The generalized natural-oscillation method in diffraction theory

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CONTENTS

1. CHARACTERISTICS OF THE METHOD

A. The fundamental idea

The generalized natural-oscillation method (GNOM) consists in expanding the diffracted field in a series in terms of the eigenfunctions of the homogeneous problem, in which the chosen eigenvalue is not necessarily the frequency. The formulation of the homogeneous problem depends on the nature of the diffraction problem, while the eigenvalue therein (the spectral parameter λ) is a quantity that can be introduced in varied ways in different variants of the method. It can enter either into the equation, or into the boundary condition at the surface of the object or at infinity. For example, for the scalar problem of diffraction at the surface S,

$$\Delta u + k^2 u = f, \quad u|_S = 0 \tag{1.1}$$

with the radiation condition at infinity, the solution u is represented in the form

$$u = u^0 + \sum_n^\infty A_n u_n. \tag{1.2}$$

Here u^0 is the field of all the sources f in a vacuum (the incident field) or the field arising from diffraction by any other object, while the u_n are the eigenfunctions of the homogeneous problem for the surface S. The formulation of this homogeneous problem differs in different variants of the method, but the u_n are always real and orthogonal, while explicit expressions are found for the A_n have the resonance factors

$$A_n \sim \frac{1}{\lambda^0 - \lambda_n}, \tag{1.3}$$

where the λ_n are the eigenvalues, while λ^0 is the value that the spectral parameter has in the diffraction problem. This can be the dielectric constant of the object (Chap. 2), the impedance of the wall (Sec. A of Chap. 2), its transparency (Sec. B of Chap. 3), or a quantity that amounts to a reflection coefficient for a convergent wave (Sec. F of Chap. 3), etc. For resonators of high Q-factor, one of the terms is much larger than the others in the series of (1.2) near a resonance. Then the field has the form

$$u \approx u^0 + A_m u_m. \tag{1.4}$$

B. Comparison of the GNOM with the natural-frequency method

In the theory of resonators, people apply the natural-frequency method (NFM). This consists in seeking the solution of the inhomogeneous problem in the form of a series in the orthogonal system of eigenfunctions of the auxiliary homogeneous problem in which the eigenvalue is the frequency. The EFM came into the theory of resonators from the theory of oscillations of systems having a finite number of degrees of freedom. The NFM can be considered to be one of the variants of the GNOM. The spectral parameter λ in it is the frequency (more exactly, the square of the wavenumber k^2; k differs from the frequency by the factor $1/c$, where c is the speed of light in a vacuum). Just as in the NFM, one doesn't first solve the diffraction problem in the GNOM, but the homogeneous problem, and the eigenelements (eigenvalues and eigenfunctions) of the object are determined. They do not depend on the excitation, i.e., on the right-hand side of an inhomogeneous problem like (1.1). If we know the eigenelements, we can then solve the problem of diffraction of any field by this object. This article treats the variants of the GNOM in which the characteristic λ is not the frequency (with the exception of Sec. H of Chap. 3). In the studied variants, there is another difference from the NFM: the isolation of the term u^0. These differences lead to definite advantages of the presented methods. We shall note those among them that are realized in solving the concrete problems given below.

In these variants, the functions u_n in which the diffraction field is expanded satisfy the radiation condition at infinity. In the NFM (for all objects but closed resonators), the eigenfunctions increase (exponentially) as we go to infinity; this causes a number of theoretical and calculational difficulties. Moreover, it gives rise to a continuous spectrum. The spectrum is always discrete in the described variants, i.e., an integral does not figure in (1.2). The behavior of the eigenfunctions at infinity permits one to use simple stationary functionals of the Rayleigh type to determine λ_n and u_n (Secs. B and F of Chap. 2). Open and closed resonators are studied by the very same apparatus.

One can simplify the coefficients of the series in (1.2) by choosing the field u^0 in a suitable way (Secs. C and F of Chap. 2). The isolation of u^0 improves the convergence in (1.2) (Sec. B of Chap. 3). The algorithms for calculating λ_n are usually substantially simpler than for calculating the eigenfrequencies (Secs. C and F of Chap. 3). In the variants of the GNOM in which λ_n is introduced into the boundary condition (Chap. 3), the dimensionality of the series is less by one than in the NFM. If all the losses are concentrated at only one site (only in the dielectric (Sec. D of Chap. 2), only in the walls (Sec. A of Chap. 3), or only in the radiation (Sec. F and G of Chap. 3), then we can use a variant of the GNOM in which the eigenelements λ_n and u_n are real. Here the complex-valued nature of the solution of the diffraction problem will enter into the coefficients A_n via the quantity λ^0, which is complex under these conditions, rather than via the λ_n.

On the other hand, an advantage of the NFM over the other variants of the GNOM is manifested in those problems concerning resonators of high Q-factor in which one must trace the frequency-dependence of the field. In the NFM, Eq. (1.3) has the form

$$A_n \sim \frac{1}{k-k_n}, \tag{1.5}$$

Here k_n is the eigenfrequency, k is the frequency in the diffraction problem, and if we know one complex number k_n, we can immediately determine the frequency dependence. In order to do this in the GNOM, we must find the function $\lambda_n(k)$. Yet if we must analyze the dependence of the field on λ^0, rather than on the frequency, then (1.3) gives an immediate answer, while the NFM requires calculating the function $k_n(\lambda^0)$. This is precisely the formulation that arises in a number of physical problems (e.g., in studying resonance properties as depending on the parameters of the material in the measurement technique).

C. Problems to which the GNOM can be expediently applied

Generally, the described variants of the method are formally applicable to any diffraction problems whatever in acoustics and in electrodynamics, and to elastic-scattering problems in nonrelativistic quantum mechanics, etc. Just like the NFM, they are primarily applicable to resonators of high Q-factor, since the solution for them has the form of (1.4). They permit one to study both open and closed resonators; they are especially convenient for trap-type resonators. One can apply them to two-dimensional resonators to find the complex propagation constants of leaky waves in open waveguides.

We give below the solutions of some problems that illustrate the application of some variants of the generalized method, and we describe these variants. The aim of this article is to show by examples the potentialities of the natural oscillation method that arise when we refrain from introducing the natural frequencies, and to show the effectiveness of this approach. The article does not present variants of the GNOM whose effectiveness has not yet been illustrated with non-trivial examples (with the exception of Sec. A of Chap. 3), even if the formal apparatus of these methods has been worked out in detail (e.g., the methods of Chap. 2 for vector problems, objects having complicated boundary conditions, the variational apparatus for the methods of Chap. 3, etc.). The GNOM would seem to be applicable in elasticity theory, in the theory of chains, and in the theory of oscillations of systems having a finite number of degrees of freedom.

At the end of the article in the review of the literature, we shall cite some studies close to the theme of this article, and compare them with the known results. We shall cite in the same place some studies that contain a partial mathematical justification of the formal procedures that are to be carried out below. We shall point out at the beginning of each section the studies in which the pertinent results have been presented in greater detail.

2. INTRODUCING THE SPECTRAL PARAMETER INTO THE EQUATION

A. A dielectric object made of a homogeneous material[1]

Let us formulate the problem of the diffraction of a field created by the sources f by an object occupying the volume V^+ with the limited surface S:

$$\Delta u + k^2\varepsilon^0 u = f \quad \text{in } V^+, \quad \Delta u + k^2 u = f \quad \text{in } V^-, \tag{2.1}$$

$$u^+ - u^- |_S = 0, \tag{2.2a}$$

$$\left.\frac{\partial u^+}{\partial N} - \frac{\partial u^-}{\partial N}\right|_S = 0. \tag{2.2b}$$

Moreover, u must satisfy the radiation condition. In all the homogeneous and inhomogeneous problems studied below (apart from that of a closed resonator (Sec. D of Chap. 2) and the homogeneous auxiliary problem (Sec. H of Chap. 3)), the fields must satisfy this condition, and we shall henceforth assume it.

In (2.1) and (2.2), the signs "+" and "-" refer to the regions inside and outside the object. We shall arbitrarily call (2.1) and (2.2) the problem of diffraction by a dielectric object, though, strictly speaking, (2.1) and (2.2) describe such a diffraction only for a two-dimensional problem and for E-polarization. Even for a two-dimensional problem with H-polarization, the condition (2.2b) must be replaced by the condition

$$\left.\frac{1}{\varepsilon^0}\frac{\partial u^+}{\partial N} - \frac{\partial u^-}{\partial N}\right|_S = 0. \tag{2.3}$$

Yet in the three-dimensional case, the electromagnetic diffraction problem generally does not reduce to a scalar problem, and it is more convenient to operate directly with the Maxwell equations than with the wave equations. However, we can present the fundamental features of this variant of the GNOM with the simple problem of (2.1) and (2.2).

We shall introduce the spectral parameter λ into the homogeneous problem in place of ε^0. The value of λ in the diffraction problem, i.e., λ^0, is ε^0. We shall denote the eigenvalue of λ_n by ε_n. The homogeneous problem is

$$\Delta u_n + k^2\varepsilon_n u_n = 0 \quad \text{in } V^+, \quad \Delta u_n + k^2 u_n = 0 \quad \text{in } V^-, \qquad u_n^+ - u_n^-|_S = 0, \quad \frac{\partial u_n^+}{\partial N} - \frac{\partial u_n^-}{\partial N}\Big|_S = 0. \tag{2.4}$$

It has an independent physical meaning: it describes the free, undamped oscillations of an object (of the same form as V^+) having the dielectric constant ε_n that occur at the given frequency k. In contrast to the NFM, the spectral parameter is lacking in the equation within V^-, and u_n satisfies the same equation in V^- as does the diffracted field.

Just as in all the methods of Chap. 2, n is a triple index, and triple sum figures in (1.2).

We can easily derive the physically evident property of the eigenvalues

$$\operatorname{Im} \varepsilon_n > 0 \tag{2.5}$$

(for the chosen time-dependence $\exp(i\omega t)$) that undamped oscillations in the absence of sources are possible only for an object that releases energy when placed in an electric field. The eigenfunctions u_n are real-orthogonal, i.e.,

$$\int_{V^+} u_n u_m \, dV = 0, \quad n \neq m. \tag{2.6}$$

Let us take as u^0 the field that arises from the same sources f in the absence of the object, i.e., in a vacuum. The series (1.2) satisfies all the boundary conditions and the equation in V^- term by term. Precisely where this does not happen (in V^+), the eigenfunctions are orthogonal, and we get the following for A_n:

$$A_n = \frac{1-\varepsilon^0}{\varepsilon^0 - \varepsilon_n} \frac{\int_{V^+} u^0 u_n \, dV}{\int_{V^+} u_n^2 \, dV}. \tag{2.7a}$$

We can eliminate the function u^0 from (2.7a), and express A_n directly in terms of f:

$$A_n = \frac{1-\varepsilon^0}{\varepsilon^0 - \varepsilon_n} \frac{1}{k^2(1-\varepsilon_n)} \frac{\int_{V^+ + V^-} u_n f \, dV}{\int_{V} u_n^2 \, dV}. \tag{2.7b}$$

For the total field in V^-, we must isolate u^0 in (1.2) (one need not do this only if all the sources lie in V^+), and the function u^0 is generally not expandable in terms of the u_n in V^-. Such an expansion exists in V^+, and we can write instead of (1.2):

$$u = \sum_n \frac{1-\varepsilon_n}{1-\varepsilon^0} A_n u_n, \tag{2.7c}$$

The formulas (2.7) give the solution of the problem of (2.1) and (2.2).

In Secs. C and D of this Chapter, we shall treat problems in which a dielectric object lies in an open (Sec. C) or closed (Sec. D) resonator, rather than in a vacuum. All of the above-derived formulas continue to hold here, except only that u^0 and u_n must satisfy the same extra conditions at the surface of the resonator as does the total field. In a closed resonator without losses in the walls, we shall have $\operatorname{Im} \varepsilon_n = 0$ instead of (2.5).

The effectiveness of the entire apparatus is determined primarily by how one actually finds the ε_n from (2.4). We shall describe in the next section the variational method of calculating ε_n, while in Secs. C and D we shall apply it to two concrete problems.

B. The variational method[2]

We shall first formulate the variational method of determining the eigenvalue ε_n of the problem (2.4) by assuming that a dielectric object is placed in a closed resonator so that the field u_n vanishes on some surface S_0 that surrounds S. In the NFM for a closed resonator having a dielectric insert, people widely use stationary functionals of the Rayleigh type. (That is, in contrast to functionals of the Schwinger type, they do not contain double integrals of the kernels of the corresponding integral equations.) For (2.4), we shall use the functional

$$L(u) = \int_{V^+ + V^-} (\nabla u)^2 \, dV - k^2 \int_{V^-} u^2 \, dV - \varepsilon k^2 \int_{V^+} u^2 \, dV. \tag{2.8}$$

It is stationary over the solutions of (2.4). That is, when one substitutes for u_n therein the similar function $u_n + \mu\varphi$, the following equation holds:

$$L(u_n + \mu\varphi) = L(u_n) + O(\mu^2). \tag{2.9}$$

The functional of (2.8), which is usually used to determine k_n, is also directly applicable for determining ε_n from (2.4). The situation proved to be so simple because stationarity, i.e., the property (2.9), requires only that φ should be continuous, and the value of ε does not enter into the condition for the admissible functions. If the conditions for the admissible functions themselves were to contain ε, then the corresponding functional could not be applied for finding ε_n. For example, the functional (2.8) is stationary over the solutions of the homogeneous problem having the boundary condition (2.3) instead of (2.2b) only if the admissible functions themselves satisfy (2.3). For the problem having the condition (2.3), we would have to use the following functional to determine ε_n:

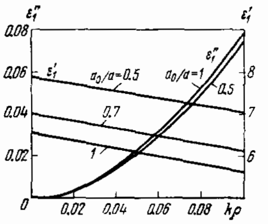


FIG. 1. Plot of ε''_1 and ε_1'' vs $k\rho$ for a resonator made of a semitransparent circular envelope containing a dielectric ellipse.

$$L(u)=\frac{1}{\varepsilon}\int_{V^+}(\nabla u)^2\,dV+\int_{V^-}(\nabla u)^2\,dV-k^2\int_{V^++V^-}u^2\,dV, \tag{2.10}$$

The condition (2.3) is natural for this functional, i.e., it need not be imposed on the admissible functions. There is a regular way of constructing stationary functionals for determining λ_n also for the homogeneous equations of Chap. 3.

For the GNOM, the extension of the result of (2.9) to an infinite region is carried out automatically. Since the eigenfunctions satisfy the radiation conditions at infinity, the admissible functions must also satisfy it (and some of them may decline more rapidly). Hence, in order that the integrals in (2.8) in V^- should converge also for an infinite region, it suffices in calculating them to assume that $\operatorname{Im} k=-0$. This procedure can be rigorously justified.

Of course, $L(u)$ in (2.8) does not have the property of extremality (L is a complex-valued functional). However, according to (2.9), we can formally apply the Ritz method to it. As we know, this method consists in seeking u in the form of a series over some basis functions v_n, L becomes a quadratic form of the coefficients of this series, the derivatives of L with respect to them are equated to zero, a homogeneous system of linear equations arises, and the vanishing of its determinant is the sought equation for calculating ε_n.

C. A resonator consisting of a semitransparent envelope containing a dielectric object[3]

Let us find the eigenvalues ε_n of the homogeneous problem (2.1)–(2.2) supplemented by the conditions

$$u_n^+-u_n^-|_{S_1}=0,\quad \frac{\partial u_n^+}{\partial N}-\frac{\partial u_n^-}{\partial N}-\frac{1}{\rho}u_n=0|_{S_1} \tag{2.11}$$

on some surface S_1 surrounding the surface S of the object. The number ρ characterizes the transparency of the envelope S_1 within which the dielectric object has been placed. If ρ is small ($k\rho\ll 1$), then the system is an open resonator of high Q-factor. When $\rho=0$, the resonator becomes closed, while there is no envelope when $\rho=\infty$.

We might supplement $L(u)$ in (2.8) with a surface integral over S_1 in such a way that the conditions (2.11) become natural, i.e., one need not impose them on the basis functions. However, we shall restrict ourselves to a simple form of the contour S_1 (the environment), and shall take the basis functions in the form

$$\begin{gathered} v_{nm}=H_n^{(2)}(kr)\cos n\varphi\quad (r\geqslant a),\\ v_{nm}=\left[A_{nm}J_n\left(\mu_{nm}\frac{r}{a}\right)+B_nJ_n\left(\nu_{n0}\frac{r}{a}\right)\right]\cos n\varphi\quad (r<a), \end{gathered} \tag{2.12}$$

Here $J_n(\mu_{nm})=0$, $J_n'(\nu_{n0})=0$, a is the radius of S_1, and A_{nm} and B_n are found in such a way that (2.11) is satisfied for the v_{nm}. The surface S will be an ellipse. Thus, one is studying a dielectric ellipse in a circular semitransparent envelope. This problem cannot be solved by the method of isolating the variables.

Figure 1 shows the real and imaginary components of the first eigenvalue $\varepsilon_1=\varepsilon_1'+\varepsilon_1''$ as functions of the transparency of the envelope $k\rho$. The semiaxes of the ellipse are denoted as a_0 and b_0; for all of the curves, $ka=1$, and $kb_0=1/2$. The calculations were performed for four values of the ratio of the long axis of the ellipse to the envelope: $a_0/a=0.5$; 0.6; 0.8; and 1.0. Figure 2 shows ε_1' as a function of the radius of the envelope.

According to (2.7) (cf. (1.3)), the dependence of the field on ε^0 is mainly determined by the factor $1/(\varepsilon^0-\varepsilon_1)$. The graph of the function $F(\varepsilon^0)=|\varepsilon^0-\varepsilon_1|^{-1}$ gives the resonance curve. Its maximum lies at $\varepsilon^0=\varepsilon_1'$, while the half-width is equal to the sum of two quantities: the imaginary component of ε_1 (i.e., ε_1''), which describes the radiation losses, and $-\operatorname{Im}\varepsilon^0$, which describes the losses in the dielectric. If $\operatorname{Im}\varepsilon^0=0$, then the increment of ε^0 with respect to ε_1' at which the amplitude of the field falls by a factor of two is ε_1''. We can define the Q-factor Q_ε with respect to the dielectric constant as the ratio of $\operatorname{Re}\varepsilon^0$ at which the resonance curve reaches its maximum to its half-width. When $\operatorname{Im}\varepsilon^0=0$,

$$Q_\varepsilon=\frac{\varepsilon_1'}{\varepsilon_1''}. \tag{2.13}$$

If $\operatorname{Im}\varepsilon^0\neq 0$, then $1/Q_\varepsilon$ equals the sum of two reciprocal Q-factors arising from the two types of losses.

For example, in the treated example with $k\rho=0.05$, $ka=1$, $ka_0=1$ (the ellipse touches the envelope), and $b_0/a_0=1/2$, Fig. 1 gives: $\varepsilon_1'=6.1$, $\varepsilon_1''=0.0224$, i.e., $Q_\varepsilon=280$. Q_ε increases approximately as $1/\rho^2$ with decreasing transparency.

The quantity in (2.13) differs from the Q-factor Q_k with respect to the frequency that is introduced in the NFM, since Q_ε describes an experiment in which $k=\text{const.}$, while Q_k describes an experiment in which $\varepsilon^0=\text{const.}$ The quantities Q_k and Q_ε are related by the equation

$$\frac{Q_k}{Q_\varepsilon}=\frac{k_n'}{\varepsilon_n'}\frac{d\varepsilon_n'}{dk}\bigg|_{k=k_n'}, \tag{2.14}$$

Here $k_n=k_n'+ik_n''$ is the complex natural frequency, i.e., the complex root of the equation $\varepsilon_n(k)=\varepsilon^0$.

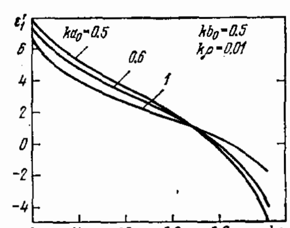


FIG. 2. Plot of ε_1 vs the radius of the envelope.

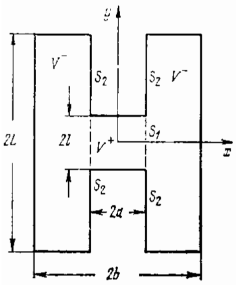


FIG. 3. Closed resonator of complicated shape containing a dielectric insert.

D. A closed resonator of complicated shape containing a dielectric object[3]

A dielectric object in the form of a $2l\times 2a$ rectangle is placed in the resonator depicted in Fig. 3. The conditions of (2.4) must be satisfied at the boundary of the dielectric, and the condition $u=0$ at the boundary of the resonator. The problem consists in finding ε_n.

Since it is hard to find a basis system of functions for such a complex figure that satisfies all the boundary conditions, we shall supplement the functional of (2.8) with contour integrals in such a way that these conditions become natural, and we need not impose them on the admissible functions. The functional possessing this property that we shall use in this problem is the sum of the functional of (2.8) and the integrals

$$\int\limits_{S_1}(u^+-u^-)\left(\frac{\partial u^+}{\partial N}+\frac{\partial u^-}{\partial N}\right)dS-2\int\limits_{S_2}u\,\frac{\partial u}{\partial N}\,dS. \tag{2.15}$$

The basis functions were taken in the form

$$\begin{aligned}v_{nm}&=\cos\frac{\pi nx}{2a}\cos\frac{\pi(m+1/2)\,y}{l}\ (\text{in } V^+),\\ v_{nm}&=\sin\frac{\pi n\,(b-x)}{2\,(b-a)}\cos\frac{\pi(m+1/2)\,y}{L}\ (\text{in } V^-).\end{aligned} \tag{2.16}$$

For them, the boundary conditions are violated on the lines $|x|=a$, and the integrals of (2.15) are taken along these lines.

Figure 4 shows the relationships of ε_n ($n=1$, 2, or 3) to l. The process of successive trimming of the determinant that arises in the Ritz method can be considered to be established when 10 functions have been taken in V^+, and 15 functions in V^-.

All of the ε_n in this problem are real, regardless of whether ε^0 is real or complex. The quantities ε_n do not depend at all on ε^0. A complex-valued field that involves losses in the dielectric arises in the diffraction problem only when one substitutes a complex ε^0 into A_n (primarily in the factor $1/(\varepsilon^0-\varepsilon_n)$).

E. A dielectric object made of inhomogeneous material[1,4]

Let us treat the problem of solving the equation

$$\Delta u+k^2\varepsilon^0(\mathbf{r})\,u=f, \tag{2.17}$$

where $\varepsilon^0(\mathbf{r})$ is a function of the coordinates. We can consider it to be continuous, and derive the case with discontinuous functions (i.e., when there is a phase boundary) by taking a limit. Outside some finite region, $\varepsilon^0=1$. Just as arbitrarily as in Sec. 3, we shall call (2.17) the problem of diffraction by an object made of an inhomogeneous dielectric. This is just the form of equation that we shall need in the next section. The same arguments that are developed below for (2.17) can be used to solve the vector electromagnetic problem of diffraction by an inhomogeneous object ($\varepsilon=\varepsilon(\mathbf{r})$ and $\mu=\mu(\mathbf{r})$) or for the scalar "second-polarization" equation

$$\nabla\left(\frac{1}{\varepsilon}\nabla u\right)+k^2u=f. \tag{2.18}$$

For the problem of (2.1) and (2.2), the spectral parameter was introduced in place of the number ε^0. In (2.17), the spectral parameter must be introduced into the function $\varepsilon^0(\mathbf{r})$. One can show two ways of introducing it in which the fundamental features of the apparatus are preserved, in particular, the orthogonality of the eigenfunctions. We shall use one of these methods.

Let us treat the homogeneous problem

$$\Delta u+k^2\{\alpha(\mathbf{r})+\lambda\,[\varepsilon^0(\mathbf{r})-\alpha(\mathbf{r})]\}\,u=0, \tag{2.19}$$

where $\alpha(\mathbf{r})$ is generally an arbitrary function that differs from $\varepsilon^0(\mathbf{r})$, and is equal to unity in the region in which $\varepsilon^0(\mathbf{r})=1$. The eigenvalue in (2.19) is λ; in the diffraction problem, $\lambda^0=1$. We can treat the problem of (2.19) as being that of the natural oscillations of an object having the dielectric constant

$$\varepsilon_n(\mathbf{r})=\alpha(\mathbf{r})+\lambda_n\,[\varepsilon^0(\mathbf{r})-\alpha(\mathbf{r})], \tag{2.20}$$

that occur at the assigned frequency. If there are losses (e.g., by radiation), then ε_n must be complex, and $\mathrm{Im}\,\lambda_n>0$. The eigenfunctions of (2.19) are orthogonal with the weight $\varepsilon^0-\alpha$:

$$\int(\varepsilon^0-\alpha)\,u_nu_m\,dV=0\ \text{when}\ n\neq m. \tag{2.21}$$

The two last formulas are the fundamental result of this section. We shall not give the formulas similar to (2.7) for the coefficients in (1.2). One can easily derive them by substituting (1.2) into (2.17), while using (2.21). Of course, these coefficients have the denominator $1-\lambda_n$. We can write a stationary functional for the λ_n; we shall restrict ourselves in the next section to writing it out in the one-dimensional case.

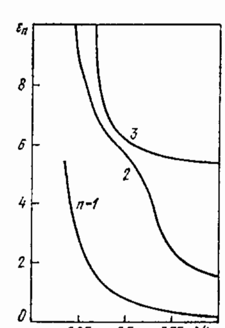


FIG. 4. Plot of the eigenvalues ε_n ($n=1$, 3, 4) vs the dimension l.

F. The quantum-mechanical problem of elastic scattering by a quasistationary level[4]

Let us imagine a spherically symmetrical potential $V(r)$ that forms a barrier. That is, it is small (or zero) at small r, it reaches a maximum value $V_{\max}$ at some finite $r = r_{\max}$, and then vanishes or approaches zero as $r \to \infty$. A flux of particles of energy k^2 less than $V_{\max}$ falls on this potential barrier. The potential forms a trap. For almost all k, only a few particles penetrate within, while the rest are reflected from the outer part of the barrier. However, at some energies, the weak tunneling infiltration causes a considerable effect, and a resonance arises.

This phenomenon has been studied for a long time in detail, and methods have been developed for calculating the scattering matrix S_l from its complex poles in the k plane. This apparatus is an application of the NFM. We shall apply to the formulated problem the variant of the GNOM that was developed above. It would seem to permit one very simply to find S_l for any barriers whatever; one could also use the method of Sec. F of Chap. 3. Mathematically, the problem of solving the Schrödinger equation for a flux of particles of energy k^2 incident on the potential $V(\mathbf{r})$ is identical to the problem of solving the Helmholtz equation for diffraction of a plane wave by an object having a dielectric constant that is a function of the coordinates:

$$\varepsilon^0(\mathbf{r}) = 1 - \frac{V(\mathbf{r})}{k^2}. \tag{2.22}$$

In the symmetrical case, this object is a spherical layer having a dielectric constant that varies with the radius, being negative in part of the layer, and unity outside the object. The field of the incident plane wave is expanded in spherical harmonics, and the problem is reduced to a set of one-dimensional diffraction problems. As we know, they can be formulated as follows: to determine the numbers S_l ($|S_l| = 1$), with $l = 0, 1, 2, \ldots$, from the equations

$$\frac{d^2u^l}{dr^2} + \left\{k^2 - \left[V(r) + \frac{l(l+1)}{r^2}\right]\right\} u^l = 0 \tag{2.23}$$

with the boundary conditions

$$u^l(0) = 0, \tag{2.24a}$$

$$u^l(r)_{r\to\infty} = \frac{-i^{l-1}}{2}\left[e^{-ikr} - (-1)^l S_l e^{ikr}\right]. \tag{2.24b}$$

The time-dependence in this section has the form $\exp(-i\omega t)$.

Let us use the apparatus of the preceding section to solve this inhomogeneous problem. We shall assume that $l = 0$, and not write out this index; the generalization to $l \neq 0$ can be done in an elementary way.

In the studied problem of an externally-excited resonator of high Q-factor, we can conveniently use the above-noted possibility of introducing u^0 into (1.2) in different ways. Let us define the auxiliary potential $V^0(r)$, which equals $V(r)$ when $r > r_{\max}$, and equals $V_{\max}$ when $r < r_{\max}$. The term u^0, which arises when the plane wave is incident on the barrier $V^0(r)$, does not have a resonance nature. We shall denote the value of the coefficient S that corresponds to it as S^0; S^0 varies slowly with varying k. We shall also relate the function α that figures in the fundamental formulas of the preceding section to V^0. That is, we shall assume that $\alpha = 1 - V^0/k^2$. In other words, analogously to (2.19), we shall introduce the system of eigenfunctions u_n by the equation

$$\frac{d^2u_n}{dr^2} + [k^2 - \lambda_n V(r) - (1-\lambda_n)V^0(r)]u_n = 0 \tag{2.25}$$

having the homogeneous boundary conditions

$$u_n(0) = 0, \tag{2.26a}$$

$$u_n(r)_{r\to\infty} = \exp(ikr); \tag{2.26b}$$

Here λ_n is the eigenvalue; in the diffraction problem (i.e., in the inhomogeneous problem of (2.23) and (2.24)), $\lambda = 1$.

The functions u_n and u^0 are connected by two relationships. The first is the orthogonality condition (2.21), while we can easily derive the second from the equations for u_n and u^0 by using the fact that all the conditions at infinity in this method contain the very same real number k:

$$\int_0^{r_{\max}} (V - V^0) u_n u_m \, dr = 0 \text{ when } n \neq m, \tag{2.27a}$$

$$\int_0^{r_{\max}} (V - V^0) u_n u^0 \, dr = -\frac{\lambda_n''}{\lambda_n} \int_0^{r_{\max}} (V - V^0)|u_n|^2 \, dr. \tag{2.27b}$$

Now we can easily find the required expression for S. Upon substituting (1.2) into (2.24b) and into (2.23), we get

$$S = S^0 + 2i\sum_n A_n, \tag{2.28a}$$

$$A_n = \frac{\lambda_n''}{1-\lambda_n} \frac{\int_0^{r_{\max}} (V - V^0)|u_n|^2 \, dr}{\lambda_n \int_0^{r_{\max}} (V - V^0) u_n^2 \, dr}. \tag{2.28b}$$

We have used (2.27b) in deriving the second of these formulas.

The resonance nature of the scattering is manifested in the fact that some interval of k values contains one eigenvalue (we shall call it λ_m) for which $|1 - \lambda_m'| \ll |\lambda_m''|$. Near the resonance, the second factor in (2.28b) is approximately equal to S^0. For all the terms of the series of (2.28a), $|A_n| \ll 1$ when $n \neq m$; we must keep in the series only the term $A_m = [\lambda_m''/(1-\lambda_m)]S^0$, and

$$S = S^0 \frac{1-\lambda_m^*}{1-\lambda_m}. \tag{2.29}$$

This is the final formula for calculating the scattering matrix. If we denote $S/S^0 = \exp(2i\delta)$, then the relation of δ to k is a resonance curve, while the root $\bar{k}$ of the

TABLE 1.

	N				Exact value
	3	4	5	6	
$\bar{k}_1 a$	2.499	2.499	2.498	2.4961	2.4952
$2\hat{\delta}k_1 a$	0.224	0.2276	0.2275	0.2286	0.2289

equation $\delta=\pi/2$ (i.e., $\lambda'_m=1$) is a quasistationary level (real). The width of the resonance $2\hat{\delta}k$ (the band of k values where δ varies strongly) is determined by the roots of the equation $|1-\lambda'_m|=|\lambda''_m|$ (i.e., $\delta=\pi/2\pm\pi/4$).

The actual determination of $\lambda_m(k)$ from (2.25) and (2.26) was performed by the Ritz method, which was applied to the stationary functional

$$L(u)=\int\limits_0^\infty\left(\frac{du}{dr}\right)^2 dr-\int\limits_0^\infty(k^2-V)\,u^2\,dr+\lambda\int\limits_0^{r_{\max}}(V-V^0)\,u^2 dr. \qquad (2.30)$$

This functional is derived from (2.8) by treating ε in it as a variable, introducing it inside the integral, subsitiuting according to (2.20), and replacing the functions $\varepsilon(r)$ and $\alpha(r)$ by $1-(V/k^2)$ and $1-(V^0/k^2)$. Of course, the stationary nature of (2.30) over the solution of the problem of (2.25) and (2.26) can also be proved directly.

In the calculations whose results are given below, we have used the basis functions

$$v_n=\left(\frac{r}{b}\right)^{2n-1}e^{ikb}\quad(r<b),\quad v_n=e^{ikr}\quad(r>b).$$

Here b is the outer boundary of the potential ($V(r)=0$ when $r>b$). The condition that such a boundary should exist is not necessary for application of the method.

Table 1 gives the results for a rectangular barrier that occupies the segment from $r=a$ to $r=b$ ($b/a=3.2$), with the height $V_{\max}=20/a^2$, as functions of the number N of basis functions taken for calculation. The last column gives the exact values of $\bar{k}a$ and $2\hat{\delta}ka$, which one can find by a transcendental equation for this shape of barrier. The table illustrates the convergence of the method.

Figure 5 shows three shapes of barriers (2, 3, and 4) that begin at $r=a$ and end at $r=b$, with $b/a=3/2$, and a rectangular barrier (1) for which $b/a=5/4$. As we know, the problem can be solved exactly for the rectangular barrier. Figure 6 shows the resonance curves for these barriers when $V_{\max}a^2=20$. The points mark the boundaries of the regions ($\delta=\pi/4$ and $\delta=3\pi/4$) and the quasistationary level $\bar{k}(\delta=\pi/2)$. For the same shapes of barriers, Fig. 7 shows the relationship of

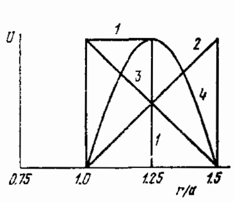


FIG. 5. Shapes of potential barriers.

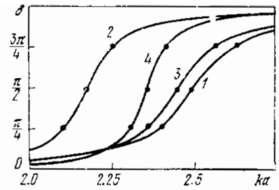


FIG. 6. Resonance curves for the potential barriers in Fig. 5.

the resonance characteristics to the height of the barrier. With increasing height, resonance is reached at higher energies of the incident particles, but the relative height of the quasistationary level ($\bar{k}^2/V_{\max}$) declines. Therefore the transparency of the barrier decreases at the resonance energy, and the resonance becomes narrower.

It takes about two minutes of machine time on a Minsk-32 computer to calculate one resonance curve.

3. INTRODUCING THE SPECTRAL PARAMETER INTO THE BOUNDARY CONDITIONS

A. The impedance method[5]

We shall illustrate it with the example of the vector problem of diffraction by a closed surface S. The fields $\mathbf{E}$ and $\mathbf{H}$ must satisfy the Maxwell equations

$$\operatorname{rot}\mathbf{H}-ik\mathbf{E}=\frac{4\pi}{c}\mathbf{j}^{(e)},\quad \operatorname{rot}\mathbf{E}+ik\mathbf{H}=-\frac{4\pi}{c}\mathbf{j}^{(m)}, \qquad (3.1a)$$

Here $\mathbf{j}^{(e)}$ and $\mathbf{j}^{(m)}$ are the given currents, and the following conditions must be satisfied at the surface S:

$$E_t-iw^0H_\tau=0;\quad E_\tau+iw^0H_t=0. \qquad (3.1b)$$

Moreover, the radiation conditions must be fulfilled. In the relationships of (3.1b), t and τ are two unit vectors tangent to S, w^0 is a given number, which is the impedance of the surface. For an ideal metal, $w^0=0$. We shall assume that $w^0=\text{const}$. The generalization to a variable impedance is carried out in about the same way as in the last section for a variable $\varepsilon(r)$.

Let us define the eigenfunctions e_n and h_n as being the solutions of the homogeneous equations

$$\operatorname{rot}h_n-ike_n=0,\quad \operatorname{rot}e_n+ikh_n=0, \qquad (3.2a)$$

that satisfy on S the conditions

$$e_{nt}-iw_nh_{n\tau}=0,\quad e_{n\tau}+iw_nh_{n\tau}=0 \qquad (3.2b)$$

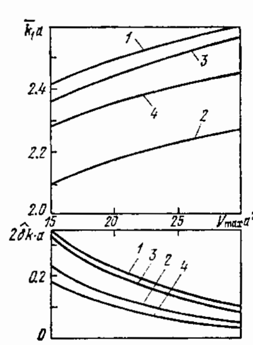


FIG. 7. Plot of the resonance characteristics vs the height of the barriers.

and the radiation condition. The index n has a dimensionality one less than that of the diffraction problem. This is inherent in all the methods of Chap. 3. The number w_n is the spectral parameter of the homogeneous problem (3.2). We can easily derive from the law of conservation of energy that $\operatorname{Im} w_n \geq 0$. The equality holds for closed resonators. The fields $\mathbf{e}_n$ and $\mathbf{h}_n$ are othogonal on S:

$$\int_S (h_{nt}h_{mt} + h_{n\tau}h_{m\tau})\, dS = 0, \quad n \neq m. \tag{3.3}$$

The solutions of the problem (3.1) can be written in the form

$$\mathbf{E} = \mathbf{E}^0 + \sum_n A_n \mathbf{e}_n, \quad \mathbf{H} = \mathbf{H}^0 + \sum_n A_n \mathbf{h}_n, \tag{3.4}$$

where $\mathbf{E}^0$ and $\mathbf{H}^0$ are the fields of the same sources in a vacuum. The coefficients A_n have the structure of (1.3), and are equal to

$$A_n = \frac{-i}{w^0 - w_n} \frac{\int_S [E_t^0 h_{n\tau} - E_\tau^0 h_{nt} - iw^0 (H_t^0 h_{nt} + H_\tau^0 h_{n\tau})]\, dS}{\int_S (h_{nt}^2 + h_{n\tau}^2)\, dS}. \tag{3.5}$$

We emphasize that the solution is represented by a double, rather than a triple series, that the sums lack the gradient terms (they are automatically included in $\mathbf{E}^0$ and $\mathbf{H}^0$), and that the coefficients of the expansions for $\mathbf{E}$ and $\mathbf{H}$ are identical. These are the differences of the apparatus proposed here from the commonly accepted one.

B. Coupling conditions[6]

Let us study with the example of scalar problems the features of the method in the case where the spectral parameter λ is introduced into the boundary conditions, which amount to coupling conditions at the boundary between the regions.

The method is applicable to closed and open diffraction problems with metallic and dielectric objects, and also on infinitesimally thin surfaces that can be, in particular, semitransparent.

Let us study here the construction of the solution of the two-dimensional problem of diffraction by an unclosed, infinitesimally thin shield. The complement $\bar{S}$ (the slit) constitutes together with the shield S a complete contour that separates all space into inner (V^+) and outer (V^-) regions. We must find the solution of the equation

$$\Delta u + k^2 u = f, \tag{3.6}$$

that satisfies on the shield S either the condition

$$u|_S = 0 \qquad (E\text{-polarization}, \ u = E_z), \tag{3.7}$$

or the condition

$$\left.\frac{\partial u}{\partial N}\right|_S = 0 \qquad (H\text{-polarization}, \ u = H_z). \tag{3.8}$$

The solution must also satisfy the appropriate conditions near sharp edges and the radiation condition. On the complement $\bar{S}$, the overall field and its normal derivative must be continuous:

$$u^+ - u^-|_{\bar{S}} = 0, \tag{3.9a}$$

$$\left.\frac{\partial u^+}{\partial N} - \frac{\partial u^-}{\partial N}\right|_{\bar{S}} = 0. \tag{3.9b}$$

The eigenfunctions in which we shall expand the diffracted field of the problems (3.6)–(3.9) must satisfy the homogeneous equation

$$\Delta u_n + k^2 u_n = 0, \tag{3.10}$$

the radiation condition, and the condition that the energy should be finite near the edges. At the boundary of the regions V^+ and V^-, we shall subject the eigenfunctions to the coupling conditions, into which the spectral parameter $\lambda = \rho_n$ is introduced: for E-polarization,

$$u_n^+ - u_n^- = 0, \quad \frac{\partial u_n^+}{\partial N} - \frac{\partial u_n^-}{\partial N} - \frac{u_n}{\alpha \rho_n} = 0; \tag{3.11}$$

and for H-polarization,

$$\frac{\partial u_n^+}{\partial N} - \frac{\partial u_n^-}{\partial N} = 0, \quad u_n^+ - u_n^- + \frac{\alpha}{\rho_n}\frac{\partial u_n}{\partial N} = 0. \tag{3.12}$$

Here α is some function of the coordinate on the contour, which is selected in solving the concrete problem.

These conditions can be set both on the contour S (the shield in the diffraction problem): (3.11) instead of (3.7) or (3.12) instead of (3.8) (here the conditions (3.9) must hold on the complement $\bar{S}$), and on the complement $\bar{S}$, and then the corresponding conditions (3.7) or (3.8) for the eigenfunctions hold on the shield S.

Orthogonality conditions hold on the part of the contour where the eigenvalue has been introduced (on S or on $\bar{S}$): for E-polarization,

$$\int \frac{1}{\alpha} u_n u_m\, dS = 0 \qquad (n \neq m), \tag{3.13}$$

or for H-polarization,

$$\int \alpha \frac{\partial u_n}{\partial N} \frac{\partial u_m}{\partial N}\, dS = 0 \qquad (n \neq m). \tag{3.14}$$

The physical treatment of the auxiliary homogeneous problems consists in the idea that they describe undamped natural oscillations (occurring at a real frequency k). The coupling conditions (3.11) and (3.12) describe an active semitransparent film in which energy is released to compensate the radiation losses ($\operatorname{Im}\rho_n > 0$).

The formal solution of the initial diffraction problems is represented in all cases by the series of (1.2). The system of eigenfunctions in (1.2) and the isolated term are determined by the polarization, and they depend on the part of the contour in which the coupling conditions that contain the spectral parameter are established.

The field u^0 is the solution of the inhomogeneous equation (3.6). When one introduces the eigenvalue on S, u^0 can be made quite simple: the field of the sources f in free space. Yet it is expedient to take a more complicated u^0 in a number of cases. For example, in the problem of excitation of a resonator consisting of a pair of mirrors, we should add to the field f of the sources in free space the sum of the diffracted fields from each mirror taken individually. This u^0 contains all the non-resonance background, and the resonance phenomenon will be well described in all space by one term of the series in u_n that substantially exceeds the others.

It is convenient to introduce the spectral parameter on the complement $\bar{S}$ if it constitutes a small fraction of the total contour (the shield is almost closed). Then u^0 is the diffraction field f of the sources at the metallized total contour. That is, it also includes all of the non-resonant background, while the resonance is given by one term of the summation. This approach is used (Sec. D) in the problem of excitation of resonators having a small aperture.

If the spectral parameter is introduced on the shield S, then the representation (1.2) having arbitrary coefficients A_n fails to satisfy only the boundary condition on the shield S ((3.7) or (3.8)). When we require that the entire series should satisfy them, we can use the corresponding orthogonality conditions to get an expression for these coefficients: for E-polarization,

$$A_n = -\frac{1}{\rho_n}\frac{\int_S u^0 v_n\,dS}{\int_S \alpha v_n^2\,dS}, \quad v_n = \left(\frac{\partial u_n^+}{\partial N} - \frac{\partial u_n^-}{\partial N}\right)\Big|_S, \tag{3.15}$$

or for H-polarization

$$A_n = \frac{1}{\rho_n}\frac{\int_S (\partial u^0/\partial N)\, v_n\,dS}{\int_S (1/\alpha)\, v_n^2\,dS}, \quad v_n = (u_n^+ - u_n^-)|_S. \tag{3.16}$$

Under resonance conditions, one of the eigenvalues becomes small in modulus, and the corresponding term of the series predominates.

Yet if the spectral parameter has been introduced on the complement $\bar{S}$, then (1.2) for arbitrary A_n fails to satisfy only one of the conditions (3.9) (depending on the polarization—(3.9a) or (3.9b)). If we make the entire series satisfy this condition, we find by using orthogonality: for E-polarization,

$$A_n = \rho_n \frac{\int_{\bar{S}} (\partial u^0/\partial N)\, u_n\,dS}{\int_{\bar{S}} (1/\alpha)\, u_n^2\,dS}, \tag{3.17}$$

or for H-polarization,

$$A_n = -\rho_n \frac{\int_{\bar{S}} u^0 (\partial u_n/\partial N)\,dS}{\int_{\bar{S}} \alpha\, (\partial u_n/\partial N)^2\,dS}. \tag{3.18}$$

The eigenvalue increases in modulus at resonance.

The diffraction problem is reduced by the formulas (1.2) and (3.15)–(3.18) to finding the field u^0 and the eigenelements of the homogeneous problems. In the next two sections, we shall give and illustrate by concrete examples the ways of finding these eigenelements.

The formal application of the method for constructing the solution of the problem of diffraction by a dielectric object faces no theoretical difficulties, and it is carried out by an analogous scheme (Sec. E) with small changes.

C. A resonator consisting of a pair of mirrors

When we introduce the spectral parameter on the shield S, the homogeneous problem for E-polarization can easily be reduced to a simple integral equation:

$$-\rho_n u_n = \int_S u_n \frac{1}{\alpha} G\,dS; \tag{3.19}$$

Here G is the Green's function for free space for the Helmholtz equation; in the two-dimensional problem $G = (i/4)\, H_0^{(2)}(kR)$, where R is the distance between the observation and integration points.

Equation (3.19) is an effective algorithm for calculating open resonators of arbitrary shape. It has been used to study the resonance properties of resonators made of a pair of mirrors (Fig. 8), which play an important role in microwave technology. The well-known asymptotic (quasioptical) theory of these resonators[7] assumes that the conditions $ka \gg 1$, $L/a \gg 1$ are satisfied. The apparatus presented above permits one to calculate these resonators rigorously, and in particular, to determine the limits of applicability of the quasioptical theory.

Equation (3.19) (in which the function α was taken to be unity) has been solved numerically for resonators with plane and confocal mirrors. Figure 9 shows the frequency-dependence of the modulus of the first eigenvalue, which gives the resonance frequencies and also permits one to calculate the Q-factor of resonators by the simple formula

$$Q = \frac{kL}{2\delta(kL)}, \tag{3.20}$$

Here $\delta(kL)$ is the half-width of the resonance peak. For example, for a resonator with plane mirrors having $L/a = 2$, $kL = 40.89$, the Q-factor is $Q = 680$. Table 2 gives the resonance frequencies and the width of the resonance peaks of the first (fundamental) natural oscillation for a fixed value of the quasioptical parameter ka^2/L with different values of L/a. These results imply that the quasioptical theory gives satisfactory ac-

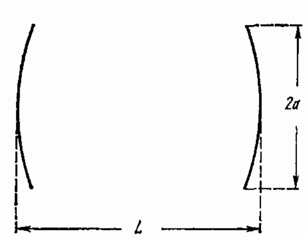


FIG. 8. A double-mirror resonator.

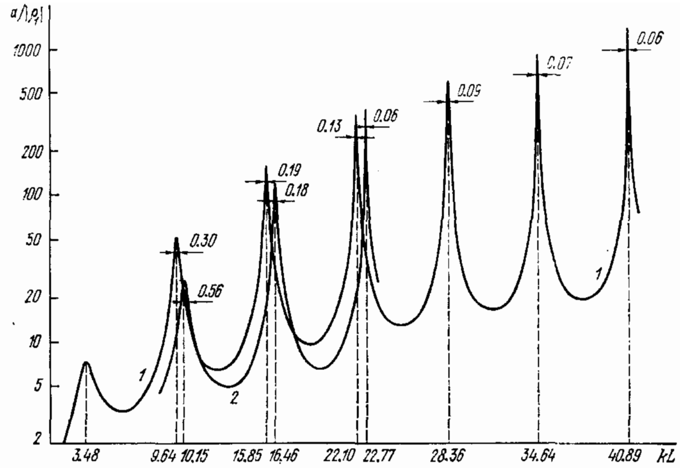


FIG. 9. The frequency-dependence of $|\rho_1|$. Curve 1—plane mirrors with $L/a=1$; curve 2—confocal mirrors with $L/a=3$.

curacy when $L/a \geqslant 4$ (1% in determining the resonance frequency, and of the order of 10% in determining the width of the resonances).

We note that the same eigenvalues ρ_n also simultaneously describe resonators having semitransparent mirrors (e.g., consisting of a closely-spaced grating). In this case it suffices to construct the curve $1/|\rho^0 - \rho_n|$, where ρ^0 is the transparency of the mirrors (see also Sec. F)). For mirrors consisting of a closely-spaced grating made of metallic strips, $\rho^0 = (p/2\pi) \times \ln \sin(\pi q/2)$, where p is the period, while q is the covering coefficient of the grating.[8]

D. The two-dimensional problem of diffraction by an arbitrary cylinder having a longitudinal slit[9]

Let us study the problem of excitation of an open resonator consisting of a cylinder of arbitrary cross section with a longitudinal slit cut in it (Fig. 10). We shall assume the width $2l$ of the slit to be small in comparison with the wavelength, while the wall is infinitely thin. In its mathematical formulation, this is the problem of (3.6)–(3.9) for the case of a small slit $\bar{S}$.

Let us introduce the spectral parameter on the slit S. Then the corresponding homogeneous problems prove to be analytically solvable, and thus, according to the formal apparatus of Sec. B, one can write the solution of the original problem of excitation of the resonator in closed form.

The method of solving the homogeneous problems consists in reducing them to equations over the slit $\bar{S}$ (an integral equation for H-polarization and an integro-differential equation for E-polarization). If we assume that the function α that enters into the coupling conditions (3.11) and (3.12) is

$$\alpha = \sqrt{l^2 - s^2} \tag{3.21}$$

(s is the coordinate referred to the middle of the slit), then the eigenvalues and eigenfunctions of these equations can be written in explicit form. Here it turns out that we must retain in the total field only one resonance term that corresponds to the first eigenvalue, in addition to u^0. All the remaining terms are not of resonance type, and they are negligibly small.

TABLE 2.

Plane mirrors, $ka^2/L = 1.1\pi$			Confocal mirrors, $ka^2/L = 1.2$		
L/a	kL	$2\delta(kL)$	L/a	kL	$2\delta(kL)$
1.58	9.58	0.210	2.74	16.49	0.122
2.04	15.86	0.192	3.22	22.77	0.110
2.42	22.15	0.186	3.63	29.05	0.122
2.74	28.43	0.179	4.67	49.70	0.115
3.03	34.70	0.175	5.24	60.47	0.123
3.98	59.85	0.164	5.76	73.03	0.117
Asymptotic value:		0.147			0.110

Since the slit is small, evidently, the resonance frequencies of the open resonator are close to the eigenfrequencies k_{nm} of the corresponding closed resonator. Hence it suffices in studying the resonance phenomena to know the structure of the solution near the frequencies k_{nm}. In these frequency ranges in the case of H-polarization, for instance, under *external* excitation, one can show that the current at the wall of the resonator far from the slit (if we consider it to be exponentially narrow) has the following structure:

$$\begin{aligned} u^-(s) &\approx u^0(s) + \frac{2Lu^0(0)\,G^-(0,s)}{[1/(k-k_{nm})A] + (1/\bar{\rho}_0) + iB}, \\ u^+(s) &\approx -\frac{2Lu^0(0)\,G^+(0,s)}{[1/(k-k_{nm})A] + (1/\bar{\rho}_0) + iB}; \end{aligned} \tag{3.22}$$

Here $2L$ is the length of the overall contour of the cross section of $(S+\bar{S})$. G^+ and G^- are the Green's functions for the inner (V^+) and outer (V^-) regions of the resonator under the condition $\partial G^{\pm}/\partial N|_{S+\bar{S}} = 0$:

$$\bar{\rho}^0 = \frac{\pi}{4L} \frac{1}{\ln(\pi l/2L)} \tag{3.23}$$

is a parameter that characterizes the coupling across the slit (the equivalent transparency),

$$B = 2L \operatorname{Im} G^-(0,0), \tag{3.24}$$

$$A = k_{nm} \int_{V^+} v_{nm}^2\, dV / L v_{nm}^2(0); \tag{3.25}$$

the v_{nm} are the eigenfunctions that correspond to the natural frequencies k_{nm} of the closed resonator under the condition $\partial v_{nm}/\partial N = 0$.

For the sake of simplicity, we have treated only resonators of shapes symmetrical about the slit, and solutions that are even with respect to the coordinate s.

According to the obtained results, the field within the resonator, and also the term complementary to u^0 outside it are almost always small (they are proportional

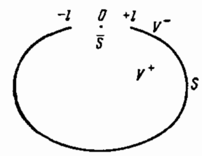


FIG. 10. Cross section of a cylinder having a longitudinal slit.

to $1/\ln(\pi l/2L)$). Resonance is manifested in the mutual compensation at certain frequencies of the first two (large) terms in the denominators of (3.22). Here the scattered field outside the resonator undergoes a finite perturbation (the term complementary to u^0 is of the order of unity), while the field inside becomes large and proportional to $\ln(\pi l/2L)$. These formulas also imply that the field inside is no longer large (of the order of unity) at the eigenfrequencies k_{nm} of the closed resonator, while it equals u^0 outside, since the second term in (3.22) vanishes here.

One can derive from the resonance condition (vanishing of the real component of the denominator of (3.22)) a simple formula for the shift in the resonance frequencies arising from the cutting of the slit:

$$\Delta k_{nm} \approx -\frac{\bar{\rho}^0}{A}. \tag{3.26}$$

The half-width of the resonance curve is much smaller, being of the order of $B(\bar{\rho}^0)^2$. In contrast to Δk_{nm}, the half-width depends on the sort of objects that lie near the resonator.

In the case of E-polarization, the resonance denominator in the expression for the total field has the same form as in (3.22), and the shift of the resonance frequencies is given by the same formula (3.26), except that the quantities A, $\bar{\rho}^0$, and B are determined in this case by the formulas

$$\bar{\rho}^0 = \frac{\pi l^2}{8L}, \tag{3.27a}$$

$$A = \frac{k_{nm} \int\limits_{V^+} v_{nm}^2 \, dV}{L\, [\partial v_{nm}(0)/\partial N]^2}, \tag{3.27b}$$

$$B = 2L \operatorname{Im} \frac{\partial^2 G^-(0, 0)}{\partial N_S \, \partial N_{S'}}. \tag{3.27c}$$

Here k_{nm} and v_{nm} are the natural frequencies and the corresponding eigenfunctions of the closed resonator under the condition that $v_{nm}|_{S+\bar{S}} = 0$; $\bar{G}$ is the Green's function for the exterior of the resonator under the same boundary condition. Actually one calculates B (and also A for complicated cross sections) by the method of Sec. H, rather than by (3.24), (3.25), and (3.27).

According to these formulas, the solutions for the two polarizations have analogous resonance properties, except that the characteristic small parameter for E-polarization is $(l/L)^2$ rather than $1/\ln(\pi l/2L)$.

The corresponding analysis of *internal* excitation for the two polarizations shows that the field is large at resonance, both within the resonator (of the order of $1/(\bar{\rho}^0)^2$) and outside it (of the order of $1/\bar{\rho}^0$). Far from the resonance frequencies, the total field in the outer region and the term complementary to u^0 on the inside are equally small (of the order of $\bar{\rho}^0$).

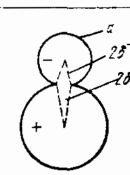
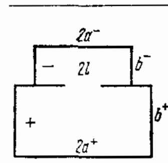
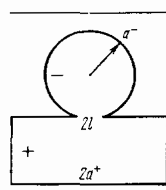
The presented apparatus can be extended in a trivial way for calculating closed resonators that are coupled through a small slit, with the obvious changes in the formulations of the problems (the radiation condition is omitted). Table 3 gives for H-polarization the shifts Δk_{nm} of the resonance frequencies for several concrete forms of coupled resonators.

E. A resonator of arbitrary shape made of material haveing $\epsilon \gg 1$[6,10]

In this section we shall briefly describe the fundamental physical phenomena that arise in diffraction by an open resonator in the form of an object having $\varepsilon \gg 1$. This is also a "trap," and application of the apparatus of Sec. B to this problem also permits one to study the qualitative side of the resonance phenomenon. We shall give only the results of this study, without writing out the formulas, neither for the diffraction problem itself (a scalar wave equation that is homogeneous within the object, but inhomogeneous outside; in the case of E-polarization, e.g., continuity of the field and its normal derivative at the boundary S of the object, and the radiation condition), nor for the eigenfunctions.

As before, the field u^0 is the result of diffraction of the same sources by a metallic resonator of the same shape (with the boundary condition $u|_S = 0$). The total field is written in the form of the series of (1.2); the coefficients A_n are proportional to the eigenvalues ρ_n. These quantities are small at all frequencies (of the order of $1/\sqrt{\varepsilon}$) except for the neighborhoods of the resonance frequencies. These latter frequencies are close to the eigenfrequencies k_{nm} of the internal problem (a shielded volume filled with a dielectric) with the boundary condition $\partial v_{nm}/\partial N_S = 0$ (which differs from the condition for $u^0|_S$). The shift of the center of the resonance curve and its width are of the same order of magnitude here ($\sim 1/\varepsilon$), and both of these characteristics depend on the type of bodies that lie near the resonator. Even at the maximum of the resonance curve, the field inside does not become large, but remains finite (yet the energy stored within the object, which is proportional to ε, is large). The actual calculation of all the parameters that determine $\rho_n(k)$ requires (just as in the problem of Sec. D) solving the boundary problem in V^- (the

TABLE 3.

	k	Δk
	k_{0m} $J_1(k_{0m}a) = 0$	$\dfrac{-1}{2a^2 k_{0m} \ln(\delta^+\delta^-/4)}$
	$\dfrac{\pi}{b^-}$	$\dfrac{-1}{2a^- \ln(\pi^2 l^2/4a^+a^-)}$
	$J_1(k_{0m}a^-) = 0$ k_{0m} $\dfrac{\pi}{b}$	$\dfrac{-1}{2a^{-2} k_{0m} \ln(\pi l^2/4a^+a^-)}$ $\dfrac{-1}{2a^+ \ln(\pi l^2/4a^+a^-)}$

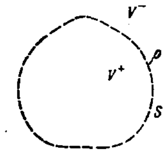


FIG. 11. Cross section of a cylinder having a semitransparent wall.

definition of u_n^- in V^- in terms of $u_n^-|_S = v_{nm}|_S$). Hence, here also one can obtain the quantitative results more easily from integral equations analogous to those to be given in the next section.

F. The spectral parameter in the conditions at infinity[11,12]

The method presented in this section for solving diffraction problems is essentially a variant of the boundary method. Its advantage over the already studied variants of the GNOM is manifested in studying open problems where losses occur only by radiation. It consists in the idea that the homogeneous problem of this method is real-valued, and it reduces to a simple integral equation having real eigenfunctions and eigenvalues. As we have noted in Chap. 1, this advantage arises from the circumstance that here the spectral parameter is introduced precisely in that region of space where the losses occur, i.e., at infinity. The energy source in the auxiliary homogeneous problem is a natural wave that converges from infinity. The solution of the homogeneous problem also contains the eigenwave scattered by the object being studied. The angular dependences of the converging and diverging waves, which are determined by the shape and properties of the object, coincide apart from taking the complex conjugate. The spectral parameter is the amplitude of the scattered natural wave.

This method is applicable for solving problems of diffraction by objects and surfaces having the most varied properties. Here we shall present it with the example of a two-dimensional scalar problem of diffraction by some semitransparent surface S (Fig. 11) for the case of E-polarization. In particular, the surface S can be open.

Assume that we need to find the solution of the equation

$$\Delta u + k^2 u = f, \tag{3.28}$$

that satisfies the radiation conditions and the boundary conditions on S:

$$u^+ - u^- = 0, \tag{3.29a}$$

$$\rho^0\left(\frac{\partial u^+}{\partial N} - \frac{\partial u^-}{\partial N}\right) - u = 0. \tag{3.29b}$$

The parameter ρ^0 characterizes the transparency of the surface S (see the end of Sec. C). For a metallic surface, $\rho^0 = 0$, and $u|_S = 0$.

In order to solve the formulated problem, we shall introduce the eigenfunctions u_n of the auxiliary homogeneous problem that satisfy Eq. (3.10), the boundary conditions (3.29) on S, and the following condition at infinity (in which we have introduced the spectral parameter $\lambda = s_n$):

$$u_n \underset{r\to\infty}{\approx} \frac{1}{1+s_n}\left(\Phi_n^*(\theta)\frac{e^{ikr}}{\sqrt{r}} + s_n\Phi_n(\theta)\frac{e^{-ikr}}{\sqrt{r}}\right) \tag{3.30}$$

with the requirement on $\Phi_n(\theta)$ that

$$\Phi_n^*(\theta) = i\Phi_n(\pi+\theta). \tag{3.31}$$

In the absence of losses at the surface S, $|s_n| = 1$. The set of numbers s_n constitutes the scattering matrix.

This homogeneous problem is reduced in the standard way to a real integral equation over the boundary S:

$$4\rho^0 u_n = \int_S u_n[N_0(kR) - \varkappa_n J_0(kR)]\,dS, \tag{3.32}$$

Here R is the distance between the integration and observation points; the real spectral parameter $\varkappa_n$ of this equation is related simply to the spectral parameter s_n:

$$\varkappa_n = i\frac{1-s_n}{1+s_n}. \tag{3.33}$$

If the observation point lies outside S, then (3.32) defines u_n throughout space from its values on S.

The formal solution of the original diffraction problem is represented in the form

$$u = u^0 + \sum_n A_n u_n^s, \tag{3.34}$$

where u_n^s is defined by the formula

$$u_n^s = \frac{-i}{4}\int_S u_n H_0^{(2)}(kR)\,dS. \tag{3.35}$$

(We note that the expansion is not carried out in terms of u_n in this variant of the GNOM.) According to this definition, the functions u_n^s satisfy Eq. (3.10), the radiation condition, and the boundary condition (3.29a) (owing to the property of continuity of the potential of a simple layer). Upon comparing (3.32) and (3.35), we get a relationship that holds at any point in space:

$$-\rho^0 u_n = \operatorname{Re} u_n^s - \varkappa_n \operatorname{Im} u_n^s. \tag{3.36}$$

In particular, this implies the biorthogonality condition

$$\int_S u_m \operatorname{Im} u_n^s\,dS = 0 \qquad (n \neq m). \tag{3.37}$$

The field u^0 that is isolated in (3.14) can be selected in various ways: the coefficients A_n depend on this choice. Here we shall study two methods. In one of them, u^0 is a very simple field (that of the same sources f in a vacuum). In the other, which is effective in analyzing resonators of high Q-factor having a closed boundary, u^0 is the diffraction field for a completely metallized resonator. In both cases, (3.34) satisfies termwise all the conditions of the original diffraction problem except (3.29b). With the use of the orthogonality of (3.27), the imposition of this condition gives an ex-

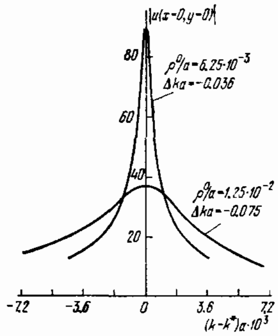


FIG. 12. Frequency dependence of the field at the center of a rectangular resonator having a semitransparent wall. Resonance in the 1st natural oscillation.

pression for the coefficients A_n. If u^0 is the field of the sources f in a vacuum, then

$$A_n = -\frac{1}{i+\varkappa_n}\frac{\int\limits_S u^0 u_n\, dS}{\int\limits_S u_n \operatorname{Im} u_n^s\, dS}, \tag{3.38a}$$

while if u^0 is the diffraction field for a metallized resonator, then

$$A_n = -\frac{\rho^0}{i+\varkappa_n}\frac{\int\limits_S (\partial u^0/\partial N)\, u_n\, dS}{\int\limits_S u_n \operatorname{Im} u_n^s\, dS}. \tag{3.38b}$$

Thus, the original diffraction problem is reduced to solving the integral equation (3.32) and calculating the field u^0.

The presented method is applied in the next section for analyzing the resonance properties of a resonator having a closed semitransparent boundary.

G. The two-dimensional problem of diffraction by a semitransparent cylinder[10,12]

Let the transparency of the wall of the resonator be small ($k\rho^0 \ll 1$), and the excitation be external. For such an open (trap-type) resonator, just as for the cylinder with a small slit that was treated in Sec. D, the resonance frequencies are close to the natural frequencies k_{nm} of the corresponding closed resonator, and we must find the structure of the eigenvalues $\varkappa_n$ and of the coefficients A_n near these frequencies.

We can show that in these frequency ranges,

$$\varkappa_n(k) \approx -\frac{\rho^0 + (k-k_{nm})A}{(k-k_{nm})^2 A^2 B}. \tag{3.39}$$

The quantity A that figures here is expressed in terms of k_{nm} and the corresponding eigenfunctions v_{nm} of the closed resonator under the condition $v_{nm}|_S = 0$;

$$A = 2k_{nm}\int\limits_{V^+} v_{mn}^2\, dV \Big/ \int\limits_S (\partial v_{nm}/\partial N)^2 dS. \tag{3.40}$$

The quantity B also does not depend on the frequency nor on the transparency. It characterizes the Q-factor of the resonator, and like (3.27), it can be expressed in terms of the Green's function of the region outside the resonator. The following formulas hold in the same frequency ranges:

$$\begin{aligned} u_n^2(k) &\approx -v_{nm} + (k-k_{nm})\frac{\partial u_n^s}{\partial k}(k_{nm}) \text{ in } V^+,\\ u_n^s(k) &\approx (k-k_{nm})\frac{\partial u_n^s}{\partial k}(k_{nm}) \text{ in } V^-. \end{aligned} \tag{3.41}$$

In the studied problem one can conveniently choose as u^0 the diffraction field for a metallized resonator.

Upon substituting (3.39) into (3.38b), we get after simple transformations

$$A_n \approx -\frac{1/(k-k_{nm})}{[1/(k-k_{nm})A] + (1/\rho^0) + iB}\,\frac{\int\limits_S (\partial u^0/\partial N)(\partial v_{nm}/\partial N)\, dS}{2k_{nm}\int\limits_{V^+} v_{nm}^2\, dV}. \tag{3.42}$$

This gives rise to a formula of the same form as in Sec. D for the shift of the resonance frequency k^* with respect to k_{nm}: $\Delta k_{nm} = -\rho^0/A$.

According to (3.39), the eigenvalue $\varkappa_n$ vanishes at this frequency:

$$\varkappa_n(k^*) = 0. \tag{3.43}$$

The derived formulas imply that the resonance properties of the studied resonator are analogous to those of a resonator with a small slit (Sec. D). The field in V^+ and all the terms complementary to u^0 in V^- far from resonances are small (of the order of ρ^0). At resonance, the total field in V^- undergoes a finite perturbation (of the order of unity), while the field inside becomes large, and proportional to $1/\rho^0$. Figure 12 shows the field at the center of a rectangular resonator having semitransparent walls ($a/b = 2$) when excited by a plane wave incident on the broad wall. Figure 13 illustrates the relationship of this field at the resonance frequency k^* to the transparency of the walls. The calculation was performed by using the integral equation (3.32) and the formulas (3.39) and (3.42).

H. Calculation of the velocity and attenuation of leaky *E*-waves in waveguides of arbitrary cross section having semitransparent walls or a slit

As we know, only the so-called leaky waves can propagate along open waveguides of the studied type. One can expand the total field in the excitation problem in a series in terms of them. These waves have complex propagation constants. The coupling between the inner and outer regions (the slit or the transparency of the wall) causes attenuation of the wave and a difference between its phase velocity and the velocity in the corresponding closed waveguide.

The propagation constants $\hat{h}_{nm} = \hat{h}'_{nm} + i\hat{h}''_{nm}$ are expressed in terms of the transverse wavenumber, or the complex natural frequency $\hat{k}_{nm}$ of the corresponding

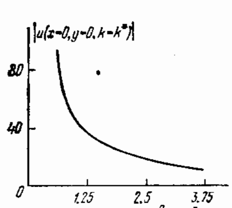


FIG. 13. Relationship of the field at resonance to the transparency of the walls.

two-dimensional resonator that amounts to the cross section:

$$\hat{h}^2_{nm} = k^2 - \hat{k}^2_{nm}. \tag{3.44}$$

Here k is the given real frequency of the generator.

Thus the problem is reduced to finding the eigenvalues of the homogeneous problem:

$$\left.\begin{array}{l} \text{a) } \Delta u_{nm} + \hat{k}^2_{nm} u_{nm} = 0, \\ \text{b) } u^+_{nm} - u^-_{nm}\big|_S = 0, \\ \text{c) } \left.\dfrac{\partial u^+_{nm}}{\partial N} - \dfrac{\partial u^-_{nm}}{\partial N} - \dfrac{u_{nm}}{\rho^0}\right|_S = 0. \end{array}\right\} \tag{3.45}$$

At the radial infinity, u_{nm} must constitute an outgoing and exponentially increasing wave. S is the contour of the cross section of the waveguide. For a metallic waveguide with a slit, $\rho^0 = 0$, and S is an open contour.

Let us compare the problem (3.45) with the homogeneous problems (3.10), (3.30), and (3.31). If we generalize the latter to the region of complex frequencies (of course, here now $|s_n| \neq 1$, and $\varkappa_n$ is complex), then the condition (3.30) transforms at the frequencies at which $s_n = \infty$ into the condition at infinity of the problem of (3.45), while the condition (3.31) drops out owing to absence of the incoming wave. Since the equation and the boundary conditions on S are the same in these problems, both formulations are fully identical at these frequencies, and we get the usual result that the complex natural frequency is a pole of the function $s_n(k)$. That is, $s_n(\hat{k}_{nm}) = \infty$, or equivalently,

$$\varkappa_n(\hat{k}_{nm}) = -i. \tag{3.46}$$

If we assume that $\varkappa_n$ is an analytic function of k, and expand it in a Taylor's series in the neighborhood of the resonance frequency k^*, we have the following with account taken of (3.43):

$$\varkappa_n(k) \approx (k - k^*) \frac{d\varkappa_n}{dk}(k^*). \tag{3.47}$$

For resonators of high Q-factor ($\operatorname{Im}\hat{k}_{nm}$ bing small), (3.46) and (3.47) give a simple formula for k_{nm}:

$$\hat{k}_{nm} \approx k^* - \frac{i}{d\varkappa_n/dk\,(k^*)}. \tag{3.48}$$

Thus the complex natural frequency is expressed solely in terms of the real quantities k^* and $d\varkappa_n/dk(k^*)$, which can be determined by solving the real integral equation

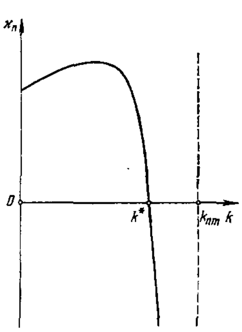


FIG. 14. Characteristic frequency dependence of the eigenvalue $\varkappa_n$ in the vicinity of the spectrum of the closed problem.

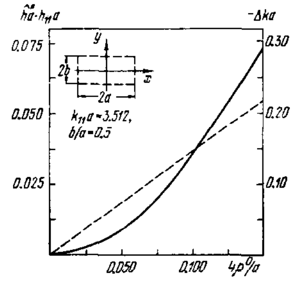


FIG. 15. Characteristics of a semitransparent waveguide of rectangular cross section.

(3.32). The formula (3.48) is general in nature, and it proves to hold for any resonators or waveguides of high Q-factor having slightly leaky waves, including even those that are not nearly closed.

For resonators and waveguides having semitransparent walls (or with a small slit), the calculation of the quantities that figure in (3.48) is substantially simplified if one uses the structure of $\varkappa_n(k)$ (3.39) (or the analogous structure for a resonator with a slit), which is valid in a wider frequency range.[1)] Then Eq. (3.48) is transformed into

$$\hat{k}_{nm} \approx k_{nm} - \frac{\rho^0}{A} + i\frac{(\rho^0)^2}{A} B. \tag{3.49}$$

The quantities ρ^0, A, and B differ for waveguides having a longitudinal slit (see Sec. D) and those having semitransparent walls (see Sec. G). We obtain, respectively, for the attenuation and the phase velocity of the leaky waves far from the critical frequencies of the waveguide:

$$\begin{aligned} \hat{h}''_{nm} &= -(\rho^0)^2 \frac{B}{A} \frac{k_{nm}}{h_{nm}}, \\ \hat{h}'^{nm} &= h_{nm} + \frac{\rho^0}{A} \frac{k_{nm}}{h_{nm}}, \end{aligned} \tag{3.50}$$

where $h_{nm} = \sqrt{k^2 - k^2_{nm}}$ is the propagation constant for the closed waveguide. For a waveguide with a slit, the equivalent transparency is defined by Eq. (3.27a).

The unknown quantities A, B, and k_{nm} are found by using (3.39) (or its equivalent for a waveguide with a slit) from the $\varkappa_n(k)$ curve that is obtained by numerical solution of the corresponding integral equation. This curve has the standard form in the frequency range of interest to us. It is depicted in Fig. 14 for a waveguide having a closed semitransparent boundary.

Figures 15–19 show quantities proportional to the attenuation (solid curves), and to the correction to the phase velocity (dotted curves) of the first leaky E-wave in waveguides of varying cross sections having semitransparent walls and having slits, as calculated by the above-described method. According to what we have

[1)]If we assume that $\varkappa_n = -i$ in (3.32), we get an equation for the complex natural frequency k_{nm} that figures in the argument of the kernel of the integral equation. Thus, using the described method permits one to solve this equation, while staying in the region of real k, and it gives a groundwork for Eq. (3.48), and as is especially important for the sake of calculation, for Eq. (3.49).

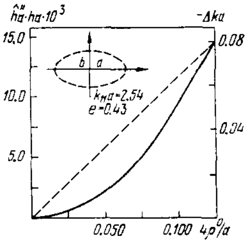


FIG. 16. Characteristics of a semitransparent waveguide of elliptical cross section.

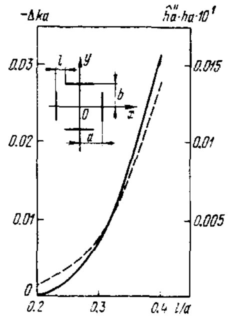


FIG. 18. Characteristics of a rectangular waveguide having slits at the corners.

said, this method does not require a knowledge of the Green's function of the outer region for a metallized waveguide. Therefore the sought quantities can be found without the usual assumption that the slit is filled with an infinite flange.

One of the first studies in which the field is represented in the form of a discrete series without bringing in a continuous spectrum is[16]. Here the solution of the Schrödinger equation is expanded in terms of the eigenfunctions of the self-adjoint problem (Sturm functions), which corresponds to a negative total energy. These functions decline at infinity. In our notation, this implies that $k^2 < 0$ in (2.25) and (2.26b). The eigenvalue is the coefficient of the potential energy (coupling constant). The later literature on this method is given in[17]. It was assumed that $k > 0$ in[18], and as in the GNOM, the eigenfunctions obey the same condition for outgoing waves at infinity as the scattered field does.

We have formulated all the problems above in differential formulation. One can convert them to integral equations by introducing the Green's functions. The coupling constant is the eigenvalue of the Lippmann–Schwinger equation, while the Sturm functions are the eigenfunctions of this equation. This equation for complex k not lying on the semiaxis $k > 0$ and the Neumann series for it have been studied in detail in[19]. The functions used above in the expansions are the eigenfunctions of the other equation (the Lippmann–Schwinger equation in the so-called distorted-wave method[20]). Here the kernel is not the Green's function for a vacuum (as in the ordinary Lippmann–Schwinger equation), but the Green's function of the auxiliary dielectric object or the auxiliary potential $V^0(\mathrm{r})$. Moreover, in contrast to the cited studies, one does not expand the total scattered field in a series in the above-described method, but the difference between it and the field that arises upon scattering by the same potential V^0. This expansion holds throughout space, while the resonance scattering, just like the resonance in closed resonators, is described by a single term.

The NFM is presented in many textboods (see, e.g., e.g., [21]). The ordinary quantum-mechanical theory of scattering with a continuous energy spectrum, and with eigenfunctions that increase at infinity for energy values that correspond to the poles of the scattering matrix, is an example of its application for open systems. It is applied in[22] to open electrodynamic systems. Variational methods in the theory of closed resonators have been treated in[23].

V. A. Steklov and other authors (see, e.g., [24]) have studied in detail the scalar problem having the spectral parameter in the form of a factor in the third-type boundary condition for the inner region. An inhomogeneous integral equation of the first type with the same simple kernel as in (3.19) (with $\alpha \equiv 1$) is often used in diffraction problems. An asymptotic boundary condition analogous to (3.30) was introduced in[22], and then used in[25] and independently in[26]. An asymptotic theory of open resonators that lacks the restriction $L/a \gg 1$ has been presented in[27,28]. A theory of resonators and waveguides having narrow slits (but necessarily having a flange) has been developed, e.g., in[29]. Other expansions for the field at the surface of metallic objects have been proposed and studied in[30].

The convergence of infinite series of the type of (1.2) and the validity of the formal procedures performed on them have been studied in[31]. The functions u_n and the numbers λ_n are the eigenelements of the non-self-adjoint operators for the infinite-region problem. The spectral properties of these operators are more complicated than for the self-adjoint operators that arise in closed-resonator problems (in particular, in the NFM). Yet, in many respects, the operators of diffraction theory are close to self-adjoint. By using the

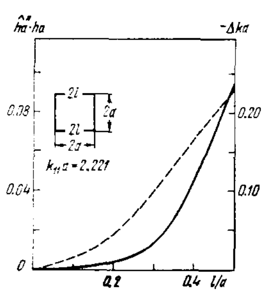


FIG. 17. Characteristics of a rectangular waveguide having longitudinal slits. The slits are in the centers of the walls.

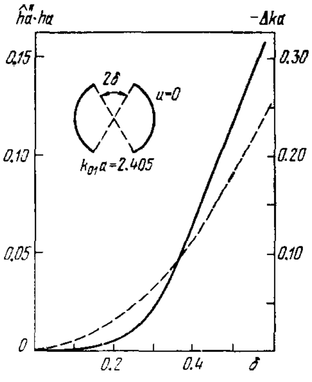


FIG. 19. Characteristics of a circular waveguide having longitudinal slits.

theorems from[32], it has been possible for all the fundamental problems to prove the completeness of their system of eigenfunctions and the summability of the corresponding series[2)], and to find the asymptotics of the eigenvalues, etc. However, there is as yet no full mathematical justification for the correctness of all the procedures performed above for the most general diffraction problems. Yet the problem of the convergence of the direct variational methods (e.g., the Ritz method) as applied to complex-valued functionals of the type of (2.8) that are generated by these operators has apparently not been studied theoretically.

[2)]More exactly, the characteristic and adjoint series. Examples are constructed in which adjoint series exist. For simplicity of notation, we have assumed that they do not exist, so that, in particular, integrals of the type of (2.6) for $n=m$ differ from zero. If the adjoint functions exist, then the number of them for each eigenvalue is finite. They must be included in series of the type of (1.2). The coefficients A_n that correspond to these eigenvalues will be determined by a finite system of linear algebraic equations.

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