Stresses produced in gasses by temperature and concentration inhomogeneities. New types of free convection

M. N. Kogan, V. S. Galkin, and O. G. Fridlender

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The main results of theoretical investigation of slow (Reynolds number $\mathbf{Re}\sim 1$) non-isothermal (temperature drop in the gas $\theta = \Delta T/T \sim 1$) are reported. These flows are described by equations that differ from the classical Navier-Stokes equations for a compressible liquid in that the momentum equation contains besides the viscous-stress tensor, also a temperature-stress tensor of the same order of magnitude. The question of the influence of temperature stresses on the motion of the gas are analyzed, as are the forces acting on bodies placed in the gas. This question was first raised long ago by J. Maxwell, who used implicitly linearization in θ and reached the conclusion that the temperature stresses cause neither motion of the gas nor forces. However, when θ is not small, a new type of convection of the gas appears in the absence of external forces (e.g., of gravitation), namely, the temperature stresses cause the gas to move near uniformly heated (cooled) bodies; some examples of this convection are presented. In addition, for the case of small θ, an electrostatic analogy is established, describing the force interaction between these bodies as a result of the temperature stresses. The problem of the flow around a uniformly heated sphere at $\mathbf{Re}_\infty \ll 1$ (the Stokes problem) is solved: the temperature stresses exert an ever increasing influence on the resistance of the sphere with increasing sphere temperature. Analogous phenomena, produced in gas mixtures by concentration (diffusion) stresses, are indicated.

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CONTENTS

1. INTRODUCTION

The first part of the title of this article is the same as that of a known paper published by Maxwell in 1879.[1] In that paper he formulated and investigated, for the first time, the existence of temperature stresses in a gas and of the gas motion due to these stresses.

In an analysis of the radiometric effect observed by Crooks,[2] Maxwell advanced a hypothesis that one of the possible causes of this effect are temperature stresses. Using the kinetic theory developed by him[3] to obtain an expression for the temperature stresses and taking these stresses into account in the equations of motion, Maxwell, however, arrived at the conclusion that these stresses can cause neither motion of the gas nor forces exerted by the gas on heated bodies. On the other hand, analyzing the boundary conditions for the equations describing the microscopic motion of the gas, he developed a theory of temperature slip, to which he attributed the radiometric effect (this is valid if the gas is dense enough).

However, the conclusion that no motion can be produced by temperature stresses turned out to be in general incorrect. As will be shown below, this conclusion is the result of the linearized expression used by Maxwell for the temperature stresses. It appears that it is precisely because of this conclusion by Maxwell that for decades after the exact expressions were derived for the temperature stresses[4,5] no attention was called to the possible gas convection caused by these stresses.

The Navier-Stokes equations which are universally used to describe gas flow do not contain temperature stresses. Used to an equal degree for gasses and liquids, these equations were obtained under the assumption that the stress tensors and the strain rates are linearly connected (Newton's law) as are the heat-flow vectors with the temperature gradient (Fourier's law).

These linear relations follow both from phenomenological considerations and from the thermodynamics of irreversible processes under conditions of small deviation of the state of the medium from thermodynamic equilibrium. However, in order to "establish which deviations can be regarded as small" and "determine in fact the limits of applicability of theory of irreversible processes, it is necessary to have a more general theory, from which the present theory would be obtained as a particular case. This more general theory is the kinetic theory of gasses."[6]

We shall use below the equations of motions obtained on the basis of the kinetic theory of gasses.[7,8] These equations, together with the Navier-Stokes stress tensor, which is proportional to the viscosity coefficient and to the first derivatives of the elastic components with respect to the coordinates, contain nonlinear terms proportional to the square of the viscosity coefficient, which are usually assumed to be small in comparison with the Navier-Stokes terms and are customarily neglected.

It has been shown in[9,10], however, that for non-isothermal gas flows at Mach numbers $\mathbf{M} \ll 1$ and at Reynolds numbers $\mathbf{Re} \sim 1$ ("slow" flows) the Navier-Stokes equations are insufficient, and account must be taken of some of the aforementioned nonlinear terms. The purpose of the present review is to cast light on certain results obtained in this new "non-Navier-Stokes" gas-dynamics. In particular, we shall consider a new type of gas convection in the absence of external forces, due to temperature stresses (thermal-stress convection). In Sec. 6 we shall describe analogous phenomena occurring in gas mixtures.

2. EQUATIONS AND BOUNDARY CONDITIONS OF SLOW NON-ISOTHERMAL FLOWS OF A SIMPLE GAS

We consider below gas flow in the "continuous medium" regime. That is, flows with small Knudsen number:

$$\mathrm{Kn}=\frac{\lambda_*}{L_*}=\frac{\mu_*}{\rho_* L_*}\left(\frac{k}{m}T_*\right)^{-1/2}\approx\frac{\mathrm{M}}{\mathrm{Re}}\ll 1, \tag{2.1}$$

where $\mathbf{M}=u_*/a_*$, $\mathbf{Re}=\rho_* u_* L_*/\mu_*$ are the Mach and Reynolds numbers, while λ_*, L_*, μ_*, ρ_*, T_*, u_*, a_* are the characteristic values of the mean free path of the molecules, of the flow dimension, of the shear-viscosity coefficient, the density, the temperature, and the gas and sound velocities.

The equations of continuity, state, momentum, and energy of a simple gas are respectively

$$\left.\begin{aligned}&\frac{\partial\rho u_i}{\partial x_i}=0,\qquad p=\frac{k}{m}\rho T,\\&\rho u_j\frac{\partial u_i}{\partial x_j}+\frac{\partial P_{ij}}{\partial x_j}=0,\qquad P_{ij}=p_{ij}+p\delta_{ij},\\&\frac{3}{2}\rho u_i\frac{\partial\ln T}{\partial x_i}+\frac{\partial q_i}{\partial x_i}+P_{ij}\frac{\partial u_i}{\partial x_j}=0,\end{aligned}\right| \tag{2.2}$$

where P_{ij} is the stress tensor, $\mathbf{q}$ is the heat-flux vector, k/m is the gas constant, and p is the pressure.

In the phenomenological mechanics of continuous media, the system (2.2) is made closed by using Newton's and Fourier's laws for the stresses and for the heat flux

$$p_{ij}^{(1)}=-2\mu\left[\frac{\partial u_i}{\partial x_j}\right],\qquad \mathbf{q}^{(1)}=-\eta\nabla T, \tag{2.3}$$

where $[A_{ij}]=(1/2)(A_{ij}+A_{ji})-(1/3)\,\delta_{ij}A_{kk}$. Similar linear expressions are given by the thermodynamics of irreversible processes. The transport coefficients μ and η in (2.3) must be obtained from experiment. An analogous situation obtains in the phenomenological theories aimed at constructing more general (nonlinear) models of continuous media: to construct generalized equations and, in particular, to determine the coefficients contained in them, additional information must be obtained from experiment concerning the properties of the medium.

Since we are interested in effects for which experimental data are still unavailable, it appears that the only possibility lies in the kinetic theory. It is then, of course, necessary to know the laws governing the interaction between the molecules. These laws, however, can be established in independent experiments (e.g., with molecular beams), or else obtained on the basis of already investigated phenomena (viscosity, thermal conductivity, from the virial coefficients, etc.). In any case, given the molecule-interaction law, to obtain the nonlinear terms of the stress tensor and of the heat-flux vector, no additional assumptions (or information) are necessary, other than those used in the derivation of the Navier-Stokes equations.

In dimensionless form, the Boltzmann kinetic equation is

$$\frac{\partial f}{\partial t}+\xi_r\frac{\partial f}{\partial x_r}=\frac{1}{\mathrm{Kn}}J, \tag{2.4}$$

where J is the collision integral, and the distribution function f is referred to $\rho_* a_*^{-3}$. The well known Chapman-Enskog method yields an asymptotic expansion of the solution of this equation in powers of the Knudsen number $\mathbf{Kn}\to 0$:

$$f\sim\sum_{k=0}^{\infty}f^{(k)}.$$

To each term of this expansion correspond terms of the expressions for the stresses and for the heat flux

$$\begin{aligned}&P_{ij}=p\delta_{ij}+p_{ij}^{(1)}+p_{ij}^{(2)}+\ldots,\quad p_{ij}^{(2)}=p_{ij}^{(T)}+p_{ij}^{(*)},\\&q_i=0+q_i^{(1)}+q_i^{(2)}+\ldots,\quad q_i^{(2)}=\alpha_1\frac{\mu^2}{\rho T}\frac{\partial u_j}{\partial x_j}\frac{\partial T}{\partial x_i}+\ldots\end{aligned} \tag{2.5}$$

The first terms of these expansions correspond to the Euler approximation, the first two correspond to the Navier-Stokes approximation, the first three to the so called Burnett approximations,[7,8] etc. In relation (2.5), the temperature stresses

$$p_{ij}^{(T)}=K_2\frac{\mu^2}{\rho T}\left[\frac{\partial^2 T}{\partial x_i\,\partial x_j}\right]+K_3\frac{\mu^2}{\rho T^2}\left[\frac{\partial T}{\partial x_i}\frac{\partial T}{\partial x_j}\right] \tag{2.6}$$

are separated from the Burnett stresses, and the rest contain derivatives of the velocity and pressure components

$$p_{ij}^{(*)}=K_1\frac{\mu^2}{p}\frac{\partial u_k}{\partial x_k}\left[\frac{\partial u_i}{\partial x_j}\right]+\ldots$$

The coefficients K_i and α_i are of the order of unity and depend on the type of molecule. In particular, for molecules whose interaction potential is given by a power law, we have $\mu\sim T^s$, $K_2=\omega_1$ and $K_3=\omega_1 s-\omega_3$, where ω_1 and ω_3 are positive numbers[7] (for Maxwellian molecules $s=1$, $\omega_1=3$, $\omega_3=0$, and for elastic-sphere molecules we have $s=1/2$, $\omega_1=2.418$, ω_3

=0.990).

Since $\mu_* \approx \rho_* a_* \lambda_*$, the following estimates hold true:

$$\left.\begin{array}{ll} p_{ij}^{(1)} \sim \mathrm{Kn}\,\rho_* a_* u_*, & p_{ij}^{(T)} \sim \mathrm{Kn}^2\,\rho_* a_*^2 \theta, \\ p_{ij}^{(*)} \sim \mathrm{Kn}^2 \rho_* u_*^2, & p_{ij}^{(k\geqslant 3)} \leqslant \mathrm{Kn}^3\,\rho_* a_* u_* \theta, \\ q^{(1)} \sim \mathrm{Kn}\,\rho_* a_*^3, & q^{(2)} \sim \mathrm{Kn}^2\,\rho_* a_*^2 u_*. \end{array}\right\} \tag{2.7}$$

Here θ is the characteristic value of the dimensionless temperature drop in the gas

$$\theta = \frac{(\Delta T)_*}{T_*}. \tag{2.8}$$

Putting in (2.2) $P_{ij} = p\delta_{ij} + p_{ij}^{(1)} p_{ij}^{(2)}$ and $q_i = q_i^{(1)} + q_i^{(2)}$, we obtain Burnett's equations,[5] which were regarded for a long time as necessary for the description of gas motion at those values of the Knudsen numbers at which the Navier-Stokes equations no longer are valid.[1)] Indeed, in a number of cases (ultrasonic oscillations,[11] exact solutions of the Boltzmann equation for special flows of the dispersal type[12]), Burnett's equations made it possible to advance into the region of large Knudsen numbers, i.e., to obtain agreement between the solutions of the macroscopic equations and the solutions of Boltzmann's equation in a wider range of the Knudsen numbers. In the general case, however, the considered series in powers of $\mathbf{Kn}$ do not converge in the usual sense: one can count only on asymptotic convergence as $\mathbf{Kn} \to 0$.[8,13] Therefore the inclusion of the next term of the series does not generally increase the region of applicability of the Knudsen-number series, and the Burnett approximation refines the Navier-Stokes approximation only negligibly as $\mathbf{Kn} \to 0$. This is why interest in Burnett's equations has waned considerably and it has been suggested that these equations are useless.[14]

In this connection, the following question was raised in[9,10]: are there such flows of a gas as a continuous medium (i.e., flows as $\mathbf{Kn} \to 0$), for the description of which in the principal approximation it is necessary to take into account some of the Burnett terms of the transport equations, i.e., are the Navier-Stokes equations universal equations of gasdynamics?

The concept of a gas as a continuous medium is realized at $\mathbf{Kn} \approx \mathbf{M}/\mathbf{Re} \to 0$. i.e., for two principal limiting cases: 1) $\mathbf{M} = 0(1)$, $\mathbf{Re} \to \infty$ ("fast" flows described by the Euler equations and by the Prandtl boundary-layer equations), and 2) $\mathbf{Re} = 0(1)$, $\mathbf{M} \approx \mathbf{KnRe} \to 0$ ("slow" flows). The present paper is devoted to an analysis of the second case.

The characteristic velocity of slow flows is $u_* \approx a_* \mathbf{M} \approx a_* \mathbf{KnRe}$. If the temperature changes in such a flow are due to conversion of the kinetic energy of the gas into thermal energy as a result of dissipative processes, then the characteristic temperature drop is $\theta \sim \mathbf{M}^2 \sim \mathbf{Kn}^2 \ll 1$. In this "isothermal" case, according to the estimates (2.7), all the Burnett terms of the conservation equations are small in comparison with the Navier-Stokes terms (their ratio is of the order of $\mathbf{Kn}^2$), and can be neglected. Consequently, the Navier-Stokes equations are correct for the description of isothermal flows.

A flow is called non-isothermal if $\theta \sim 1$. Such flows take place, e.g., if bodies that are warmer or colder than the gas are placed in the gas. Then according (2.7), we have $p_{ij}^{(1)} \sim \mathbf{Kn}^2 \mathbf{Re}\, \rho_* a_*^2$, $p_{ij}^{(T)} \sim \mathbf{Kn}^2 \rho_* a_*^2$, and the temperature stresses are of the same order of magnitude as the ordinary viscous stresses that enter in the Navier-Stokes equations, since $\mathbf{Re} \sim 1$. The remaining terms p_{ij} are of order $\mathbf{Kn}^2$ in comparison with $p_{ij}^{(T)}$. Subject to the same error, we have $\mathbf{q} = -\eta \nabla T$.

Thus, in the case of slow non-isothermal flows, the Navier-Stokes equations are not correct: it is necessary to take into account the temperature stresses in the momentum equation. The reason is not that the Burnett stresses are large, but that the velocities are low; since $\mathbf{Re} = 0(1)$, the velocities are of the order of the "viscous" velocity

$$u_V = \frac{\mu_*}{\rho_* L_*} \sim a_* \,\mathrm{Re}\,\mathrm{Kn}. \tag{2.9}$$

Consequently, the pressure perturbations are small, namely

$$p = p_* \left(1 + \mathrm{Kn}^2 P\right), \tag{2.10}$$

where $p = 0(1)$ is the dimensionless alternating part of the pressure.

Changing over to the dimensionless variables

$$T^* = \frac{T}{T_*}, \quad \rho^* = \frac{\rho}{\rho_*}, \quad p^* = \frac{p}{p_*}, \quad \mu^* = \frac{\mu}{\mu_*}, \quad x_i^* = \frac{x_i}{L_*}, \quad \mathrm{v} = \frac{\mathrm{u}}{u_V},$$

omitting henceforth the superior asterisk marking the dimensionless quantities, and discarding the terms $O(\mathbf{Kn}^2)$ in comparison with the principal terms, we obtain the sought equations of non-isothermal stationary flows of a simple gas:

$$\left.\begin{array}{c} \rho = \dfrac{1}{T}, \\ \nabla \mathbf{v} = \mathbf{v} \nabla \ln T, \\ \dfrac{1}{T} v_j \dfrac{\partial v_i}{\partial x_j} + \dfrac{\partial P}{\partial x_i} = \dfrac{\partial}{\partial x_j} 2\mu \left[\dfrac{\partial v_i}{\partial x_j}\right] \\ \qquad - \dfrac{\partial}{\partial x_j} \mu^2 \left(K_2 \left[\dfrac{\partial^2 T}{\partial x_i\, \partial x_j}\right] + \dfrac{K_3}{T} \left[\dfrac{\partial T}{\partial x_i} \dfrac{\partial T}{\partial x_j}\right]\right), \\ \dfrac{2}{3} \mathbf{v} \cdot \nabla \ln T = \nabla_i^{\prime} (\mu \nabla T). \end{array}\right\} \tag{2.11}$$

We emphasize that the difference between (2.11) and the corresponding equations in the Navier-Stokes approximation consists in the presence of the last two terms in the right-hand side of the momentum equation, with coefficients K_2 and K_3.

For the sake of clarity, we shall write down the equations for a simple monatomic gas, when the ratio of the

1) Solutions of problems in the dynamics of a rarefied gas with the aid of Burnett's equations have been the subject of a rather large number of papers, in which shock-wave structure, propagation of ultrasonic oscillations, planar and cylindrical Couette flows, etc. were investigated. A critical review of the main results was presented by Scherman and Tolbot.[11]

specific heats is $\varkappa = 5/3$ and the Prandtl number is $\mathbf{Pr} = C_p \mu_* / \eta_* = 2/3$. More general forms of (2.11) are contained in[10, 15, 28]. In the case of a polyatomic gas, the structure of the equations remains unchanged, and only the values of the coefficient change.

Equations (2.11) are at first glance of higher order than the system of transport equations in the Navier-Stokes approximation. In fact, however, the temperature stresses do not increase the order of the system of the equations, and the number of boundary conditions for (2.11) is the same as for the Navier-Stokes equations. This can be easily verified by recognizing that the highest-order derivative in $\partial p_{ij}^{(T)}/\partial x_j$ is proportional to $\partial \nabla^2 T / \partial x_i$, and by eliminating $\nabla^2 T$ with the aid of the energy equation (the last equation of (2.11)).

The indicated transformations reduce the momentum equation to the form

$$\left.\begin{aligned} &\frac{1}{T}(\mathbf{v}\nabla)\mathbf{v} + \nabla\Pi = \Pi^{(1)} - \omega T^{2s-2}(\nabla T)^2 \nabla T + \frac{4\omega}{3s}(\mathbf{v}\nabla \ln T)\nabla T^s, \\ &\Pi_i^{(1)} = \frac{\partial}{\partial x_j} T^s \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \quad \omega = \frac{1}{2}(s\omega_1 + \omega_3), \\ &\Pi = P + \frac{1}{2}\left(s\omega_1 - \frac{\omega_3}{s}\right) T^{2s-1}(\nabla T)^2 + \frac{2}{3}\left(1 + \frac{2}{3}\omega_1\right) T^{s-1}(\mathbf{v}\nabla T). \end{aligned}\right\} \tag{2.12}$$

It is important to emphasize that part of the temperature and Navier-Stokes stress tensor is contained in the quantity Π, which can be regarded as a generalized pressure, and its structure must be known only when the forces are calculated. Equation (2.12) was written down for the particular case $\mu = \eta = T^s$, when the structure of the Burnett transport coefficients is known.[7] A more general form of the equations of slow flows, which is valid also for a polyatomic gas, is given in[15].[2)]

We proceed to the boundary conditions. From the kinetic theory we have the boundary slip conditions[8, 16]

$$\begin{aligned} &v_n = O(\mathrm{Kn}^2), \quad v_\tau = \beta_1 \frac{\partial T_w}{\partial x_\tau} + \beta_2 \,\mathrm{Kn} \left.\frac{\partial v_\tau}{\partial x_n}\right|_{n=0} + O(\mathrm{Kn}^2), \\ &T_{n=0} = T_w + \beta_3 \,\mathrm{Kn} \left.\frac{\partial T}{\partial x_n}\right|_{n=0} + O(\mathrm{Kn}^2), \quad \beta_i = O(1), \end{aligned} \tag{2.13}$$

where n and τ are the normal and tangent to the wall, and it is assumed that $\theta = O(1)$. It follows from (2.13) that the usual non-flow conditions $v_n = 0$ and the conditions that the gas and wall temperatures be equal $T = T_w$ are valid with a relative error $O(\mathbf{Kn})$; for the tangential component of the velocity, the principal condition is the temperature slip condition

$$v_\tau = \beta \frac{\partial T_w}{\partial x_\tau} \qquad (\text{at} \quad n = 0) \tag{2.13a}$$

(if the change in the wall temperature T_w along the surface of the body is on the order of unity, i.e., $\theta_w \sim 1$).

In accordance with the classification given in[17], this boundary condition is strong, since it can cause autonomous motion of the gas, in contrast to the other conditions in (2.13), which result only in corrections to the main flow due to other causes.[16]

It follows from (2.13a) that at $\theta_w \sim 1$ the temperature slip gives rise to flows with velocities on the order of u_V and must be taken into account together with the temperature stresses, i.e., the temperature stresses and the temperature slips should produce effects of the same order.

The equations written out above are valid outside a Knudsen wall layer of thickness on the order λ. It is therefore necessary to estimate the changes of the stress tensor and of the heat-flow vector across this layer. This is usually done[9] by integrating the complete conservation equations across the Knudsen layer. It turns out that these changes can be neglected with a relative error $O(\mathbf{Kn})$.

Thus, the momentum and heat fluxes into the body can be calculated in the usual manner, i.e., by using the values of the stresses and heat flux directly on the wall, and by neglecting the presence of the Knudsen layer. This completes the formulation of the problem. Thus, the Navier-Stokes equations cannot be used to describe slow non-isothermal gas flows: the gas dynamic equations of such flows are Eqs. (2.11) and (2.12) with boundary conditions $v_n = 0$, $T_{n=0} = T_w$, and the equation of temperature slip on the wall (2.13a).

3. THERMAL STRESS (TEMPERATURE) CONVECTION

The fundamental question is whether temperature stresses give rise to motion of the gas, i.e., whether they cause a new type of gas convection in the absence of external forces.

In order for the gas near heated (cooled) bodies to be at rest in the case when the usual sources of stimulated and free convections are absent, and the wall surfaces S_i bordering on the gas are isothermal (otherwise $\mathbf{v} \neq 0$ because of the temperature slip), it is necessary to have

$$\nabla(\eta\nabla T) = 0, \quad \mathrm{rot}\,[Y_T(\nabla T)^2 \nabla T] = 0 \tag{3.1}$$

(where Y_T is a known function of T), with conditions $T = T_{wi}$ on S_i and $T = T_\infty$ as $x \to \infty$, where T_{wi} are given constants. An analysis of these conditions (which consists essentially of integrating the second equation of (3.1) over the isothermal surfaces or else along normals to them) shows[9] that they are satisfied for an isothermal sphere, for concentric spheres, for coaxial circular cylinders and for parallel planes heated to different temperatures, when the temperature stresses can become balanced by the pressure.

In all the remaining cases, in a gas bounded by walls that are uniformly heated (cooled) to different temperatures, a new type of convection takes place, namely thermal-stress (temperature) convection, i.e., gas motion caused by temperature stresses.

In the general case, the velocities of the thermal-stress convections are of the order of u_V, and this convection is described by Eqs. (2.11) and (2.12) with the adhesion conditions. This system of equations is quite complicated, at any rate more complicated than the Navier-Stokes equations. As a result, only very sim-

2) Expression (1.7) for Y_T in the cited reference contains a misprint: the symbols > should be replaced by minus signs.

ple examples of convections have been constructed so far.[10,18] We shall describe some of them.

If two semi-infinite plates meeting at an angle α (Fig. 1) are heated to different temperatures, then in accordance with classical gasdynamics the gas between them should be at rest, and a certain temperature distribution should be established in it. However, owing to the existence of temperature stresses, the gas moves between the plates. This problem is the analog of the well known Hamel problem for a viscous liquid (see, e.g.,[16] p. 72 of original). In the latter, however, the motion is due to escape of liquid from the vertex of the corner (the presence of outflow), but in our case it is due to temperature stresses and takes place at zero outflow. The problem is self-similar: in a cylindrical coordinate system the variables are sought in the form $T=T(\varphi)$, $v_r=V(\varphi)/r$, $v_\varphi=0$, $\Pi=\pi(\varphi)r^{-2}$. For T, V, and π we obtain a system of ordinary differential equations, the analytic solution of which was obtained at $\theta \ll 1$ or at $\alpha \ll 1$. A numerical analysis is given in[18]. The flow picture is shown in Fig. 1, where the profile of $V(\varphi)$ is given.

A similar self-similar solution is the solution of the problem of the thermal-stress flow between coaxial conical surfaces.[18]

Definite information on the character of the flows due to temperature stresses can be obtained by a perturbation method.[10,18] We point out, for example, thermal-stress convection between slightly bent parallel plates heated to different temperatures, shown schematically in Fig. 2. In this case the thermal-stress convection causes only weak perturbations in the temperature distribution, and consequently also in the heat flow, compared with the distributions obtained within the framework of the Navier-Stokes equations.

However, if the curvature of the walls is not small (as, e.g., when real rough surfaces of bodies come close together), then the temperature distribution and the heat transfer are changed in order of magnitude, and this can turn out to be essential for the correct understanding of contact heat transfer.

If $\theta \ll 1$, then the velocities of the thermal-stress convection are $u_T \ll u_V$, and consequently also $\mathbf{Re}=u_T/u_V \ll 1$. As a result, the flow will be of the "Stokes" type and the initial equations take the form

$$\nabla \mathbf{v} = 0, \quad \nabla^2 T = 0, \tag{3.2}$$

$$\nabla \Pi - \nabla \mathbf{v} = Y_{T=1}(\nabla T)^2 \nabla T, \quad \Pi = P + X_{T=1}(\nabla T)^2, \tag{3.3}$$

where Y and X are known functions of T, and the tensor of the local (on the wall) temperature stresses is

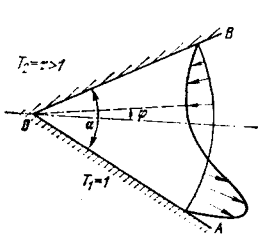

FIG. 1. Thermal-stress flow in a corner.

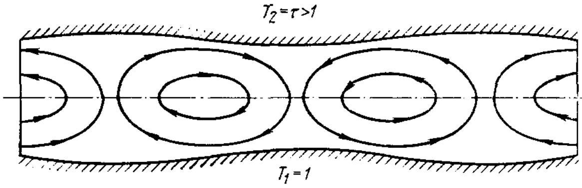

FIG. 2. Thermal-stress flow between slightly bent walls.

$$p_{ij}^T \approx K_2 \mu^2 \frac{\partial^2 T}{\partial x_i\, \partial x_j}. \tag{3.4}$$

It follows therefore that at $\theta \ll 1$ we have

$$v \sim \theta^3, \quad P \sim \theta^2. \tag{3.5}$$

The "temperature" problem was separated from the "velocity" problem. After finding the temperature from the Laplace equation, which the temperature satisfies at $\theta \ll 1$, the right-hand side of the momentum equation (3.3) becomes known. It can be interpreted as an external force proportional to $|\nabla T|^3$ directed opposite ($Y_T < 0$) to the temperature gradient; this enables us to obtain quite simply a qualitative picture of the streamlines (Fig. 3).

The foregoing calls for two remarks.

The flows of the class considered here were regarded above as slow. The velocities in such flows, however, need not necessarily be very small. Indeed, assume that a body with temperature $T_* \gg T_\infty$ is placed in a gas that is at rest at infinity. Near the body, the Knudsen number is

$$\mathrm{Kn} \approx \mathrm{Kn}_\infty \left(\frac{T_*}{T_\infty}\right)^{s+(1/2)} \ll 1 \qquad (\mu \sim T^s).$$

The velocities of the thermal-stress convection near the body are of the order of the viscous velocity, and the characteristic number is

$$\mathrm{M}_* = \frac{u_V}{a_*} = \frac{\mu_*}{a_* \rho_* L_*} \approx \mathrm{Kn} \ll 1.$$

In this sense the flow is slow. However, the Mach number calculated from the speed of sound at infinity is

$$\mathrm{M} = \frac{u_V}{a_\infty} = \mathrm{M}_* \frac{a_*}{a_\infty} \approx \mathrm{Kn} \sqrt{\frac{T_*}{T_\infty}}$$

and can be large at $T_*/T \gg 1$. Instead of a body, we can consider a volume of gas heated to a high temperature

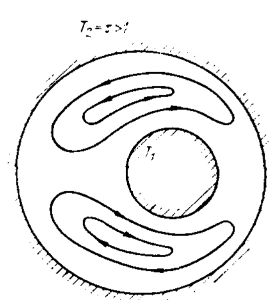

FIG. 3. Example of thermal-stress convection at $(\tau-1) \ll 1$.

by an external source of heat, e.g., by radiation. If $\mathbf{Kn} = 0.1$, $T_\infty = 300$ °K, and the temperature of this gas volume is $T_* = 3\times 10^4$ °K, then $\mathbf{M} = 1$ and the velocity of the thermal-stress convection is of the order of 10^2 m/sec.

It should be noted in general that the velocity of the thermal-stress convection at $\mathbf{Re} = 0(1)$ is much larger than the characteristic velocities of the known Stokes flow corresponding to Reynolds numbers $\mathbf{Re} \ll 1$.

So far we have disregarded the influence of external forces. The question is: under what conditions can the influence of free gravitational convection be neglected in comparison with the effects considered here? The characteristic velocity $u_g = L_*^2 \rho_* \mu_*^{-1} g$ will be smaller than u_V if[10]

$$L \ll L_*, \quad L_* = \mu_*^{2/3} (g\rho_*^2 \theta)^{-1/3}.$$

Therefore the influence of gravitational convection will be negligible for particles of small dimensions. Under normal atmospheric conditions $L_* = 3\times 10^{-2}$ cm, and the corresponding Knudsen number is 2×10^{-4}. Heated particles with such dimensions are extensively encountered in practice. In the analysis of their motion it is necessary to take into account the thermal-stress phenomena. L_* increases with decreasing ρ_*. Thus, at a height $H \approx 50$ km, where the density is smaller by a factor 10^3 than under normal conditions, we have $L_* \approx 1$ cm, corresponding to a Knudsen number $\mathbf{Kn} \approx 10^{-3}$. The fact that the role of the thermal-stress convection increases with decreasing density may suggest that this phenomenon is due to rarefraction and is typical of flows at $\mathbf{Kn} \to \infty$. It is therefore important to emphasize that the thermal-stress convection phenomenon takes place as $\mathbf{Kn} \to 0$. To the contrary, as $\mathbf{Kn} \to \infty$ (i.e., in the free-molecular limit), the gas is at rest near heated (cooled) bodies of arbitrary shape (under the usual laws of interaction between gas molecules and surfaces).

As already noted in the Introduction, the first to call attention to the existence of temperature stresses was Maxwell. He obtained the expressions for the transport properties from the kinetic-moment equations for a gas made up of Maxwellian molecules, imperfectly assuming $\theta \ll 1$. As a result he obtained expression (3.4) for $p_{ij}^{(T)}$. Inasmuch as at $\theta \ll 1$ the temperature satisfies the Laplace equation, while μ and K_2 are constant, it follows that

$$\frac{\partial p_{ij}^T}{\partial x_j} \approx \frac{\partial}{\partial x_j} K_2 \mu^2 \frac{\partial^2 T}{\partial x_i \partial x_j} = K_2 \mu^2 \frac{\partial}{\partial x_i} \nabla^2 T = 0$$

and the temperature stresses drop out of the momentum equation. This led Maxwell to the conclusion that these stresses cause no motion of the gas. The question of the existence of thermal-stress convection was apparently not raised subsequently any more.

4. FORCES DUE TO TEMPERATURE STRESSES. ELECTROSTATIC ANALOGY

We consider the question of forces acting on isothermal bodies. It was shown in[15] by direct integration over the surface of the body that the contribution of local temperature stresses to the resultant force exerted on some isothermal body by a finite system of k bodies is equal to

$$F_i^T = \frac{2}{3} \omega_3 T_w^{2s-1} \int (\nabla\tau)^2 (\mathbf{n}e_i)\, dS_k, \quad \tau = T - 1. \tag{4.1}$$

The forces in (4.1) and (4.3) are referred to μ_*^2/ρ_*; when writing down the coefficients (just as in (2.12)) we confine ourselves to the case of a power-law dependence of μ or η on the temperature. At $\theta \ll 1$ in the approximation linear in θ the temperature stresses, as indicated above, drop out of the momentum equation, and consequently $\mathbf{v} = 0$ and $P = 0$. And since the expression for F_i^T is quadratic in θ, it follows that the force acting on an arbitrary body is also equal to zero in the approximation linear in θ. In the next-order approximation, $\mathbf{v}$ is also equal to zero, but in accordance with (3.3) and (2.12) we have

$$P = -\frac{1}{2} \left(\omega_1 s - \frac{\omega_3}{3} \right) (\nabla\tau)^2. \tag{4.2}$$

Taking (4.1) and (4.2) into account, we find that the force acting on the body at $\theta \ll 1$ and due to the temperature stresses is

$$F_i = \frac{1}{2} (s\omega_1 + \omega_3) \int (\nabla\tau)^2 (\mathbf{n}e_i)\, dS_k, \tag{4.3}$$

where τ is a solution of the following problem:

$$\nabla^2 \tau = 0, \quad \tau|_{S_k} = \tau_k, \quad \tau(x \to \infty) \to 0, \tag{4.4}$$

where τ_k are specified constants. Similar relations hold for the electrostatic potential in the field of charge conductors, and expression (4.3) for the force differs only by a numerical factor from the expression for the ponderomotive force acting on a conductor.

Thus, there is an electrostatic analogy. From this analogy follow a number of new phenomena. Thus, for example, heated bodies ($T_w > T_\infty$) repel each other, but if one of them is cooled ($T_{w1} < T_\infty$) and the other is heated ($T_{w2} > T_\infty$), then they are attracted (an analogue of Coulomb's law); a body whose temperature is different from the screen temperature $T_{scr} = T_\infty$ lands on the screen, etc.

At the same time, a unit body placed in an infinite volume of gas is not acted upon by any force. This conclusion was proved only in an approximation quadratic in θ. The fact that the force acting on the body is zero in the approximation linear in θ was indicated already by Maxwell; for a unit body this statement was proved more rigorously by Epstein.[19] An attempt to take into account the terms nonlinear in θ when determining the radiometric force acting between parallel plates heated to different temperatures was made by Einstein.[20] Although by that time Enskog has already calculated p_{ij}^T exactly, Einstein, using the maximum-probability principle, obtained not only incorrect coefficients, but also an inexact structure of the expression for the stress tensor: in place of the divergence-free tensor

$[\nabla_i T \nabla_j T]$ she obtained the tensor $\nabla_i T \nabla_j T - (1/2)\nabla_k T \nabla_k \times T\delta_{ij}$, which leads to an incorrect equation of state. We note also a study[30] of the influence of temperature stresses on the pressure distribution between infinite parallel plates heated to different temperatures; the conclusions of this study are valid, strictly speaking, only for the case of Maxwellian molecules.

The question whether a unit body will be acted upon by a force at an arbitrary value of the surface temperature T_w still remains open. A partial (and furthermore negative) answer was obtained by an analysis of surfaces close to spherical.[15] A system of perturbed equations of first-order approximation in δ, where δ characterizes the degree of deviation of the shape of the surface from spherical, was solved by using expansions in spherical harmonics.

Finally, within the framework of the validity of the electrostatic analogy, a new form of thermophoresis takes place. It consists in the fact that a body of infinite conductivity (consequently, $T_w = \text{const}$), situated in an inhomogeneous temperature field, and having a surface temperature T_w that does not coincide with the local temperature of the gas, is acted upon by a force of the order of $\mu_*^2 \theta^2/\rho_*$, and the direction of the force is determined by the temperature difference between the body and the gas and by the gas-temperature gradient.

Ordinary thermophoresis is due to temperature slip. If the thermal conductivity of the particle material is finite, then, owing to the temperature slip of first order, a force $F_1 \sim \mu_*^2/\rho_* \theta$ acts on the particle and is directed opposite to the gas temperature gradient (see, e.g., [16,21]). There exists also a different type of thermophoresis, due to second-order temperature slip, which was also observed already by Maxwell[1]

$$v_\tau = A\,\mathrm{Kn}\,\frac{\partial^2 T}{\partial x_\tau\,\partial x_n}, \quad A = O(1). \tag{4.5}$$

If the body is isothermal (there is no slip of first order), and $\theta \ll 1$, then the slip of second order leads to a force $F_2 \sim \mu_*^2/\rho_*\,\mathbf{Kn}\,\theta$. The corresponding theory was developed by Sone,[22] although this effect was implicitly contained also in[23]. One of the two types of thermophoresis predominates, depending on the ratio of the thermal conductivities of the gas and of the particle.

5. "STOKES" FLOW AROUND A UNIFORMLY HEATED SPHERE

In practice it is frequently necessary to deal with flow around heated (cooled) bodies at small numbers $\mathbf{Re}_\infty = \rho_\infty u_\infty L/\mu_\infty$, where L is the dimension of the body and u_∞ is the velocity of the gas relative to the body. This situation is observed, for example, in the motion of suspended particles. Although the velocities and Reynolds numbers of the flow can on the whole be large, the gas velocity relative to the suspended particles is small, and consequently the corresponding numbers $\mathbf{Re}_\infty$ for a flow around the particles are small. The particles can be heated either by irradiation or by reactions occurring in the particles or on their surfaces.

If the flow incident on a uniformly heated or cooled body is such that $\mathbf{Re}_\infty \ll 1$, then the velocity field near the body will be determined in the general case by the thermal-stress convection, and the incident flow exerts only a perturbing action (the Reynolds number of the thermal-stress convection is $\mathbf{Re} = \rho_\infty u_T L/\mu_\infty = O(1)$, and consequently its velocity is $u_T \gg u_\infty$). An exception is an isothermal sphere, inasmuch as at $u_\infty = 0$, owing to the symmetry, the temperature stresses are balanced here by the pressure and $u_T = 0$. The problem of flow around an isothermal sphere at $\mathbf{Re}_\infty \ll 1$ was solved in[24]. The gas temperature can be represented in the form of the series $T = T^{(0)} + \mathbf{Re}_\infty T^{(1)} + \ldots$, where T_0 satisfies the equation $\nabla^2 T^{(0)2} = 0$ (the case $\mu = T$ was considered), the corresponding stresses are balanced by the pressure, and to calculate $T^{(1)}$ it is necessary to resort to the method of merging asymptotic expansions. The temperature stresses due to $T^{(1)}$ turn out to be of the principal order of magnitude (the velocities caused by them are of the order of u_∞). The equations written out above for slow non-isothermal flows can be reduced by separation of the variables to a system of ordinary differential equations, which has been solved numerically.

It has turned out that with increasing sphere temperature T_w the temperature stresses decrease more and more the force F acting on the sphere, in comparison with the value calculated within the framework of the Navier-Stokes equations for a compressible liquid (the dashed line in Fig. 4; the solid curve represents the values obtained with allowance for the temperature stresses).

We emphasize that until recently problems of this type were solved only within the framework of the Navier-Stokes approximation.[25,26]

6. GASDYNAMICS OF SLOW FLOWS OF GAS MIXTURES

The temperature stresses in a simple gas constitute a "crossover" effect due to changes in the components of the thermal flux,[1] inasmuch as $p_{ij}^T \approx \lambda[\partial q_i/\partial x_j]$ (the last relation is exact for Maxwellian molecules). Anal-

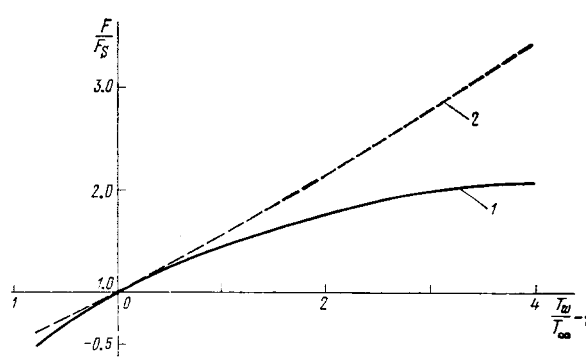

FIG. 4. Dependence of the sphere resistance on its temperature at $\mathbf{Re}_\infty \ll 1$. 1) Results of calculations with allowance for the temperature stresses; 2) results obtained with the aid of the Navier-Stokes equations for a compressible liquid; F_S—resistance force in accordance with the Stokes law.

ogously, the Burnett tensor of the stresses in gas mixtures contains sums of the terms $\lambda[\partial q_{Ni}/\partial x_j]$ summed over all the components of the mixtures, as well as products of the diffusion velocities. This result is particularly lucid if the $p_{ij}^{(2)}$ are derived by Grad's method (see[27,28]; the most general expressions for the transport properties of the Burnett approximation in gas mixtures were obtained by the Chapman-Enskog method in[29].

In gas mixtures, the partial thermal fluxes $\mathbf{q}_N$ depend not only on the temperature gradient but also on the concentration gradients $y_N = n_N/n$, where n_N is the number of N particles per unit volume and n is the summary numerical density. As a result, the values of $p_{ij}^{(2)}$ for a gas mixture contain, besides the "velocity" and "temperature" terms, also sums (over N and M) of terms proportional to the expressions

$$\lambda^2\left[\frac{\partial^2 y_N}{\partial x_i\,\partial x_j}\right],\quad \lambda^2\left[\frac{\partial y_N}{\partial x_i}\frac{\partial y_M}{\partial x_j}\right],\quad \lambda^2\left[\frac{\partial y_N}{\partial x_i}\frac{\partial T}{\partial x_j}\right],$$

the coefficients of which are complicated functions of the concentrations, of the mass ratios, and of the molecule-collision cross sections.

Thus, concentration (diffusion) stresses proportional to the second derivatives and to products of the first derivatives of the concentration with respect to the coordinates are present in gas mixtures. If the concentration drops specified by the boundary conditions in the mixture are of the order of unity, then the diffusion stresses, in analogy with the temperature stresses, will be of principal order of magnitude as the case of slow flows, and should be taken into account in the momentum equation. The influence exerted on the gas by these stresses is analogous to the influence of the temperature stresses, and they produce effects analogous to those described above. In particular, if the concentrations are constant over the surfaces of the bodies (there is no diffusion slip) and the convection "stimulators" indicated above are missing, then there exists one more type of convection, namely concentration-stress convection due to the indicated stresses.

The effects due to diffusion stresses manifest themselves particularly clearly in the case of a binary isothermal gas mixture.[15,28] If the concentration drops are of the order of unity, i.e., $\theta_c = |\nabla y_1| = O(1)$, then it is necessary to take into account in the momentum equation (besides the Navier-Stokes stresses) also the stresses

$$p_{ij}^{(c)} = \alpha\lambda^2\left[\frac{\partial^2 y_1}{\partial x_i\,\partial x_j}\right] + \beta\lambda^2\left[\frac{\partial y_1}{\partial x_i}\frac{\partial y_1}{\partial x_j}\right]. \qquad (6.1)$$

After carrying out a transformation of $\partial p_{ij}^{(c)}/\partial x_j$ analogous to that indicated above, the structure of the momentum equation turns out to be exactly the same as the structure of (2.12). Instead of the temperature-slip condition we have here the diffusion-slip condition $u_\tau = A\lambda\partial y_{1w}/\partial x_\tau$. At $\theta_c \lesssim 1$ the velocities of the concentration-stress convection is $u_c = u_V O(\theta_c^3)$, there is an electrostatic analogy, etc.[15]

Particles on the surfaces of which chemical reactions take place, are extensively encountered in various technological processes and in the atmosphere. To describe their behavior it is necessary to take into account the established effect, since appreciable concentration gradients can arise near the surfaces of the particles. In general, in the presence of a concentration gradient, the mixture, besides diffusing, also moves under the influence of the diffusion stresses, with velocities of the same order as the diffusion velocity. These concentration-stress motions must therefore be taken into account in technological processes, particularly in experiments aimed at determining the diffusion coefficients, etc.

7. CONCLUSION

The described phenomena have not yet been confirmed by experiment. The difficulties in setting up "pure" experiments lies in the need for separating the phenomena due to temperature or diffusion stresses from other effects of the same order.

To make the influence of the gravitational convection negligibly small under normal conditions, as indicated in Sec. 3, it is necessary to deal with particles of small dimensions ($\sim 10^{-2}$ cm). It is necessary here to ensure homogeneity of the particle-surface temperature and of the reservoir temperature, for otherwise the temperature slip can cause gas flow around the particle (or a force acting on the particle) of the same order of magnitude. But the dimensions L_* of the investigated flows can be greatly increased by decreasing the pressure. Thus, when the pressure is lowered to 10^{-1} mm Hg, this dimension increases to several centimeters, and the measured velocities of the thermal-stress convection are increased to several meters per second at $\theta \sim 1$ (temperature drop ~ 300 °K). Here, too, there are certain difficulties with thermostatically controlling the walls and with the measurement of the flow velocities of low-density gas, although these difficulties can apparently be overcome. The most interesting and important are experiments with artificial satellites, inasmuch as the thermal-stress convection is the only form of free convection under weightlessness conditions. Analogous difficulties arise also when it comes to demonstrate experimentally the concentration-stress convection in gas mixtures.

At the same time, the theoretical validity of the described phenomena is subject to no doubt. Indeed, Burnett's initial equations are obtained from the Boltzmann equation by the same Chapman-Enskog method as the Navier-Stokes equation, without any additional assumptions.[3)]

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3)The equations of slow flows were obtained above by simplifying the Burnett equations, with account taken of the smallness of the characteristic velocity $u_* = a_*\mathbf{Kn}$, with $\mathbf{Kn} \ll 1$. The latter circumstance, however, can be taken in account beforehand when constructing the asymptotic expansion of the solution of the system of Boltzmann's equations for a gas mixture.[31] As a result, the sought equations for a slow flow are obtained directly, in the principal approximation, and naturally coincide with the equations derived by the method described here.

The conclusions obtained by classical methods of kinetic theory were invariably confirmed by experiment (the discovery of thermal diffusion in gasses, the dependence of the transport coefficients on the gas parameters and molecule parameters, etc.). We emphasize that in gas mixtures there have already been observed effects due to the influence of certain Burnett terms of the expressions for the diffusion velocities; the most interesting is the influence of "viscous momentum transfer" on diffusion: under certain conditions some "velocity" Burnett terms of the diffusion-velocity vector turn out to be of the order of its barodiffusion term.[27]

On the other hand, slow flows ($\mathbf{Re} = O(1)$, $\mathbf{M} = \mathbf{KnRe} \ll 1$) with appreciable temperature and concentration gradients are so widely present in nature and in practice, that the need for further development of the "non-Navier-Stokes" nonlinear gasdynamics described above is subject to no doubt.

One can expect, generally speaking, temperature and diffusion stresses, and consequently all the phenomena caused by them, to occur also in certain liquids. However, unlike the kinetic theory of gasses, the kinetic theory of liquids does not provide us as yet with reliable information on the nonlinear properties of the stress tensor. The planning of appropriate experiments is therefore of considerable interest.

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