# Expressions for the energy density and evolved heat in the electrodynamics of a dispersive and absorptive medium

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Usp. Fiz. Nauk. **118**, 523–537 (March 1976)

The question of the expressions for the energy density $W$ and evolved heat (dissipation) $Q$ in the electrodynamics of a dispersive and absorptive medium is discussed. Attention is concentrated on explaining the fact that $W$ and $Q$ are not expressed, generally speaking, in terms of the complex dielectric permittivity $\epsilon(\omega)$. This statement is illustrated with the example of a medium consisting of a collection of oscillators and with the example of the simplest model of a plasma. A convenient expression for the energy density of a field with arbitrary time dependence in a transparent medium is obtained in the Appendix. A derivation of the high-frequency average of $(1/4\pi)\partial \mathbf{D}/\partial t$ for a quasi-monochromatic field in an absorptive dispersive medium is also given there.

PACS numbers: 03.50.Jj

Up to comparatively recent times, in the discussion of the energy relations in electrodynamics and optics courses, the medium has been assumed to be nondispersive. In this case, and in the absence of absorption all the terms appearing in the Poynting relation can be interpreted unambiguously and we can assume that the expressions for the energy density $W$ of the electromagnetic field and for the energy flux $\mathbf{S}$ are known. In fact, however, all media possess dispersion and, whereas we can usually neglect the spatial dispersion in the low-frequency region, and, partially, in optics too, it is often absolutely necessary to take the frequency dispersion into account. In the presence of dispersion the question of the energy relations in macroscopic electrodynamics is, in certain respects, not so simple, especially when absorption is taken into account (at the same time, as follows from the dispersion relations, a dispersive medium is always absorptive, although the absorption and dispersion may be dominant in different spectral regions).

The energy relations in a dispersive medium have already been considered more than once, and, in particular, have been elucidated in the monographs[1] (Sec. 22) and[2] (Sec. 3), where the other literature is also indicated. Nevertheless, in the presence of absorption the question evidently remains insufficiently clear (or at least insufficiently well-known), as indidicated, e.g., by the appearance in 1975 of an article[3] that is incorrect. The latter circumstance stimulated the authors to publish a critical comment[4] and to analyze certain questions in somewhat more detail than in[1,2]. We hope that the account of the corresponding material in the present article is sound and will be useful.

1. We write the equations for the field in the medium in the form

$$\operatorname{curl}\mathbf{B}=\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}+\frac{4\pi}{c}\mathbf{j}_{ext},\quad \operatorname{curl}\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t},\qquad \operatorname{div}\mathbf{D}=4\pi\rho_{ext}\quad \operatorname{div}\mathbf{B}=0. \tag{1}$$

In this form the properties of the medium are obviously reflected only in the relationship between the generalized induction $D$ and the electric field $\mathbf{E}$ (for more detail, see, e.g.,[2]). From (1), by the well-known route, we arrive at the Poynting theorem

$$\frac{1}{4\pi}\left(\frac{\partial\mathbf{D}}{\partial t}\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}\mathbf{B}\right)+\mathbf{j}_{ext}\mathbf{E}=-\frac{c}{4\pi}\operatorname{div}[\mathbf{E}\times\mathbf{B}]\,. \tag{2}$$

Below we shall consider only a linear medium, at rest and not changing in time. If, in addition, the medium is isotropic, nonmagnetic and nonabsorptive and does not possess dispersion, then $\mathbf{D}=\varepsilon'\mathbf{E}$, $\mathbf{B}=\mathbf{H}$, and $\varepsilon'=\varepsilon$ is a real quantity. Then the relation (2) takes the form

$$\frac{\partial}{\partial t}\left(\frac{\varepsilon' E^2+H^2}{8\pi}\right)+\mathbf{j}_{ext}\mathbf{E}=-\frac{c}{4\pi}\operatorname{div}[\mathbf{E}\times\mathbf{H}]\,. \tag{3}$$

In this simplest case, which we have already mentioned, the quantity $W=(\varepsilon' E^2+H^2)/8\pi$ is immediately identified with the energy density, and the vector $\mathbf{S}=(c/4\pi)\mathbf{E}\times\mathbf{H}$ has the meaning of the total energy flux through unit surface. The generalizations which pertain to taking the magnetic permeability $\mu$ or the anisotropy into account are obvious and do not involve fundamental difficulties. The same can be said about the transition to a moving medium (so long as its velocity $\mathbf{u}$ can be assumed to be constant).[1)] The relation (2) is valid in all these cases, and, in the absence of dispersion, the values of $\mathbf{D}$, $\mathbf{B}$, $\mathbf{H}$ and $\mathbf{E}$ taken at the same point $\mathbf{r}$ and at the same time $t$ are mutually related (in linear electrodynamics, which we have in mind here, this relationship is linear). In addition, Eqs. (1) and the relation (2) are also valid in a dispersive absorptive medium, but in this case $\mathbf{D}$ is a linear functional of $\mathbf{E}$, or, to be more specific, the value of $\mathbf{D}(\mathbf{r}, t)$ is determined by the field $\mathbf{E}(\mathbf{r}', t')$ at times $t'\leqslant t$ and at points $\mathbf{r}'$ situated in a certain region about the point $\mathbf{r}$. For a linear stationary medium, uniform in space and unchanging in time,

[1)]True, a moving medium possesses the feature that a term corresponding to the work done by the force acting on the medium appears when we go from (2) to a relation of the type (3) (cf., e.g.,[5]).

the dispersion and absorption are taken into account using the relationship

$$D_i(\omega, \mathbf{k}) = \varepsilon_{ij}(\omega, \mathbf{k}) E_j(\omega, \mathbf{k}),$$

where

$$\mathbf{E}(\omega, \mathbf{k}) = \frac{1}{(2\pi)^4} \int \mathbf{E}(\mathbf{r}, t) e^{-i(\mathbf{kr}-\omega t)} d\mathbf{r}\, dt$$

and analogously for $\mathbf{D}$, where $\varepsilon_{ij}(\omega, \mathbf{k})$ is the complex permittivity tensor. The dependence of $\varepsilon_{ij}$ on $\omega$ corresponds to the frequency dispersion and the dependence on $\mathbf{k}$ corresponds to the spatial dispersion.

In the presence of spatial dispersion, besides the energy flux $\mathbf{S} = (c/4\pi)\mathbf{E}\times\mathbf{H}$ there appears, generally speaking, an additional energy flux $\mathbf{S}^{(1)}$ (cf. [2]). We, however, shall not consider media with spatial dispersion but confine ourselves to the simplest class of dispersive media—an isotropic and nonmagnetic medium with frequency dispersion describable by the permittivity $\varepsilon(\omega) = \varepsilon' + i\varepsilon''$, $\varepsilon' = \mathrm{Re}\,\varepsilon$, $\varepsilon'' = \mathrm{Im}\,\varepsilon$. As in the absence of dispersion, with only frequency dispersion the generalization to a magnetic and anisotropic medium also presents no difficulty and only leads to more cumbersome expressions (for convenience, however, some of these expressions are given in the Appendix).

2. For the medium just mentioned, characterized by a complex permittivity $\varepsilon(\omega)$, the Poynting relation (2) can be written in the form

$$\frac{\partial(W_E+W_M)}{\partial t} + Q = -\mathbf{j}_{\mathrm{ext}}\mathbf{E} - \frac{c}{4\pi}\operatorname{div}[\mathbf{E}\times\mathbf{H}],$$
$$\frac{\partial W_E}{\partial t} + Q = \frac{1}{4\pi}\frac{\partial \mathbf{D}}{\partial t}\mathbf{E}, \quad W_M = \frac{H^2}{8\pi}, \tag{4}$$

$$\mathbf{D}(t, \mathbf{r}) = \int_{-\infty}^{+\infty} \varepsilon(\omega, \mathbf{r}) \mathbf{E}(\omega, \mathbf{r}) e^{-i\omega t} d\omega,$$
$$\mathbf{E}(t, \mathbf{r}) = \int_{-\infty}^{+\infty} \mathbf{E}(\omega, \mathbf{r}) e^{-i\omega t} d\omega, \quad \mathbf{E}(-\omega, \mathbf{r}) = \mathbf{E}^*(\omega, \mathbf{r}), \tag{5}$$

where the argument $\mathbf{r}$ will be omitted in the following, since, in the assumed absence of spatial dispersion, it appears only as a parameter. The relation $\mathbf{E}(-\omega) = \mathbf{E}^*(\omega)$ reflects the fact that the field $\mathbf{E}$ is real; from the reality of $\mathbf{D}$ it then follows that

$$\varepsilon(-\omega) = \varepsilon^*(\omega), \ \mathrm{Re}\,\varepsilon(-\omega) \equiv \varepsilon'(-\omega) = \varepsilon'(\omega),$$
$$\mathrm{Im}\,\varepsilon(-\omega) \equiv \varepsilon''(-\omega) = -\varepsilon''(\omega). \tag{6}$$

In the case of a field with an abritrary time dependence, the expression for $[\partial(W_E + W_M)/\partial t] + Q$ can be written in the form of an integral over frequencies, but it is then impossible to carry out the time integration in general form. However, the latter can be done for a nonabsorptive medium (see the Appendix). For an absorptive medium it is possible to obtain certain general results only by making the time dependence of the field $\mathbf{E}$ specific. The most important such case is a quasi-monochromatic field

$$\mathbf{E}(t) = \frac{1}{2}[\mathbf{E}_0(t) e^{-i\omega t} + \mathbf{E}_0^*(t) e^{i\omega t}],$$
$$\mathbf{H}(t) = \frac{1}{2}[\mathbf{H}_0(t) e^{-i\omega t} + \mathbf{H}_0^*(t) e^{i\omega t}], \tag{7}$$

where the quasi-monochromatic character of the field is manifested in the fact that the functions $\mathbf{E}_0(t)$ and $\mathbf{H}_0(t)$ vary very slowly in the time $T = 2\pi/\omega$. Below it will also be assumed that $\mathbf{E}_0(-\infty) = 0$ and $\mathbf{H}_0(-\infty) = 0$. This condition is obviously not satisfied by a monochromatic field, and this prevents unrestricted use of it.

We substitute the fields (7) into (4) and carry out averaging over high frequencies $\omega$; this is equivalent to neglecting the terms containing factors $e^{\pm 2i\omega t}$ (such averages are denoted below by a bar). We then obtain the following result (cf., [6] and the Appendix):

$$\frac{1}{4\pi}\overline{\frac{\partial \mathbf{D}(t)}{\partial t}\mathbf{E}(t)} = \frac{1}{16\pi}\frac{d(\omega\varepsilon'(\omega))}{d\omega}\frac{\partial}{\partial t}[\mathbf{E}_0(t)\mathbf{E}_0^*(t)] + \frac{\omega\varepsilon''(\omega)}{8\pi}\mathbf{E}_0(t)\mathbf{E}_0^*(t)$$
$$+ \frac{i}{16\pi}\frac{d(\omega\varepsilon''(\omega))}{d\omega}\left(\frac{\partial \mathbf{E}_0(t)}{\partial t}\mathbf{E}_0^*(t) - \frac{\partial \mathbf{E}_0^*(t)}{\partial t}\mathbf{E}_0(t)\right), \tag{8}$$

where, as below, the frequency derivatives are taken at the "carrier frequency" $\omega$ appearing in (7) (in the Appendix, this frequency is denoted by $\omega_0$).

In the absence of absorption, when $\varepsilon''(\omega) = 0$ and $\varepsilon''(\omega) = \varepsilon(\omega)$, obviously,

$$\overline{\frac{\partial W_E(t)}{\partial t}} = \frac{1}{4\pi}\overline{\frac{\partial \mathbf{D}(t)}{\partial t}\mathbf{E}(t)} = \frac{1}{16\pi}\frac{d(\omega\varepsilon(\omega))}{d\omega}\frac{\partial|\mathbf{E}_0(t)|^2}{\partial t},$$
$$\overline{W}_E = \frac{d(\omega\varepsilon(\omega))}{d\omega}\frac{|\mathbf{E}_0|^2}{16\pi}. \tag{9}$$

This widely known expression is given in a large number of textbooks and monographs (cf., e.g., [1,2,7,8]), and is obtained in different ways. In the Appendix we give yet another derivation of the relation (9).

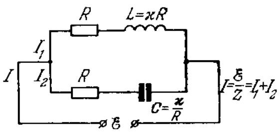
If there is no absorption, then, as is clear from (4) and (9), the interpretation of the quantity $\overline{W}_E$ as the average energy density of the electric field raises no doubts. But what is the situation in an absorptive medium?

At first sight it seems that in an absorptive medium the average energy density has the form

$$\tilde{\overline{W}}_E = \frac{d(\omega\varepsilon'(\omega))}{d\omega}\frac{|\mathbf{E}_0|^2}{16\pi}, \tag{10}$$

since just this expression appears in (8), in which the other terms depend on $\varepsilon''(\omega)$ and vanish in the absence of absorption so that it is natural to associate them with the evolved heat $Q$. There are, however, insufficient grounds for this conclusion, since the separation of the given sum into unknown terms is manifestly nonunique. Moreover, the expression (10) in the general case is certainly not the energy density of the electric field. Below this will be shown using examples which, at the same time, indicate that the densities $W_E$, $\overline{W}_E$ and $Q$ can in no way be expressed directly in terms of the permittivity $\varepsilon(\omega)$ in the general case.[2] Such a conclusion is natural even from extremely general considerations. The permittivity $\varepsilon(\omega)$ determines the linear "response" of the medium, i.e., the induction $\mathbf{D}$ that

[2]In a state of thermodynamic equilibrium there are no average losses and, therefore, even in an absorptive medium, the average electromagnetic-energy density, being a thermodynamic quantity in the familiar sense (cf. [9]), can be expressed in terms of the dielectric permittivity of the medium.




arises under the influence of the field **E**. There are no reasons why, for a sufficiently complicated absorptive medium, this "response" should also uniquely determine a quantity quadratic in the field, namely, the energy density. The nonunique correspondence between the linear "response" and the energy stored in the system is demonstrated particularly prominently by the example of discrete electric circuits. We shall consider, e.g., the circuit depicted in the figure, which is well-known from the literature. If to such a circuit (a two-terminal network) we apply a voltage $\mathscr{E}=\mathscr{E}_0 e^{-i\omega t}$, the current will be equal to $I=I_0 e^{-i\omega t}=\mathscr{E}/Z(\omega)$, where $Z=R$ for any values of the parameter $\varkappa$ if the self-inductance $L=\varkappa R$ and the capacitance $C=\varkappa/R^*$.[3] At the same time, the energy concentrated in the circuit is equal to

$$\frac{LI_1^2}{2}+\frac{\left(\int I_2\,dt\right)^2}{2C}$$

and depends, of course, on the values of $L$ and $C$.

Of course, it certainly does not follow from the arguments given that it is completely impossible to obtain any expressions for the energy or dissipation separately for an absorptive medium. The simplest example of this kind is the expression for the amount of heat, averaged over a period, in the case of a monochromatic field. For a strictly monochromatic field, obviously, $\mathbf{E}_0=\text{const}$ (cf. (7)). Furthermore, it is clear that in this case the energy averaged over a period ($\overline{W_E(t)}$) is constant in time; therefore, from (4) and (8) we obtain

$$\frac{\partial \overline{W}_E}{\partial t}+\overline{Q}=\overline{Q}=\frac{1}{4\pi}\overline{\frac{\partial \mathbf{D}(t)}{\partial t}\mathbf{E}(t)}=\frac{\omega\varepsilon''(\omega)}{8\pi}|\mathbf{E}_0|^2. \tag{11}$$

We shall consider now the rather instructive case of an absorptive medium (called a "medium without dispersion") in which

$$\frac{\partial \mathbf{D}}{\partial t}=\varepsilon'\frac{\partial \mathbf{E}}{\partial t}+4\pi\sigma\mathbf{E},$$

where $\varepsilon'$ and $\sigma$ are real and frequency-independent quantities. Of course, the dielectric permittivity $\varepsilon(\omega)=\varepsilon'+i(4\pi\sigma/\omega)$ of such a medium possesses obvious frequency dispersion, but, nevertheless, from a physical point of view, the term used ("absorptive medium without dispersion") is reasonable and, after what has been said, completely clear. The relation (2) in the given case takes the form

$$\frac{\partial}{\partial t}\left(\frac{\varepsilon'\mathbf{E}^2(t)+\mathbf{H}^2(t)}{8\pi}\right)+\sigma\mathbf{E}^2+\mathbf{j}_{ext}\mathbf{E}=-\frac{c}{4\pi}\operatorname{div}[\mathbf{E}\times\mathbf{H}]. \tag{12}$$

At first sight, it follows from (12) and (4) that the quantities $W'(t)=(\varepsilon' E^2+H^2)/8\pi$ and $Q(t)=\sigma E^2(t)$ can be identified unambiguously with the energy density and loss density, but in fact this is incorrect. Only in the case of a field varying sufficiently slowly in time does the expression for the heat (and, in the general case, only this expression) take the form written out above: $Q=\sigma E^2$.

In fact, in the presence of dispersion of the permittivity $\varepsilon(\omega)$, i.e., for a relationship between the quantities considered and the field that is nonlocal in time, the terms $\partial W_E/\partial t$ and $Q$ in the relation (4)

$$\frac{\partial W_E}{\partial t}+Q=\frac{1}{4\pi}\frac{\partial \mathbf{D}}{\partial t}\mathbf{E} \tag{4$'$}$$

can be represented in the form of series of the type

$$a_1\mathbf{E}^2(t)+a_2\mathbf{E}(t)\frac{\partial \mathbf{E}}{\partial t}+a_{31}\mathbf{E}(t)\frac{\partial^2\mathbf{E}}{\partial t^2}+a_{32}\left(\frac{\partial \mathbf{E}}{\partial t}\right)^2+\ldots$$

In the case of a medium with $\varepsilon'=\text{const}$ and $\sigma=\text{const}$, only the terms containing $\mathbf{E}^2(t)$ or $\partial\mathbf{E}^2(t)/\partial t$ will appear in the right-hand side of (4′). Therefore, we see from (4) that the expressions for the energy and heat in a medium without dispersion of the quantities $\varepsilon'$ and $\sigma$, for an arbitrary time dependence of the field, have, generally speaking, the form

$$W_E(t)=a\frac{\mathbf{E}^2(t)}{8\pi}+\ldots, \tag{13}$$

$$Q(t)=\sigma\mathbf{E}^2(t)+c\frac{1}{8\pi}\frac{\partial\mathbf{E}^2(t)}{\partial t}+\ldots, \tag{14}$$

with

$$a+c=\varepsilon', \tag{15}$$

but, generally speaking, $a$ and $c$ cannot be expressed separately in terms of the dielectric permittivity, and the coefficients in the terms not written out are adjusted such that in the expression for $(\partial W_E/\partial t)+Q$ all such terms containing time derivatives of the field that are higher than first-order cancel. If the electric field varies sufficiently slowly with time, so that $|c|/T\ll\sigma$, where $T$ is the characteristic time for the variation of the field, the expression for the heat evolved, as can be seen from (14), takes the form

$$Q(t)=\sigma\mathbf{E}^2(t). \tag{16}$$

As regards the expression for the energy of the field in an "absorptive medium without dispersion," i.e., for $\varepsilon'(\omega)=\text{const}$ and $\sigma(\omega)=\text{const}$, only under the condition $|a|\gg|c|$ does it take the form $W_E(t)=[\varepsilon' E^2(t)/8\pi]=W_E'(t)$. However, it is by no means certain that the above condition $|a|\gg|c|$ will be fulfilled and it is perfectly possible that $|a|\lesssim|c|$; we shall see this below in a specific example (cf. (29), (30)). Of course, the possibility of the appearance in (13) and (14) of terms containing time derivatives of the field is due to the fact

[3] As is clear from the Figure, $\mathscr{E}=I_1R-i\omega LI_1\equiv Z_1I_1=I_2R-(I_2/i\omega C)\equiv Z_2I_2=Z(I_1+I_2)=ZI$. From this follows the well-known relation for parallel circuits: $1/Z=(1/Z_1)+(1/Z_2)$. For the circuit considered we indeed have $1/Z=[1/(R-i\omega L)+\{1/[R-(1/i\omega C)]\}=1/R$ for all values of $\varkappa$. We note that this same circuit recently figured in the pages of Usp. Fiz. Nauk (Soviet Physics Uspekhi)[10]. The very small overlap between the present article and[10] is further justified, it seems to us, by the fact that the articles as a whole are devoted to completely different topics.

that the medium under consideration possesses frequency dispersion of the permittivity ($\varepsilon(\omega)=\varepsilon'+i(4\pi\sigma/\omega)$, $\varepsilon'=\text{const}$, $\sigma=\text{const}$).

3. Since we cannot express $W_E$ and $Q$ in terms of $\varepsilon$ in the general case, to find these quantities it is necessary to turn to the analysis of one or another specific medium or model of the medium. It is also natural to do the same to elucidate the situation as a whole. As is well-known, an extremely general model of a medium is the model reducing to a set of oscillators with masses $m_k$, eigenfrequencies $\omega_k$ (we are concerned with frequencies in the absence of absorption) and effective collision numbers $\nu_k$ ($m_k\nu_k$ is the coefficient in the frictional force). The equation of motion for such an oscillator, of the type $k$, has the form

$$\ddot{\mathbf{r}}_k+\nu_k\dot{\mathbf{r}}_k+\omega_k^2\mathbf{r}_k=\frac{e_k}{m_k}\mathbf{E}, \tag{17}$$

where $e_k$ is the charge ($e_k\mathbf{r}_k$ is the dipole moment of the oscillator) and $\mathbf{E}$ is the field acting on the oscillator. Below, in order not to complicate the model unnecessarily, the field $\mathbf{E}$ will be identified with the average macroscopic field. This assumption is, generally speaking, particular or approximate. But, e.g., for a plasma (in this case $\omega_k=0$), it is practically completely justified (cf.,[1] Sec. 3). Applied to a plasma, Eq. (17) with $\omega_k=0$ has an extremely wide range of applicability, as is clear from a more general analysis on the basis of the kinetic equation.[1,6] As regards the application of the classical model of oscillators to atomic or molecular gases and to certain other media, this finds its justification on the basis of quantum theory (the oscillator model in the scheme discussed below was mentioned in[1,6] and was considered in more detail in[11]).

In the field $\mathbf{E}=\mathbf{E}_0e^{-i\omega t}$ the forced solution of Eq. (17) has the form

$$\mathbf{r}_k=-\frac{e_k}{m_k}\frac{\mathbf{E}_0e^{-i\omega t}}{(\omega^2-\omega_k^2)+i\omega\nu_k}. \tag{17'}$$

Since the polarization of the medium is $\mathbf{P}=\sum_k e_kN_k\mathbf{r}_k$ and, by definition, for the field under consideration, $\mathbf{D}=\mathbf{E}+4\pi P=\varepsilon(\omega)\mathbf{E}$, we have[4]

$$\varepsilon(\omega)=1-\sum_k\frac{\Omega_k^2}{(\omega^2-\omega_k^2)+i\omega\nu_k},\quad \Omega_k^2=\frac{4\pi e_k^2N_k}{m_k}, \tag{18}$$

where $N_k$ is the concentration of oscillators of the type $k$. For a plasma, when $\omega_k=0$,

$$\varepsilon(\omega)=1-\frac{\Omega^2}{\omega^2+i\omega\nu},\quad \varepsilon'(\omega)=1-\frac{\Omega^2}{\omega^2+\nu^2},$$
$$\varepsilon''(\omega)=\frac{4\pi\sigma(\omega)}{\omega}=\frac{\nu\Omega^2}{\omega(\omega^2+\nu^2)},\quad \Omega^2=\frac{4\pi e^2N}{m}, \tag{19}$$

where, for simplicity, the plasma is assumed to be a one-component plasma and the index $k$ is omitted (we do not touch upon the question of the background of, say, positive ions maintaining the quasi-neutrality of the medium).

The energy conservation law for an oscillator of type $k$ has the form

$$\frac{d}{dt}\left(\frac{m_k\dot{\mathbf{r}}_k^2}{2}+\frac{m_k\omega_k^2\mathbf{r}_k^2}{2}\right)=-m_k\nu_k\dot{\mathbf{r}}_k^2+e_k\dot{\mathbf{r}}_k\mathbf{E}. \tag{20}$$

From this it is clear that for the model of the medium under consideration,

$$K=\sum_k\frac{N_km_k\dot{\mathbf{r}}_k^2}{2},\quad U=\sum_k\frac{N_km_k\omega_k^2\mathbf{r}_k^2}{2},\quad Q=\sum_kN_km_k\nu_k\dot{\mathbf{r}}_k^2, \tag{21}$$

where $K$ is the kinetic energy associated with the field, $U$ is the potential energy and $Q$ is the heat evolved in unit time in unit volume (more precisely, $Q$ is the work done by the frictional forces, which we assume is converted to heat).

The energy density of the field and of the motion of the charges (oscillators) that is induced by the field in the medium is $W_E=(E^2/8\pi)+K+U$. Inasmuch as we are concerned with quadratic quantities, it is convenient now to consider a real field $\mathbf{E}=\mathbf{E}_0\,\mathrm{Re}\,e^{-i\omega t}=\mathbf{E}_0\cos\omega t$, $\mathbf{E}_0=\mathbf{E}_0^*=\text{const}$, and find $W_E$, $Q$ and then the values averaged over a period ($\overline{W}_E$ and $\overline{Q}$) (this means in practice that all terms containing factors $e^{\pm 2i\omega t}$ are discarded). The result of the elementary calculation is:

$$\overline{W}_E=\left[1+\sum_k\frac{\Omega_k^2(\omega^2+\omega_k^2)}{(\omega^2-\omega_k^2)^2+\omega^2\nu_k^2}\right]\frac{|\mathbf{E}_0|^2}{16\pi}, \tag{22}$$

$$\overline{Q}_E=\sum_k\frac{\Omega_k^2\nu_k\omega^2}{(\omega^2-\omega_k^2)^2+\omega^2\nu_k^2}\frac{|\mathbf{E}_0|^2}{8\pi}=\omega\varepsilon''(\omega)\frac{|\mathbf{E}_0|^2}{8\pi}, \tag{23}$$

since, according to (18),

$$\varepsilon''(\omega)=\sum_k\frac{\omega\nu_k\Omega_k^2}{(\omega^2-\omega_k^2)^2+\omega^2\nu_k^2}. \tag{24}$$

At the same time,

$$\varepsilon'(\omega)=1-\sum_k\frac{\Omega_k^2(\omega^2-\omega_k^2)}{(\omega^2-\omega_k^2)^2+\omega^2\nu_k^2}, \tag{24'}$$

and, therefore, $\overline{W}_E$ is not expressed in terms of $\varepsilon'(\omega)$ (see also below). In the particular case of the plasma model (19) already mentioned,

$$\overline{W}_E=\left(1+\frac{\Omega^2}{\omega^2+\nu^2}\right)\frac{|\mathbf{E}_0|^2}{16\pi}=(2-\varepsilon'(\omega))\frac{|\mathbf{E}_0|^2}{16\pi},$$
$$\overline{Q}_E=\frac{\nu\Omega^2}{\omega^2+\nu^2}\frac{|\mathbf{E}_0|^2}{8\pi}=\omega\varepsilon''(\omega)\frac{|\mathbf{E}_0|^2}{8\pi}, \tag{25}$$

i.e., not only is $\overline{Q}_E$ expressed in terms of $\varepsilon(\omega)$ but so too is $\overline{W}_E$ (specifically, in terms of $\varepsilon'(\omega)=\mathrm{Re}\,\varepsilon(\omega)$). This case, however, is clearly special. Moreover, the actual values of $W_E(t)$ and $Q(t)$, as distinct from the corresponding averages, are not expressed directly in terms of $\varepsilon(\omega)$, even for a plasma; they have the form

$$W_E(t)=\left\{\left[1-\frac{\Omega^2(\omega^2-\nu^2)}{(\omega^2+\nu^2)^2}\right]\cos^2\omega t+\frac{\nu\omega\Omega^2}{(\omega^2+\nu^2)^2}\sin 2\omega t+\frac{\omega^2\Omega^2}{(\omega^2+\nu^2)^2}\right\}\frac{\mathbf{E}_0^2}{8\pi}, \tag{26}$$

$$Q(t)=\nu\Omega^2\left[\frac{1}{\omega^2+\nu^2}-\frac{\omega^2-\nu^2}{(\omega^2+\nu^2)^2}\cos 2\omega t+2\frac{\nu\omega}{(\omega^2+\nu^2)^2}\sin 2\omega t\right]\frac{\mathbf{E}_0^2}{8\pi}. \tag{26'}$$

[4]We recall that, with the choice of time dependence in the form $e^{-i\omega t}$, by definition, $\mathbf{D}_0e^{-i\omega t}=\varepsilon(\omega)\mathbf{E}_0e^{-i\omega t}$.

For a system of oscillators the quantity defined by (10) is

$$\tilde{\overline{W}}_E=\frac{d\,(\omega\varepsilon'(\omega))}{d\omega}\frac{|\mathbf{E}_0|^2}{16\pi}=\left\{1+\sum_k\frac{\Omega_k^2(\omega^2+\omega_k^2)\,[(\omega^2-\omega_k^2)^2-\omega^2\nu_k^2]}{[(\omega^2-\omega_k^2)^2+\omega^2\nu_k^2]^2}\right\}\frac{|\mathbf{E}_0|^2}{16\pi} \tag{27}$$

and for a plasma

$$\tilde{\overline{W}}_E=\frac{d\,(\omega\varepsilon'(\omega))}{d\omega}\frac{|\mathbf{E}_0|^2}{16\pi}=\left[1+\frac{\Omega^2(\omega^2-\nu^2)}{(\omega^2+\nu^2)^2}\right]\frac{|\mathbf{E}_0|^2}{16\pi}. \tag{28}$$

It is obvious that, in the presence of absorption, $\overline{W}_E \neq \tilde{\overline{W}}_E$ in both cases (cf. (22), (25), (27) and (28)), and $\overline{W}_E=\tilde{\overline{W}}_E$ only in the absence of absorption (i.e., for $\nu_k=0$), i.e., the average energy density $\overline{W}_E$ is correctly defined by formula (10), which goes over in this case into (9). This could not be otherwise, since in the absence of absorption

$$\frac{1}{4\pi}\frac{\partial\mathbf{D}}{\partial t}\mathbf{E}=\frac{\partial W_E}{\partial t}$$

and we do not need to divide the quantity $(1/4\pi)(\partial\mathbf{D}/\partial t)\mathbf{E}$ into parts $\partial W_E/\partial t$ and $Q$ (cf. (4)).

The examples given (which, incidentally, have an extremely general character) do not leave any doubts about the fact that the quantity

$$\tilde{\overline{W}}_E=\frac{d\,(\omega\,\mathrm{Re}\,\varepsilon(\omega))}{d\omega}\frac{|\mathbf{E}_0|^2}{16\pi}$$

(cf. (10)) is not, generally speaking, the energy density of the electric field in the medium. From (27), (28) it is also clear that $\tilde{\overline{W}}_E$ can be negative (e.g., in (28) $\tilde{\overline{W}}_E<0$ if $\Omega^2\nu^2>\Omega^2\omega^2+(\omega^2+\nu^2)^2$; in the limiting case $\nu^2\gg\omega^2$ this reduces to the condition $\Omega^2>\nu^2$). But the quantity $\overline{W}_E$, as is clear from (22) or (25), is always positive, as should be the case for the quantity $\overline{W}_E=\overline{(\mathbf{E}^2/8\pi)+K}$. We note that in the article[3] already mentioned it is the quantity $\tilde{\overline{W}}_E$ that is unjustifiably assumed to be the average energy density; this is only masked by the notation (for more detail see[4]).

For the region of frequencies $\omega^2\ll\nu^2$, according to (19), $\varepsilon'=1-(\Omega^2/\nu^2)$ and $\sigma=\Omega^2/4\pi\nu$, and this means that in the given case the plasma is an example of an absorptive medium without dispersion, already discussed above[5)]. In this case, from (26), (26′), in complete correspondence with (13)–(15), we have (we recall that $\mathbf{E}(t)=\mathbf{E}_0\cos\omega t$ in (26), (26′))

$$W_E(t)=\left(1+\frac{\Omega^2}{\nu^2}\right)\frac{\mathbf{E}^2(t)}{8\pi}, \tag{29}$$

$$Q(t)=\frac{\Omega^2}{4\pi\nu}\mathbf{E}^2(t)-2\frac{\Omega^2}{\nu^2}\frac{1}{8\pi}\frac{\partial\mathbf{E}^2(t)}{\partial t}, \tag{30}$$

so that $a=1+\Omega^2/\nu^2$, $c=-2\Omega^2/\nu^2$, and $a+c=1-\Omega^2/\nu^2=\varepsilon'$. The relationship between $a$ and $|c|$ is determined, as can be seen from the formulas, by the parameter $\Omega^2/\nu^2$, and for $\Omega^2/\nu^2>1$ we have $|c|>|a|$.

[5)]We note that for $\sigma>0$ and $\varepsilon'<0$ such a medium is unstable in the absence of external sources[6], by virtue of which the case of a plasma with $\omega^2\ll\nu^2$ and $\Omega^2>\nu^2$ must be treated with the usual care (it is necessary to lean on the general expressions (19), which indicate the stability of the corresponding model of the plasma). In this connection, we note that, in essence, nowhere do we rely on the inequality $\omega^2\ll\nu^2$.

4. Although, generally speaking, the quantities $W_E(t)$ and $Q(t)$ in general form cannot be expressed in terms of $\varepsilon(\omega)$, for the sum

$$\frac{\partial W_E(t)}{\partial t}+Q(t)=\frac{1}{4\pi}\frac{\partial\mathbf{D}}{\partial t}\mathbf{E}$$

this is, of course, possible. From this it is clear that both the term $\partial W_E(t)/\partial t$ and the term $Q(t)$ make a contribution to the terms with $\varepsilon'(\omega)$ and with $\varepsilon''(\omega)$ appearing in the expression for $(1/4\pi)(\partial\mathbf{D}/\partial t)\mathbf{E}$. Nevertheless, it is fairly instructive to convince oneself of this using a specific example. Of course, a medium consisting of oscillators is suitable for this, but we shall confine ourselves to the plasma model considered above.

We note that in this case it is also particularly easy to check the validity of the relation

$$\frac{\partial W_E}{\partial t}+Q=\frac{1}{4\pi}\frac{\partial\mathbf{D}}{\partial t}\mathbf{E}, \tag{4′}$$

itself, which, it can be said, has been written out previously on the basis of general considerations. In fact, for the plasma model under discussion,

$$eN\dot{\mathbf{r}}=\frac{\partial\mathbf{P}}{\partial t}=\frac{1}{4\pi}\frac{\partial(\mathbf{D}-\mathbf{E})}{\partial t},$$

where $\mathbf{P}$ is the total polarization of the medium (if the conduction current $\mathbf{j}$ and polarization $\mathbf{P}$ are introduced separately, the form $eN\dot{\mathbf{r}}=\mathbf{j}+\partial\mathbf{P}/\partial t$ is used; cf., e.g., [6]). On the other hand, according to the equation of motion, $m\ddot{\mathbf{r}}+m\nu\dot{\mathbf{r}}=e\mathbf{E}$, and therefore

$$\frac{\partial}{\partial t}\left(Nm\frac{\dot{\mathbf{r}}^2}{2}\right)+Nm\nu\dot{\mathbf{r}}^2=Ne\dot{\mathbf{r}}\mathbf{E}=\frac{1}{4\pi}\frac{\partial\mathbf{D}}{\partial t}\mathbf{E}-\frac{1}{4\pi}\frac{\partial\mathbf{E}}{\partial t}\mathbf{E}.$$

Hence,

$$\frac{\partial W_E}{\partial t}+Q=\frac{1}{4\pi}\frac{\partial\mathbf{D}}{\partial t}\mathbf{E},\quad W_E=K+\frac{\mathbf{E}^2}{8\pi},\quad K=\frac{Nm\dot{\mathbf{r}}^2}{2},\quad Q=mN\nu\dot{\mathbf{r}}^2,$$

as should be the case.

For a monochromatic field $\mathbf{E}=\mathbf{E}_0\cos\omega t$ the values of $W_E$ and $Q$ have already been written out (cf. (26), (26′)) and, thus,

$$\begin{aligned}\frac{\partial W_E}{\partial t}+Q&=\left\{-\omega\sin 2\omega t+\frac{\omega\Omega^2(\omega^2-\nu^2)}{(\omega^2+\nu^2)^2}\sin 2\omega t+2\frac{\omega^2\Omega^2\nu}{(\omega^2+\nu^2)^2}\cos 2\omega t\right\}\frac{\mathbf{E}_0^2}{8\pi}\\&\quad+\left[\frac{\nu\Omega^2}{\omega^2+\nu^2}-\frac{\nu\Omega^2(\omega^2-\nu^2)}{(\omega^2+\nu^2)^2}\cos 2\omega t+2\frac{\Omega^2\nu^2\omega}{(\omega^2+\nu^2)^2}\sin 2\omega t\right]\frac{\mathbf{E}_0^2}{8\pi}\\&=\frac{1}{4\pi}\frac{\partial\mathbf{D}}{\partial t}\mathbf{E}=[-\omega\varepsilon'(\omega)\sin 2\omega t+\omega\varepsilon''(\omega)+\omega\varepsilon''(\omega)\cos 2\omega t]\frac{\mathbf{E}_0^2}{8\pi}\\&=\left(-\omega\sin 2\omega t+\frac{\omega\Omega^2}{\omega^2+\nu^2}\sin 2\omega t+\frac{\nu\Omega^2}{\omega^2+\nu^2}+\frac{\nu\Omega^2}{\omega^2+\nu^2}\cos 2\omega t\right)\frac{\mathbf{E}_0^2}{8\pi},\end{aligned} \tag{31}$$

where the last expression is obtained by substituting the expressions (19) for $\varepsilon'(\omega)$ and $\varepsilon''(\omega)$; as regards the penultimate expression in (31), it is obtained immediately when the relationship between the field $\mathbf{E}=\mathbf{E}_0(e^{i\omega t}+e^{-i\omega t})/2$ and the induction $\mathbf{D}=(\mathbf{E}_0/2)\times(\varepsilon(-\omega)e^{i\omega t}+\varepsilon(\omega)e^{-i\omega t})$ is taken into account, since, when the relation (6) is taken into account, we then obtain $\mathbf{D}=\varepsilon'(\omega)\mathbf{E}_0\cos\omega t+\varepsilon''(\omega)\mathbf{E}_0\sin\omega t$.

From a comparison of the different terms in (31) it is clear, e.g., that the term

$$-\omega\varepsilon'(\omega)\sin 2\omega t = -\omega\sin 2\omega t + \frac{\Omega^2\omega}{\omega^2+\nu^2}\sin 2\omega t$$

is formed (or, if preferred, arises) both from $\partial W_E/\partial t$ and from $Q$. This, obviously, also applies to the term

$$\omega\varepsilon''(\omega)\cos 2\omega t = \frac{\nu\Omega^2}{\omega^2+\nu^2}\cos 2\omega t.$$

Only the time-independent term

$$\frac{\nu\Omega^2}{\omega^2+\nu^2}\frac{\mathbf{E}_0^2}{8\pi}$$

is due to the dissipation alone. The latter is not surprising if we take into account that we are considering a monochromatic field. It is entirely obvious that analogous conclusions remain valid after integrating over the time. We remark here that in integrating (31) over the time and determining the quantity $W_E(t) + \int Q(t)\,dt$ in this way, it is necessary to exercise some caution and, in essence, to return first to the expressions (26), (26′). In fact, from (31) we obtain

$$W_E(t) + \int Q(t)\,dt = \varepsilon'(\omega)\frac{\mathbf{E}_0^2\cos 2\omega t}{16\pi} + \omega\varepsilon''(\omega)\frac{\mathbf{E}_0^2}{8\pi}t + \omega\varepsilon''(\omega)\frac{\mathbf{E}_0^2}{8\pi}\frac{\sin 2\omega t}{2\omega} + \text{const} = \frac{1}{4\pi}\int\frac{\partial\mathbf{D}(t)}{\partial t}\mathbf{E}(t)\,dt. \tag{32}$$

As is clear from the discussion in Secs. 2 and 3 and the comments that follow below, generally speaking, the integration constant appearing in (32) in the general case cannot be expressed in terms of $\varepsilon$. But in the absence of absorption, by comparing (32) with (9) it can be seen that

$$\text{const} = \frac{d(\omega\varepsilon(\omega))}{d\omega}\frac{|\mathbf{E}_0|^2}{16\pi}.$$

On the other hand, for the model of a medium, and, specifically, for the model of a plasma, determining the quantity $W_E(t) + \int Q(t)dt$ does not present any difficulty even in the presence of absorption. In fact, we already know the expression for $W_E(t)$ in the case of the plasma (cf. (26)), and here, taking into account that $\cos^2\omega t = (1+\cos 2\omega t)/2$, it is convenient to write it in the form

$$W_E(t) = \left\{\left(1+\frac{\Omega^2}{\omega^2+\nu^2}\right) + \frac{2\omega\Omega^2\nu}{(\omega^2+\nu^2)^2}\sin 2\omega t + \left[1-\frac{\Omega^2(\omega^2-\nu^2)}{(\omega^2+\nu^2)^2}\right]\cos 2\omega t\right\}\frac{\mathbf{E}_0^2}{16\pi}. \tag{33}$$

Next, integrating (26′) over the time, we find

$$\int Q(t)\,dt = \nu\Omega^2\left[\frac{t}{\omega^2+\nu^2} - \frac{\omega^2-\nu^2}{(\omega^2+\nu^2)^2}\frac{\sin 2\omega t}{2\omega} - \frac{\nu\cos 2\omega t}{(\omega^2+\nu^2)^2}\right]\frac{\mathbf{E}_0^2}{8\pi}, \tag{34}$$

where the integration constant is chosen in such a way that, for the period-averaged energy dissipation over a time interval $t$, with monochromatic behavior of the field, we have $\overline{\int Q(t)dt} = \int\overline{Q(t)}dt \sim t$. Thus, combining (33) and (34) and equating with (32), we obtain

$$W_E(t) + \int Q(t)\,dt$$
$$= \left[\frac{\cos 2\omega t}{2} - \frac{\Omega^2(\omega^2-\nu^2)}{(\omega^2+\nu^2)^2}\frac{\cos 2\omega t}{2} + \frac{\nu\omega\Omega^2}{(\omega^2+\nu^2)^2}\sin 2\omega t + \frac{1}{2}\left(1+\frac{\Omega^2}{\omega^2+\nu^2}\right)\right]\frac{\mathbf{E}_0^2}{8\pi} + \left[\frac{\nu\Omega^2 t}{\omega^2+\nu^2} - \frac{\nu\Omega^2(\omega^2-\nu^2)}{(\omega^2+\nu^2)^2}\frac{\sin 2\omega t}{2\omega} - \frac{\nu^2\Omega^2\cos 2\omega t}{(\omega^2+\nu^2)^2}\right]\frac{\mathbf{E}_0^2}{8\pi} = \frac{1}{4\pi}\int\frac{\partial\mathbf{D}}{\partial t}\mathbf{E}\,dt$$
$$= \left[\varepsilon'(\omega)\frac{\cos 2\omega t}{2} + \omega\varepsilon''(\omega)t + \varepsilon''(\omega)\frac{\sin 2\omega t}{2} + \text{const}\right]\frac{\mathbf{E}_0^2}{8\pi}$$
$$= \left[\frac{\cos 2\omega t}{2} - \frac{\Omega^2\cos 2\omega t}{2(\omega^2+\nu^2)} + \frac{\nu\Omega^2 t}{\omega^2+\nu^2} + \frac{\nu\Omega^2\sin 2\omega t}{2\omega(\omega^2+\nu^2)} + \text{const}\right]\frac{\mathbf{E}_0^2}{8\pi}, \tag{35}$$

where all expressions are written in complete analogy with (31). It is clear that, even in the presence of absorption, the integration constant exactly determines the expression for $\overline{W}_E(t)$ (cf. (33), (35) and (19)).

We also emphasize here that, even for a quasi-monochromatic field, it is impossible to integrate the Poynting relation for an absorptive medium over the time in general form, and, in this sense, the use of the initial conditions $\mathbf{E}(-\infty) = 0$ and $\mathbf{H}(-\infty) = 0$ does not lead to a solution of the problem. In fact, it can already be seen from (8) that the terms containing $\varepsilon''(\omega)$ cannot be represented in the form of total time derivatives of certain expressions. The latter fact is not surprising, since, as is well-known, the heat evolved is not a function of the state of the system, and $\delta Q$, as in ordinary thermodynamics, is thus not an exact differential. Therefore, depending on the way in which the field $\mathbf{E}_0(t)$ varies in time from $\mathbf{E}_0(-\infty) = 0$ to the value $\mathbf{E}_0$, we can obtain different answers on integrating the relation (8) over the time for an absorptive medium.

The reasons why we considered it appropriate to discuss such simple calculations in detail have already been mentioned in the Introduction. All that remains for us to note is that the discussion of the energy relations in an absorptive medium situated in an electromagnetic field is not only useful for understanding the mechanism and character of the absorption and relaxation but is also used in the calculation of the "energy velocity"—the rate of energy transfer in electromagnetic waves propagating in an absorptive medium (cf. [1,2,8,10,11]).

The authors are grateful to L. A. Vaĭnshteĭn for comments he made on reading the manuscript.

## APPENDIX

We use the Fourier expansion

$$\mathbf{E}(t,\mathbf{r}) = \int_{-\infty}^{\infty} d\omega e^{-i\omega t}\mathbf{E}(\omega,\mathbf{r}),$$
$$D_i(t,\mathbf{r}) = \int_{-\infty}^{\infty} d\omega\varepsilon_{ij}(\omega,\mathbf{r})E_j(\omega,\mathbf{r})e^{-i\omega t}. \tag{A.1}$$

For an isotropic medium, (A.1) goes over, of course, into (5). Substituting (A.1) into the expression for $(1/4\pi)(\partial\mathbf{D}/\partial t)\mathbf{E}$ (cf. (4)) and integrating over the time, we obtain

$$\frac{1}{4\pi}\int\mathbf{E}(t,\mathbf{r})\frac{\partial\mathbf{D}(t,\mathbf{r})}{\partial t}dt = \frac{1}{4\pi}\int_{-\infty}^{\infty}d\omega_1\int_{-\infty}^{\infty}d\omega_2 e^{-i(\omega_1-\omega_2)t}\frac{\omega_1}{\omega_1-\omega_2}E_i^*(\omega_2,\mathbf{r})\varepsilon_{ij}(\omega_1,\mathbf{r})E_j(\omega_1,\mathbf{r}) + \text{const}. \tag{A.2}$$

In a nonabsorptive medium, as is well-known (cf., e.g., [2]),

$$\varepsilon_{ij}(\omega)=\varepsilon^*_{ji}(\omega) \qquad (A.3)$$

and, in addition, since the fields $\mathbf{E}(t, \mathbf{r})$ and $\mathbf{D}(t, \mathbf{r})$ are real in any medium, we have, in analogy with (6)

$$\varepsilon_{ij}(-\omega)=\varepsilon^*_{ij}(\omega). \qquad (A.4)$$

Using (A.3) and (A.4) it is not difficult to see that for a nonabsorptive medium the singularity of the integrand in (A.2) at $\omega_1=\omega_2$ has a "fictitious" character, i.e., its contribution is cancelled on integration because of the appropriate symmetry of the integrand. In fact, we shall replace the integration variables in (A.2): $\omega_1 \to -\omega_2$, $\omega_2 \to -\omega_1$. Using (A.3) and (A.4) and taking into account that $\mathbf{E}(-\omega, \mathbf{r})=\mathbf{E}^*(\omega, \mathbf{r})$, we obtain an expression, the half-sum of which with (A.2) gives

$$\frac{1}{4\pi}\int dt \mathbf{E}(t, \mathbf{r})\frac{\partial \mathbf{D}(t, \mathbf{r})}{\partial t}=W_E(t, \mathbf{r})$$
$$=\frac{1}{8\pi}\int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} d\omega_2 e^{-i(\omega_1-\omega_2)t}E^*_i(\omega_2, \mathbf{r})\frac{\omega_1\varepsilon_{ij}(\omega_1, \mathbf{r})-\omega_2\varepsilon_{ij}(\omega_2, \mathbf{r})}{\omega_1-\omega_2}E_j(\omega_1, \mathbf{r})+\text{const}. \qquad (A.5)$$

Suppose that $\mathbf{E}(t, \mathbf{r})\to 0$ as $t\to -\infty$; this imposes certain restrictions on $\mathbf{E}(\omega, \mathbf{r})$. Since the quantity

$$\lim_{\omega_2\to\omega_1}\frac{\omega_1\varepsilon_{ij}(\omega_1, \mathbf{r})-\omega_2\varepsilon_{ij}(\omega_2, \mathbf{r})}{\omega_1-\omega_2}=\frac{d(\omega_1\varepsilon_{ij}(\omega_1, \mathbf{r}))}{d\omega_1} \qquad (A.6)$$

is finite, it is clear that the requirement

$$\lim_{t\to-\infty}\frac{1}{4\pi}\int dt \mathbf{E}(t, \mathbf{r})\frac{\partial \mathbf{D}(t, \mathbf{r})}{\partial t}=0 \qquad (A.7)$$

leads to the determination of the integration constant: const = 0. Thus, in a nonabsorptive medium the expression for the energy density of the electromagnetic field, for an arbitrary time dependence of the field, has the form

$$W_E(t, \mathbf{r})$$
$$=\frac{1}{8\pi}\int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} d\omega_2 e^{-i(\omega_1-\omega_2)t}\frac{\omega_1\varepsilon_{ij}(\omega_1, \mathbf{r})-\omega_2\varepsilon_{ij}(\omega_2, \mathbf{r})}{\omega_1-\omega_2}E^*_i(\omega_2, \mathbf{r})E_j(\omega_1, \mathbf{r}). \qquad (A.8)$$

For a real medium with absorption, for (A.8) to be applicable it is, of course, required that the spectral components of the field differ substantially from zero only in those regions of the spectrum in which the absorption can be neglected.

As a particular case, we now consider a monochromatic time dependence of the field (cf. (7), in which we assume that $\mathbf{E}_0=\text{const}$), when

$$\mathbf{E}(\omega, \mathbf{r})=\frac{1}{2}[\mathbf{E}_0(\mathbf{r})\delta(\omega-\Omega)+\mathbf{E}^*_0(\mathbf{r})\delta(\omega+\Omega)]. \qquad (A.9)$$

The deviations from monochromaticity, associated with the condition $E(t\to-\infty)=0$, will be taken into account automatically when formula (A.8) is used.

Substituting (A.9) into (A.8) and taking (A.3), (A.4) and (A.6) into account, we obtain

$$W_E(t, \mathbf{r})=\frac{1}{16\pi}\frac{d(\omega\varepsilon_{ij}(\omega, \mathbf{r}))}{d\omega}E^*_{0i}(\mathbf{r})E_{0j}(\mathbf{r})$$
$$+\frac{1}{16\pi}\operatorname{Re}[E_{0i}(\mathbf{r})E_{0j}(\mathbf{r})\varepsilon_{ij}(\omega, \mathbf{r})e^{-2i\omega t}], \qquad (A.10)$$

and for this quantity averaged over a period we have

$$\overline{W_E(t, \mathbf{r})}=\frac{1}{16\pi}\frac{d(\omega\varepsilon_{ij}(\omega, \mathbf{r}))}{d\omega}E^*_{0i}(\mathbf{r})E_{0j}(\mathbf{r}). \qquad (A.11)$$

For an isotropic medium, from (A.11) we thus obtain formula (9).

We now reproduce the deviation of formula (8). For this we expand the field (7) in a Fourier integral

$$\mathbf{E}(t)=\frac{1}{2}[\mathbf{E}_0(t)e^{-i\omega_0 t}+\mathbf{E}^*_0(t)e^{i\omega_0 t}]=\int_0^{\infty}[\mathbf{g}(\omega)e^{-i\omega t}+\mathbf{g}^*(\omega)e^{i\omega t}]d\omega. \qquad (A.12)$$

In the language of the spectral quantities $\mathbf{g}(\omega)$ the quasi-monochromatic character of the field means that at $\omega=\pm\omega_0$ the function $\mathbf{g}(\omega)$ has sharp and extremely large (but finite; cf. (A.9)) maxima, and as we move away from these points $\mathbf{g}(\omega)$ tends, sufficiently rapidly, to zero. For the electric induction we have

$$\mathbf{D}(t)=\int_0^{\infty}\varepsilon'(\omega)[\mathbf{g}(\omega)e^{-i\omega t}+\mathbf{g}^*(\omega)e^{i\omega t}]d\omega$$
$$+i\int_0^{\infty}\varepsilon''(\omega)[\mathbf{g}(\omega)g^{-i\omega t}-\mathbf{g}^*(\omega)e^{i\omega t}]d\omega. \qquad (A.13)$$

Thus,

$$\frac{1}{4\pi}\mathbf{E}\frac{\partial \mathbf{D}}{\partial t}=\frac{1}{4\pi}\int_0^{\infty}\int(-i\omega)\varepsilon'(\omega)[\mathbf{g}(\omega)\mathbf{g}(\omega')e^{-i(\omega+\omega')t}$$
$$+\mathbf{g}(\omega)\mathbf{g}^*(\omega')e^{-i(\omega-\omega')t}-\mathbf{g}^*(\omega)\mathbf{g}(\omega')e^{i(\omega-\omega')t}-\mathbf{g}^*(\omega)\mathbf{g}^*(\omega')e^{i(\omega+\omega')t}]d\omega d\omega'$$
$$+\frac{1}{4\pi}\int_0^{\infty}\int\omega\varepsilon''(\omega)[\mathbf{g}(\omega)\mathbf{g}(\omega')e^{-i(\omega+\omega')t}+\mathbf{g}(\omega)\mathbf{g}^*(\omega')e^{-i(\omega-\omega')t}$$
$$+\mathbf{g}^*(\omega)\mathbf{g}(\omega')e^{i(\omega-\omega')t}+\mathbf{g}^*(\omega)\mathbf{g}^*(\omega')e^{i(\omega+\omega')t}]d\omega d\omega'. \qquad (A.14)$$

We now find the high-frequency average of the expression (A.14), i.e., the average over a time longer than $2\pi/\omega_0$ but short compared with the characteristic time of variation of the amplitude $\mathbf{E}_0(t)$. It is not difficult to convince oneself that this averaging is equivalent to discarding terms with $e^{\pm i(\omega+\omega')t}$ as compared with terms containing $e^{\pm i(\omega-\omega')t}$. Furthermore, in carrying out this averaging we take into account the aforementioned character of the behavior of the quantities $\mathbf{g}(\omega)$, because of which, in the first approximation in (A.14), we put

$$\omega\varepsilon'(\omega)=\omega_0\varepsilon'(\omega_0)+\frac{d(\omega_0\varepsilon'(\omega_0))}{d\omega_0}(\omega-\omega_0),$$
$$\omega\varepsilon''(\omega)=\omega_0\varepsilon''(\omega_0)+\frac{d(\omega_0\varepsilon''(\omega_0))}{d\omega_0}(\omega-\omega_0). \qquad (A.15)$$

By next making use of the symmetry of some of the integrands in (A.14) under interchange of the integration variables $(\omega\to\omega', \omega'\to\omega)$, after extremely simple calculations we arrive at the expression

$$\overline{\frac{1}{4\pi}\mathbf{E}(t)\frac{\partial \mathbf{D}(t)}{\partial t}}=\frac{1}{4\pi}\frac{d(\omega_0\varepsilon'(\omega_0))}{d\omega_0}\int_0^{\infty}\int(-i)(\omega-\omega')\mathbf{g}(\omega)\mathbf{g}^*(\omega')e^{-i(\omega-\omega')t}d\omega d\omega'$$
$$+2\sigma(\omega_0)\int_0^{\infty}\int\mathbf{g}(\omega)\mathbf{g}^*(\omega')e^{-i(\omega-\omega')t}d\omega d\omega'$$
$$+\frac{d\sigma(\omega_0)}{d\omega_0}\int_0^{\infty}\int[(\omega-\omega_0)+(\omega'-\omega_0)]\mathbf{g}(\omega)\mathbf{g}(\omega')e^{-i(\omega-\omega_0)t}e^{-i(\omega'-\omega_0)t}d\omega d\omega'. \qquad (A.16)$$

Taking into account that

$$E_0(t)=2\int_0^\infty d\omega g(\omega)\, e^{-i(\omega-\omega_0)t},$$

$$\overline{E^2(t)}=\frac{E_0(t)E_0^*(t)}{2}=2\int_0^\infty\int g(\omega)g^*(\omega')\,e^{-i(\omega-\omega')t}\,d\omega\,d\omega', \qquad (A.17)$$

from (A.16) we arrive at formula (8). When the anisotropy and magnetic properties are taken into account, calculations completely analogous to those described, but taking into account in addition the hermiticity of the quantities $\varepsilon'_{ij}(\omega)$, $\varepsilon''_{ij}(\omega)$, $\mu'_{ij}(\omega)$ and $\mu''_{ij}(\omega)$, where $\varepsilon_{ij}(\omega)=\varepsilon'_{ij}(\omega)+i\varepsilon''_{ij}(\omega)$ and $\mu_{ij}(\omega)=\mu'_{ij}(\omega)+i\mu''_{ij}(\omega)$, lead to the relation[6]

$$\frac{1}{4\pi}\overline{\frac{\partial \mathbf{D}}{\partial t}\mathbf{E}(t)}+\frac{1}{4\pi}\overline{\frac{\partial \mathbf{B}}{\partial t}\mathbf{H}}$$
$$=\frac{1}{16\pi}\frac{\partial}{\partial t}\left[\frac{d(\omega_0\varepsilon'_{ij}(\omega_0))}{d\omega_0}E^*_{0i}(t)E_{0j}(t)+\frac{d(\omega_0\mu'_{ij}(\omega_0))}{d\omega_0}H^*_{0i}(t)H_{0j}(t)\right]$$
$$+\frac{\omega_0\varepsilon''_{ij}(\omega_0)}{8\pi}E^*_{0i}(t)E_{0j}(t)+\frac{\omega_0\mu''_{ij}(\omega_0)}{8\pi}H^*_{0i}(t)H_{0j}(t)$$
$$+\frac{i}{16\pi}\frac{d(\omega_0\varepsilon''_{ij}(\omega_0))}{d\omega_0}\left(\frac{\partial E_{0j}}{\partial t}E^*_{0i}-\frac{\partial E^*_{0j}}{\partial t}E_{0i}\right)$$
$$+\frac{i}{16\pi}\frac{d(\omega_0\mu''_{ij}(\omega_0))}{d\omega_0}\left(\frac{\partial H_{0j}}{\partial t}H^*_{0i}-\frac{\partial H^*_{0j}}{\partial t}H_{0i}\right). \qquad (A.18)$$

In conclusion we shall consider the case when, in a nonabsorptive medium (i.e., for $\varepsilon'_{ij}=\varepsilon_{ij}$), a quasi-monochromatic field (packet) has, nevertheless, a linewidth $\Delta\omega$ such that it is necessary to take corrections to formula (A.11) into account. To find the correction terms we turn to the general formula (A.8). Using the expansions

$$\left.\begin{aligned}\frac{\omega_1\varepsilon_{ij}(\omega_1,\mathbf{r})-\omega_2\varepsilon_{ij}(\omega_2,\mathbf{r})}{\omega_1-\omega_2}&=\sum_{n=0}^{\infty}\frac{d^{n+1}(\omega_2\varepsilon_{ij}(\omega_2,\mathbf{r}))}{d\omega_2^{n+1}}\frac{(\omega_1-\omega_2)^n}{(n+1)!},\\ \frac{d^n(\omega_2\varepsilon_{ij}(\omega_2,\mathbf{r}))}{d\omega_2^n}&=\sum_{k=0}^{\infty}\frac{d^{n+k}(\omega_0\varepsilon_{ij}(\omega_0,\mathbf{r}))}{d\omega_0^{n+k}}\frac{(\omega_2-\omega_0)^k}{k!},\end{aligned}\right\} \qquad (A.19)$$

for the high-frequency-averaged energy density we obtain from (A.8)

$$\overline{W_E(t,\mathbf{r})}=\frac{1}{16\pi}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(i)^{n+k}(-1)^k}{(n+1)!\,k!}$$
$$\times\frac{d^{n+k+1}(\omega_0\varepsilon_{ij}(\omega_0,\mathbf{r}))}{d\omega_0^{n+k+1}}\frac{d^n}{dt^n}\left[E_{0j}(t,\mathbf{r})\frac{d^kE^*_{0i}(t,\mathbf{r})}{dt^k}\right], \qquad (A.20)$$

where $\omega_0$ and $\mathbf{E}_0(t)$ are defined in (A.12) and (A.17). Using formula (A.20) we now write out an expression which is analogous to (A.11) but takes into account the first two corrections corresponding to the nonmonochromatic character of the time dependence of the field (A.12):

$$\overline{W_E(t,\mathbf{r})}=\frac{1}{16\pi}\frac{d(\omega_0\varepsilon_{ij}(\omega_0,\mathbf{r}))}{d\omega_0}E_{0j}(t,\mathbf{r})E^*_{0i}(t,\mathbf{r})$$
$$+\frac{i}{32\pi}\frac{d^2(\omega_0\varepsilon_{ij}(\omega_0,\mathbf{r}))}{d\omega_0^2}\left[\frac{dE_{0j}(t,\mathbf{r})}{dt}E^*_{0i}(t,\mathbf{r})-E_{0j}(t,\mathbf{r})\frac{dE^*_{0i}(t,\mathbf{r})}{dt}\right]$$
$$-\frac{1}{96\pi}\frac{d^3(\omega_0\varepsilon_{ij}(\omega_0,\mathbf{r}))}{d\omega_0^3}\left[\frac{d^2}{dt^2}(E_{0j}(t,\mathbf{r})E^*_{0i}(t,\mathbf{r}))-3\frac{dE_{0j}(t,\mathbf{r})}{dt}\frac{dE^*_{0i}(t,\mathbf{r})}{dt}\right]. \qquad (A.21)$$

We note that the reality of the corrections of each order of smallness in (A.21), as in (A.20) also, is guaranteed when the condition (A.3) for the dielectric permittivity in a nonabsorptive medium is taken into account. For an isotropic medium, formula (A.21) was obtained by another method in[12].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Translated by P. J. Shepherd