ ,  $\mu^* = \infty$  and  $\mu^* = (2/n) - 1$ . The last one is stable when  $0 < n \leq 2$ , and the first when  $2 \leq n < 4$ . For  $n > 4$  the exponent  $\alpha$  becomes negative and only the point  $\mu^* = \infty$  is stable. From (13.22) we find

$$\frac{d \ln \Gamma}{d\xi} = - \begin{cases} 2 + (2\alpha/n\nu), & 0 < n \leq 2, \\ 2 + (\alpha/\nu), & 2 < n < 4, \\ 2, & n \geq 4. \end{cases} \quad (13.28)$$

The second result corresponds to the situation in which it is the energy which relaxes the most slowly. In this case, the order parameter relaxes with the same frequency. The third result, if we write it with the necessary accuracy (to  $\varepsilon^2$ ) is transformed to  $2 + c\eta$ . In this case the order parameter relaxes more slowly than the energy, since  $\alpha < 0$  for  $n > 4$ . Of interest is the second case, when a new relaxation time, intermediate between the first and third, arises. Unfortunately, it is not clear whether this result has any relation to real sys-

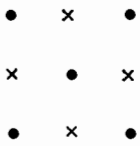


FIG. 7.

tems. Apart from the general problem of the applicability of the  $\epsilon$ -expansion, the question of the adequacy of the model arises.

The investigation of the problem in a space of  $4 - \epsilon$  dimensions is due to Halperin, Hohenberg and Ma.<sup>[11]</sup> We regard this investigation as useful, since it raises important questions about the role of the conservation laws in dynamics and shows how dynamical indices that are not determined by the static indices can arise. This possibility was first pointed out by Polyakov.<sup>[55]</sup> Unfortunately, the model character of the theory prevents us from relating its variants unambiguously to definite experimental situations.

Recently, Abrahams and Tsuneto<sup>[57]</sup> and Abrahams, Grest and Zawadowsky<sup>[58]</sup> have proposed a theory of the dynamical phenomena in helium, based on an application of the ideas of Polyakov<sup>[55]</sup> in combination with the  $\epsilon$ -expansion. Their results can be formulated as follows: in the hydrodynamic regime the dynamic-scaling theory of Halperin and Hohenberg is correct, but the dynamical exponent of the fluctuations is not determined by the static indices.

#### 14. NUMERICAL METHODS OF CALCULATION USING THE RENORMALIZATION GROUP

In principle, a direct method of calculation should consist in the following. We write the most general Hamiltonian, e.g., in the form of an expansion in an integro-power series. We carry out an infinitesimal renormalization-group transformation and seek the fixed points  $\mathbf{g}^*$  of this transformation (cf. Sec. 3). We then find the eigenvalues of the equations linearized about  $\mathbf{g}^*$ , and the "fields" corresponding to these eigenvalues. Of course, all this is in real three-dimensional space. Unfortunately, such a program is easier to formulate than to carry out. Approximate methods are necessary. First of all, it is necessary to restrict the number of components of the vector  $\mathbf{g}$  that are considered, assuming that all but the first few are unimportant. Furthermore, it is more convenient to effect the renormalization in finite steps.

All the numerical calculations that have been carried out up to now<sup>[21, 59, 60, 61]</sup> have used a renormalization operation not in momentum space but in coordinate space, or, more precisely, on lattice systems.

Suppose that we are given a lattice system defined by the values of the variables  $\varphi_i = \varphi(x_i)$  at the sites  $x_i$  in a  $d$ -dimensional space. The Hamiltonian  $\mathcal{H}$  is a function of all the  $\varphi_i$  that is invariant under the group  $G$  of all transformations that do not change the form of the lattice. Therefore, it can be represented in the form of a series

$$\mathcal{H} = \sum g_k I_k(\varphi_i), \quad (14.1)$$

where the  $I_k(\varphi_i)$  are all possible invariants of the group  $G$ .

We integrate over a fraction of the variables ( $\varphi_\mu$ ) and change to new variables  $\psi_j$ , taking the same values as the  $\varphi_i$ , in a new lattice that differs from the old only in size:

$$\mathcal{H}'(\psi_j) = R(\mathcal{H}) = -\ln \int \prod_\mu d\varphi_\mu e^{-\mathcal{H}(\varphi_i)} \quad (14.2)$$

This transformation is equivalent to a nonlinear transformation of the vector  $\mathbf{g}$ :

$$\mathbf{g}' = R(\mathbf{g}). \quad (14.3)$$

The form of the fixed point depends essentially on the choice of the variables  $\psi_j$  and  $\varphi_\mu$ , but the eigenvalues  $\Delta_k$  are invariant. The arbitrariness that remains can be used to improve the convergence.

We shall give some very simple examples. In a simple "cubic" lattice we introduce two sublattices (Fig. 7). We integrate over the "spins" of one sublattice. We then obtain a Hamiltonian  $\mathcal{H}'$  that depends only on the "spins" of the second sublattice. Another way of choosing the variables is shown in Fig. 8. Here the summation is performed for fixed values of the sums of the "spins" surrounding the cells marked by the crosses. Such a calculation was performed by Kadanoff and Houghton,<sup>[21]</sup> who tested their method on the exactly soluble planar Ising model. Of all the invariants of the planar Ising model they kept a small number, corresponding to not-too-distant and not-too-numerous spin couplings (12 in the first variant), and the majority of these were taken into account only as a perturbation. It is remarkable that even such a crude method of calculation gave, with a suitable choice of initial conditions, agreement with the known values of the exponents ( $\nu$  and  $\beta$ ) up to the fourth significant figure.

We cannot go into the details of the calculation here. But we mention, in addition, the variational method formulated by Kadanoff.<sup>[62]</sup> An approximate Hamiltonian is proposed (in place of the exact one), such that, on the one hand, it gives the possibility of carrying out the renormalization procedure explicitly, and, on the other hand, it gives a free energy that is either always larger or always smaller than the exact free energy. It is well known that it is fairly simple to obtain a free energy greater than the exact value. It is much more complicated to obtain a lower bound. It turns out to be sufficient that the difference between the exact and the approximate Hamiltonian be represented by a sum of terms

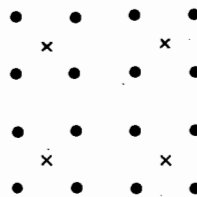


FIG. 8.



each of which changes sign under at least one transformation of the group. All approximations that give an overestimate for the free energy are extremely crude, while those that give a lower bound for the free energy lead to strikingly accurate values for the indices, at least in the two-dimensional Ising model. Apparently, the reason is that the Kramers-Wannier symmetry is respected at each stage of the calculation. Kadanoff, Houghton and Yalabik<sup>[61]</sup> have also found the indices of the three-dimensional Ising model, but it is not yet clear whether the values obtained by them are better than the results of the calculation by means of high-temperature series.

An extremely crude but effective method of calculation for many-component systems has been proposed by A. A. Migdal,<sup>[63]</sup> who confirmed the existence of a phase transition in a planar system of two-component classical spins.

In principle, one can increase the accuracy of the numerical methods; this is impossible in the framework of the  $\epsilon$ -expansion. Unfortunately, the error in the results obtained is not known beforehand.

*Note added in proof:* The idea of the fluctuational nature of first-order phase transitions in complex antiferromagnets has received experimental confirmation in the work of Bloch *et al.*,<sup>[66]</sup> who have measured the antiferromagnetic order in MnO samples subjected to strong uniaxial compression along the [111] axis. In the ordinary state MnO has cubic symmetry. The star of the vectors along which the density modulation occurs consists of the four diagonals of the cube. The twelve components of the corresponding four spin-density vectors decompose into two irreducible representations. A previous neutron scattering analysis<sup>[45]</sup> had shown that an 8-component representation composed of the components of the spin-density vectors perpendicular to the corresponding diagonals of the cube is realized in MnO. In this case the theory predicts a first-order phase transition.<sup>[46]</sup> The application of stress along the [111] axis lowers the symmetry sharply. The largest possible dimensionality of a representation becomes equal to 3. In this situation, for not-too-strong anisotropy, the theory gives a second-order phase transition,<sup>[67]</sup> and this has been observed experimentally.

<sup>1</sup>L. D. Landau and E. M. Lifshitz, *Statisticheskaya fizika* (Statistical Physics), Nauka, M., 1964 (English translation published by Pergamon Press, Oxford, 1969).

<sup>2</sup>A. Z. Patashinskii and V. L. Pokrovskii, *Fluktuatsionnaya teoriya fazovykh perekhodov* (Fluctuation Theory of Phase Transitions), Nauka, M., 1975.

<sup>3</sup>W. M. Fairbank, C. Buckingham, and C. F. Kellers, in: *Proc. of 5th Intern. Conference on Low Temperature Physics*, Madison, Wisconsin, 1957.

<sup>4</sup>M. N. Bagatskii, A. V. Voronel', and V. G. Gusak, *Zh. Eksp. Teor. Fiz.* 43, 728 (1962) [Sov. Phys. JETP 16, 517 (1963)].

<sup>5</sup>A. Z. Patashinskii and V. L. Pokrovskii, *Zh. Eksp. Teor. Fiz.* 50, 439 (1966) [Sov. Phys. JETP 23, 292 (1966)].

<sup>6</sup>L. P. Kadanoff, *Physics* 2, 263 (1966).

<sup>7</sup>V. L. Pokrovskii, *Usp. Fiz. Nauk* 94, 127 (1968) [Sov. Phys. Usp. 11, 66 (1968)].

<sup>8</sup>M. A. Anisimov, *Usp. Fiz. Nauk* 114, 249 (1974) [Sov. Phys. Usp. 17, 722 (1975)].

<sup>9</sup>K. G. Wilson, *Phys. Rev.* B4, 3174, 3184 (1971).

<sup>10</sup>K. G. Wilson and M. E. Fisher, *Phys. Rev. Lett.* 28, 240 (1972).

<sup>11</sup>B. I. Halperin, P. C. Hohenberg, and S.-K. Ma, *ibid.* 29, 1548 (1972).

<sup>12</sup>M. E. Fisher and A. Aharony, *ibid.* 30, 559 (1973); A. D. Bruce and A. Aharony, *Phys. Rev.* B10, 2078 (1974).

<sup>13</sup>K. G. Wilson and J. Kogut, *Phys. Rept.* C12, 75 (1974).

<sup>14</sup>M. E. Fisher, *Rev. Mod. Phys.* 46, 597 (1974).

<sup>15</sup>A. Z. Patashinskii and V. L. Pokrovskii, *Zh. Eksp. Teor. Fiz.* 46, 994 (1964) [Sov. Phys. JETP 19, 677 (1964)].

<sup>16</sup>A. M. Polyakov, *Zh. Eksp. Teor. Fiz.* 55, 1026 (1968) [Sov. Phys. JETP 28, 533 (1969)]; *Pis'ma Zh. Eksp. Teor. Fiz.* 12, 538 (1970) [JETP Lett. 12, 381 (1970)].

<sup>17</sup>A. A. Migdal, *Zh. Eksp. Teor. Fiz.* 55, 1964 (1968) [Sov. Phys. JETP 28, 1036 (1969)].

<sup>18</sup>B. Widom, *J. Chem. Phys.* 43, 3892, 3898 (1965).

<sup>19</sup>C. Domb and D. L. Hunter, *Proc. Phys. Soc.* 86, 1147 (1965).

<sup>20</sup>M. Gell-Mann and F. E. Low, *Phys. Rev.* 95, 1300 (1954).

<sup>21</sup>L. P. Kadanoff and A. Houghton, *ibid.* B11, 377 (1975).

<sup>22</sup>A. Z. Patashinskii, *Zh. Eksp. Teor. Fiz.* 53, 1987 (1967) [Sov. Phys. JETP 26, 1126 (1968)].

<sup>23</sup>A. M. Polyakov, *Zh. Eksp. Teor. Fiz.* 57, 271 (1969) [Sov. Phys. JETP 30, 151 (1970)]; L. P. Kadanoff, *Phys. Rev. Lett.* 23, 1430 (1969).

<sup>24</sup>L. D. Landau, A. A. Abrikosov, and I. M. Khalatnikov, *Dokl. Akad. Nauk SSSR* 95, 497, 773, 1117 (1954) (English translations in "Collected Works of L. D. Landau," Gordon and Breach, N.Y., 1965); V. V. Sudakov, *Zh. Eksp. Teor. Fiz.* 30, 87 (1956) [Sov. Phys. JETP 3, 65 (1956)].

<sup>25</sup>A. I. Larkin and D. E. Khmel'nitskii, *Zh. Eksp. Teor. Fiz.* 56, 2087 (1969) [Sov. Phys. JETP 29, 1123 (1969)].

<sup>26</sup>T. Tsuneto and A. Abrahams, *Phys. Rev. Lett.* 30, 217 (1973).

<sup>27</sup>V. I. Arnol'd, *Obyknoennye differentsial'nye uravneniya* (Ordinary Differential Equations), Nauka, M., 1971 (English translation published by MIT Press, Cambridge, Mass., 1973).

<sup>28</sup>N. N. Bogolyubov and D. V. Shirkov, *Vvedenie v teoriyu kvantovannykh polei* (Introduction to the Theory of Quantized Fields), Nauka, M., 1973 (English translation of earlier edition published by Interscience, N.Y., 1959).

<sup>29</sup>F. J. Wegner, *Phys. Rev.* B5, 4529 (1972).

<sup>30</sup>A. A. Migdal, *Zh. Eksp. Teor. Fiz.* 59, 1015 (1970) [Sov. Phys. JETP 32, 552 (1971)].

<sup>31</sup>K. G. Wilson, *Phys. Rev. Lett.* 28, 548 (1972).

<sup>32</sup>G. M. Avdeeva and A. A. Migdal, *Pis'ma Zh. Eksp. Teor. Fiz.* 16, 253 (1972) [JETP Lett. 16, 178 (1972)]; G. M. Avdeeva, *Zh. Eksp. Teor. Fiz.* 64, 741 (1973) [Sov. Phys. JETP 37, 377 (1973)].

<sup>33</sup>E. Brezin, D. J. Wallace, and K. G. Wilson, *Phys. Rev. Lett.* 29, 591 (1972); *Phys. Rev.* B7, 232 (1973).

<sup>34</sup>A. A. Migdal, *Zh. Eksp. Teor. Fiz.* 62, 1559 (1972) [Sov. Phys. JETP 35, 816 (1972)]; cf. also, P. Schofield, J. D. Litster, and J. T. Ho, *Phys. Rev. Lett.* 23, 1098 (1969).

<sup>35</sup>A. Z. Patashinskii and V. L. Pokrovskii, *Zh. Eksp. Teor. Fiz.* 64, 1445 (1973) [Sov. Phys. JETP 37, 733 (1973)].

<sup>36</sup>E. Brezin and D. J. Wallace, *Phys. Rev.* B7, 1967 (1973).

<sup>37</sup>I. F. Lyuksyutov and V. L. Pokrovskii, *Pis'ma Zh. Eksp. Teor. Fiz.* 21, 22 (1975) [JETP Lett. 21, 9 (1975)].

<sup>38</sup>I. F. Lyuksyutov, V. L. Pokrovskii, and D. E. Khmel'nitskii, *Zh. Eksp. Teor. Fiz.* 69, 1817 (1975) [Sov. Phys. JETP 42, 923 (1975)].

<sup>39</sup>D. R. Nelson, J. M. Kosterlitz, and M. E. Fisher, *Phys. Rev. Lett.* 33, 813 (1974).

<sup>40</sup>D. Mukamel, *ibid.* 34, 481 (1975).

<sup>41</sup>S. Coleman and E. Weinberg, *Phys. Rev.* D7, 1888 (1973).

<sup>42</sup>I. F. Lyuksyutov, *Phys. Lett.* 56A, 135 (1976).

- <sup>43</sup>A. I. Larkin and S. A. Pikin, Zh. Eksp. Teor. Fiz. 56, 1664 (1969) [Sov. Phys. JETP 29, 891 (1969)].
- <sup>44</sup>S. A. Brazovskii and I. E. Dzyaloshinskii, Pis'ma Zh. Eksp. Teor. Fiz. 21, 360 (1975) [JETP Lett. 21, 164 (1975)].
- <sup>45</sup>D. Bloch, R. Maury, C. Vettier, and W. B. Yelon, Phys. Lett. 49A, 354 (1974).
- <sup>46</sup>B. I. Halperin, T. C. Lubensky, and S.-K. Ma, Phys. Rev. Lett. 32, 292 (1974).
- <sup>47</sup>P. B. Vigman, A. I. Larkin, and V. M. Filev, Zh. Eksp. Teor. Fiz. 68, 1883 (1975) [Sov. Phys. JETP 41, 944].
- <sup>48</sup>P. B. Vigman, A. I. Larkin, and V. M. Filev, Yad. Fiz. 22, 1225 (1975) [Sov. J. Nucl. Phys. 22, 636 (1975)].
- <sup>49</sup>D. Mukamel and S. Krinsky, Phys. Rev. B13, 5065, 5078 (1975). P. Bak and D. Mukamel, *ibid.*, p. 586.
- <sup>50</sup>S. A. Brazovskii, I. E. Dzyaloshinskii, and B. G. Kukhareenko, Zh. Eksp. Teor. Fiz. 70, 2257 (1976) [Sov. Phys. JETP 43, 1178 (1976)].
- <sup>51</sup>M. Archibald, J. M. Mochel, and L. Weaver, Phys. Rev. Lett. 21, 1156 (1968).
- <sup>52</sup>G. B. Benedek, in: Polarisation, matiere et rayonnement, P., Presses Universitaires de France, 1969.
- <sup>53</sup>K. Kawasaki, Ann. Phys. (N.Y.) 61, 1 (1970).
- <sup>54</sup>L. P. Kadanoff and J. Swift, Phys. Rev. 166, 89 (1968).
- <sup>55</sup>A. M. Polyakov, Zh. Eksp. Teor. Fiz. 57, 2144 (1969) [Sov. Phys. JETP 30, 1164 (1970)].
- <sup>56</sup>B. I. Halperin and P. C. Hohenberg, Phys. Rev. 177, 952 (1969).
- <sup>57</sup>E. Abrahams and T. Tsuneto, *ibid.* B11, 4498 (1975).
- <sup>58</sup>E. Abrahams, G. S. Grest, and A. Zawadowsky, Preprint Rutgers Univ., 1975.
- <sup>59</sup>K. G. Wilson, in: Proc. of the Cargese Summer School on Field Theory and Critical Phenomena, N. Y., Gordon and Breach, 1973.
- <sup>60</sup>S.-C. Hsu, Th. Niemeijer, and J. D. Gunton, Phys. Rev. B11, 2699 (1975).
- <sup>61</sup>L. P. Kadanoff, A. Houghton, and M. C. Yalabik, Preprint Brown University, 1975.
- <sup>62</sup>L. P. Kadanoff, Phys. Rev. Lett. 34, 1005 (1975).
- <sup>63</sup>A. A. Migdal, Zh. Eksp. Teor. Fiz. 69, 1457 (1975) [Sov. Phys. JETP 42, 743].
- <sup>64</sup>a) A. P. Levanyuk, Zh. Eksp. Teor. Fiz. 36, 810 (1959) [Sov. Phys. JETP 9, 571 (1959)]; b) V. L. Ginzburg, Flz. Tverd. Tela (Leningrad) 2, 2031 (1960) [Sov. Phys. Solid State 2, 1824 (1961)].
- <sup>65</sup>E. Brezin, J. C. le Guillou, and J. Zinn-Justin, Phys. Rev. B10, 892 (1974).
- <sup>66</sup>D. Bloch, D. Hermann-Ronzaud, C. Vettier, W. B. Yelon, and R. Alben, Phys. Rev. Lett. 35, 963 (1975).
- <sup>67</sup>P. Bak, S. Krinsky, and D. Mukamel, Phys. Rev. Lett. 36, 829 (1976).

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