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A "paradox" of electrodynamics

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1. The application of Gauss's theorem to a nonstationary spherically symmetric charge distribution leads to an unexpected result. Suppose that the charge is bounded by a sphere of radius R, and the charge density inside the sphere is determined by the function $\rho(\mathbf{r}, t)$. If we seek the field outside the charged sphere, by Gauss's theorem we obtain

$$4\pi r^2 \varepsilon E(r, t) = 4\pi Q(t), \tag{1}$$

where $Q(t)$ is the total charge of the sphere at time t. From (1) it follows immediately that

$$E(r, t) = \frac{Q(t)}{r^2}, \tag{2}$$

and for the potential $\varphi(\mathbf{E} = -\nabla\varphi)$

$$\varphi(r, t) = -\frac{Q(t)}{r} \qquad (r > R). \tag{3}$$

The results (2) and (3) are, of course, surprising. Gauss's theorem is, in essence, the Maxwell equation div $\mathbf{D} = 4\pi\rho$, and in the Maxwell theory the speed of propagation of interactions (the speed of propagation of the field) is finite and equal to the speed of light $1/\sqrt{\varepsilon\mu}$. From (2) and (3) it is possible to conclude that the field propagates instantaneously. Instantaneous propagation of the field manifestly contradicts the special theory of relativity.

The fact that the expression (3) corresponds to infinitely fast propagation of the interaction can also be seen, in particular, from the fact that D'Alembert's equation (for the vacuum)

$$\Delta\varphi-\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}=-4\pi\rho(\mathbf{r},t) \tag{4}$$

goes over into Poisson's equation

$$\Delta\varphi=-4\pi\rho(\mathbf{r},t) \tag{5}$$

when $c\to\infty$. The expression (3) is just a particular solution of Poisson's equation. The analogous solution of Eq. (4) has the form

$$\varphi(r,t)=\frac{Q(t-(r/c))}{r}\qquad (r\gg R), \tag{6}$$

which corresponds to propagation of the interaction with speed c. Perhaps Gauss's theorem is not always valid? But the answer is much simpler. Any change in the charge density should obey the charge conservation law

$$\frac{\partial\rho}{\partial t}+\operatorname{div}\mathbf{j}=0, \tag{7}$$

where $\mathbf{j}$ is the current density.

From the conditions of the problem there are no charges, i.e., $\rho=0$, outside the sphere of radius R. Therefore, for any spherical surface drawn outside the sphere of radius R, we obtain that $\mathbf{j}\equiv\rho\mathbf{v}=0$. Consequently, the total charge inside such a sphere remains constant:

$$Q=\int\rho(t)\,dV=\text{const.} \tag{8}$$

But in this case there is no difference between (4) and (6). The answer has turned out to be rather unexpected: the charge conservation law does not allow us to obtain a varying total charge inside a certain spherical region if $\rho=0$ outside it. The spherical symmetry is, of course, not essential for this conclusion.

2. The result (2) follows, of course, not only from Gauss's theorem but also from the exact solution of the field equations. If we start from the system of D'Alembert equations for the potentials:

$$\Delta\varphi-\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}=-4\pi\rho(\mathbf{r},t),\qquad \Delta A-\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2}=-\frac{4\pi}{c}\mathbf{j}(\mathbf{r},t), \tag{9}$$

it is simplest to expand the fields, charges and currents $\mathbf{E},\mathbf{H},\mathbf{A},\varphi,\rho$ and $\mathbf{j}$ in Fourier integrals, i.e.,

$$\mathbf{A}(\mathbf{r},t)=\int\mathbf{A}_{\mathbf{k},\omega}e^{i\mathbf{kr}-i\omega t}\,d\mathbf{k}d\omega,$$

etc. As is well known, the differential relations between the quantities of interest to us in this case go over into algebraic relations between the Fourier components. For example, (7) becomes

$$\mathbf{kj}_{\mathbf{k},\omega}-\omega\rho_{\mathbf{k},\omega}=0. \tag{10}$$

Hence,

$$\rho_{\mathbf{k},\omega}=\frac{1}{\omega}\mathbf{kj}_{\mathbf{k},\omega}=\frac{k}{\omega}\mathbf{j}_{\mathbf{k},\omega}. \tag{11}$$

The latter equality is written with allowance for the spherical symmetry of the problem: $\mathbf{j}_{\mathbf{k},\omega}$ can point only along $\mathbf{k}$. Obviously, we can also write

$$\mathbf{j}_{\mathbf{k},\omega}=\frac{\mathbf{k}}{k^2}\rho_{\mathbf{k},\omega}. \tag{12}$$

Eqs. (9) are written in terms of Fourier components as follows:

$$\begin{aligned}&\left(k^2-\frac{\omega^2}{c^2}\right)\varphi_{\mathbf{k},\omega}=4\pi\rho_{\mathbf{k},\omega},\\&\left(k^2-\frac{\omega^2}{c^2}\right)\mathbf{A}_{\mathbf{k},\omega}=\frac{4\pi}{c}\mathbf{j}_{\mathbf{k},\omega},\end{aligned} \tag{13}$$

whence it follows immediately that

$$\varphi_{\mathbf{k},\omega}=\frac{4\pi\rho_{\mathbf{k},\omega}}{k^2-(\omega^2/c^2)},\qquad \mathbf{A}_{\mathbf{k},\omega}=\frac{4\pi}{c}\frac{\mathbf{j}_{\mathbf{k},\omega}}{k^2-(\omega^2/c^2)}. \tag{14}$$

The expressions (14) for $\varphi_{\mathbf{k},\omega}$ and $\mathbf{A}_{\mathbf{k},\omega}$ have poles at $k^2=\omega^2/c^2$. Usually the presence of a pole leads to the emission of electromagnetic waves but in the case under consideration there is no radiation at all. This can be seen immediately after going over from the potentials to the fields. Making use of the relations $\mathbf{E}=-\nabla\varphi-(1/c)\dot{\mathbf{A}}$, $\mathbf{H}=\operatorname{curl}\mathbf{A}$, we obtain

$$\begin{aligned}&\mathbf{E}_{\mathbf{k},\omega}=-i\mathbf{k}\varphi_{\mathbf{k},\omega}+\frac{i\omega}{c}\mathbf{A}_{\mathbf{k},\omega},\\&\mathbf{H}_{\mathbf{k},\omega}=i[k\mathbf{A}_{\mathbf{k},\omega}].\end{aligned} \tag{15}$$

From (14) it can be seen that $\mathbf{A}_{\mathbf{k},\omega}$ points in the direction of $\mathbf{j}_{\mathbf{k},\omega}$, and $\mathbf{j}_{\mathbf{k},\omega}$, as can be seen from (12), points in the direction of the vector $\mathbf{k}$. Therefore, it follows from (15) that $\mathbf{H}_{\mathbf{k},\omega}=0$. The spherical symmetry of the problem has led to the disappearance of the magnetic field. From this the absence of radiation is obvious.

We now find the electric field. From (15),

$$\mathbf{E}_{\mathbf{k},\omega}=-4\pi i\frac{\mathbf{k}\rho_{\mathbf{k},\omega}-(\omega/c^2)\mathbf{j}_{\mathbf{k},\omega}}{k^2-(\omega^2/c^2)}. \tag{16}$$

We shall transform the numerator of (16) with the aid of (11) and (12):

$$\mathbf{k}\rho_{\mathbf{k},\omega}-\frac{\omega}{c^2}\mathbf{j}_{\mathbf{k},\omega}=\frac{\mathbf{k}}{k^2}\left(k^2-\frac{\omega^2}{c^2}\right)\rho_{\mathbf{k},\omega}=\frac{1}{\omega}\left(k^2-\frac{\omega^2}{c^2}\right)\mathbf{j}_{\mathbf{k},\omega}. \tag{17}$$

Substituting (17) into (16), we see that the pole that exists in the expression for the potentials disappears in the expression for the electric field. As a result, we have from (16)

$$\mathbf{E}_{\mathbf{k},\omega}=-4\pi i\frac{\mathbf{k}}{k^2}\rho_{\mathbf{k},\omega}=\frac{4\pi i}{\omega}\mathbf{j}_{\mathbf{k},\omega}. \tag{18}$$

If we substitute this value of the Fourier component into the expression

$$\mathbf{E}(\mathbf{r},t)=\int\mathbf{E}_{\mathbf{k},\omega}e^{i\mathbf{kr}-i\omega t}\,d\mathbf{k}\,d\omega$$

and perform the integration, we obtain formula (2). In conclusion, we note only that the absence of radiation in the case of a spherically symmetric charge distribution also follows from the fact that in this case all the multipole moments are constant.

3. One can point to yet another case in which, at first glance, an infinitely fast signal appears.[1] We shall consider a sphere, the material of which has conductivity σ and dielectric premittivity ε. From the Maxwell equations there follows the dependence for the variation of the volume charge density at any point:

$$\rho(t) = \rho(0)\, e^{-\sigma t/\varepsilon} \tag{19}$$

It can be seen from (19) that at any point where $\rho(0) = 0$ charge density cannot appear. We shall suppose now that at time $t = 0$, inside a very small sphere concentric with the original sphere, a volume (or even a surface) charge density is created. This charge immediately begins to decay in accordance with (19); again in accordance with (19), it cannot appear anywhere inside the sphere, and, consequently, by the charge conservation law, it must appear immediately on the surface of the sphere in the form of surface charge, irrespective of the radius of the sphere. The appearance of the charge on the surface of the sphere could serve as a signal, and, as we see, an infinitely fast one.

But it is simply impossible to send such a signal. First, as we have seen in considering the previous case, it is impossible to create the charge instantaneously in the interior region: the charge conservation law would be violated by this. However, we can postulate one further assumption. Suppose that a charge was introduced into the very small sphere concentric with the original sphere at some much earlier time. To prevent it from flowing away we confine it in a thin shell of an ideal dielectric. This situation is stationary; then, at a certain time, we take away the insulating shell. Then, on the basis of our discussions, it would seem that the charge should appear without delay on the surface of the large sphere. It is not difficult to find the error in this argument. The charge situated inside the original sphere and separated from it by the insulating shell gives rise to an electrostatic displacement in the material of the sphere. Free charges appear at the boundary with the insulating shell in the dielectric. There will also be such charges on the outer boundary of the dielectric sphere. The removal of the insulating shell inside the sphere only gives rise to compensation of the "added" charge and the free charge on the interior boundary of the original sphere (these charges are equal in absolute magnitude). As regards the charge on the exterior boundary of the sphere, it simply remains unchanged.

Note added in proof. As is frequently the case, something which appeared to be a "paradox" for some raises no question for others. Having become acquainted with this note, Ya. B. Zel'dovich and I. A. Yakovlev remarked that the charge conservation law forbids the very formulation of the problem. But it is precisely this which forms the content of the explanation given in the note.

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