Propagation of pulses

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The propagation of pulsed wave fields in a homogeneous medium or in a homogeneous line with known dispersion law and damping can be regarded from the ray, wave, and energy points of view. The review describes the most significant theoretical results on one-dimensional pulse propagation, using all three approaches. Greatest attention is paid to the principal part of a high-frequency pulse, the part governed by frequencies closest to the pulse carrier frequency. Paradoxes pertaining to superluminal and negative "group velocities" are resolved, and questions connected with amplification or attenuation of waves in wave beams and active media are discussed.

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CONTENTS

"What beasts and birds haven't I seen there [in the museum]. What flies, butterflies, cockroaches, little bits of beetles!—some like emeralds, others like corals. And what tiny cochineal insects! Why, really some of them are smaller than a pin's head."

"But did you see the elephant? What did you think it looked like? I'll be bound you felt as if you were looking at a mountain."

"Are you sure it's there?"

"Quite sure."

"Well brother, you mustn't be hard on me; but to tell the truth, I didn't remark the elephant."

Krylov's fable "The Inquisitive Man"
[Transl. by W. R. S. Ralston, London, 1869].

INTRODUCTION

This review is devoted to the propagation of pulses—principally narrow-band high-frequency pulses—in homogeneous media or homogeneous lines. We have in mind linear electromagnetic waves, for which the natural velocity limit is the velocity c of light in vacuum, although a number of results can be extended to waves of other type, and some results can also be used for nonlinear waves.

We confine ourselves to one-dimensional wave fields that depend on one coordinate z and on the time, and thus consider the simplest *kinematics* of fields. The refinement and systematic exposition of this kinematics is at present of particular interest for the following reasons:

1. In many problems pertaining to diffraction and propagation of waves, the waves cannot be regarded as monochromatic and it is necessary to take into account their pulsed character. It is easiest to show how this is done by using one-dimensional waves as an example.

2. A number of new devices have been developed in recent years, [1] in which transmission lines with dis-

persion give rise to transformation (particularly, compression) of pulses, and this makes it possible to increase the resolution of radar stations and to carry out more effectively a spectral analysis of the signals. More extensive use is made of short (including nanosecond) pulses, which become strongly deformed even after propagation over relatively short distances.

3. A number of questions (sometimes paradoxical) have recently been raised, pertaining to the propagation of waves in active (that is, unstable, non-equilibrium) systems, including masers and lasers, and also in systems with electron beams. Many of these questions are easy to answer if one considers not the propagation of monochromatic waves but the propagation of pulses, and some of the paradoxes are common to active and passive media.

We shall attempt to formulate the difference between our exposition and the one that can be found in the classical works of Sommerfeld[2] and Brillouin[3] and in the later papers that elaborate on them (see, e.g., [4-6]). In these papers, accurate results were obtained for the front and for the percursor (due to the very high frequencies in the pulse spectrum, $\omega \gg \omega_0$, where ω_0 is the carrier frequency), other field bursts (due, generally speaking, to frequencies different from ω_0) were investigated, and some data were obtained concerning the slow "trail" or "tail" of the pulse (due as a rule to the very low frequencies, $\omega \ll \omega_0$). However, a detailed investigation of the *principal part of the pulse*, due to the frequencies $\omega \approx \omega_0$, was initiated relatively recently (by Bliokh[7] and by others[8,9]), mainly in the absence of attenuation and amplification. On the other hand, if frequency-dependent attenuation is present (and strictly speaking it is always present), then the behavior of the principal part becomes, as we shall show, far from trivial, and additional complications arise in active (amplifying) systems. We recall in this connection Krylov's fable given in the epigraph. Our principal attention will be to the "elephant"—the principal part of the pulse, and the minute details that arise in pulse propagation will be dealt with only briefly (see Chaps. 2 and 3).

By virtue of the linearity of the problem, the propagation of the pulses is usually investigated with the aid of an ordinary (Chap. 1) or modified (Chap. 6) Fourier integral; in this way we obtain the wave kinematics of the propagating fields, while the simpler ray kinematics (or the space-time geometrical optics), as shown in a number of papers, [10-12] can be obtained from the equation

$$\frac{\partial \tau}{\partial t}+\frac{\partial \zeta}{\partial z}=0, \tag{I.1}$$

which connect the two quantities τ and ζ, between which a functional relation exists: $\tau=\tau(\zeta)$ or $\zeta=\zeta(\tau)$. Equation (I.1) has as solutions

$$\zeta=F(z-vt) \text{ and } \zeta=G\left(t-\frac{z}{v}\right) \quad \left(v=\frac{d\zeta}{d\tau}\right), \tag{I.2}$$

where F and G are differentiable functions, and v (the propagation velocity) depends in the general case on ζ. The first solution determines the evolution of the quantity ζ, which is specified at $t=0$ and at all z by the expression $\zeta=F(z)$; the second solution corresponds to a specified $\zeta=G(t)$ at $z=0$ and at all t. (We shall henceforth be interested only in solutions of the second type, since we seek the field at $z>0$ and specify the field at $z=0$.)

The solutions (I.2), with τ and ζ properly interpreted, determine the bunching of the electrons in a klystron[13] and the formation of shock waves in a gas[10-13] and in transport streams.[10] We shall apply them to a harmonic wave (frequency ω_0) modulated in amplitude or in frequency in a sufficiently slow manner, so that the width $\Delta\omega$ of its spectrum can be regarded as infinitesimally small. In the case of amplitude modulation we can put $\tau=W$ and $\zeta=S_z$, where W is the energy density and S_z is the component of the Umov-Poynting vector, which we can represent in the form $S_z=Wv_e$, where v_e is the energy transport velocity at the frequency ω_0. In the absence of losses, W and S_z satisfy Eq. (I.1), which represents in this case the energy-conservation law in differential form, and in formulas (I.2) the velocity $v=v_e$ turns out to be constant. In the case of frequency modulation, we can put $\tau=h=\partial\Psi/\partial z$ and $\zeta=\omega=-\partial\Psi/\partial t$, where Ψ is the total phase of the wave, h is the instantaneous wave number of the wave, ω is the instantaneous frequency (if h and ω is constant we have $\Psi=hz-\omega t$). Equation (I.1) is satisfied, since $\partial^2\Psi/\partial z\partial t=\partial^2\Psi/\partial t\partial z$, and the second solution in (I.2) takes the form

$$\omega=G\left(t-\frac{z}{v(\omega)}\right), \quad v(\omega)=\frac{d\omega}{dh}, \tag{I.3}$$

and as $\Delta\omega\to 0$ we can replace $v(\omega)$ under the sign of the function G by $v(\omega_0)$, which will be the constant propagation velocity in this case (the velocities v_e and $v(\omega_0)$ coincide, see Chaps. 3 and 5).

On the other hand, if $\Delta\omega$ is constant, then (I.3) is an equation that defines[11] ω as a function of z and t; it is easiest to investigate this equation graphically, breaking it up into two relations

$$\omega=G(\theta), \quad t=\theta+\frac{z}{v(G(\theta))} \tag{I.4}$$

and plotting on the (z, t) plane the straight lines $\theta=\text{const}$ (space-time rays). In Fig. 1 they are plotted under the assumption that

$$G(\theta)=\omega_0+b\theta \quad \text{at} \quad -\theta_0<\theta<\theta_0 \tag{I.5}$$

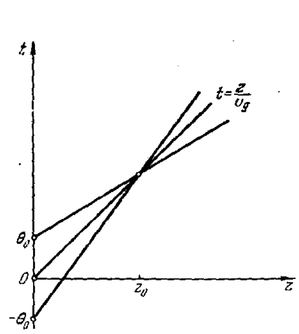

FIG. 1. Ray kinematics: formation of focus and contraction of the pulse.

and

$$\frac{1}{v(\omega)}=\frac{1}{v_g}-\frac{\theta}{z_0},\quad v_g=v(\omega_0),\quad z_0=\frac{v^2(\omega_0)}{v'(\omega_0)}>0\ (-\theta_0<\theta<\theta_0), \tag{I.6}$$

when the second relation of (I.4) takes the form

$$t=\theta\left(1-\frac{z}{z_0}\right)+\frac{z}{v_g} \tag{I.7}$$

and shows that in this case all the rays intersect at the point $z=z_0$, $t=z_0/v_g$. Thus, in the ray approximation, the pulse, which has at $z=0$ a finite duration $2\theta_0$, is contracted to zero duration at $z=z_0$: a space-time focus is produced (see Fig. 1).

Formulas (I.4) can be supplemented[11] by an expression for the field energy density $W=W(z,t)$, which is expressed in terms of the energy density $W_0(\theta)$ and $z=0$ as follows:

$$W=W_0(\theta)\left.\frac{\partial\theta}{\partial t}\right|_{z=\mathrm{const}}. \tag{I.8}$$

This is the energy-conservation law in the space-time ray tubes: as they become narrower the energy density should increase.

The ray approach, as always, gives only the "skeleton" of the wave field and should be supplemented by the wave approach, to which we now proceed.

1. DISPERSION LAW AND FOURIER INTEGRAL

In investigations of plane waves in an infinite homogeneous medium it is customary to consider first monochromatic waves, the dependence of the field components for which (i.e., for the electromagnetic wave components E_x, E_y, H_x, or H_y) on the coordinate z and on the time t takes the form (in complex notation) $e^{i[h(\omega)z-\omega t]}$. If we have a waveguide that is homogeneous in the direction of the z axis, then for each monochromatic wave in the waveguide the dependence on z and on t is the same, but is supplemented with a dependence on the transverse coordinates x and y, which usually does not influence the dependence on z and t.

The function $h(\omega)$ determines the law of dispersion and attenuation of the waves in a given medium or a line; we shall call this function simply the dispersion law. The properties of the function $h(\omega)$ can be most readily explained with examples. The first example is

$$h(\omega)=\frac{\omega}{c}\sqrt{1+\frac{2i\alpha}{\omega}},\qquad \alpha=\mathrm{const}>0,\quad c=\mathrm{const}>0 \tag{1.1}$$

and corresponds to a conducting medium with conductivity $\sigma=\alpha/2\pi$ that does not depend on the frequency, or else to a long line in vacuum, with conductors whose resistance does not depend on the frequency. The second example

$$h(\omega)=\frac{\omega+i\alpha}{c}\sqrt{1+\frac{\beta^2}{(\omega+i\alpha)^2}}\qquad (\alpha\geqslant\beta\geqslant 0) \tag{1.2}$$

corresponds to a long line with losses in both the conductors and in the medium between the conductors; at $\alpha=\beta$ formula (1.2) goes over into formula (1.1).

The third example

$$h(\omega)=\frac{\omega}{c}\sqrt{1-\frac{\omega_p^2}{\omega(\omega+i\nu)}}\qquad (\omega_p=\mathrm{const}>0,\ \nu=\mathrm{const}>0) \tag{1.3}$$

corresponds to a cold plasma with plasma frequency ω_p and collision frequency ν or else (at $\nu=0$) to an ideal waveguide with cutoff frequency ω_p; at $\nu>0$ the same formula accounts approximately for the properties of a wave in a waveguide with losses, at least in a sufficiently narrow frequency band. The fourth example

$$h(\omega)=\frac{\omega}{c}\sqrt{1-\frac{\omega_p^2}{\omega^2+i\nu\omega-\omega_r^2}}\qquad (\omega_r=\mathrm{const}>0) \tag{1.4}$$

corresponds to a medium with elastically bound electrons, having a natural frequency ω_r and a bandwidth $\omega_r<\omega<\sqrt{\omega_r^2+\omega_p^2}$ (also referred to as the anomalous dispersion band); the same formula corresponds approximately to transmission lines near an isolated rejection band or in this band. If there is a set of resonant frequencies ω_{rj}, then

$$h(\omega)=\frac{\omega}{c}\sqrt{1-\sum_j\frac{\omega_{pj}^2}{\omega^2+i\nu_j\omega-\omega_{rj}^2}}. \tag{1.5}$$

The sixth example

$$h(\omega)=\frac{\omega}{c}\sqrt{1+\frac{\beta^2}{\omega^2}} \tag{1.6}$$

is due to Ehrenfest[14] and corresponds to an unstable (active) system. The dispersion law (1.6) is obtained from (1.2) at $\alpha=0$ when one form of losses is negative and cancels out the losses of the other form.

In the first five examples, corresponding to *passive* systems, the function $h(\omega)$ at real ω has a positive imaginary part (the wave attenuates with increasing z), admits of analytic continuation to complex values of ω, and is a holomorphic (analytic) function of ω in the upper half-plane $\mathrm{Im}\,\omega\geqslant 0$; its asymptotic expansion (as $|\omega|\to\infty$) is

$$h(\omega)=\frac{\omega}{c}+b_0-\frac{b_1}{\omega+i\nu}+\dots, \tag{1.7}$$

where b_0 and b_1 are constants, and $\nu=0$ in the examples (1.1) and (1.6). In all the examples, c is the positive velocity limit

$$c=\lim_{|\omega|\to\infty}\frac{\omega}{h(\omega)}=\lim_{|\omega|\to\infty}\frac{1}{h'(\omega)}; \tag{1.8}$$

for electromagnetic waves this is as a rule the velocity of light in vacuum, which in general is the velocity limit of signal propagation and of material-particle motion. Of course, for waves of other type the velocity of light in vacuum is also the limiting velocity, but usually the dispersion law for these waves is chosen in such a form that the velocity limit (1.8) for these waves exists and is less than that of light.

The transition to pulses is obtained with the aid of the Fourier integral

$$f(z,t)=\int_{-\infty}^{\infty}A(\omega)\,e^{i[h(\omega)z-\omega t]}\,d\omega, \tag{1.9}$$

which can be regarded as superposition of the monochromatic waves referred to above. The function $A(\omega)$ is the spectral amplitude of the corresponding wave, and is complex in the general case; for high-frequency pulses with a carrier frequency ω_0 the function $A(\omega)$ has the largest value at $\omega=\omega_0$, and when ω moves away from ω_0 its absolute value decreases. The function $f(z,t)$ in (1.9) is in the general case complex, and a physical meaning (for example, that of the electric-field component at certain z and t) is possessed by the quantity $\mathrm{Re} f(z,t)$ or $\mathrm{Im} f(z,t)$.

The integral (1.9) corresponds to the following formulation of the problem; we are given the function f at $z=0$ and $-\infty<t<\infty$. In other words, a certain radiator is located in the initial section $z=0$ and thus determines

$$f(0,\ t)=\int_{-\infty}^{\infty} A(\omega)\, e^{-i\omega t}\, d\omega \text{ and } A(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(0,\ t)\, e^{i\omega t}\, dt. \tag{1.10}$$

We assume that the function $h(\omega)$ has the properties listed above. It follows then from the integral (1.9) that the wave front moves with the limiting velocity c: in other words,

$$\text{if } f(0,\ t)=0 \quad \text{at} \quad t<0, \text{ then } f(z,\ t)=0 \quad \text{at} \quad t<\frac{z}{c}. \tag{1.11}$$

The proof of relation (1.11) and the general analysis of the propagation of pulses become easier if we introduce the function

$$g(z,\ t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i[h(\omega)z-\omega t]}\, d\omega, \tag{1.12}$$

which satisfies the initial condition

$$g(0,\ t)=\delta(t), \tag{1.13}$$

where $\delta(t)$ is the Dirac delta function, formally represented in the form of the Fourier integral

$$\delta(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\, d\omega. \tag{1.14}$$

With the aid of the function $g(z,t)$, we can transform (1.9) into

$$f(z,\ t)=\int_{-\infty}^{\infty} g(z,\ t-\tilde{t})\, f(0,\ \tilde{t})\, d\tilde{t}, \tag{1.15}$$

this being done by substituting the second expression of (1.10) in the integral (1.9) and changing the order of the integration. The function g can be called the space-time Green's function, and if z is given it can be called the reaction or the response of the given system to a delta pulse (1.13). Inasmuch as any function $f(0,t)$ can be represented as a superposition of delta pulses, expression (1.15) becomes self-evident.

2. PROPERTIES OF THE FUNCTION $g(z,t)$

Even in the first studies[2,3] the integral investigated differed from (1.12) by an additional factor $1/(\omega-\omega_0)$ under the integral sign (its appearance was due to the fact that a semi-infinite sinusoid was used instead of a delta function). The integral (1.12) was investigated analogously; in particular we have

$$g(z,\ t)=0 \quad \text{at} \quad t<\frac{z}{c}, \tag{2.1}$$

and expression (1.15) yields immediately relation (1.11). As to the relation (2.1) itself, it follows from the holomorphism of the function $h(\omega)$ above the integration contour (1.9), and from the asymptotic expansion (1.7), owing to which the integrand of (1.12) tends to zero as $\mathrm{Im}\,\omega\to\infty$ and at $t<z/c$; we shift the integration contour upwards and prove (2.1).

At $t\geqslant z/c$, the function $g(z,t)$ differs from zero and in some cases can be calculated in explicit form. For the dispersion law (1.2) we have

$$g(z,\ t)=\left[\delta\left(t-\frac{z}{c}\right)+\frac{\beta z}{c\sqrt{t^2-(z^2/c^2)}}\, I_1\left(\beta\sqrt{t^2-\frac{z^2}{c^2}}\right)\right]e^{-\alpha t}, \tag{2.2}$$

whence, putting $\alpha=\beta$ and $\alpha=0$, we obtain $g(z,t)$ for the dispersion laws (1.1) and (1.6); $I_1(x)$ is a modified Bessel function that increases exponentially as $x\to\infty$, so that for $\alpha=0$ and $t\to\infty$ we have $g(z,t)\to\infty$, indicating that a system with dispersion law (1.6) is unstable. At $\beta=0$ we have

$$h(\omega)=\frac{\omega+i\alpha}{c}, \quad g(z,\ t)=\delta\left(t-\frac{z}{c}\right)e^{-\alpha t} \tag{2.3}$$

and then

$$f(z,\ t)=f\left(0,\ t-\frac{z}{c}\right)e^{-(\alpha/c)z}, \tag{2.4}$$

that is, a pulse of any waveform propagates without distortion with velocity c, and experiences at the same time an attenuation that does not depend on the pulse waveform. At $\beta>0$ the pulse propagates over long line in a more complicated manner, as seen from formula (2.2), namely, a "tail" or "trail" is superimposed on the undistorted pulse (2.4) and attenuates, as seen from (2.2), more slowly than $e^{-\alpha t}$. When signals are transmitted over the line, this leads to a distortion superposition of the signals, and it was therefore concluded even in the past century that losses in telegraph cables are least harmful at $\beta=0$.

For the dispersion law (1.3) we obtain at $\nu=0$, putting $\beta=-i\omega_p$ and $\alpha=0$ in (2.2),

$$g(z,\ t)=\delta\left(t-\frac{z}{c}\right)-\frac{\omega_p z}{c\sqrt{t^2-(z^2/c^2)}}\, J_1\left(\omega_p\sqrt{t^2-\frac{z^2}{c^2}}\right). \tag{2.5}$$

The trail determined by the Bessel function J_1 has an oscillating character in this case.

In other cases the integral (1.12) cannot be expressed in terms of known functions, but can be approximately calculated at large z and at different ratios $z/t\leqslant c$. Thus, from the asymptotic expansion of (1.7) we can obtain the approximate expression

$$g(z,\ t)=\left[\delta\left(t-\frac{z}{c}\right)-\sqrt{\frac{b_1 z}{t-(z/c)}}\, J_1\left(2\sqrt{b_1 z\left(t-\frac{z}{c}\right)}\right)e^{-\nu[t-(z/c)]}\right]e^{ib_0 z}, \tag{2.6}$$

which can be used at small nonnegative $t - z/c$ such that the argument of the Bessel function J_1 is small or finite. At $z=0$ expression (2.6) agrees with the initial condition (1.13), and at arbitrary z it agrees with the exact expressions (2.2) and (2.5).

It is easiest to derive (2.2), (2.5), and (2.6) by introducing the functions

$$\Gamma_n(z, t)=\frac{1}{2\pi}\int\limits_{i\sigma-\infty}^{i\sigma+\infty} e^{i[h(\omega)z-\omega t]}\frac{d\omega}{(-i\omega)^{n+1}},$$

$$\Gamma(z, t)=\frac{1}{2\pi}\int\limits_{i\sigma-\infty}^{i\sigma+\infty} e^{i[h(\omega)z-\omega t]}\frac{d\omega}{ih(\omega)},$$

where $n=0, 1, 2, \ldots$ and $\sigma>0$ (see Chap. 6). It is easily shown that we have the identities

$$g(z, t)=\frac{\partial^{n+1}}{\partial t^{n+1}}\Gamma_n(z, t) \text{ and } g(z, t)=\frac{\partial}{\partial z}\Gamma(z, t). \tag{2.7}$$

The last identity yields immediately expression (2.2) at $z<ct$: for this purpose the integration contour must be shifted downward and this leads to an integral over a cut and we obtain

$$\Gamma(z, t)=-cI_0\left(\beta\sqrt{t^2-\frac{z^2}{c^2}}\right)e^{-\alpha t}.$$

Differentiation with respect to z together with the function I_1 yields a delta function, since $\Gamma(z,t)=0$ at $z>ct$ and there is a discontinuity at $z=ct$. The functions $\Gamma_n(z,t)$ at $z>ct$ are also equal to zero, and at $z<ct$ they can be reduced to an integral along a circle

$$\omega+iv=i\Omega e^{-i\varphi},\quad -\pi<\varphi<\pi,\quad \Omega=\sqrt{\frac{b_1 z}{t-(z/c)}},$$

on which the function $h(\omega)$ can be replaced at large z and small $t-z/c$ by the written-out terms of the expansion (1.7). We obtain

$$\Gamma_n(z, t)=\left[\frac{t-(z/c)}{b_1 z}\right]^{n/2} J_n\left(2\sqrt{b_1 z\left(t-\frac{z}{c}\right)}\right)e^{ib_0 z-\nu[t-(z/c)]}, \tag{2.8}$$

for which follows immediately at $n=0$ the expression (2.6), for that part of the field which is excited by the delta pulse and moves with the limiting velocity c.

The remaining parts of the field, with velocity less than c, move behind the field whose velocity is c. They can be calculated by the saddle-point method, by deforming the initial integration contour (the real axis) into a contour passing in the upper half-plane $\mathrm{Im}\,\omega \gg 0$ through one or several saddle points $\bar{\omega}$ defined by the equation

$$h'(\bar{\omega})=\frac{1}{v}\qquad(0<v<c), \tag{2.9}$$

where the prime denotes differentiation with respect to ω. In the vicinity of each point $\bar{\omega}$ we can write down the Taylor expansion

$$h(\omega)=h(\bar{\omega})+h'(\bar{\omega})(\omega-\bar{\omega})+\frac{1}{2}h''(\bar{\omega})(\omega-\bar{\omega})^2+\frac{1}{6}h'''(\bar{\omega})(\omega-\bar{\omega})^3+\ldots \tag{2.10}$$

The asymptotic expression for the Green's function

$$g(z, t)=\sum_{\bar{\omega}} G(z, t, \bar{\omega})\, e^{i[h(\bar{\omega})z-\bar{\omega}t]} \tag{2.11}$$

breaks up into a sum of contributions of each point $\bar{\omega}$. If we confine ourselves near the point $\bar{\omega}$ to the first three terms of the expansion (2.10), then we obtain for G the expression

$$G(z, t, \bar{\omega})=\frac{1}{\sqrt{-2i\pi h''(\bar{\omega})z}}e^{-i\tau^2/2},\qquad \tau=\frac{t-(z/v)}{\sqrt{h''(\bar{\omega})z}}, \tag{2.12}$$

which is valid at $t\approx z/v$, and more accurately at $\tau\sim 1$. If only $h''(\bar{\omega})$ is not too small, then this expression is a good approximation of $G(z,t,\bar{\omega})$ at large z, more accurately at $z\gg[h'''(\bar{\omega})]^2/[h''(\bar{\omega})]^3$ (this is the condition for the smallness of the first discarded term on the principal section of the integration path). On the other hand if $h''(\bar{\omega})$ is small, then we can easily shift $\bar{\omega}$ in such a way that we get exactly $h''(\bar{\omega})=0$, and take into account the next (cubic) term in the expansion (2.10). We thus arrive at the equation

$$G(z, t, \bar{\omega})=\frac{V(-\tau)}{\sqrt{\pi}\sqrt[3]{h'''(\bar{\omega})z/2}},\qquad \tau=\frac{t-(z/v)}{\sqrt[3]{h'''(\bar{\omega})z/2}}, \tag{2.13}$$

where

$$V(s)=\frac{1}{2\sqrt{\pi}}\int\limits_{-\infty}^{\infty} e^{i[(x^3/3)+sx]}\,dx \tag{2.14}$$

is an Airy function[15] which decreases exponentially at $s>0$ and oscillates at $s<0$.

It can be easily shown that for the dispersion laws (1.1) and (1.2) there will be only one saddle point $\bar{\omega}$ on the imaginary axis, and this leads to a single-term asymptotic expression for G, which agrees with (2.2) at $\beta\sqrt{t^2-(z^2/c^2)}\gg 1$. For the dispersion law (1.3) there will be two points $\bar{\omega}$, symmetrical relative to the real axis and giving an oscillatory dependence of G, which agrees with (2.5) at $\omega_p\sqrt{t^2-(z^2/c^2)}\gg 1$. For dispersion laws (1.4) and (1.5) there exists a large number of points ω, and the corresponding saddle contours and contributions were considered in detail in Baerwald's article.[4] We present here only a general analysis of the function $g(z,t)$. Whereas at $z=0$ there is a delta pulse (1.13), at $z>0$ a complicated pulse $g(z,t)$ is observed. Considering $g(z,t)$ at $z=0$, we again have $g=0$ (the shaded sector in Fig. 2). At the instant $t=z/c$ and at the succeeding instant (that is, above the limiting line $t=z/c$ in Fig. 2), expression (2.6) is valid; if the

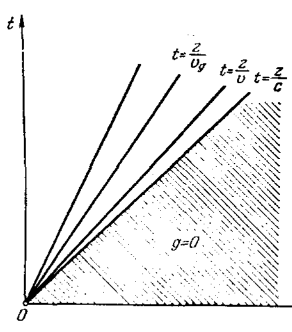

FIG. 2. The function g on the (z,t) plane.

argument of the function J_1 in this expression is large, then it is necessary to go over to expressions (2.11) and (2.12). If furthermore $h''(\overline{\omega})$ vanishes in one of the points $\overline{\omega}$ at a certain value of v, this means[4] that two points $\overline{\omega}$ appear at smaller values of v. In this situation, expression (2.13) is valid and must be used at $\tau \sim 1$: the corresponding line at $t = h'(\overline{\omega})z$ is the space-time caustic at $h''(\overline{\omega}) = 0$.

Comparing Fig. 2 with Fig. 1, we see that the delta pulse, which contains in accordance with (1.14) all the frequencies in equal measure, generates a bundle of rays, whereas slow frequency modulation generates at $z = 0$ only one ray at each instant. It should be noted, however, that in the wave approximation the field becomes deformed as it propagates, and this deformation is taken into account by expressions (2.6), (2.11), (2.12), and (2.13). We consider first a pulse for which $f(0, t) = 0$ at $t < 0$, and the function $f(0, t)$ and all its derivatives with respect to t are continuous for all t with the exception of $t = 0$. Then the field $f(z, t)$ immediately behind the front—the so-called precursor—can be obtained from (1.15) and from the first identity of (2.7) by integrating by parts, in the form

$$f(z, t) = \sum_{n=0}^{\infty} \Gamma_n(z, t) \frac{\partial^n f}{\partial t^n}(0, +0), \tag{2.15}$$

in which the functions $\Gamma_n(z, t)$ are determined by formula (2.8). They determined the law of propagation of the *discontinuities* of the function $f(0, t)$ and its derivatives. We see that these discontinuities propagate with the limiting velocity (1.8), and in the absence of losses (at $\mathrm{Im}\, b_0 = 0$) the absolute values of these discontinuities are conserved.

However, the structure of the field that follows the discontinuity is entirely different as $z = 0$ and $z \to \infty$. We take by way of example a semi-infinite harmonic oscillation

$$f(0, t) = e^{-i\omega_0 t} \quad \text{at} \quad t > 0, \; f(0, t) = 0 \quad \text{at} \quad t < 0, \tag{2.16}$$

for which series (2.15) takes the form

$$f(z, t) = \Gamma_0(z, t) - i\omega_0 \Gamma_1(z, t) - \omega_0^2 \Gamma_2(z, t) + \ldots, \tag{2.17}$$

and there is nothing that looks like (2.16); the principal part of the field, which leads to establishment of the harmonic oscillation, moves with velocity $v_g < c$ (see Chap. 3).

In the field moving behind the precursor (2.15) there exists, besides the principal part of the field, also parts moving at all possible velocities $v(0 < v < c)$; notice should be taken, among them, of field bursts due to the "caustic" values of v (see above). They are sometimes also called precursors (second, third, etc.), although their velocity in some cases may be lower than the velocity v_g of the principal parts; they can be calculated with the aid of expression (2.13), as shown at the end of Chap. 3.

3. PRINCIPAL PART OF HIGH-FREQUENCY PULSE IN THE ABSENCE OF DAMPING AND AMPLIFICATION

What is usually of interest is not the propagation of a delta pulse, but the propagation of a high-frequency pulse with carrier frequency ω_0, for which the function $A(\omega)$ in the integral (1.9) has a sharp maximum at the point $\omega = \omega_0$, with $|A(\omega)| \sim |A(\omega_0)|$ at $|\omega - \omega_0| \lesssim \Delta\omega$. The law of propagation of the *principal* part of the high-frequency pulse (at sufficiently small $\Delta\omega$) is obtained by expanding the function $h(\omega)$ and the integral (1.9) in a Taylor series analogous to the series (2.10). Assuming the wave number $h(\omega)$ to be *real* in the vicinity of the point ω_0 and representing $f(0, t)$ in the form

$$f(0, t) = F(0, t)\, e^{-i\omega_0 t}, \tag{3.1}$$

where $F(0, t)$ is a slowly varying function of t (the complex envelope of the pulse at $z = 0$) we obtain, confining ourselves to the first two terms in the Taylor expansion, the following form for $f(z, t)$

$$f(z, t) = F(z, t)\, e^{i[h(\omega_0)z - \omega_0 t]}, \tag{3.2}$$

where the complex envelope $F(z, t)$ is expressed in terms of $F(0, t)$ in the following manner:

$$F(z, t) = F(0, t - h'(\omega_0) z). \tag{3.3}$$

Thus, in this approximation the complex envelope F (that is, the usual envelope $|F|$ due to the amplitude modulation, and the additional phase $\arg F$ connected with the frequency or phase modulation), move at the so-called group velocity

$$v_g = \frac{1}{h'(\omega_0)} = \frac{d\omega}{dh}\bigg|_{\omega=\omega_0}, \tag{3.4}$$

whereas the high-frequency carrier of the pulse moves, as seen from (3.2) at the phase velocity $\omega_0/h(\omega_0)$: it is determined by the same factor $e^{i[h(\omega_0)z - \omega_0 t]}$ as for the monochromatic wave with frequency ω_0.

With the aid of the Fourier integral it is easy to refine formula (3.3) and to determine the limits of its applicability. To this end it suffices to take into account in the Taylor expansion one more (third) term, and at $h''(\omega_0) = 0$ the fourth term. The resultant expressions, as well as the simple formula (3.3), can be written in a unified form

$$F(z, t) = \int_{-\infty}^{\infty} G(z, t - \tilde{t}, \omega_0) F(0, \tilde{t})\, d\tilde{t}, \tag{3.5}$$

which connects directly the complex envelopes $F(0, t)$ and $F(z, t)$. Formula (3.3) is then obtained by putting

$$G(z, t - \tilde{t}, \omega_0) = \delta(\tilde{t} - t + h'(\omega_0) z); \tag{3.6}$$

if the third form in the Taylor expansion is taken into account we have

$$G(z, t - \tilde{t}, \omega_0) = \frac{1}{\sqrt{-2i\pi}\,\Delta} \exp\left[-\frac{i}{2}\left(\frac{\tilde{t} - t + h'(\omega_0) z}{\Delta}\right)^2\right], \quad \Delta = \sqrt{h''(\omega_0) z}, \tag{3.7}$$

and at $h''(\omega_0)=0$ we get with allowance for the fourth term

$$G(z, t-\tilde{t}, \omega_0)=\frac{1}{\sqrt{\pi}\,\Delta} V\left(\frac{\tilde{t}-t+h'(\omega_0)z}{\Delta}\right), \quad \Delta=\sqrt[3]{\frac{h'''(\omega_0)z}{2}}. \qquad (3.8)$$

Formula (3.5) determines the propagation of the *principal* part of the high-frequency pulse—more accurately, the propagation of its *complex* envelope F. We note that expressions (3.7) and (3.8) are exact analogs of expressions (2.12) and (2.13): however, expressions (2.12) and (2.13) are asymptotic (they are suitable only at sufficiently large z), whereas expressions (3.7) and (3.8) are suitable for *any* z, including $z\to 0$, when they go over into the delta function (3.6), which for our purpose can be defined by the relations

$$\int_{t_1}^{t_2}\delta(t)\,dt=0 \quad \text{at} \quad t_1<0, \quad t_2<0 \text{ and } t_1>0, \quad t_2>0,$$

$$\int_{t_1}^{t_2}\delta(t)\,dt=1 \quad \text{at} \quad t_1<0, \quad t_2>0,$$

and then it leads to expression (3.3).

The function G determines the deformation of the complex envelope F with increasing z. This deformation is determined by the second derivative $h''(\omega_0)$, and at $h''(\omega_0)=0$ it is determined by the third derivative $h'''(\omega_0)$, that is, by the *variability of the group velocity* $v(\omega)=1/h'(\omega)$ *within the frequency band* $|\omega-\omega_0|\lesssim\Delta\omega$ *occupied by the pulse*. If the following condition is satisfied

$$\frac{1}{2}h''(\omega_0)\Delta\omega^2 z\ll 1, \qquad (3.9)$$

and at $h''(\omega_0)=0$ the condition

$$\frac{1}{6}h'''(\omega_0)\Delta\omega^3 z\ll 1, \qquad (3.10)$$

then the deformation of F can be neglected and formula (3.3) can be used. Thus, the narrower the frequency band occupied by the pulse, the larger the distance covered by it without a noticeable distortion. In other words, the slower the time variation of the complex envelope, the slower its deformation in space.

It should be noted[7,8] that the law governing the deformation of the complex envelope coincides, according to formulas (3.6) and (3.7), with the law of propagation of two-dimensional wave beams of paraxial type. Assume that we have a monochromatic wave field (frequency ω_0, time dependence $\exp[-i\omega_0 t]$), the complex amplitude of which $u(z,x)$ satisfies the two-dimensional Helmholtz equation

$$\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 u}{\partial x^2}+k_0^2 u=0 \qquad \left(k_0=\frac{\omega_0}{c}\right). \qquad (3.11)$$

In the case when the wave field has the character of a quasi-plane wave propagating in the direction of the z axis (that is, we have a paraxial wave beam), the function u can be represented in the form

$$u=U(z,x)\,e^{ik_0 z}, \qquad (3.12)$$

where the slowly varying function $U(z,x)$ is connected with $U(0,x)$ in the same manner as $F(z,t)$ is connected with $F(0,t)$, if $t-h'(\omega_0)z$ is replaced by x/c and $h''(\omega_0)$ is replaced by $-1/k_0c^2$. In other words, $F(z,t)$ satisfies the parabolic equation

$$\frac{\partial F}{\partial z}=-\frac{ih''(\omega_0)}{2}\frac{\partial^2 F}{\partial\theta^2}, \quad \theta=t-h'(\omega_0)z=t-\frac{z}{v_g}, \qquad (3.13)$$

and when (3.7) is replaced by (3.8) it satisfies the more complicated equation

$$\frac{\partial F}{\partial z}=-\frac{h'''(\omega_0)}{6}\frac{\partial^3 F}{\partial\theta^3}. \qquad (3.14)$$

These equations determine F on the (z,t) plane near the "principal ray" $t=z/v_g$ (Fig. 2).

The analogy between the paraxial beams and the high-frequency pulses enables us to distinguish for the latter between the near, far, and intermediate zones. In the near zone, formula (3.3) is valid and is determined by the condition (3.9) and (3.10). If these conditions are reversed, then we deal with the far zone, in which

$$F(z,t)=2\pi A(\omega_0)\,G(z,t,\omega_0). \qquad (3.15)$$

In the far zone, the pulse is so deformed that its field is proportional to $1/\sqrt{z}$ or $1/z^{1/3}$, and is determined by the spectral amplitude $A(\omega)$, just as the field of an antenna is determined not by the distribution of the field at the radiating aperture, but by the Fourier transform of this distribution. In the *intermediate* zone, which joins the near and far zones, a gradual deformation of the pulse takes place and, in particular, the pulse becomes compressed (focused).

If the initial pulse is a semi-infinite harmonic oscillation (2.16), then expression (3.7) for the function G yields in this case[4]

$$F(z,t)=\frac{e^{i\pi/4}}{\sqrt{2\pi}}\int_{-\infty}^{\theta/\Delta}e^{-i\tau^2/2}\,d\tau,$$
$$\Delta=\sqrt{h''(\omega_0)z}, \qquad (3.16)$$

and expression (3.8) yields

$$F(z,t)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\theta/\Delta}V(-\tau)\,d\tau,$$
$$\Delta=\sqrt[3]{\frac{h'''(\omega_0)z}{2}}, \qquad (3.17)$$

while in both expressions we have $\theta=t-h'(\omega_0)z$. Figure 3 shows the values of $|F|$ calculated from these formulas. We see that the wave front, which at first is infinitely steep, becomes more gently sloping with increasing z. In this case we deal not with the front of the pulse, the structure of the field near which was investigated at the end of Chap. 3, but with the front of its principal part, moving with velocity $v_g<c$. Formulas (3.16) and (3.17) remain in force also at $\omega_0=0$, that is, for a pulse without a high-frequency carrier. They then determine the establishment of a *constant* field applied at the instant $t=0$ to the point $z=0$. After the front of the *principal part* has passed through this point with velocity $v_g=v(0)$, a constant field is estab-

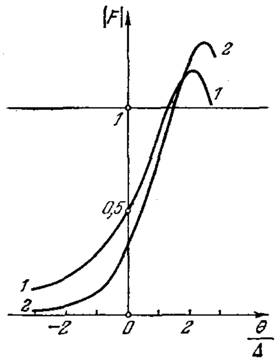

FIG. 3. Front of principal part of semi-infinite harmonic oscillation in accordance with formulas (3.16) (1) and (3.17) (2).

lished at this point, in accordance with Fig. 3.

As a second example, we take a Gaussian pulse with quadratic phase modulation [or linear frequency modulation, see formula (I.5)], when

$$F(0,\ t)=e^{-(a+ib)t^2/2},\quad A(\omega)=\frac{1}{\sqrt{2\pi(a+ib)}}\exp\left[-\frac{(\omega-\omega_0)^2}{2(a+ib)}\right]\qquad (a>0). \tag{3.18}$$

For such a pulse we obtain simple expressions. In particular, instead of formula (3.16) we have the expression

$$F(z,\ t)=\frac{1}{\sqrt{1-i(a+ib)h''(\omega_0)z}}\exp\left\{-\frac{(a+ib)[t-h'(\omega_0)z]^2}{2[1-i(a+ib)h''(\omega_0)z]}\right\}, \tag{3.19}$$

which makes it easy to trace the deformation of the principal part of the pulse as it emerges to the near zone (where $(a+ib)h''(\omega_0)z \ll 1$). We see, in particular, that the duration of the pulse is minimal at the point

$$z_0=-\frac{b}{(a^2+b^2)h''(\omega_0)}, \tag{3.20}$$

with

$$F(z_0,\ t)=\sqrt{-i\frac{b}{a}}\exp\left\{-\frac{(a^2+b^2)[t-h'(\omega_0)z]^2}{2a}\right\}.$$

If $z_0>0$, then at $0<z<z_0$ the pulse becomes compressed, and at $z>z_0$ it expands. It is easily seen that at $a \ll |b|$ the values of z_0 given by formulas (I.6) and (3.20) coincide, and the effective pulse duration $\Delta t=\sqrt{a/(a^2+b^2)}$ at the point z_0 is much less than the initial duration, equal to $1/\sqrt{a}$. Thus, with the aid of the wave treatment we determine more precisely the pulse contraction process shown in Fig. 1.

At sufficiently large z, the lateral part of the pulse, which moves with velocity v, can be calculated with the aid of expression (2.11) for the function g. We obtain the formula

$$f(z,\ t)=\sum_{\bar\omega}F(z,\ t,\ \bar\omega)\,e^{i[h(\bar\omega)z-\bar\omega t]}, \tag{3.21}$$

where the slowly varying function

$$F(z,\ t,\ \bar\omega)=\int_{-\infty}^{\infty}G(z,\ t-\tilde t,\ \bar\omega)\,F(0,\ \tilde t)\,e^{i(\bar\omega-\omega_0)\tilde t}\,d\tilde t \tag{3.22}$$

is transformed into $F(z,t)$ at $\bar\omega=\omega_0$. If $\bar\omega$ differs strongly from ω_0, then, owing to the oscillating factor $e^{i(\bar\omega-\omega_0)t}$ we obtain $|F(z,t,\bar\omega)| \ll |F(z,t)|$. In the far zone, the integral (3.22) leads to the expression

$$F(z,\ t,\ \bar\omega)=2\pi A(\bar\omega)\,G(z,\ t,\ \bar\omega), \tag{3.23}$$

which also proves the relative smallness of the sidebands at real $\bar\omega$ that differ greatly from ω_0. In the case of complex $\bar\omega$, the smallness of the sidebands follows from the smallness of the exponentials in (3.21).

The foregoing can be supplemented in some respects. First, formula (3.5) was used by many authors without taking into account the dispersion deformation,[1,16,17] and also in more complicated cases, for example, for reflection of pulses from an inhomogeneous ionosphere,[18] so that the number of examples could be increased. Second, in some papers[9,19] formula (3.5) is made more accurate by taking into account additional terms in the Taylor expansion for the function $h(\omega)$ at $\omega\approx\omega_0$. This makes it possible not so much to refine the function G, as to estimate the limits of applicability of the simple expressions (3.7) and (3.8). The situation here is the same as in paraxial physical optics, where use is made of the Huygens–Fresnel principle, which leads to Fresnel integrals of the type (3.16) and corresponds in the differential formulation to a parabolic equation of the type (3.13); if this principle is not convenient, then it is usually expressed more precisely, and one resorts to more rigorous methods.

Third, the saddle-point method can be used[20] not only for the integral (1.12) but also for the integral (1.9), after first specifying concretely the form of the function $A(\omega)$, for example by means of formula (3.18); then the saddle point $\bar\omega$ is determined by the equivalent equations

$$\bar\omega=\omega_0+i(a+ib)[h'(\bar\omega)z-t],\quad h'(\bar\omega)=\frac{1}{z}\left(t+i\frac{\bar\omega-\omega_0}{a+ib}\right), \tag{3.24}$$

which yield $\bar\omega\to\omega_0$ at $a+ib\to 0$ and go over into (2.9) at $a+ib\to\infty$. Equation (2.9) turns out to be valid also for large t—for the tail of the pulse, which is well accounted for[21a] by formula (3.23). At $a=0$ the saddle point turns out to be real, the saddle-point method leads[20a] to the ray kinematics (see the Introduction) as refined by the wave treatment. At $a>0$, the point $\bar\omega$ is complex, and the results are interpreted[20b] with the aid of complex rays (see below).

In addition, for the semi-infinite harmonic oscillation (2.16) expression (3.23) determines, at sufficiently large values of z, the so-called Brillouin precursor[3]

$$f(z,\ t)=\frac{iV(-\tau)}{\sqrt{\pi}\sqrt[3]{h'''(\bar\omega)z/2}\,(\bar\omega-\omega_0)}\,e^{i[h(\bar\omega)z-\bar\omega t]},\qquad \tau=\frac{t-h'(\bar\omega)z}{\sqrt[3]{h'''(\bar\omega)z/2}}, \tag{3.25}$$

which corresponds to caustic values of the frequency $\bar\omega$ and to a velocity $v=1/h'(\bar\omega)$; $0<v<c$, $h''(\bar\omega)=0$. These precursors (second, third, etc.) were referred to at the end of Chap. 2. The same result is obtained by integrating by parts formula (3.22) and discarding the inte-

gral proportional to $z^{-2/3}$, or else by asymptotic calculation of the initial integral (1.9). The first precursor determined by formula (2.17) is also called the Sommerfeld precursor.[2]

According to (3.25), the field at large negative τ is exponentially small, since

$$V(s)=\frac{1}{2}s^{-1/4}\exp\left(-\frac{2}{3}s^{3/2}\right) \qquad (s\gg 1)$$

at $\tau\sim 1$ the field increases, and at large positive τ the function $V(-\tau)$ oscillates and decreases slowly, since

$$V(-\tau)=\tau^{-1/4}\sin\left(\frac{2}{3}\tau^{3/2}+\frac{\pi}{4}\right) \qquad (\tau\gg 1).$$

This behavior is similar (albeit qualitatively) to the behavior of the first precursor.

Precursors were investigated experimentally[22] at the dispersion law (1.4) and at carrier frequencies in the decimeter and centimeter bands. Such investigations enable us to evaluate[23] the dispersion law for frequencies that differ strongly from the carrier; however, the larger $|\omega-\omega_0|$, the more accurately should the envelope realized in experiment coincide with the theoretical one.

4. PRINCIPAL PART OF A PROPAGATING DAMPED PULSE

Formulas (3.1) to (3.3) are valid under condition (3.9) or (3.10) and whenever $\operatorname{Im}h(\omega)>0$ within the frequency band occupied by the high-frequency pulse. In this case, however, the group velocity (3.4) turns out to be complex—this means that the envelope moves without distortion only if

$$t=h'(\omega_0)z+\theta, \tag{4.1}$$

where θ is a real quantity. If $h'(\omega_0)$ is complex, relation (2.1) can be satisfied only for complex z or t, that is, on a complex ray. What is the result in the case of real z and t? It can be shown that a complex v_g causes a strong additional deformation of the pulse,[17,24] and the deformation is different for different complex envelopes $F(0,t)$.

By way of example we take the pulse (3.18) and, putting

$$h'(\omega_0)=\xi+i\eta, \tag{4.2}$$

we write down

$$F(0,\ t-h'(\omega_0)z)=e^{-(1/2)(a+ib)(\xi+i\eta)^2z^2}e^{-[(1/2)at^2-(a\xi-b\eta)zt]}\times e^{-i[(1/2)bt^2-(a\eta+b\xi)zt]}. \tag{4.3}$$

At a given z, the first factor in the right-hand part is constant, the second shows that the maximum of the amplitude (the amplitude center of the pulse) moves with velocity

$$v_1=\frac{a}{a\xi-b\eta}, \tag{4.4}$$

and the third is the point at which the instantaneous velocity is equal to ω_0 (the phase center) and moves with velocity

$$v_2=\frac{b}{b\xi+a\eta}. \tag{4.5}$$

The velocities v_1 and v_2 are different (this is how the deformation of the given pulse manifests itself) and depend on the parameters a and b. Only at $\eta=0$ do they coincide with each other and with the group velocity $1/\xi$, but then we return to propagation without attenuation or else with attenuation that does not depend on the frequency. The velocities v_1 and v_2 can be arbitrarily large or negative; in particular, values $v_1>c$ and $v_1<0$ are possible even without frequency modulation, that is, at $b=0$ (examples based on the dispersion laws (1.1) and (1.3) were considered in our papers[21]). There are published[25] examples of calculations by formulas (3.5), (3.19), and (4.3) in the presence of damping, and a comparison of the results with those obtained by exact calculation of the integral (1.9). The agreement is satisfactory for moderate values of z and becomes worse with increasing z. Such a comparison was carried out[21] for the "exotic" values $v_1=1/\xi$ (at $b=0$) that is, for $v_1>c$ and $v_1<0$, with the following conclusion: the more exotic the velocity v_1, the less stable it is, that is, the shorter the distance over which the pulse moves with this velocity. Further propagation either changes v_1 to "normal" values $(0<v_1<c)$, or distorts the pulse until it becomes unrecognizable.

In the absence of attenuation, as we have seen in Chap. 3, the main cause of the deformation of the pulse is the phase distortion due to variability of the group velocity where then the frequency band $|\omega-\omega_0|\lesssim\Delta\omega$ occupied by the pulse. In the presence of frequency-dependent attenuation $\operatorname{Im}h(\omega)$, a new phenomenon arises, namely distortion of the energy spectrum of the pulse as it propagates. Indeed, at $z=0$ the spectrum is proportional to $|A(\omega)|^2$, and at $z>0$ we already have $|A(\omega)|^2\times e^{-2\operatorname{Im}h(\omega)z}$: the pulse, first, undergoes attenuation determined by the factor $e^{-2\operatorname{Im}h(\omega_0)z}$, and second, experiences a change in shape, as determined by the formula

$$S(\omega,\ z)=|A(\omega)|^2e^{-2\operatorname{Im}[h(\omega)-h(\omega_0)]z}, \tag{4.6}$$

and, in particular, the effective carrier frequency ω_m, corresponding to the maximum of $S(\omega,z)$ changes, namely, ω_m shifts from ω_0 towards larger values of the exponential, that is, towards a smaller attenuation coefficient. Inasmuch as ω_m determines the carrier frequency, a change takes place in the pulse velocity and, as can be readily seen, its attenuation decreases (in comparison with a monochromatic wave of frequency ω_0). The start of this process can be traced with formula (4.3). Let, for example, $b=0$, and then at $z=0$ the pulse with carrier frequency ω_0 is modulated only in amplitude. At $z>0$ the carrier frequency is equal to

$$\omega_m=\omega_0-a\eta z,\quad \eta=\operatorname{Im}h'(\omega_0), \tag{4.7}$$

that is, it is smaller than ω_0 at $\eta>0$ and larger at $\eta<0$, in accordance with the statements made above. The

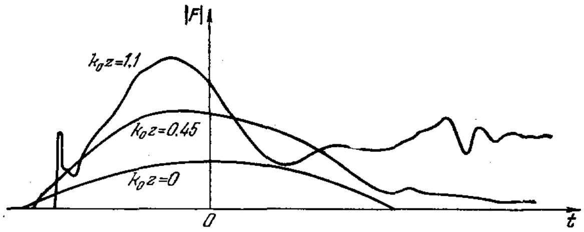

FIG. 4. Propagation of cosinusoidal pulse with negative velocity v_1: $F(0,t)=\cos(2T/\pi t)$ at $|t|\leqslant T$, $F(0,t)=0$ at $|t|\geqslant T$, dispersion law (1.3); $\omega_0=0.2\ \omega p$, $\nu=0.18\ \omega p$, $v_1=-1.7\ c$, $\omega_0 T=2\pi$.

subsequent shift of the carrier is no longer linear as in formula (4.7); it can cause the high-frequency pulse to be transformed into a pulse without a carrier[21]—this will be the case if the minimum attenuation corresponds to the frequency $\omega=0$.

Of course, in the case of an appreciable displacement of the carrier, the pulse itself, propagating with attenuation, is strongly attenuated and greatly deformed. However, following an attenuation of 40–70 dB the pulse can be received and somehow utilized (to be sure, this is difficult at too large a deformation, see below).

As a result of the change of the form of the spectrum with distance, formulas (3.5) and (3.19) actually give us that part of the pulse which has a frequency ω_0, but *this part is principal only for moderate z, at which the shape of the spectrum still differs little from the initial form* $S(\omega,0)=|A(\omega)|^2$, and, in particular, the effective carrier differs little from ω_0 (say $|\omega_m-\omega_0|\lesssim\Delta\omega$). From the practical point of view, the only important values of z are those at which the pulse is similar to the initial one and is detected by a receiver operating in the frequency band of the initial pulse. As we observe (with the aid of a broadband receiver) our pulse at large distances z, where the form of its spectrum is entirely different, we encounter the following phenomenon: what we receive is not at all similar to what was transmitted, since the received pulse is determined by those properties of the radiation pulse (its spectrum at $|\omega-\omega_0|\gg\Delta\omega$) which are not controlled in the course of the radiation.

The change in the form of the spectrum and the deformation of the pulse occur with increasing z slower the smaller $\Delta\omega$ (this is seen, in particular, from formula (4.7), where $a\sim\Delta\omega^2$), that is, the slower the change in the complex envelope $F(0,t)$ with time (see Chap. 3). However, even beyond the limit of the band $|\omega-\omega_0|\lesssim\Delta\omega$, there is always present a certain power that initiates the sideband parts of the field (Chap. 3). If the attenuation depends on the frequency, then these sideband parts can become predominant. Using our epigraph, we can state that the "elephant" can become smaller than the "fly," and in amplifying systems (Chap. 6) the "fly" can become larger than the "elephant." We note that variability of the pulse velocity is due to the change of the effective carrier frequency: each elementary spectral interval ω, $\omega+d\omega$ is characterized by a different velocity, and at a given z there is realized a velocity (the ratio of z to the propagation time) which corresponds to the dominant frequency ω_m.

In connection with the statements made above, it is natural to raise the question: *what is the signal and what is its velocity*? So far we did not use the word signal (the transporter of the information!); this word is not appropriate for the pulse (3.18), for it has neither beginning nor end, in analogy to an infinite harmonic wave. It is not surprising that the velocities v_1 and v_2 for such a pulse can be superluminal or negative, in analogy with the phase velocity $\omega_0/\mathrm{Re}\,h(\omega_0)$ of a harmonic wave. In general, one should not assume that a pulse with a holomorphic (analytic) complex envelope $F(0,t)$ is not a signal in the proper sense of this word, since such a pulse is infinite in time and lacks the element of suddenness: if we observe it in a small interval Δt, then everything is uniquely defined. The signal, for example, can be a rectangular pulse or else a pulse whose envelope is defined by formula (3.18) at $-T<t<T$ and is equal to zero at $t<-T$ and $t>T$; no matter how large T and how small $e^{-(1/2)aT^2}$, such a pulse is not a holomorphic function of t and up to the instant $t=-T$ one cannot predict whether a pulse will exist or not. Thus, expression (3.3) with a group velocity (3.4) *has no meaning* for a *signal*, and of fundamental significance for it is the velocity limit c with which the front of the pulse moves (or any sudden change of the pulse, see the end of Chap. 2). The remaining parts of the pulse move behind the front, each with its own velocity. The principal part of the pulse is calculated from (3.5) and (3.7), which determine the wave field in the vicinity of the principal complex ray (4.1). Over a finite interval of distances, certain characteristic attributes, for example the maximum of the envelope $|F(z,t)|$, can move with superluminal or even negative velocity, but this does not contradict relativistic causality or common sense.[1)] Thus, if the maximum velocity is $v_1>c$, this means that the maximum approaches the front (which is similar to the process of contraction of the pulse), but an unlimited approach is impossible, since the spectrum (4.6) changes and with it the velocity v_1. It is more difficult to visualize the case $v_1<0$, and therefore Fig. 4 shows results obtained by calculating[21c] the integral (1.9) for a pulse with a cosine envelope: with increasing z, the front arrives later and the maximum earlier, but the pulse collapses completely already at small z.

We note that it is meaningful to speak of holomorphy or nonholomorphy of the envelope only if the noise in the system and in the receiver is neglected; allowance for the noise, however, is a complicated statistical problem that calls for a separate analysis.

In the Introduction, the propagation of the pulses was

[1)]It must be stated that transmission of information with phase velocity is impossible principally because this is the velocity of the internal motion in a pulse whose front moves with velocity c, and the principal part moves with velocity $v_g<c$ (if there is no attenuation). As to the velocity of the maximum in the presence of attenuation, this is also the velocity of internal motion, which furthermore is of more or less short duration.

investigated within the framework of the ray kinematics, which was then supplemented by complex rays. The subsequent analysis was within the framework of the wave kinematics, and a distinction must be made between the rigorous wave treatment (for example formulas (1.9), (1.12), (1.15), and (2.1)–(2.5), on the one hand, and the approximate relations (3.1)–(3.8) for the principal part of the pulse, analogous to the paraxial wave optics, on the other. A third approach is also possible, which can be naturally called *energetic kinematics*; it makes it possible to compare a more exact physical representation of pulse propagation with amplitude modulation.

5. ENERGETIC KINEMATICS

Brillouin[26] was the first to point out the connection between the propagation of electromagnetic pulses and the rate of energy transport at the carrier frequency

$$v_e = \frac{S_z}{W}|. \tag{5.1}$$

As shown above (see the Introduction), in the ray approximation this velocity is the velocity of the pulses (in the case of pure amplitude modulation and a sufficiently narrow bandwidth $\Delta\omega$), and must therefore coincide with the group velocity (3.4). A direct proof of the equality $v_g = v_e$ for non-absorbing media was given by Rytov[27] (see also[28]). It is interesting, that in hydrodynamics the equality of the velocity of a group of waves to the energy-transport velocity was noticed already in 1877.[29]

On the other hand, if we consider systems with losses, then it turns out that for real media or lines we always obtain[26,28] v_e in the range $0 \leq v_e \leq c$; it is therefore natural to assume that the "true velocity of motion of the pulse is equal to v_e." The present author has previously adhered[28] to this point of view, which seems to be reinforced by the fact that the pulse is a bundle of energy, the energy is equivalent to mass, and mass cannot move with superluminal velocity. This point of view, however, is incorrect for the following reasons:

a) It was shown in Chap. 4 that a pulse velocity v_1 (3.18) can be arbitrary ($v_1 > c$ and $v_1 < 0$); therefore $v_1 \neq v_e$, and the relation between v_1 and v_e is more complicated (see below). In particular, in active systems a case is possible when $v_e < 0$ for a pulse propagating in the positive direction (Chap. 6). All this shows that in presence of attenuation or amplification it is impossible to compare a pulse to a material body. If that attenuation has a pure reactive character and $h(\omega)$ is pure imaginary at $\omega \approx \omega_0$, then at $b = 0$, in accordance with (4.4), we obtain $v_1 = 1/\xi = \infty$, and in accordance with (5.1) we get $v_e = 0$, since there is no energy flux.

b) To calculate the energy density W in the presence of losses it is not sufficient to know the macroscopic characteristics of the medium, for example the complex dielectric constant $\varepsilon(\omega)$ and permeability $\mu(\omega)$, but it is necessary to have some information on its microscopic properties, namely, it is necessary to know the "equation of motion" for it[30] or to have additional experimental information. Thus, different values of W and v_e can be obtained for a given dispersion law $h(\omega)$ that defines uniquely the propagation of the pulse.

c) In general, the question of pulse velocity cannot be answered with a mere definition. Thus, a velocity was recently introduced[30] in accordance with formula (5.1), where W denotes the *vacuum* part of the electromagnetic energy: this velocity has a definite meaning, but has no bearing whatever on the propagation of pulses.[2)]

Let us explain the statements made in Sec. (b). The energy density W, the energy flux density S_z (the component of the Poynting vector), and the loss power density P are connected by the relation

$$\frac{\partial W}{\partial t} + \frac{\partial S_z}{\partial z} + P = 0. \tag{5.2}$$

If W, S_z, and P are taken to mean quantities averaged over the time (for example, over the period $2\pi/\omega_0$ of the carrier frequency), then for passive media we have $W \geq 0$, $S_z \geq 0$, and $P \geq 0$, while for active media W, S_z, and P can be negative. On the other hand, if we consider not a plane wave in a homogeneous medium, but a wave in a transmission line, then the relation (5.2) is valid for the quantities W, S_z, and P averaged also over the cross section of the line.

For an ideal medium, $P \equiv 0$ and the energy density W is calculated from the macroscopic characteristics without additional information. In electric circuit theory[31,32] this corresponds to the fact that the energy of a reactive two-terminal network having at a time dependence $e^{-i\omega t}$ an impedance $Z(\omega) = -iX(\omega)$, is defined uniquely, mainly, it is proportional to the derivative $X'(\omega)$. This yields immediately expressions for the electric and magnetic energy in a lossless medium, assuming this medium to be filled with a capacitance C or an inductance L, and obtaining in the former case $X(\omega) = -1/\omega\varepsilon(\omega)C$, and in the latter $X(\omega) = \omega\mu(\omega)L$.

The formula for the energy is easiest to derive by considering oscillations with frequencies $\omega \pm \Delta\omega$, when the current and the voltage of the two-terminal network are given by

$$I(t) = \operatorname{Re}(I_0 e^{-i\omega t}\cos\Delta\omega t) = \operatorname{Re}\left[\frac{I_0}{2}\left(e^{-i(\omega+\Delta\omega)t} + e^{-i(\omega-\Delta\omega)t}\right)\right], \qquad I_0 = \mathrm{const},$$
$$U(t) = \operatorname{Re}\left\{-i\frac{I_0}{2}\left[X(\omega+\Delta\omega)e^{-i(\omega+\Delta\omega)t} + X(\omega-\Delta\omega)e^{-i(\omega-\Delta\omega)t}\right]\right\}$$
$$= \operatorname{Im}(X(\omega) I_0 e^{-i\omega t}\cos\Delta\omega t) - \Delta\omega \operatorname{Re}(X'(\omega) I_0 e^{-i\omega t}\sin\Delta\omega t) \quad \text{at} \quad \Delta\omega \to 0.$$

The energy of the two-terminal network at the instant $t = 0$ is obviously

$$W_0 = -\lim_{\Delta\omega \to 0} \int_0^{\pi/2\Delta\omega} I(t)\, U(t)\, dt,$$

that is, the energy delivered at $0 < t < \pi/2\Delta\omega$ to an external load (at $t \sim \pi/2\Delta\omega$ and $\Delta\omega \to 0$ we have practically $I(t) = 0$ and $U(t) = 0$). Assuming $I_0 = |I_0| e^{i\varphi}$ and averaging also over the phase φ, we obtain the sought formula

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2) The same can be said concerning the definitions given in earlier papers[3,4] (see also the review[6]).

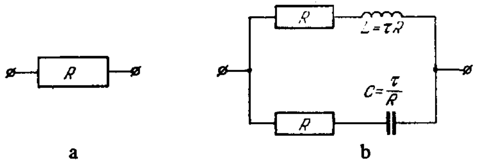

FIG. 5. Two-terminal networks having one and the same impedance $Z(\omega) = R = \text{const}$.

$$W = \frac{1}{4} X'(\omega) |I_0|^2.$$

The same result is obtained by considering a slightly damped harmonic oscillation, that is, an oscillation with a complex frequency $\omega - i\gamma$.

In the presence of losses it is difficult to distinguish between the energy that still preserves its electromagnetic nature and the energy contained in different forms, that is, to distinguish between the terms $\partial W/\partial t$ and P in (5.2). In circuit theory this corresponds to the fact that the energy of an arbitrary two-terminal network with impedance $Z(\omega) = R(\omega) - iX(\omega)$ is not determined by the functions $R(\omega)$ and $X(\omega)$: to calculate this energy it is necessary to know the internal structure of the two-terminal network. The foregoing is illustrated by the classical example[33] of a two-terminal network with $Z(\omega) = R = \text{const}$; this can be either a pure resistor R (Fig. 5a) with $W = 0$, or a more complicated network (Fig. 5b), in which the energy is stored in elements C and L and dissipated after the terminal voltage is turned off in the elements R within a time on the order of τ; the longer τ, the higher the energy stored in the network. Formulas that determine the minimum value of W and the maximum value of v_e have been derived in the literature[17,26,28,30] for the dispersion laws (1.1)–(1.7).

From Fig. 5 we can draw two additional conclusions.[33] First, it is seen that the Joule–Lenz law, according to which the heat power produced in the resistor R by the current I is equal to RI^2 is not a trivial consequence of Ohm's law, but is a meaningful experimental result, since Ohm's law admits also of the circuit of Fig. 5b, in which this power is not equal to $RI.^2$ Second, the elements R in Fig. 5b can be replaced by similar complicated circuits, and in these circuits it is possible to carry out in turn a similar replacement, that is, we obtain in this manner an infinite purely reactive system having dissipative properties—just as in kinetic theory of gases a conservative system has dissipative properties.

We turn to the pulse propagation. We formulate the problem in the same manner as before: at a certain $z > 0$ we observe a pulse transmitted from the point $z = 0$. The pulse brings with it an energy flux S_z and releases in the vicinity of the point z a power proportional to P and having a pulsed character. If we introduce t_0 and t_1 with the aid of the relations[28,34]

$$t_0 = \frac{\int_{-\infty}^{\infty} tS_z\,dt}{\int_{-\infty}^{\infty} S_z\,dt}, \quad t_1 = \frac{\int_{-\infty}^{\infty} tP\,dt}{\int_{-\infty}^{\infty} P\,dt}, \tag{5.3}$$

then t_0 will determine the instant at which the center of the pulsed energy flux arrives at the point z, while t_1 determines the analogous instant corresponding to the loss power. From (5.2) follows the identity

$$\frac{dt_0}{dz} = \frac{1 + \gamma(t_0 - t_1)}{v_e}, \tag{5.4}$$

where

$$v_e = \frac{\int_{-\infty}^{\infty} S_z\,dt}{\int_{-\infty}^{\infty} W\,dt}, \quad \gamma = \frac{\int_{-\infty}^{\infty} P\,dt}{\int_{-\infty}^{\infty} W\,dt}. \tag{5.5}$$

At sufficiently slow amplitude modulation, v_e coincides with (5.1), and γ is equal to the ratio P/W at $\omega = \omega_0$.

Formula (5.4) shows once more that in the absence of damping v_e coincides with the group velocity (3.4). Indeed, in this case $P = 0$ and $\gamma = 0$, and consequently $dt_0/dz = 1/v_e$, in accordance with Chap. 3 we have in the near zone $dt_0/dz = 1/v_g$, therefore $v_e = v_g$. But formula (5.4) with $\gamma = 0$ tells us something more than (3.4): in the intermediate and far zones, where the pulse is deformed, its center continues to move with the same velocity as in the near zone.

We proceed now to propagation with damping due to losses, and consider a pulse determined in the near zone by formula (4.3) at $b = 0$. We obviously have $dt_0/dz = 1/v_1$, and therefore

$$v_1 = \frac{v_e}{1 + \gamma(t_0 - t_1)}, \tag{5.6}$$

and we see that at $\gamma \neq 0$ the energy center of the pulse can move with a velocity different from v_e and dependent on the shape of the pulse (since it determines the difference $t_0 - t_1$): $v_1 > v_e$ at $0 < \gamma(t_1 - t_0) < 1$ and $v_1 < 0$ at $\gamma(t_1 - t_0) > 1$. In an absorbing medium we have $\gamma > 0$ and both inequalities mean that the energy-flux pulse leads the loss-power pulse (Fig. 6a); the losses "eat away" predominantly the trailing part of the pulse, which either increases the velocity of its maximum, pushing the pulse, as it were, or else (at larger values of $\gamma(t_1 - t_0)$) again cause the pulse to move in a negative direction.

A similar process occurs[35,36] when a light pulse propagates in an amplifying (laser) medium: the leading part of the high-power pulse passes through the medium with the largest population inversion and is amplified, whereas the trailing part passes through a depleted medium and is therefore hardly amplified ($\gamma < 0$, $t_1 < t_0$; see Fig. 6b), consequently the maximum of the pulse

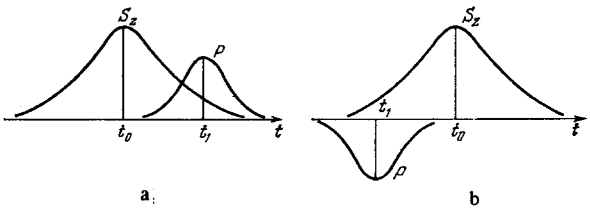

FIG. 6. Energy-flux pulse and loss-power pulse at $v_1 > v_e$ in a passive (absorbing) medium (a) and in an active medium (b).

moves with superluminal velocity. Formula (5.4) remains in force for this superluminal process: superluminal propagation of a pulse in an absorbing linear medium and in an amplifying nonlinear medium is due to similar causes.

The energy interpretation of the superluminal propagation of pulses shows once more that it is due the deformation of the pulses and cannot be used to transmit a signal with a velocity exceeding the velocity of light in vacuum, just as negative values of v_t cannot be used to get around the causality principle.

6. LINEAR RELATIONS FOR ACTIVE SYSTEMS

It was shown in Chaps. 4 and 5 that the presence of damping greatly complicates the pulse-propagation process. Even more complicated phenomena occur when we consider active (unstable, amplifying) systems or media. For such media $\mathrm{Im}h(\omega)<0$ at certain real ω or else $h(\omega)$ is not a holomorphic function of ω at $\mathrm{Im}\,\omega \geqslant 0$, as for example $h(\omega)$ in accordance with formula (1.6).

However, the principle of relativistic causality (1.11) must be satisfied also for amplifying systems, by virtue of which the Fourier integral (1.9) must be modified in the following manner:

$$f(z,\ t)=\int\limits_{i\sigma-\infty}^{i\sigma+\infty} A(\omega)\, e^{i[h(\omega)z-\omega t]}\, d\omega, \tag{6.1}$$

where the positive parameter σ is chosen such that at $\mathrm{Im}\,\omega \geqslant \sigma$ the function $h(\omega)$ is holomorphic, and in particular, it is necessary to choose $\sigma>\beta$ for the dispersion law (1.6). Since we are interested in the principal part of the pulse, we can deform the integration contour and make it pass through the point ω_0 on the real axis. In the vicinity of this point we can again apply formulas (3.1)–(3.4) and we obtain for the dispersion law (1.6) the group velocity

$$v_g=c\sqrt{1+\frac{\beta^2}{\omega_0^2}}>c, \tag{6.2}$$

whereas the phase velocity is smaller than c.

This result was noted by Ehrenfest as a paradox, but it can be properly explained on the basis of Chap. 4. Of course, the principal part corresponding to the frequency ω_0 moves with superluminal velocity (6.2), but for a pulse with a front—and only such a front can be a signal—it cannot overtake the front and, in addition, it gradually vanishes, being lost in the exponentially growing sideband part which is obtained from formula (2.2) at $\alpha=0$.

In the theory of active systems it is sometimes difficult to distinguish between damped waves and amplified ones. When as a result of a theoretical analysis one obtains a wave for which $\mathrm{Im}h(\omega_0)<0$, then the question arises: does this correspond to amplification of a wave traveling in the direction of increasing z, or to the damping of a wave traveling in a direction of decreasing z? In passive systems there are no such difficulties, at $\mathrm{Im}h(\omega_0)>0$ we always have a wave propagating in a positive direction, at $\mathrm{Im}h(\omega)<0$ it propagates in a negative direction, and at $\mathrm{Im}h(\omega_0)=0$ everything is determined by the sign of the group velocity (3.4) or the limiting velocity (1.8). For active systems there were proposed[37–39] a number of criteria which make it possible to answer this question, but the simplest and the physically most natural answer is connected with consideration of the integral (6.1) for a high-frequency pulse with carrier frequency ω_0. This procedure seems all the more natural, since a monochromatic wave is an abstraction, and in practice we always deal with pulses that are more or less long. We shall consider below concrete examples in which the determination of the propagation direction is not so simple.

For active systems, just as for passive systems, the wave propagation is determined by the function $h(\omega)$, and it is meaningless to develop a theory for an arbitrary function $h(\omega)$, but one must bear in mind real media and lines as well as those limitations that are naturally superimposed on the dispersion $h(\omega)$. If we have a plane wave in a homogeneous medium with a complex dielectric constant $\varepsilon(\omega)$, then at $\mu=1$ the function $h(\omega)$ is given by

$$h(\omega)=\frac{\omega}{c}\sqrt{\varepsilon(\omega)}, \tag{6.3}$$

and for a passive medium $\varepsilon(\omega)$ is a function that is holomorphic in the upper half-plane $\mathrm{Im}\,\omega \geqslant 0$ and tends to unity as $|\omega|\to\infty$. It is easily seen that for linear *active* media this property of the function $\varepsilon(\omega)$ remains in force if the upper half-plane is taken to mean the half-plane $\mathrm{Im}\,\omega \geqslant \sigma$, where σ is a positive parameter that enters in the integral (6.1).

In fact, let us denote by $A(\omega)$ the complex spectral amplitude of the electric field at a given point, and by $E(t)$ and $D(t)$ the electric field itself and the corresponding electric induction[3)] at the same point (we do not distinguish between the displacement currents and the conduction current, which we include in D). Then, representing $E(t)$ and $D(t)$ in the form

$$E(t)=\int\limits_{i\sigma-\infty}^{i\sigma+\infty} A(\omega)\, e^{-i\omega t}\, d\omega,\quad D(t)=\int\limits_{i\sigma-\infty}^{i\sigma+\infty} \varepsilon(\omega)\, A(\omega)\, e^{-i\omega t}\, d\omega, \tag{6.4}$$

we cannot use the ordinary Fourier integrals but must, as in (6.1), shift the integration contour, since the usual Fourier integrals taken on the real ω axis can represent only functions $E(t)$ and $D(t)$ that vanish as $t\to\pm\infty$, while the modified Fourier integrals (6.4), as can be readily shown with the aid of the change of variable $\omega=\omega'+i\sigma$, require only the vanishing of the products $E(t)\,e^{-\sigma t}$ and $D(t)\,e^{-\sigma t}$ as $t\to\pm\infty$. In the active medium, the perturbations can increase in time after the cessation of the external actions, therefore by choosing σ sufficiently large we can write down the expressions in (6.4); the inverse of the first of these expressions is

3)By E and D we mean here one of the Cartesian components of the intensity and of the induction.

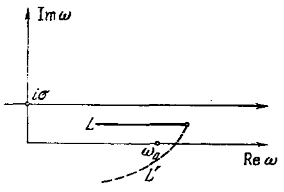

FIG. 7. Calculation of the principal part in the case when the point ω_0 lies under the cut L.

$$A(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}E(t)\,e^{i\omega t}\,dt,\quad \mathrm{Im}\,\omega\geqslant\sigma. \tag{6.5}$$

The causality principle calls for satisfaction of the condition

$$\text{if } E(t)=0 \quad\text{at}\quad t<0,\ \text{then } D(t)=0 \quad\text{at}\quad t<0, \tag{6.6}$$

and then expression (6.5) leads to holomorphy of the function $A(\omega)$ in the half-plane $\mathrm{Im}\,\omega\geqslant\sigma$, while the condition (6.6) is satisfied only if $\varepsilon(\omega)$ is holomorphic in this half-plane.

The dispersion laws (1.4) and (1.5) correspond to the dielectric constant

$$\varepsilon(\omega)=\bar{\varepsilon}-\frac{\omega_p^2}{\omega^2+i\nu\omega-\omega_r^2}, \tag{6.7}$$

in which the term $\bar{\varepsilon}$ at $\omega\approx\pm\omega_r$ can be regarded as a constant positive number, and all the other resonant frequencies are far enough from ω_r. Population inversion leads[40] to the negative value $\omega_p^2=-\beta^2$ and to a function $\varepsilon(\omega)$ that is holomorphic in the half-plane $\mathrm{Im}\,\omega\geqslant 0$; only in the limiting case $\nu=\omega_r=0$ do we arrive at the dispersion law (1.6), for which holomorphy is obtained only at $\mathrm{Im}\,\omega>\beta$. From the point of view of condition (6.6), it will be possible to have an active medium with dielectric constant (6.7), in which $\omega_p^2>0$, but then $\nu<0$. Such media are unknown, and probably do not exist, but the dispersion laws (1.3) and (1.4) at $\omega_p^2>0$ and $\nu<0$ correspond approximately[41] to a traveling-wave parametric quantum amplifier (maser) consisting of a transmission line with a rejection band $0<\omega<\omega_p$ or $\omega_r<\omega<\sqrt{\omega_r^2+\omega_p^2}$, and active elements in which the negative losses exceed the positive losses in the line. At $\omega_p^2>0$ and $\nu<0$ the function $h(\omega)$ is holomorphic only at $\mathrm{Im}\,\omega\geqslant\sigma>|\nu|/2$ and the cut of the function $h(\omega)$ lies *above* the real axis.

On the other hand, in the first case, that is, at $\omega_p^2<0$ and $\nu>0$, the cut of the function $h(\omega)$ lies below the real axis, the integral (6.1) is taken along the real axis, and the principal part of the pulse is amplified. In the second case the integral (6.1) is taken along a contour located above the cut and joining the points

$$\omega_1=\omega_r-i\frac{\nu}{2}\quad\text{and}\quad\omega_2=\omega_r+\frac{\omega_p^2}{2\omega_r}-i\frac{\nu}{2}\qquad(\nu<0,\ \omega_p^2\ll\omega_r^2),$$

and amplification will occur only if ω_0 does *not* lie *under* the cut: it is then possible to draw the integration contour through the point ω_0, to calculate the principal part, and to verify that for this part $\mathrm{Im}\,h(\omega_0)>0$. On the other hand, if the point ω_0 lies under the cut (Fig. 7), the situation is different: we can deform the initial integration contour and the contour passing through the point ω_0 only by moving the cut L to the position L'. Therefore in formula (3.2) for the principal part we have $\mathrm{Im}\,h(\omega_0)>0$, inasmuch as between the curves L and L' the value of $h(\omega_0)$ is taken with a different sign than at the point ω_0 under the cut L. In other words, what takes place at $\nu<0$ is the same as at $\nu=0$, when the wave attenuates, if its frequency falls in the rejection band, and the active elements cannot "revive" it and convert it into a growing wave.

Thus, the distribution of the field in the system will be determined by the damped exponential both at $\nu\geqslant 0$ and at $\nu<0$. The sign of ν determines only the sign of the energy flux in the direction of the z axis: at $\nu>0$ this flux is positive and at $\nu<0$ negative, that is, the energy is supplied by the source of the field to the system at $\nu>0$ and is delivered from the active system to the field source at $\nu<0$, so that $S_z<0$ and in accordance with (5.1) we have $v_e<0$ inasmuch as $W>0$.

Nonetheless a pulse propagates in such a system, as expected, in a positive direction ($v_1>c$) although it loses rapidly its shape because of the enhancement of the sideband parts and attenuation of the principal part. According to (5.4), the reason is that the negative-loss pulse travels very far in advance of the pulse of the negative (to the field source) energy flux (see Fig. 8).

This example shows that the sign of v_e or S_z (as the sign of v_1 before) still does not determine the wave propagation direction. There exist two attributes that make it possible to distinguish the positive direction from the negative one. The first is absolute: the way it propagates in a positive direction at a positive limiting velocity (1.8), that is, the integral (6.1) can contain the function $h(\omega)$ only with a positive limiting velocity. Sometimes (see Chap. 7) it is impossible to go to the limit $|\omega|\to\infty$, and it becomes necessary to use a second and relative attribute: if at some finite value of ω the given value of $h(\omega)$ corresponds to a wave with positive direction, then analytic continuation yields a similar wave.

7. AMPLICATION IN TRANSMISSION OR AMPLIFICATION IN REFLECTION?

We consider the following problem[42]: a plane wave is incident at an angle φ on the interface between two media with dielectric constants ε_0 (at $z<0$) and ε (at $z>0$); the electric field of the wave is parallel to the interface (Fig. 9). The reflection coefficient of such a wave is

$$R=\frac{\sqrt{\varepsilon_0}\cos\varphi-\rho}{\sqrt{\varepsilon_0}\cos\varphi+\rho},\qquad \rho=\sqrt{\varepsilon-\varepsilon_0\sin^2\varphi}. \tag{7.1}$$

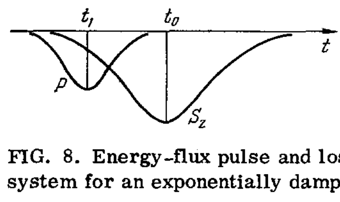

FIG. 8. Energy-flux pulse and loss-power pulse in an active system for an exponentially damped field.

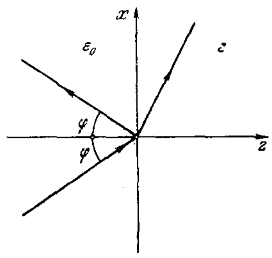

FIG. 9. Incidence of a plane wave on a plane boundary between a passive ($z<0$) medium and an active ($z>0$) medium.

We assume that ε_0 is a real positive number with $\varepsilon_0 > \mathrm{Re}\,\varepsilon$. If $\mathrm{Im}\,\varepsilon > 0$, then the sign of ρ is chosen such that the quantity

$$h=\frac{\omega}{c}\rho=\frac{\omega}{c}\sqrt{\varepsilon-\varepsilon_0\sin^2\varphi}, \tag{7.2}$$

which determines the z-dependence of the wave passing into the second medium ($z>0$) had a positive imaginary part. On the other hand, if $\mathrm{Im}\,\varepsilon<0$, that is, the second medium is active, then, generally speaking, it is necessary to choose $\mathrm{Im}\,\rho<0$ and $\mathrm{Im}\,h<0$, since the transmitted wave should become amplified in the active medium. However, at

$$\sin^2\varphi > \frac{\mathrm{Re}\,\varepsilon}{\varepsilon_0} \tag{7.3}$$

doubts arise. On the one hand, it can be assumed that small negative losses (usually $-\mathrm{Im}\,\varepsilon \ll 1$) cannot alter strongly the damped field distribution in the second medium, and one can choose ρ in this case such that $\mathrm{Im}\,\rho>0$. We then obtain

$$|R|^2=\frac{(\sqrt{\varepsilon_0}\cos\varphi-\mathrm{Re}\,\rho)^2+(\mathrm{Im}\,\rho)^2}{(\sqrt{\varepsilon_0}\cos\varphi+\mathrm{Re}\,\rho)^2+(\mathrm{Im}\,\rho)^2} =1-\frac{4\sqrt{\varepsilon_0}\cos\varphi\,\mathrm{Re}\,\rho}{(\sqrt{\varepsilon_0}\cos\varphi+\mathrm{Re}\,\rho)^2+(\mathrm{Im}\,\rho)^2}, \tag{7.4}$$

and since

$$\rho=\sqrt{\mathrm{Re}\,\varepsilon-\varepsilon_0\sin^2\varphi+i\,\mathrm{Im}\,\varepsilon} \approx i\sqrt{\varepsilon_0\sin^2\varphi-\mathrm{Re}\,\varepsilon}+\frac{\mathrm{Im}\,\varepsilon}{\sqrt{\varepsilon_0\sin^2\varphi-\mathrm{Re}\,\varepsilon}}, \tag{7.5}$$

it follows that $\mathrm{Re}\,\rho<0$ and $|R|^2>1$, that is, if the transmission is not accompanied by amplification, then there will be amplification upon reflection, due to the energy flux from the active medium towards the interface (cf. Fig. 8).

On the other hand, the preceding approach leads to a jumplike change of ρ and $|R|^2$ on going to the angle $\varphi_0=\arcsin\sqrt{\mathrm{Re}\,\varepsilon/\varepsilon 0}$. This jump calls for a verification.

It is clear from Chap. 6 above that the choice of a sign of ρ or, equivalently, the choice of the sign of h, cannot be made correctly by confining oneself only to monochromatic oscillations, and it is necessary to take a pulse and to ascertain whether it becomes amplified or attenuated, depending on the behavior of the function (7.2) in the complex ω plane. For the active media used in quantum electronics, expression (6.7) is usually simplified to the form[40]

$$\varepsilon(\omega)=\bar{\varepsilon}+\frac{\beta^2}{2\omega_r\left(\omega-\omega_r+i\frac{\nu}{2}\right)} \tag{7.6}$$

and then

$$\varepsilon(\omega)-\varepsilon_0\sin^2\varphi=\hat{\varepsilon}+\frac{\beta^2}{2\omega_r[\omega-\omega_r+i(\nu/2)]}\qquad(\hat{\varepsilon}=\bar{\varepsilon}-\varepsilon_0\sin^2\varphi), \tag{7.7}$$

where $\hat{\varepsilon}$ can assume either small values, comparable with the second term of the right-hand side of (7.7), or else finite (positive or negative) values. So long as $\hat{\varepsilon}>0$ we have $\mathrm{Im}\,h(\omega)<0$, and the wave in the active medium is amplified. The sign of $\hat{\varepsilon}$ changes on going through the critical angle $\varphi_1=\arcsin\sqrt{\bar{\varepsilon}/\varepsilon_0}$, which differs slightly from φ_0 (incidentally, $\varphi_1=\varphi_0$ at $\omega=\omega_r$).

Thus, on going through the angle φ_1, the properties of the function $h(\omega)$ change jumpwise: it is to this angle that the jumps of ρ and $|R|^2$ must correspond, and not to the angle φ_0. The correctness of the choice of the value $\mathrm{Im}\,h(\omega)>0$ at $\varphi>\varphi_1$ can be verified (see the end of Chap. 3) by taking values of $|\omega-\omega_r|$ such that $\varepsilon(\omega)\approx\bar{\varepsilon}$: we then have total internal reflection from the boundary of two ordinary dielectrics, and a damped field distribution should be produced in the second medium.

It is sometimes stated[42,43a] that the sign of ρ can be determined by choosing the active medium in the form of a layer of thickness d, solving the problem for the layer, and then taking the limit as $d\to\infty$. This procedure is illusory, since at $\mathrm{Im}\,\rho<0$ the waves produced by successive reflections from the boundaries of the layer form, at sufficiently large d, a diverging series and there is simply no solution. Nonetheless, the main results of[42] are correct (there are only inaccuracies with respect to the angle φ_0 and the value of $|R|_{\max}$), but[43a] is in error. In[41] there is an incorrect conclusion (with which the present writer unfortunately agreed) that in the case shown in Fig. 7 the value $\mathrm{Im}\,h(\omega_0)<0$ corresponds to a wave traveling in a positive direction. This conclusion was based on positiveness of S_z and v_e, and the error of this argument was demonstrated by Sturrock[44]; in fact, as shown in Chap. 6, the value $\mathrm{Im}\,h(\omega_0)<0$ for a wave propagating in the position direction is not realized, and for a final solution of the equation it is necessary to consider a pulse, that is, a transient process.

The maximum gain in the direction of the z axis, according to formulas (7.2) and (7.7), equal to

$$-\mathrm{Im}\,h=\frac{\omega_r}{c}\sqrt{\frac{\beta^2}{\omega_r\nu}}=\frac{\beta}{c}\sqrt{\frac{\omega_r}{\nu}}, \tag{7.8}$$

is realized at $\varphi=\varphi_1-0$ and $\omega=\omega_r$. It greatly exceeds the gain for normal incidence: the reason is that at $\varphi=\varphi_1-0$ the transmitted wave propagates almost perpendicular to the z axis, that is, it moves along the z axis and covers a much larger path in the active medium. At $\varphi=\varphi_1+0$ and $\omega=\omega_r$ we have $\mathrm{Im}\,h>0$, and the power reflection coefficient, equal to

$$|R|^2=\frac{1+2\xi+2\xi^2}{1-2\xi+2\xi^2},\qquad \xi=\frac{\beta}{\sqrt{2\omega_r\nu(\varepsilon_0-\bar{\varepsilon})}}, \tag{7.9}$$

assumes a maximum value[45]

$$|R|^2_{\max} = (\sqrt{2}+1)^2 = 5.82 \quad \text{at} \quad \xi = \sqrt{2}. \qquad (7.10)$$

In the experiments,[45,46] however, much larger reflection coefficients are observed. In view of the advanced opinion[43b] that this is due to the finite width of the light beam, we shall analyze this question in greater detail. Reflection of a monochromatic (time dependence $e^{-i\omega t}$) wave beam is calculated in the same manner as the propagation of pulses, except that in place of the integral (1.9) over the monochromatic waves, the solution of (3.11) with wave number $k_0 = (\omega/c)\sqrt{\varepsilon_0}$ is represented in the form of an integral over plane waves. If the wave beam is narrow enough and is incident on the interface at an angle φ, then the most important role in the integrals for the incident wave beam $u_i(z, x)$ and the reflected wave beam $u_r(z, x)$ is played by plane waves whose incidence angles are close to φ, and these integrals can be calculated in the same manner as at the beginning of Chap. 3. The details are given in the work of Brekhovskikh.[47] If we denote the reflection coefficient (7.1) by $R(\varphi) = e^{-i\Omega(\varphi)}$, then we obtain the relation

$$u_r(0, x) = R(\varphi)\, u_i(0, x-\Delta), \qquad \Delta = \frac{\Omega'(\varphi)}{k_0 \cos\varphi}, \qquad (7.11)$$

which is analogous to expression (3.3) and shows that a beam incident on the interface at an angle φ is reflected with the same reflection coefficient $R(\varphi)$ as a plane wave, and experiences a shift Δ upon reflection. In the usual case of total internal reflection $|R(\varphi)| = 1$ and the shift Δ is real,[47] but at $R(\varphi) \neq 1$ the quantity Δ is complex. If, for example,

$$u_i(0, x) = e^{-(1/2)(a+ib)x^2 + ik x \sin\varphi} \text{ and } \Delta = \xi + i\eta, \qquad (7.12)$$

then this means (see the start of Chap. 4) a shift of the amplitude center of the beam by a distance Δ_1 along the x axis, and a shift of the phase center of the beam by Δ_2, where

$$\Delta_1 = \frac{a\xi - b\eta}{a} \text{ and } \Delta_2 = \frac{b\xi + a\eta}{b}, \qquad (7.13)$$

and, in addition, at $x = \Delta_1$ we have

$$|u_r(0, \Delta_1 - \Delta)| = K\,|u_i(0, 0)|, \qquad K = \exp\left(\frac{a^2+b^2}{2a}\eta^2\right), \qquad (7.14)$$

that is, the complex shift Δ leads to an increase of the amplitude of the beam by a factor K. Since at $\varphi \approx \varphi_1$ (and $\varphi > \varphi_1$) we obtain from (7.1)

$$\Omega(\varphi) = 2\frac{\sqrt{\varepsilon_0 \sin^2\varphi - \varepsilon}}{\sqrt{\varepsilon_0}\cos\varphi}, \quad \Omega'(\varphi) = \frac{2\sqrt{\varepsilon_0}\sin\varphi}{\sqrt{\varepsilon_0 \sin^2\varphi - \varepsilon}},$$

it follows that according to (7.11), at $\varphi = \varphi_1 + 0$ and $\omega = \omega_r$, the complex shift Δ is equal to

$$\Delta = \frac{\operatorname{tg}\varphi_1}{k_r}\sqrt{\frac{2}{\delta\varepsilon}}(1-i), \quad k_r = \frac{\omega_r}{c}\sqrt{\varepsilon_0}, \quad \delta\varepsilon = \frac{\beta^2}{\omega_r \nu \varepsilon_0} \qquad (7.15)$$

and the gain of the beam is equal to

$$K = \exp\left[\frac{(a^2+b^2)\operatorname{tg}^2\varphi_1}{a k_r^2 \delta\varepsilon}\right], \qquad (7.16)$$

that is, it is larger the smaller $\delta\varepsilon$. This, however, does not mean at all that as $\delta\varepsilon \to 0$ we can obtain an arbitrarily large gain K and an arbitrarily large complex shift Δ: the point is that formula (7.11), which is analogous to (3.3), is not always valid. For formula (3.3) it is necessary to satisfy the condition (3.9), and the analogous condition for (7.11) is

$$\frac{1}{2}\Omega''(\varphi)\,\Delta\psi^2 \ll 1, \quad \Delta\psi^2 = \frac{\sqrt{a^2+b^2}}{k_0^2}, \qquad (7.17)$$

where $\Delta\psi$ determines the angle width of the beam (7.12) just as $\Delta\omega = \sqrt{a^2+b^2}$ determines the width of the frequency band occupied by the pulse with complex envelope (3.18). If account is taken of the condition (7.17) and of the smallness of $\delta\varepsilon$, then it turns out that within the limits of applicability of formula (7.16) we always have $K \approx 1$. This result could be predicted without calculations: after all, an unbounded plane wave can always be subdivided mentally into a number of parallel beams, which undergo a shift upon reflection and which add up to form the reflected plane wave. It is therefore clear that reflection of each beam should occur with the same gain as the reflection of a plane wave.

Thus, the large gain in reflection cannot be attributed to the finite width of the beam (see[43b]). It appears that in fact we have not amplification in reflection, but amplification in transmission: part of the beam goes over into the active medium, becomes amplified there with a gain on the order of (7.8), and then returns to the first medium by reflection from inhomogeneities. The amplification of part of the beam is due to the fact that its expansion in plane waves always contains waves that go over into the active medium and are amplified there. The situation is the same as in the case shown in Fig. 7: whereas the frequency ω_0 corresponds to damping, the frequencies ω lying to the right of the cut correspond to amplification, and they give rise to the growing sideband parts referred to in Chap. 6.

The examples given above pertain to quantum electronics. In "classical" electronics, that is, for electron beams and for a plasma, analogous problems arise and are considered in the previously cited papers,[37–39,44] and are at first glance more difficult, inasmuch as the complex wave numbers h have no explicit expressions, and only a characteristic equation is derived for them; if n waves interact in a given system, then this is an algebraic equation of degree n. However, if a small parameter is introduced,[38,48] this equation simplifies and becomes quadratic, that is, the situation reduces to pairwise interaction. It is now possible to apply in explicit form the general theory developed above to each wave that results from the interaction and has a function $h(\omega)$, and, in particular, to find the propagation direction. We do not report here this investigation,[49] because it has not yielded so far any new results (compared with the article[38]).

CONCLUSION

The questions considered in this article have a long history. The principles of wave kinematics were founded by Hamilton (1839), Stokes (1876), and Rayleigh (1877), while the principles of energetic kinematics were founded by N. A. Umov (1874). The number of subsequent works on these questions is truly immense and cannot be listed here. The large number is due to the fact that pulse propagation is constantly of interest in a variety of branches of physics and engineering, but it is impossible to present a complete analytic solution of this problem unless the solution is in terms of the Fourier integrals (1.9) or (6.1) together with a recommendation of calculating them with computers. Unfortunately, the number of such calculations is still very small. Even in recent years, most authors place unjustifiably large trust in the analytic formalism, and if numerical calculations are made the results are frequently not evaluated in the proper manner.

It should be noted in this connection that the relations derived above make it possible, as a rule, only to understand the main phenomena that occur in pulse propagation, and to estimate them in limiting cases, without replacing the complete calculation referred to above. A special position is occupied by the "elephant"—the principal part of the narrow-band pulse, for which the simpler expressions (3.5)–(3.8) were obtained, and the applicability limits of which were refined, particularly in Chaps. 4 and 6.

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