Quantum field theory with a chiral Lagrangian and low-energy meson physics

M. K. Volkov and V. N. Pervushin

Joint Institute for Nuclear Research
Usp. Fiz. Nauk **120**, 363–391 (November 1976)

PACS numbers: 11.10.Ef, 11.20.Fm, 11.30.Rd

CONTENTS

1. INTRODUCTION

Dispersion-relation techniques[1] have been successfully used to describe hadron physics at low energies. However, these techniques have been phenomenological in character, since their use requires a number of arbitrary parameters. By invoking dynamical principles such as chiral symmetry, it has been possible to substantially reduce the number of undetermined parameters, thereby imposing additional boundary conditions on the low-energy solutions of the dispersion relations.[1b]1) The combined use of the dispersion approach and perturbation theory in a quantum field theory with a chiral dynamical symmetry may be even more promising.

The purpose of the present review is to give a detailed account of such a quantum chiral theory, which makes it possible to obtain low-energy expansions for the amplitudes of various hadronic processes without introducing arbitrary parameters in the theory (apart from the hadron masses and the pion decay constant).

The first attempts to apply the methods of quantum field theory in describing strong interactions were made in the early 1950s, immediately after quantum electrodynamics had been formulated. However, these attempts did not lead to any significant successes, not only because of the large value of the coupling constant, but also because the simplest Lagrangians which had been proposed at that time did not in general reflect any dynamical symmetry of the strong interactions. The point is that the requirements of relativistic invariance and other so-called algebraic symmetries used to classify particles leave a considerable arbitrariness in the choice of the Lagrangian. Consequently, to specify the form of the interaction, it is necessary to postulate a larger group of transformations—a dynamical group. We recall that gauge invariance in quantum electrodynamics is an example of such a dynamical symmetry.

The currently known dynamical symmetries which are used in gravitational theory, in the unified theory of the weak and electromagnetic interactions, and in strong-interaction theory not only determine the form of the Lagrangian, but also give rise to a universality property of the interactions, in the sense that the lowest orders of the expansion in the coupling constant coincide with the lowest orders of the expansion in powers of the energy. This makes it possible to use such a field theory to obtain reasonable results at low energies, independently of the value of the coupling constant.

In the present review we attempt to describe the physics of low-energy mesonic processes according to the following scheme: 1) we assume a dynamical symmetry of the strong interactions; 2) we find an interaction Lagrangian satisfying this symmetry; 3) we deduce the physical implications of the quantum field theory in question by making use of the single-loop approximation.

Our starting point is the very fruitful idea of chiral symmetry of the strong interactions, according to which the strong interactions are approximately invariant with respect to a certain group of transformations that includes both isotopic transformations and transformations which mix states of opposite parity.

The idea of this symmetry had already been conceived in the celebrated paper of Feynman and Gell-Mann[2a] concerning the $V-A$ form of the weak interactions, where arguments along the following lines were employed. It is well known that the universality of certain interactions of different particles (for example, the fact that leptons and hadrons have the same electric charges) indicates the existence of conserved quantities (the elec-

1) In particular, this approach has been successfully used to obtain a correct description of the ρ-wave resonance in the $\pi\pi$ system and a unique expression for the mass of the ρ meson in terms of the mass and decay constant of the pion.

tromagnetic current) and hence the existence of a definite symmetry group (in our example, gauge invariance). In exactly the same way, starting from the universality of the weak interaction of the axial-vector currents of the leptons and hadrons in μ decay and in β decay of the neutron,[2)] it was proposed in[2a] that the full group of transformations for the hadrons should include transformations which mix states of opposite parity. This idea was later developed by Gell-Mann,[2b] who introduced not only the vector hadronic currents which generate the unitary symmetry SU_3, but also the axial-vector currents. The commutation relations for all these hadronic currents in[2b] became known as chiral current algebra, and the corresponding symmetry as chiral symmetry.

There are two possible types of realizations of chiral symmetry: algebraic and dynamical realizations.

An example of an algebraic realization is the classification of non-interacting massless particles (such as the neutrino) according to new quantum numbers—helicities. The realization of chiral symmetry for massless particles has been widely applied in describing lepton-hadron processes at large momentum transfers and high energies. This symmetry is consistent with models of hadrons as beams of massless non-interacting partons (or quarks) concentrated in a spatial region of dimensions of order 1 GeV^{-1}.[3]

The other, dynamical,[4] type of realization of current algebra has been extremely fruitful in the region of low energies, $\ll 1$ GeV, where hadrons can be represented approximately as point-like massive particles. To understand the essence of the dynamical realization, it is useful to consider the simple example of the axial-vector currents used to describe the β decay of the nucleon:

$$J_\mu^{(N)^i} = \overline{N}(p_1)\tau^i\gamma_5\gamma_\mu N(p_2) g_A, \quad p_{1\mu} - p_{2\mu} = q_\mu,$$

where $\overline{N}$ and N are wave functions of free nucleons, γ_μ are the Dirac matrices, τ^i are the Pauli matrices, and g_A is the coupling constant. It follows from the Dirac equations $(\hat{p}_2 - M)N(p_2) = 0$ and $\overline{N}(p_1)(\hat{p}_1 - M) = 0$, that the current $J_\mu^{(N)i}$ is not conserved if $M \neq 0$:

$$q_\mu J_\mu^{(N)^i} = 2M\overline{N}\tau^i\gamma_5 N g_A \neq 0.$$

Let us add to this current a pole term which describes the emission by the nucleon of a massless pseudoscalar particle, which has a weak decay constant F_π and a constant g/M characterizing its axial-vector interaction with the nucleon:

$$J_\mu^i = \overline{N}(p_1)\tau^i\gamma_5\left(\gamma_\mu g_A - \frac{F_\pi}{M}\ g\ \frac{\hat{q}q_\mu}{q^2}\right)N(p_2). \qquad (1.1)$$

It is easy to see that we can now ensure conservation of the axial-vector current

$$q_\mu J_\mu^i = 0 \qquad (1.2)$$

and hence the existence of a corresponding symmetry, by putting

$$g_A = \frac{F_\pi}{M} g. \qquad (1.3)$$

Chiral symmetry in this case determines the dynamics of the interaction of nucleons with the pseudoscalar particle, which is usually called a Goldstone particle. If this particle is identified with the pion, Eq. (1.3), which was first derived by Goldberger and Treiman[5] by means of dispersion relations, is satisfied to an accuracy of 7%. The pion has a non-zero mass m_π, so that the divergence of the axial-vector current, Eq. (1.2), must have the value[3)]

$$q_\mu J_\mu^i = m_\pi^2 F_\pi \pi^i, \qquad (1.4)$$

where π^i is the pion field. Since the quantities in Eq. (1.3) are slowly-varying functions of the momentum transfer q at distances of the order of the pion mass (the smoothness hypothesis), the right-hand side of Eq. (1.4) can be regarded as a small perturbation, i.e., it is "almost zero." The relation (1.4), known as the hypothesis of partial conservation of the axial-vector current (PCAC), determines the mechanism of chiral symmetry breaking. Using PCAC and the commutation relations of current algebra, it is possible to derive sum rules for the matrix elements of the weak and electromagnetic hadronic currents, which are in good agreement with the experimental data,[6] as well as a number of low-energy relations among the hadronic amplitudes with and without meson emission at zero unphysical momentum. The information obtained in this way requires a further extrapolation to physical momentum values and becomes meaningless at higher energies, since it does not satisfy the unitarity condition.

A possible method of avoiding these difficulties is the method of phenomenological Lagrangians, by which the results of current algebra can easily be reproduced at the level of the tree approximation. Chiral symmetry then ensures the self-consistency of the strong interactions, in the sense that the effective low-energy coupling constants of the weak, electromagnetic, and even the strong interactions are not renormalized as a result of higher orders in the strong interaction.[7] In other words, it is a consequence of chiral symmetry that the expansion in the strong coupling constant to the lowest orders coincides with the low-energy expansion in powers of the energy. We recall that an analogous situation occurs in quantum electrodynamics,[8] where the lowest orders of perturbation theory would contain the main information at low energies even for a large coupling constant.

During the period 1968–1971 many authors[9,10] pointed out that the method of phenomenological La-

---

[2)]While the paper of Feynman and Gell-Mann[2a] was being written in 1958, it became known that the axial-vector and vector constants in β decay are in the ratio $g_A \approx 1.3 \pm 0.1$.

[3)]Equation (1.4) is written for the current (1.1) with the substitution $1/q^2 \to 1/(q^2 - m_\pi^2)$. This equation follows from the equation of motion for the pion, $(q^2 - m_\pi^2)\pi^i = -\overline{N}\gamma_5\tau^i\hat{q}Ng/M$.

grangians not only reproduces the results of current algebra, but can also be extended to higher energies by applying the techniques of quantum field theory. During the past six years, many papers have been concerned with the quantization of chiral Lagrangians[9-12] and with the description of a large quantity of experimental data on low-energy processes in terms of quantum chiral theories.[13-24]

There are several approaches to the problem of quantizing chiral Lagrangians. Lagrangians which are renormalizable as a result of the introduction of hypothetical sigma particles were considered in[9,13]. Nonlinear Lagrangians with no hypothetical particles have been widely studied.[10-12,14-24] Such theories are nonrenormalizable and, as is well known, differ from renormalizable theories by the fact that they contain an infinite number of undetermined parameters which cannot be fixed by a renormalization of the physical quantities. In[14] the number of undetermined parameters was reduced to one, by exploiting the fact that nonlinear theories are obtained as limiting cases of linear theories when the mass of the sigma particle tends to infinity, $m_\sigma \to \infty$.

Finally, many authors[15-24] have employed the method of regularizing a quantum field theory with a nonpolynomial Lagrangian, which makes it possible to fix all the undetermined parameters. This method was proposed by one of the present authors[25] and further developed by Lehmann[26] and by Salam and Strathdee.[27] It is now known in the literature as the superpropagator (SP) method. In eliminating the divergences according to the SP method, it is necessary to consider the expression corresponding to the set of diagrams having a fixed number of vertices and an arbitrary number of internal lines as a single analytic function.

The present review is concerned with the description of low-energy mesonic processes in terms of the single-loop approximation in a quantum chiral theory. We confine ourselves here to a discussion of the SP approach, in which low-energy processes have been analyzed most fully, giving results in agreement with those of the σ-model in the limit $m_\sigma \to \infty$.[14]

The basic hypotheses and assumptions are as follows.

I. The Lagrangian is a realization of a chiral dynamical symmetry. By virtue of the smoothness hypothesis, a Lagrangian with the minimum number of derivatives is chosen. Chiral symmetry is broken by the introduction of meson masses according to the scheme of Gell-Mann, Oakes, and Renner.[28]

II. The SP method is used to calculate the meson loop diagrams, and this fixes all the undetermined parameters.

III. To calculate the single-loop baryon diagrams, it is sufficient to use standard renormalization theory, which satisfies chiral symmetry. Thus only the finite integrals corresponding to the baryon loop diagrams are considered. The average values of the virtual baryon momenta are finite and small. At low meson energies, the strong vertices in the baryon loops are therefore in the same regime as in the Goldberger-Treiman relation (1.3). This means that we should also expect the correspondence between the low-energy expansion and the expansion in the strong coupling constant to hold for the single-loop baryon diagrams.

Tests of this assumption[14,29] have shown that the corrections due to higher orders of perturbation theory in the strong coupling constant are of order 20–30%.[4)] The single-loop approximation gives essentially new information which is not contained in the original tree approximation. We shall give a brief account of the results of the single-loop approximation, indicating in parentheses which hypotheses and assumptions (I, II, III) are used to obtain these results.

The strong interactions of mesons have been considered in[14-17,22], where calculations were made of the amplitudes for $\pi\pi$ scattering[14-17,22] as well as πK and KK[14] scattering. The scattering phases were calculated by means of Padé approximants. This review contains a detailed account of the results of calculations of the $\pi\pi$ scattering amplitude; these calculations were made in the first instance for massless pions[15,16] and later for massive pions.[17,22] The scattering phases are in agreement with the experimental data and contain the ρ-meson resonance (I–III) (at an energy ~800 MeV and with a width ~150 MeV). The scattering amplitude, which satisfies the most general requirements of quantum field theory, can be used to extract information about all the scattering lengths, including those of the d wave (I, II, III) and the other higher partial waves (I). (We note that the Born approximation gives only s and p waves.) All the results are in good agreement with the established experimental data[32] and with the results of their phenomenological analysis.[33]

The electromagnetic interactions of mesons have been considered in[18,20,22,23], where calculations were made of the form factors and amplitudes for the Compton effect on pions and kaons. Within the errors, the theoretical value for the root-mean-square radius of the pion (I–III) is in agreement with the latest experimental data.[34] The values obtained for the radii of the charged (I–III) and neutral (I, II) kaons are in good agreement with the vector-dominance model. Values are predicted for the polarizability of mesons (I, III), and it is found that there should be an appreciable enhancement of the effective coefficient of polarizability for mesons in the vicinity of the two-pion production threshold (I).

The principal weak decay modes of mesons were considered in[21], where calculations were made of the weak decay constant of the pion as a function of the pion mass (I, III) and the structure constants for the decay (I, III). An effect of second order in the weak interac-

[4)]It is of interest to note that the exact calculation of the polarizability of an elementary particle in quantum electrodynamics[30] gives the same result as the calculation of this quantity in the single-loop approximation[31]; i.e., the second order of the expansion in the energy is the same as the second order of the expansion in the coupling constant (1/137).

tion—the mass difference between the neutral kaons (I, II)—was calculated in[19].

All the results obtained from the single-loop approximation satisfy the relations between the observable quantities which follow from current algebra and are in reasonable agreement with the experimental data.

The material of our review is arranged as follows. The next two sections contain an account of the main principles used to construct chiral-invariant Lagrangians and the basic calculational techniques. In the following three sections we describe the strong, electromagnetic, and weak interactions, respectively. The concluding section contains a discussion of the future prospects for current algebra and quantum chiral theory.

2. PHENOMENOLOGICAL LAGRANGIANS

In this section we describe the general method of obtaining phenomenological Lagrangians as non-linear realizations of chiral symmetry.[35,36] We first consider the symmetry $SU_2\times SU_2$.

Suppose that the Lagrangian for free non-interacting nucleons $\psi=\binom{\psi_p}{\psi_n}$ is given by

$$L_0(\psi)=i\bar{\psi}\,\hat{\partial}\psi-M\bar{\psi}\psi,\quad \hat{\partial}=\frac{\partial}{\partial x_{\mu}}\gamma_{\mu}. \tag{2.1}$$

This Lagrangian is invariant with respect to the isotopic transformations containing a parameter ω,

$$\psi'=e^{i(\tau/2)\omega}\psi, \tag{2.2}$$

as reflected in the classification of the nucleons according to the representation (1/2) of the group SU_2. Let us also consider the following transformations containing a parameter **a**, which mix states of opposite parity:

$$\psi'=e^{\tau a\gamma_5/2}\psi,\quad \gamma_5=-i\begin{pmatrix}0 & I\\ I & 0\end{pmatrix},\quad I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \tag{2.3}$$

After making the transformation (2.3), the Lagrangian (2.1) takes the form

$$L_0(\psi')=i\bar{\psi}\,\hat{\partial}\psi-M\bar{\psi}e^{\tau a\gamma_5}\psi=L_0(\psi)+M\bar{\psi}(1-e^{\tau a\gamma_5})\psi. \tag{2.4}$$

The Lagrangian (2.1) is invariant if we set the mass of the nucleon equal to zero. This chiral invariance with respect to (2.3) allows us to introduce an additional classification of the nucleons according to their helicities, i.e., according to right and left isotopic spin, corresponding to the representations (1/2, 0) and (0, 1/2) of the group $SU_2\times SU_2$. Another method of making the Lagrangian (2.1) chiral-invariant without setting $M=0$ is to introduce an interaction of the nucleons with "compensating" fields which, under the transformations (2.3), cancel the non-invariant factor that appears in the Lagrangian (2.4). Since we are concerned with transformations which change the parity, we must introduce interactions with pseudoscalar massless "pions" by making the substitution

$$M\bar{\psi}\psi\to M\bar{\psi}\exp\left(-\gamma_5\frac{\pi\tau}{F_{\pi}}\right)\psi, \tag{2.5}$$

where F_{π} is a dimensioned constant. The fields π^i must transform according to the non-linear law

$$\exp\left(-\gamma_5\frac{\pi'\tau}{F_{\pi}}\right)=\exp\left(-\gamma_5\frac{a\tau}{2}\right)\exp\left(-\gamma_5\frac{\pi\tau}{F_{\pi}}\right)\exp\left(-\gamma_5\frac{a\tau}{2}\right) \tag{2.6}$$

in such a way that the expression (2.5) is invariant with respect to the simultaneous transformations (2.3) and (2.6) of the fields ψ and π. It is easy to construct the invariant Lagrangian for the pion field itself from the matrix $\exp(\gamma_5\,\pi\cdot\tau/F_{\pi})$:

$$L(\pi)=-\frac{F_{\pi}^2}{4}\,\mathrm{Sp}\left[\partial_{\mu}\exp\left(\gamma_5\frac{\pi\tau}{F_{\pi}}\right)\partial_{\mu}\exp\left(-\gamma_5\frac{\pi\tau}{F_{\pi}}\right)\right]. \tag{2.7}$$

Thus the full invariant Lagrangian has the form

$$L(\psi,\tau)=i\bar{\psi}\hat{\partial}\psi-M\bar{\psi}\exp\left(-\gamma_5\frac{\pi\tau}{F_{\pi}}\right)\psi+L(\pi). \tag{2.8}$$

If we identify the Goldstone field that has been introduced with the real pion and the axial-vector current $J_{5\mu}=F_{\pi}\partial_{\mu}\pi+O(\pi^3)$ with the current that takes part in the weak interactions, then the constant F_{π} in the first Born approximation is equal to the weak decay constant of the pion; this gives $F_{\pi}\approx 92$ MeV. Throughout the remainder of this section, we shall employ the dimensionless system of units, putting $F_{\pi}=1$.

To formulate the standard method of describing the interaction of Goldstone particles,[35,36] it is convenient to go over from the Lagrangian (2.8) to the physically equivalent Lagrangian

$$L(N,\pi)=\bar{N}i\gamma_{\mu}\left\{\partial_{\mu}+\left[\exp\left(-\gamma_5\frac{\pi\tau}{2}\right)\times\partial_{\mu}\exp\left(\gamma_5\frac{\pi\tau}{2}\right)\right]\right\}N-M\bar{N}N+L(\pi) \tag{2.9}$$

by means of the transformation

$$N=\exp\left(-\gamma_5\frac{\pi\tau}{2}\right)\psi. \tag{2.10}$$

We note that the expression $\exp[-\gamma_5(\pi/2)\cdot\tau]$ represents a finite transformation of the group $SU_2\times SU_2$ with parameters that are identified with the Goldstone fields.

We now consider the general representation. Let I^i and K^i be the generators of the isotopic transformations and proper chiral transformations, satisfying the commutation relations

$$\left.\begin{aligned} &\text{a)} && [I_i, I_j]=i\varepsilon_{ijk}I_k,\\ &\text{b)} && [K_i, I_j]=i\varepsilon_{ijk}K_k,\\ &\text{c)} && [K_i, K_j]=i\varepsilon_{ijk}I_k. \end{aligned}\right\} \tag{2.11}$$

In particular, for the representation considered above, these generators have the form

$$I_i=\frac{\tau_i}{2},\quad K_i=-i\gamma_5\frac{\tau_i}{2}.$$

The expression

$$\exp\left(-\gamma_5\frac{\pi\tau}{2}\right)\partial_{\mu}\exp\left(\gamma_5\frac{\pi\tau}{2}\right)\equiv e^{-iK^i\pi^i}\partial_{\mu}e^{iK^i\pi^i}$$

can be expanded in terms of all the generators of the group $SU_2\times SU_2$:

$$e^{-i\mathbf{K}\pi}\partial_{\mu}e^{i\mathbf{K}\pi}=i\left(K^j\omega_{\mu}^j(\pi)+I^j\theta_{\mu}^j(\pi)\right). \tag{2.12}$$

The explicit structure of the forms ω_μ^i and θ_μ^i, which are generally known in the physics literature as the Cartan forms, [35] can readily be obtained by differentiating with respect to a parameter t. This parameter is introduced in Eq. (2.12) by making the change of variable $\pi \to t\pi$. After differentiating, the right-hand side of Eq. (2.12) gives

$$i\left(K^j \frac{\partial}{\partial t}\omega_\mu^j + I^j \frac{\partial}{\partial t}\theta_\mu^j\right), \qquad (2.13)$$

while the left-hand side, using the commutation relations (2.11), gives

$$\frac{\partial}{\partial t}\left(e^{-it\mathbf{K}\pi}\partial_\mu e^{it\mathbf{K}\pi}\right) = [\mathbf{K}\pi, (K^i\omega^i + I^i\theta^i)] + iK^i\partial_\mu\pi^i$$
$$= i(\pi^j\omega^i I^k e_{jik} + \pi^j\theta^i K^k \varepsilon_{jik} + K^i\partial_\mu\pi^i). \qquad (2.14)$$

Equating the coefficients of the corresponding generators on the right- and left-hand sides given by (2.13) and (2.14), respectively, we obtain a system of two first-order equations:

$$\begin{aligned} &\frac{\partial}{\partial t}\omega_\mu^k = \partial_\mu\pi^k + \pi^j\theta_\mu^i\varepsilon_{jik}, \quad \omega_\mu|_{t=0} = 0, \\ &\frac{\partial}{\partial t}\theta_\mu^k = \pi^j\omega_\mu^i\varepsilon_{jik}, \qquad \theta_\mu|_{t=0} = 0. \end{aligned} \qquad (2.15)$$

The solution of this system of equations has the form

$$\begin{aligned} &\omega_\mu^k|_{t=1} = \partial_\mu\pi^k + \left(\delta_{ik} - \frac{\pi^i\pi^k}{\pi^2}\right)\left(\frac{\sin z}{z} - 1\right)\partial_\mu\pi^i, \\ &\theta_\mu^k|_{t=1} = \varepsilon_{ikj}\pi^i\partial_\mu\pi^j\frac{\cos z - 1}{\pi^2}, \quad z = \sqrt{\pi^2}. \end{aligned} \qquad (2.16)$$

It is easy to see that the pion Lagrangian (2.7) is given by

$$L(\pi) = \frac{1}{2}\omega_\mu^i\omega_\mu^i = \frac{1}{2}\partial_\mu\pi^i\partial_\mu\pi^i + \frac{1}{2}\partial_\mu\pi^i\partial_\mu\pi^j\left(\delta_{ij} - \frac{\pi^i\pi^j}{\pi^2}\right)\left(\frac{\sin^2 z}{z^2} - 1\right). \qquad (2.17)$$

Thus the full Lagrangian (2.9) can be written

$$L(N, \pi) = i\overline{N}\gamma_\mu D_\mu N + i\overline{N}\frac{\tau^i}{2}\gamma_\mu\gamma_5\omega_\mu^i N - M\overline{N}N + \frac{1}{2}\omega_\mu^i\omega_\mu^i, \qquad (2.18)$$

where $D_\mu N = [\partial_\mu + i(\theta_\mu^k\tau^k/2)]N$ and $\omega_\mu \equiv D_\mu\pi$ are called the covariant derivatives of the fields N and π, respectively. All the terms in (2.18) are invariant with respect to chiral transformations; the second term is not coupled to the kinetic terms and can appear in the Lagrangian for an arbitrary value of the parameter g_A.

We now formulate the standard method of constructing the Lagrangian. Suppose that we know the full symmetry group G of the interaction in question. The noninteracting particles are classified according to the representations of a subgroup H with generators I^i (we shall denote the remaining generators of the group G by K^i).

Now, to construct a Lagrangian which is invariant with respect to the transformations of the full group G, it is sufficient to replace the ordinary derivatives of the fields in the free-field Lagrangian by the covariant derivatives, using the Cartan forms ω and θ, which can be calculated according to the equation

$$\exp(-iK^ja^j)\partial_\mu\exp(iK^ja^j) = i[K^j\omega_\mu^j(a) + I^j\theta_\mu^j(a)].$$

The parameters $a^j(x)$ are identified with the fields of the Goldstone particles.

We note that a similar procedure was used in[37] to construct Einstein's theory of gravitation. The initial group G was taken to be the 20-parameter group $A(4)$ of all linear transformations of 4-space, and the subgroup H was taken to be the 10-parameter Poincaré group $\mathcal{P}$. The remaining 10 parameters were identified with the fields of Goldstone gravitons.

Let us construct a Lagrangian which is invariant with respect to the transformations of the chiral group $SU_3 \times SU_3$. Suppose that B^i and Φ^i are the baryon and meson octets. (We henceforth adopt the notation of the reviews.[38]) A chiral-invariant Lagrangian which reproduces the theorems of current algebra has the form

$$L = \frac{1}{2}D_\mu\Phi^i D_\mu\Phi^i$$
$$+ \overline{B}_i\gamma_\mu\left[i\partial_\mu\delta_{ij} + \theta_\mu^k f_{kij} + D_\mu\Phi^k(-if_{kij}(1-\overline{\alpha}) + \overline{\alpha}d_{kij})\gamma_5\frac{g}{M} - M\delta_{ij}\right]B_j; \qquad (2.19)$$

here $\overline{\alpha} \approx 2/3$ is the mixing parameter for the F and D couplings, and the Cartan forms

$$D_\mu\Phi^i = \partial_\mu\Phi^i + O(\Phi^3), \quad \theta_\mu^j = \Phi^i f_{ijk}\partial_\mu\Phi^k + O(\Phi^4)$$

are determined by the equation

$$\exp\left(-\frac{1}{2}\gamma_5\xi\right)\partial_\mu\exp\left(\frac{1}{2}\gamma_5\xi\right) = \gamma_5\frac{\lambda^i}{2}D_\mu\Phi^i + i\frac{\lambda^k}{2}\theta_\mu^k, \quad \xi = \sum_{k=1}^{8}\lambda^k\Phi^k, \qquad (2.20)$$

where λ^1 are the (3×3) Gell-Mann matrices.

We have so far been considering massless mesons. The meson masses lead to chiral symmetry breaking. This symmetry breaking is usually chosen according to a definite representation of $SU_3 \times SU_3$. In particular, it was proposed in[28] to describe the symmetry breaking according to the simplest representation $(3, 3^*)\oplus(3^*, 3)$ of the group, which characterizes the transformation properties of the matrix

$$\exp(i\xi) = \sum_{n=0}^{8}(\lambda^n s^n + i\lambda^n p^n), \quad \lambda^0 = \sqrt{\frac{2}{3}}I,$$

where s^n and p^n are non-linear functions of the fields Φ^i.

The symmetry breaking is chosen in the form

$$\Delta L = F_\pi^2(c_1 s^0 + c_2 s^8).$$

The constants c_1 and c_2 in this expression are determined by the condition that the terms quadratic in the fields in the expansion of s^0 and s^8 are equal to the mass part of the free Lagrangian $m_K^2\overline{K}K + (1/2)m_\pi^2\pi^2$:

$$c_1 = \sqrt{\frac{2}{3}}\left(m_k^2 + \frac{1}{2}m_\pi^2\right), \quad c_2 = -\frac{2}{\sqrt{3}}(m_k^2 - m_\pi^2).$$

In conclusion, we write the explicit form of the matrix $\exp(i\xi)$:

$$\exp(i\xi) = \begin{vmatrix} \alpha^2\left[U_{ij} - \frac{2\psi_{L_i}\bar{\psi}_{R_j}}{F_\pi^2(1+\gamma)}\right] & \left(\frac{i\sqrt{2}\alpha^{-1}\psi_L}{F_\pi}\right)_{3j} \\ \left(i\frac{\overline{\psi}_R\sqrt{2}\alpha^{-1}}{F_\pi}\right)_{i3} & (\gamma\alpha^{-4})_{33} \end{vmatrix},$$

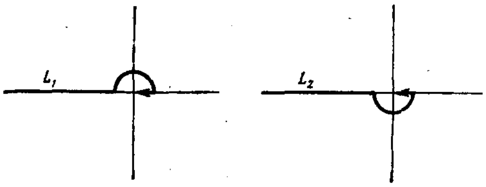

FIG. 1. The paths along which the point $x^2=0$ is approached in the integrals with the contours L_1 and L_2.

$$U=\exp\left(i\frac{\pi^i\tau^i}{F_\pi}\right),\quad \bar{\psi}_L=\exp\left(i\frac{\pi^i\tau^i}{2F_\pi}\right)\psi\equiv U^{1/2}\psi,\quad \psi_R=U^{-1/2}\psi,$$
$$\gamma=\sqrt{1-\frac{2\bar{\psi}\psi}{F_\pi^2}},\quad \alpha^2=e^{i\eta/\sqrt{3}},\quad \psi=\begin{pmatrix}K^+\\K^0\end{pmatrix}.$$

3. REGULARIZATION TECHNIQUES AND COVARIANT PERTURBATION THEORY

In constructing a quantum chiral field theory, we shall adopt the S-matrix formalism in the interaction representation, where the S-matrix is usually written in the form

$$\hat{S}=T\exp\left[i\int d^4xL_{\text{int}}(x)\right]=\sum_0^\infty\frac{i^n}{n!}T\left[\int d^4xL_{\text{int}}(x)\right]^n. \qquad (3.1)$$

To eliminate the ambiguities which occur in the theory when regularizing the divergent integrals in those cases in which we must deal with renormalizable quantum field theories (such as electrodynamics), it is sufficient to renormalize a finite number of observable physical quantities such as mass, charge, and wave functions of fields. For non-renormalizable theories, an example of which is a chiral theory, it would also be possible to apply the regularization procedure of renormalizable theories in each order of perturbation theory, but this would lead to the presence of too many undetermined parameters, which could no longer be fixed by renormalizing a finite number of observable physical quantities. In our case, we must therefore use completely different regularization techniques which are characteristic of theories involving non-polynomial Lagrangians. These methods were first proposed in[25,39].

One such method, now known as the superpropagator (SP) method, will be used in the remainder of our calculations. A detailed account of this method can be found in[25]. Here we shall give a brief description of its basic idea. The characteristic singularities of the coefficient functions on the light cone in renormalizable theories are pole-type singularities. For example, the two-vertex loop with n internal lines corresponding to massless scalar particles is of the form

$$\Pi^{(n)}(x)=i\left[\Delta^c(x)\right]^n=i\left[-\frac{i}{(2\pi)^2(x^2-i\varepsilon)}\right]^n. \qquad (3.2)$$

To make the integral convergent in constructing the Fourier transform of (3.2), it is necessary to carry out a finite number of subtractions (using, for example, the Pauli-Villars or Bogoliubov-Parasyuk regularization; see[40]). However, the removal of this intermediate regularization entails the presence of a certain number of undetermined parameters in the final expression.

We turn now to non-polynomial Lagrangians. A typical example of such a Lagrangian is one having the exponential form

$$L_{\text{exp}}=G\left[\exp\left(g\varphi(x)\right)-1\right].$$

If we consider not an individual diagram but the complete set of two-vertex diagrams obtained in second-order perturbation theory in the constant G, it is easy to derive the expression

$$\Pi_{(-)}^{(\text{exp})}(x)=iG^2\left[\exp\left(-ig^2\Delta(x)\right)-1\right]=iG^2\left\{\exp\left[-\frac{g^2}{(2\pi)^2(x^2-i\varepsilon)}\right]-1\right\}. \qquad (3.3)$$

In contrast with (3.2), we find here not pole singularities on the light cone, but an essential singularity. On the one hand, the behavior of the Green's function on the light cone becomes more singular than in renormalizable theories. But on the other hand, we can now exploit entirely new methods of regularizing the divergent integrals, which were previously inapplicable. These methods make use of the fact that for an essential singularity, unlike a pole singularity, the behavior of the Green's function (3.3) on the light cone depends strongly on the path along which we approach the singularity. If, in the integral corresponding to the Fourier transform of (3.3), we choose the contour of integration with respect to the variable $\lambda=x^2$ in such a way as to approach the light cone from the region $x^2>0$, we therefore obtain a convergent integral and a finite expression for the Green's function in momentum space. The specific choice of the contour of integration is dictated by the unitarity condition for the S-matrix.[25] 5)

These properties of the full function $\Pi^{\text{exp}}(x)$ are completely lost if we consider each term of its expansion in powers of g^2 individually (in the same way that the limit

$$\lim_{t\to\infty}e^{-t}=\lim_{t\to\infty}\sum_0^\infty\frac{(-t)^n}{n!}=0$$

differs from the sum of the limits of the individual terms of the expansion). Moreover, the function $\tilde{\Pi}^{\text{exp}}(p)$ has a logarithmic-type non-analytic dependence on the coupling constant g^2. This confirms once again that the expansion of this function in powers of g^2 is not valid.

---

5)To illustrate this point, let us consider the two functions $\Pi_{(\pm)}^{\text{exp}}(x)$, where $\Pi_{(+)}^{\text{exp}}(x)$ differs from (3.3) by the sign in the exponent. It follows from the unitarity of the S-matrix that $\text{Im}\,\tilde{\Pi}_{(\pm)}^{\text{exp}}(p)=0$ for $p^2<0$. This condition is satisfied by the simple finite expression for $\tilde{\Pi}_{(+)}^{\text{exp}}(p)$:

$$\tilde{\Pi}_{(+)}^{\text{exp}}(p)=\int_0^\infty d\lambda' I_{(+)}(|p|,\lambda'),\quad (p^2<0),$$

where $\lambda'=-x^2$, $|p|=\sqrt{-p^2}$, $I_{(\pm)}=(2\pi^2G^2/|p|)\sqrt{\lambda'}\,J_1(|p|\sqrt{\lambda'})\times[\exp(\mp g^2/(2\pi^2\lambda')-1]$, and J_1 is a Bessel function. For $\tilde{\Pi}_{(-)}^{\text{exp}}(p)$, we find the more complicated expression

$$\tilde{\Pi}_{(-)}^{\text{exp}}(p)=\frac{1}{2}\left[\int_{L_1}d\lambda' I_{(-)}(|p|,\sqrt{\lambda'})+\int_{L_2}d\lambda' I_{(-)}(|p|,\sqrt{\lambda'})\right].$$

The contours L_1 and L_2 are shown in Fig. 1.

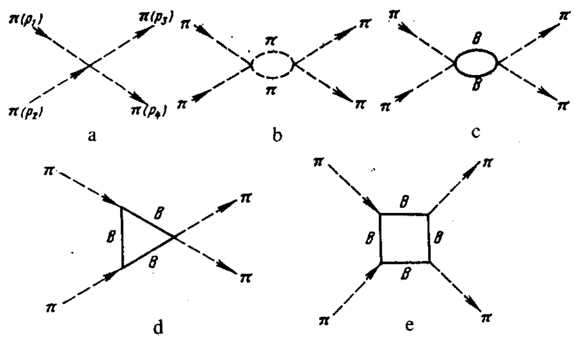

FIG. 2. Diagrams corresponding to the "tree" approximation (a) and the single-loop approximation (order F_π^{-4}) (b) for $\pi\pi$ scattering. The dashed and solid lines represent pions and baryons, respectively.

The function $\tilde{\Pi}^{\exp}(p)$ can be represented in the form of a modified expansion in g^2, in which each power g^{2n} may be associated with factors containing an additional logarithmic dependence on g^2 (i.e., $g^{2n}\ln g^2$). This is a manifestation of the fact that in obtaining a finite expression for "a given order in g^2" we took into account the effects of all the remaining terms of the expansion of the function $\Pi^{\exp}$.

We now consider perturbation theory itself. It is convenient to write the S-matrix (3.1) in another equivalent form by dividing the fields into internal fields Γ (which lead to propagators in the coefficient functions) and external fields φ:

$$\hat{S}=N_{\varphi_\Gamma}\langle 0|T_\Gamma \exp\left[i\int d^4x L_{\text{int}}(\varphi+\Gamma)\right]|0\rangle_\Gamma. \tag{3.4}$$

As we have already seen in Sec. 2, the different forms of chiral Lagrangians must lead to identical physical results. Physically equivalent Lagrangians are related to one another by point transformations

$$\varphi=\varphi' f(\varphi'),\quad f(0)=1. \tag{3.5}$$

For example, the Lagrangians

$$L=\lambda\varphi^4+\frac{1}{2}(\partial_\mu\varphi)^2, \tag{3.6a}$$

$$L=\lambda(\sin\varphi)^4+\frac{1}{2}(\partial_\mu\sin\varphi)^2 \tag{3.6b}$$

are physically equivalent. This equivalence is due to the fact that the metric properties of the space of the field itself, which are determined by the quadratic form of the derivatives of the fields, does not depend on the transformation (3.5). The choice of some particular Lagrangian from among all the equivalent Lagrangians (for example, (3.6a) instead of (3.6b)) in formulating the perturbation theory is governed entirely by considerations of simplicity of the techniques which are used. In a similar way, there exist infinitely many equivalent chiral Lagrangians. Chiral Lagrangians written in the form

$$L(\pi)=\frac{1}{2}\partial_\mu\pi^i\partial_\mu\pi^j g_{ij}(\pi) \tag{3.7}$$

have an elegant geometrical interpretation: g_{ij} is the metric tensor of a three-dimensional isospace of constant curvature characterized by the constant F_π. The transformations (2.6) correspond to a displacement of the origin on a sphere by the vector **a** and can therefore be interpreted in terms of vector addition in a curved space:

$$\pi'(\pi,\bar{a})=\pi(+)\bar{a}=\pi+\bar{a}+O\left(\frac{1}{F_\pi}\right). \tag{3.8}$$

The limit $F_\pi\to\infty$ gives the usual Euclidean isospace.

In analogy with the $\lambda\varphi^4$ theory (3.6a), the essence of our covariant perturbation theory is as follows:

1) We choose the simplest so-called normal coordinates in the space of the fields along the geodesics (this choice corresponds to the exponential parametrization $e^{iK\pi}$ used in Sec. 2 for the finite transformations of the group).

2) In separating the fields into external fields π and internal fields Γ, we make use of the operation of vector addition (3.8) along geodesics in a curved space (in particular, to calculate the Cartan forms corresponding to the Lagrangian $L(\pi(+)\Gamma)$ in (2.12), we must make the substitution $\exp(iK\pi)\to\exp(iK\pi)\exp(iK\Gamma)$[41,42]).

This choice of the fundamental fields has the following advantages: 1) the simplest combinations occur in calculating the matrix elements (compare, for example, the Lagrangians (3.6a) and (3.6b)); 2) covariant perturbation theory leads directly to a covariant separation of all the diagrams into diagrams involving fixed numbers of vertices, which is important for the application of the SP calculational technique in a form which is invariant with respect to the group $SU_2\times SU_2$. To achieve such an invariant separation in an arbitrary coordinate system, it would be necessary to make additional and very cumbersome rearrangements of the Feynman diagrams.[35,41,42]

4. STRONG INTERACTIONS ($\pi\pi$ SCATTERING)

In the remainder of our exposition, we shall try as far as possible to avoid dwelling on the details of the calculations; we shall concentrate mainly on the results that have been obtained, referring the reader who is interested in the details to the original literature.[15–23] We begin by considering the process of elastic $\pi\pi$ scattering.

The scattering amplitude has the form

$$(2\pi)^6 4\sqrt{p_1^0p_2^0p_3^0p_4^0}\langle i_1i_2|S|i_3i_4\rangle=I+i(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)\times[\delta_{i_1i_2}\delta_{i_3i_4}A(s,t,u)+\delta_{i_1i_3}\delta_{i_2i_4}A(t,s,u)+\delta_{i_1i_4}\delta_{i_2i_3}A(u,t,s)], \tag{4.1}$$

where I is the unit matrix, i_k are isotopic indices of the pion, δ_{ij} is the Kronecker delta symbol, $s=(p_1+p_2)^2$, $t=(p_1-p_3)^2$, and $u=(p_1-p_4)^2$. In Fig. 2 we show the diagrams corresponding to the single-loop approximation (the order is not higher than $1/F_\pi^4$). The diagram a corresponds to the tree approximation. The contribution to the amplitude from the diagram b is calculated by means of the SP method.[25] The contributions of the remaining diagrams are calculated by means of the

TABLE I.

$\alpha_l^{I\,(n)}$	Experiment[32]	Values from[22]	Values from[33]
a_0^0	0.10; 0.60	0.15	0.15 ± 0.02
a_0^2	−0.10; −0.03	−0.042	-0.065 ± 0.025
a_1^1	0.042; 0.040	0.031	0.0341 ± 0.0036
b_1^1		$1.14\cdot10^{-3}$	$(1.07\pm0.27)\cdot10^{-3}$
a_2^0	$1.4\cdot10^{-3}$; $1.8\cdot10^{-3}$	$1.85\cdot10^{-3}$	$(1.48\pm0.08)\cdot10^{-3}$
a_2^2	$-2\cdot10^{-4}$; $3\cdot10^{-4}$	$2.6\cdot10^{-4}$	$(-3\pm8)\cdot10^{-5}$
b_2^0		$-1.02\cdot10^{-4}$	$(-3.8\pm1.1)\cdot10^{-5}$
b_2^2		$-5.1\cdot10^{-5}$	$(-4.4\pm1.1)\cdot10^{-5}$
c_2^0		$2\cdot10^{-5}$	$(1.13\pm0.36)\cdot10^{-5}$
c_2^2		$1.06\cdot10^{-5}$	$(1.27\pm0.36)\cdot10^{-5}$
a_3^1		$1.33\cdot10^{-5}$	$(3.8\pm0.5)\cdot10^{-5}$
a_4^0		$5\cdot10^{-6}$	$(4.8\pm0.8)\cdot10^{-6}$
a_4^2		$2\cdot10^{-6}$	$(1.7\pm0.8)\cdot10^{-6}$

usual methods of renormalizable field theories, and we retain here only the quadratic terms in the variables s, t, and u, since the terms of higher order are small and are of the type $[s^2/(4\pi F_\pi)^4](s/M_N^2)$.[6)] The contributions of all the members of the baryon octet are taken into account by means of $SU(3)$ theory.[16,17] This procedure gives the following expression for $A(s, t, u)$ in the F_π^{-4} approximation[19]:

$$(4\pi)^{-2}A\,(s,\ t,\ u) = \alpha_0\,(3\bar{s}-1) + \alpha_0^2\bar{\Pi}\,(\bar{s},\ \bar{t},\ \bar{u}), \qquad (4.2)$$

$$\bar{\Pi}\,(\bar{s},\ \bar{t},\ \bar{u}) = -1.5 + 3\bar{s} + 0{,}6\bar{s}^2 + 20\,(\bar{t}^2+\bar{u}^2) - (3\bar{s}-1)^2 J\,(\bar{s})$$
$$-[3\,(\bar{u}-1)\,(\bar{u}-\bar{t}) + 3\bar{u}-1]\,J\,(\bar{u}) - [3\,(\bar{t}-1)\,(\bar{t}-\bar{u}) + 3\bar{t}-1]\,J\,(\bar{t}),$$

where

$$\bar{\xi} = \frac{\xi}{4m_\pi^2}\ (\xi = s,\ t,\ u), \qquad \alpha_0 = \frac{1}{3}\left(\frac{m_\pi}{2\pi F_\pi}\right)^2 \approx 0.02,$$

$$J\,(\bar{\xi}) = 1 - \frac{1}{2}\sum_{1}^{\infty}(4\bar{\xi})^n\,\frac{n!\,(n-1)!}{(2n+1)!}$$
$$= \begin{cases} x\,\mathrm{arctg}\,x^{-1}, & x = \sqrt{\frac{1}{\bar{\xi}}-1}, \quad 0<\bar{\xi}<1, \\ \frac{y}{2}\left[\ln\left(\frac{1+y}{1-y}\right) - i\pi\right], & y = \sqrt{1-\frac{1}{\bar{\xi}}}, \quad \bar{\xi}>1, \\ \frac{y}{2}\ln\left(\frac{y+1}{y-1}\right), & \bar{\xi}<0. \end{cases} \qquad (4.3)$$

In the region of energies much smaller than $4\pi F_\pi$, Eq. (4.2) provides a good expansion of the $\pi\pi$ scattering amplitude in the small parameter α_0. The amplitudes in the channels with isospins 0, 1, and 2 are given by the equations

$$A^0 = 3A\,(s,\ t,\ u) + A\,(t,\ s,\ u) + A\,(u,\ t,\ s),$$
$$A^1 = A\,(t,\ s,\ u) - A\,(u,\ t,\ s), \quad A^2 = A\,(t,\ s,\ u) + A\,(u,\ t,\ s).$$

Following[33], we introduce the notation

$$\alpha_l^{I(n)} = \lim_{\bar{s}\to1}\frac{1}{4^n}\frac{\partial^n}{\partial\bar{s}^n}A_l^I\,(\bar{s}),$$
$$\alpha_l^{I(0)} = a_l^I, \quad \alpha_l^{I(1)} = b_l^I, \quad \alpha_l^{I(2)} = c_l^I,$$
$$A_l^I\,(\bar{s}) = \frac{1}{2\,(\bar{s}-1)^l}\int_{-1}^{1}dx\,P_l\,(x)\,A^I\,(\bar{s},\ x)$$
$$\left(\bar{t} = -\frac{(\bar{s}-1)}{2}\,(1-x), \quad \bar{u} = -\frac{(\bar{s}-1)}{2}\,(1+x)\right);$$

here a_l^I are the scattering lengths, b_l^I and c_l^I are effective-range parameters, and $P_l(x)$ are the Legendre polynomials. The scattering lengths and effective-range parameters for the $\pi\pi$ system are then found to have the values given in the Table I.

For $l \geqslant 3$, the foregoing equations lead to the following simple expressions for the scattering lengths:

$$a_l^0 = (2l+1)\,(4l+7)\,z_l,$$
$$a_l^1 = \frac{1}{3}\,(4l^2-2l-1)\,z_l, \quad z_l = 3\pi\alpha_0^2 4^{l-1}\,\frac{(l!)^3\,(l-3)!}{[(2l+1)!]^2},$$
$$a_l^2 = (4l^2+3l+8)\,z_l.$$

The results given in the table are in good agreement with the known experimental data[32] and with the results of the phenomenological approach of Palou and Yndurain,[33] who make use of the Froissart-Gribov representation.

All the scattering lengths for $l \geqslant 3$ satisfy the inequalities

$$a_{l+2}^I \leqslant a_l^I\,\frac{(l+1)\,(l+2)}{4\,(2l+3)\,(2l+5)},$$

which were obtained in[43] from the requirements of unitarity and analyticity of the scattering amplitude.

We note that while the values of the scattering lengths of the S and P waves are determined mainly by the Born term (Fig. 2a), the Born term does not contribute at all to the scattering lengths of the higher partial waves beginning with the D wave, whose values are determined by the contribution of the pion loop diagram 2b.

By making a partial-wave expansion of the amplitude A^I and applying the formula $(\cot\delta_l^I - i)^{-1} = \sqrt{1-(1/\bar{s})}A_l^I$, we can obtain information about the behavior of the phase shifts for the $\pi\pi$ system. The corresponding graphs are shown in Figs. 3 and 4. The dashed curves in these figures show the behavior of the phase shifts in the limit $m_\pi = 0$ (the case considered in[15,16]). The P wave contains a conspicuous ρ-meson resonance at an energy ~800 MeV with a width ~150 MeV.

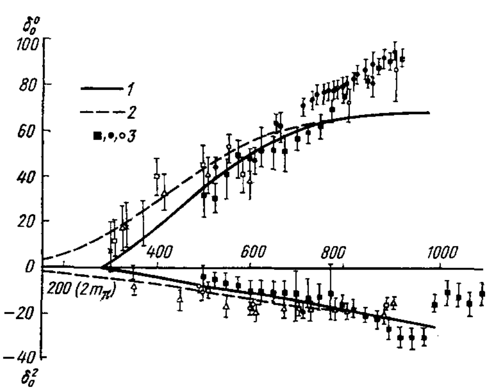

FIG. 3. Behavior of the S-wave phase shifts δ_0^0 and δ_0^2 of the $\pi\pi$ scattering amplitude. 1—theoretical curves for $m_\pi \neq 0$,[17] 2—curves for $m_\pi = 0$,[15,16] 3—experimental points from[32a] and[32b], respectively. See[16] for the remaining points ($\sqrt{s}$ is in MeV).

[6)]The contributions to the terms which are constant or linear in the variables s, t, and u from the diagrams c–e of Fig. 2 contain undetermined parameters, which can be fixed by applying the low-energy theorems requiring that the amplitude has the form $A(s, t, u) \approx s/F_\pi^2$ at low energies.

To conclude this section, we mention a number of further inequalities obtained by Martin[44] in the subthreshold region from the conditions of unitarity and crossing symmetry for the S wave of the process $\pi^0\pi^0 \to \pi^0\pi^0$:

$$f_0^{00}(\bar{s}) = \frac{1}{64\pi}\int_{-1}^{1} dx\,[A(s, t, u) + A(t, s, u) + A(u, t, s)].$$

These inequalities are as follows:

1) $f_0^{00}(\bar{s}) \leqslant f_0^{00}(1), \quad 0 \leqslant \bar{s} \leqslant 1;$

2) $\frac{df_0^{00}(\bar{s})}{d\bar{s}} > 0, \quad 0.5 \leqslant \bar{s} \leqslant 1;$

3) $f_0^{00}(\bar{s}) \geqslant 2\int_{0.5}^{1} d\bar{s} f_0^{00}(\bar{s});$

4) $f_0^{00}(0) > f_0^{00}\left(\frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right)\right);$

5) $\frac{df_0^{00}(\bar{s})}{d\bar{s}} < 0, \quad 0 \leqslant \bar{s} \leqslant \frac{1.29}{4};$

6) $\frac{df_0^{00}(\bar{s})}{d\bar{s}} > 0, \quad 1.7 \leqslant 4\bar{s} \leqslant 1.76;$

7) $f_0^{00}(0.8) > f_0^{00}\left(\frac{0.21}{4}\right) > f_0^{00}\left(\frac{2.98}{4}\right).$

Direct calculations show that the amplitude (4.2) is completely consistent with all these inequalities.

5. ELECTROMAGNETIC INTERACTIONS

The interaction with the electromagnetic field A_μ is introduced in the Lagrangian in the usual gauge-invariant manner:

$$\partial_\mu \chi^\pm \to (\partial_\mu \pm ieA_\mu)\chi^\pm, \quad \chi^\pm = (\pi^\pm, K^\pm, p, \Sigma^\pm, \Xi^\pm). \tag{5.1}$$

A. Electromagnetic interactions of pions[18,22,23]

1) *The form factor*. The matrix element for a pion in an external electromagnetic field A_μ is given by

$$\langle \pi^+ | S(A) | \pi^+ \rangle = ie \frac{p_\mu A_\mu(q)}{(2\pi)^3 2\sqrt{p_1^0 p_2^0}} \Phi_\pi(q),$$

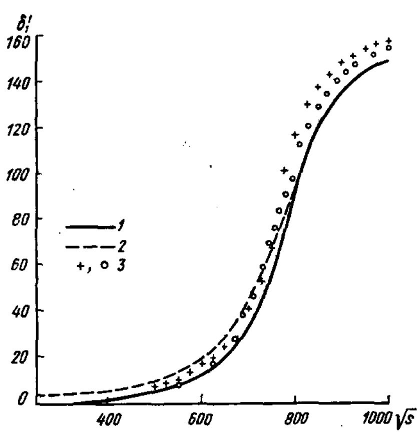

FIG. 4. Behavior of the P-wave phase shift δ_1^1 of the $\pi\pi$ scattering amplitude. 1—theoretical curve for $m_\pi \neq 0$,[17] 2—for $m_\pi = 0$,[15,16] 3—experimental points from[32a] and[32b], respectively ($\sqrt{s}$ is in MeV).

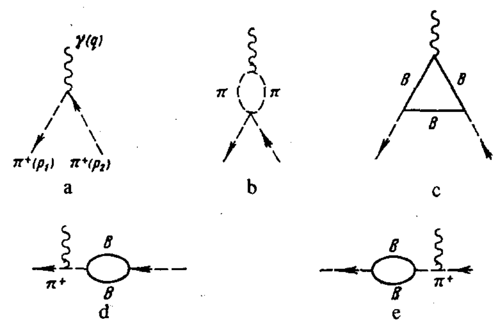

FIG. 5. Diagrams corresponding to the "tree" approximation (a) and the single-loop approximation (order e/F_π^2) (b—e) for the pion form factor. The wavy lines represent photons.

where p_1 and p_2 are the pion momenta, $p = p_1 + p_2$, $q = p_1 - p_2$, and

$$\Phi_\pi(q) = 1 + \Phi_\pi^{(\pi)}(q) + \Phi_\pi^{(b)}(q) + \ldots \tag{5.2}$$

is the pion form factor. Here $\Phi_\pi^{(\pi)}(q)$ is the contribution to the form factor from the pion diagram 5b, and $\Phi_\pi^{(b)}(q)$ is the contribution from the baryon diagrams c–e of Fig. 5 in the e/F_π^2 approximation.

We make use of the SP method to calculate the function $\Phi_\pi^{(\pi)}(q)$. This gives

$$\Phi_\pi^{(\pi)}(q) = \alpha_0\left[\bar{q}^2\left(\frac{13}{24} - \frac{3}{2}C + \ln\frac{2\pi F_\pi}{m_\pi}\right) - 1 + \frac{4}{3}\bar{q}^2 + (1 - \bar{q}^2)J(\bar{q}^2)\right], \tag{5.3}$$

where $C = 0.577\ldots$, $\bar{q}^2 = q^2/4m_\pi^2$, and α_0 and $J(\bar{q}^2)$ are the same as in Eq. (4.2). It can be seen from (5.3) that the pion-loop contribution to the radius of the pion is given by

$$\langle r^2\rangle_\pi^{(\pi)} = \frac{3}{2}\frac{\alpha_0}{m_\pi^2}\left(\frac{13}{24} - \frac{3}{2}C + \ln\frac{2\pi F_\pi}{m_\pi}\right) \approx 0.065\ \mathrm{F}^2. \tag{5.4}$$

As before, we calculate the contribution from the baryon diagrams as far as the q^2 terms, since the remaining terms are small. All the divergences in the diagrams c–e of Fig. 5 cancel among themselves, and we find that $\Phi_\pi^{(b)}(q)$ is given by the expression[7)]

$$\Phi_\pi^{(b)}(q) \approx \frac{1.7}{6(2\pi)^2} g^2 \frac{q^2}{M_N^2} \tag{5.5}$$

This gives the following contribution to the mean-square radius of the pion:

$$\langle r^2\rangle_\pi^{(b)} = 0.36\ fm^2.$$

The radius of the pion

$$\sqrt{\langle r^2\rangle_\pi} = \sqrt{\langle r^2\rangle_\pi^{(\pi)} + \langle r^2\rangle_\pi^{(b)}} \approx 0.65\ \mathrm{F} \tag{5.6}$$

is in satisfactory agreement with the latest experimental data.[34]

Substituting the functions (5.3) and (5.5) in (5.2), we

7)The factor 1.7 appears after taking into account all the members of the baryon octet (see[18]). The πK interactions contribute very little to the pion form factor.

find the following expression for the pion form factor:

$$\Phi_\pi = 1 + \alpha_0 [-1 + 8.6\bar{q}^2 + (1-\bar{q})^2 J(\bar{q}^2)].$$

This formula describes a behavior of the pion form factor at energies $\sqrt{|q^2|} < 1$ GeV which is in good agreement with experimental data recently obtained at Dubna and at Serpukhov[34] (see the graphs in Fig. 6).

It is interesting to note that the radius of the pion is determined almost entirely by the contribution of the baryon loop diagrams. The value (5.6) for the radius is close to predictions based on the ρ-dominance model ($\sqrt{\langle r^2\rangle_\pi} \sim \sqrt{6/m_\rho^2} \sim 0.64$ F).

2) *The Compton effect.* The matrix element corresponding to the Compton effect on the pion can be written in the form

$$\langle \pi^a(p_1)\, \pi^b(p_2) | S | \gamma_{\lambda_1}(q_1)\, \gamma_{\lambda_2}(q_2)\rangle = \frac{i\delta^{(4)}(p_1+p_2-p_3-p_4)\, \varepsilon^\mu_{\lambda_1}\varepsilon^\nu_{\lambda_2}}{(2\pi)^2\, 4\sqrt{p_1^0 p_2^0 q_1^0 q_2^0}}\, T^{\mu\nu}_{ab}(p_1, p_2 | q_1, q_2),$$

where q_1 and q_2 are the photon momenta, $\varepsilon^\mu_{\lambda_1}$ and $\varepsilon^\nu_{\lambda_2}$ are the polarizabilities, p_1 and p_2 are the pion momenta, and a and b are isotopic indices. We note at the outset that in the case of this process the divergences cancel in the single-loop approximation not only in the baryon loop diagrams, but also in the pion loop diagrams. There is therefore no need to use the SP method here. Without giving the general form of the covariant amplitude $T^{\mu\nu}_{ab}$, we quote here the form that is obtained in the lowest orders of perturbation theory:

$$\begin{aligned} T^{\mu\nu}_{ab} = 2e^2(\delta_{ab} - \delta_{3a}\delta_{3b}) \Big\{ g^{\mu\nu} - \frac{p_1^\mu p_2^\nu}{p_1 q_1} - \frac{p_1^\nu p_2^\mu}{p_1 q_2} \\ + (g^{\mu\nu} q_1 q_2 - q_1^\nu q_2^\mu)[\beta^{(\pi)}_\pi(q_1 q_2) + \beta^{(b)}_\pi(q_1 q_2)] \Big\} \\ + 4e^2 \delta_{3a}\delta_{3b}(g^{\mu\nu} q_1 q_2 - q_1^\nu q_2^\mu)\, \beta^{(\pi)}_\pi(q_1 q_2). \end{aligned} \quad (5.7)$$

The first three terms in the curly brackets are the Born terms (the diagrams a and b of Fig. 7), $\beta^{(\pi)}_\pi(q_1 q_2)$ is the contribution of the pion loops (c and d of Fig. 7), and $\beta^{(b)}_\pi(q_1 q_2)$ is the contribution of the baryon loops (e–h of Fig. 7). We have retained only the constant terms in $\beta^{(b)}_\pi$, since the remaining terms of the expansion in powers of $(q_1 q_2)$ are small. In deriving (5.7), we have also made use of the equalities

$$(q_1\varepsilon_1) = (q_2\varepsilon_2) = 0, \quad q_1^2 = q_2^2 = 0, \quad p_1^2 = p_2^2 = m_\pi^2.$$

If the contributions to the amplitude from the diagrams c and d of Fig. 7 are calculated together, we obtain the finite expression

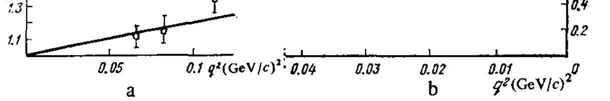

FIG. 6. Behavior of the pion form factor in the regions $q^2 > 0$ (a) and $q^2 < 0$ (b). 1—theoretical curve,[18] 2—experimental points from[34a] and[34b], respectively.

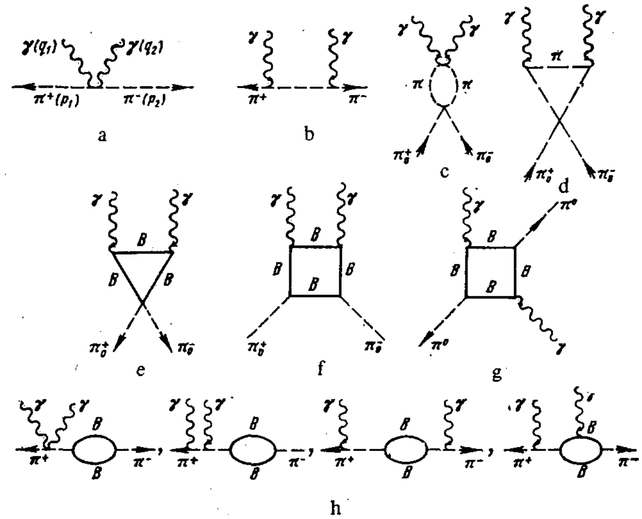

FIG. 7. Diagrams corresponding to the "tree" approximation (a, b) and the single-loop approximation (order e^2/F_π^2) (c–h) for the Compton effect on the pion.

$$\beta^{(\pi)}_\pi(q_1 q_2) = (4\pi F_\pi)^{-2}\left(1 - \frac{2}{3}\frac{m_\pi^2}{q_1 q_2}\right)\left\{\frac{2m_\pi^2}{q_1 q_2}\left[\operatorname{arctg}\left(\frac{2m_\pi^2}{q_1 q_2} - 1\right)^{-1/2}\right]^2 - 1\right\}.$$

In considering the Compton effect on the neutral pion, there is a complete cancellation among the diagrams e–g of Fig. 7. For charged pions, the contributions from the nucleon diagrams e–h of Fig. 7 are given by

$$\beta^{(N)}_\pi = \frac{2}{3}\frac{g_A^2}{(4\pi F_\pi)^2}. \quad (5.8)$$

As before, allowance for the contributions from the remaining members of the baryon octet leads to an additional factor 1.7 in (5.8).

Defining the polarizability of the pion as the coefficient of the effective interaction of the pion with an external electromagnetic field A_μ,[8]

$$V_{\text{int}} = -\frac{\alpha_\pi}{2}(\mathbf{E}^2 - \mathbf{H}^2),$$

we obtain

$$\begin{aligned} \alpha_{\pi^\pm} = \alpha_{\pi^\pm}(q_1 q_2)|_{q_1 q_2 = 0} = \frac{e^2\beta^{(\pi)}_\pi(0) + \beta^{(b)}_\pi}{m_\pi(\beta^{(\pi)}_\pi(0) + \beta^{(b)}_\pi)} = 0.33\frac{\alpha}{m_\pi^3} = 7\cdot 10^{-3}\ \text{F}^3, \\ \alpha_{\pi^0} = \alpha_{\pi^0}(q_2 q_2)|_{q_1 q_2 = 0} = \frac{2e^2\beta^{(\pi)}_\pi(0)}{m_\pi} = -0.04\frac{\alpha}{m_\pi^3} = -8\cdot 10^{-4}\ \text{F}^3. \end{aligned} \quad (5.9)$$

$$\left(\alpha = \frac{e^2}{4\pi} = \frac{1}{137}\right).$$

It is interesting to note that the function $\beta^{(\pi)}_\pi(q_1 q_2)$ varies rapidly in the threshold region. Thus, at the two-pion production threshold, we obtain

$$\alpha_{\pi^\pm}(2m_\pi^2) = 0.53\frac{\alpha}{m_\pi^3}, \quad \alpha_{\pi^0}(2m_\pi^2) = 0.35\frac{\alpha}{m_\pi^3}.$$

[8] The energy factor $(g^{\mu\nu} q_1 q_2 - q_1^\nu q_2^\mu)$, which is always present in the amplitude $T^{\mu\nu}_{ab}$ in the single-loop approximation (see (5.7)), corresponds in quantum-mechanical language to the combination $\mathbf{E}^2 - \mathbf{H}^2$. This implies that the electric and magnetic polarizabilities of the pion in this approximation are equal in magnitude but opposite in sign.

These values of $\alpha_{\pi\pm}$ are of the same order of magnitude as estimates based on current algebra[45] and on quark models,[46] differing by a factor of two from the predictions of[45]. Moreover, the value $\alpha_{\pi 0}=0$ was obtained in[45].

B. Electromagnetic interactions of kaons [20]

We turn now to the problem of calculating the electromagnetic form factor and polarizability of the kaon.

1) *The form factor.* We write the form factor of the charged kaon in the form

$$\Phi_K(q)=1+\Phi_K^{(\pi)}(q)+\Phi_K^{(K)}(q)+\Phi_K^{(b)}(q)+\ldots;$$

here $\Phi_K^{(\pi)}$ is the contribution to the form factor from the pion loop diagram shown in Fig. 5b, $\Phi_K^{(K)}$ is the contribution from the kaon loop diagram, and $\Phi_K^{(b)}$ is the contribution from the baryon loop diagrams c and d in Fig. 5 but with kaons at the ends. As before, these contributions correspond to the e/F_π^2 approximation. As in the case of the pion form factor, the contribution from the kaon loop can be neglected, and we give here only the expressions for $\Phi_K^{(\pi)}(q)$ and $\Phi_K^{(b)}(q)$; the first of these contributions is given by

$$\Phi_K^{(\pi)}(q)=\alpha_0\left\{\frac{q^2}{8m_\pi^2}\left[\ln\left(2\left(\frac{2\pi F_\pi}{m_\pi}\right)^2\right)-3C+1\right]-1+\frac{q^2}{3m_\pi^2}+\left(1-\frac{q^2}{4m_\pi^2}\right)J\left(\frac{q^2}{4m_\pi^2}\right)\right\};$$

here the constants α_0 and C and the function $J(q^2/4m_\pi^2)$ are the same as in Eqs. (4.2) and (5.3). The term in the square brackets contributes to the mean-square radius of the kaon. This term has the value

$$\langle r^2\rangle_{K\pm}^{(\pi)}=0.08\ \text{F}^2. \tag{5.10}$$

The contribution from the baryon diagrams is again substantial; it is given by[9)]

$$\Phi_K^{(b)}(q)=\frac{1.4}{6(2\pi)^2}\frac{g^2}{M_N^2}q^2. \tag{5.11}$$

It follows from (5.10) and (5.11) that the root-mean-square radius of the charged kaon has the value

$$\sqrt{\langle r^2\rangle_{K\pm}}\approx 0.61\ \text{F}.$$

For the neutral kaon, the contributions from the tree diagrams and the baryon loop diagrams are equal to zero, while the contribution from the pion loop diagram remains the same as before in magnitude but has the opposite sign. Hence we have

$$\sqrt{\langle r^2\rangle_{K^0}}\approx 0.28\ \text{F}.$$

9)The factor 1.4 appears when allowance is made for the contributions from the entire baryon octet. It is of interest to note that in the case of exact $SU(3)$ symmetry all the single-loop diagrams for the self-energy of the meson, for its form factor, and for the amplitude of the Compton effect are proportional to the function $f(\bar\alpha)=3(1-\bar\alpha)^2+(5/3)\bar\alpha^2$. The minimum of this function corresponds to the value $\bar\alpha=0.65$, in good agreement with the experimental data.

This last result is in good agreement with predictions based on the vector-dominance model (a variant involving a model of current mixing[47]).

2) *The Compton effect.* We now give a brief account of the results of calculations of the amplitude for the Compton effect on the kaon.

In addition to the diagrams shown in Fig. 7 but with kaons instead of pions in the external lines, we shall also consider the two diagrams c and d of Fig. 7 but with internal kaon lines. These diagrams gave a negligibly small contribution to the amplitude for the Compton effect on the pion, but they must now be taken into account. Dropping the Born terms, we then obtain the following expressions in the e^2/F_π^2 approximation for the amplitudes with charged or neutral external kaons:

$$T_+^{\mu\nu}=2e^2(g^{\mu\nu}q_1q_2-q_1^\nu q_2^\mu)[\beta_K^{(\pi)}(q_1q_2)+\beta_K^{(K^+)}(q_1q_2)+\beta_K^{(b)}],$$
$$T_0^{\mu\nu}=2e^2(g^{\mu\nu}q_1q_2-q_1^\nu q_2^\mu)[\beta_K^{(\pi)}(q_1q_2)+\beta_K^{(K^0)}(q_1q_2)].$$

The function $\beta_K^\pi(q_1q_2)$ corresponds to the contribution from the two diagrams containing internal pion lines (the diagrams c and d of Fig. 7). When these diagrams are calculated together, they give the finite contribution

$$\beta_K^{(\pi)}(q_1q_2)=(4\pi F_\pi)^{-2}\frac{q_1q_2}{4m_\pi^2}\mathcal{J}\left(\frac{q_1q_2}{2m_\pi^2}\right),$$

where

$$\mathcal{J}(\xi)=\frac{1}{\xi}\left\{\frac{1}{\xi}\left[\operatorname{arctg}\left(\frac{1}{\xi}-1\right)^{-1/2}\right]^2-1\right\}.$$

The function $\mathcal{J}(\xi)$ varies quite rapidly as ξ increases, so that the contribution to the amplitude $T_{\pm 0}^{\mu\nu}$ from $\beta_K^\pi(q_1q_2)$, which is equal to zero at $q_1q_2=0$, can become appreciable at sufficiently large q_1q_2.

The function $\beta_K^{(K^+)}(q_1q_2)$ corresponds to the contribution from the two diagrams containing internal kaon lines and charged external kaons:

$$\beta_K^{(K^+)}(q_1q_2)=(8\pi F_\pi)^{-2}\left(1+\frac{q_1q_2}{2m_K^2}\right)\mathcal{J}\left(\frac{q_1q_2}{2m_K^2}\right).$$

The analogous function in the case of neutral external kaons is given by

$$\beta_K^{(K^0)}(q_1q_2)=(4\pi F_\pi)^{-2}\frac{q_1q_2}{4m_K^2}\mathcal{J}\left(\frac{q_1q_2}{2m_K^2}\right).$$

These results show that the only non-zero contribution to $T_{\pm 0}^{\mu\nu}$ at $q_1q_2=0$ comes from the function $\beta_K^{(K^+)}\approx 0.08\times(4\pi F_\pi)^{-2}$.

With neutral external kaons, as in the case of neutral pions at the ends, the contribution to the amplitude $T_0^{\mu\nu}$ from the baryon loop diagrams is equal to zero. For charged external kaons, the combined contribution from the diagrams e, f, and h of Fig. 7 is given by $\beta_K^{(b)}\approx 1.4(4\pi F_\pi)^{-2}$; we have retained only the constant terms here, since the higher-order terms of the expansion in powers of (q_1q_2) (of the type $O(q_1q_2/M_B^2)$) are small. This shows that the baryon loop diagrams give the decisive contribution to the amplitude $T_+^{\mu\nu}(0)$ for the Compton effect at $(q_1q_2)=0$. For the other amplitude, we have $T_0^{\mu\nu}(0)=0$.

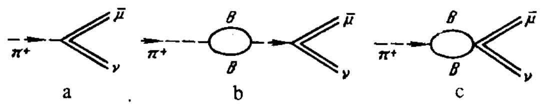

FIG. 8. Diagrams corresponding to the "tree" approximation (a) and the single-loop approximation (order G/F_π) (b, c) for the decay $\pi^+\to\bar\mu\nu$. The double lines represent leptons.

Using equations analogous to (5.9), we find that the polarizability of the kaon has the values

$$\alpha_{K^\pm}=\left(\frac{e}{4\pi F_\pi}\right)^2\cdot\frac{1.5}{m_K}\approx 1.6\cdot 10^{-3}\ \mathrm{F}^3,\qquad \alpha_{K^0}=0.$$

These values are consistent with recent theoretical estimates based on current algebra and PCAC,[48] giving $\alpha_{K^+}\sim 10^{-3}\ \mathrm{F}^3$, and with experimental data,[49] which, unfortunately, have rather poor accuracy at the present time, giving $\alpha_{K^+}^{\exp}\sim -(4\pm 11)\times 10^{-3}\ \mathrm{F}^3$.

6. WEAK INTERACTIONS

A. Decays of charged pions[21]

We shall now consider the principal decays of charged mesons and calculate the structure constants for these decays. To do this, we shall have to supplement the chiral Lagrangian with the part which is responsible for the weak interactions. We write this part in the form

$$L_{\mathrm{int}}^{(G)}=\,:L_\mu^{(+)}[-\sqrt{2}\,F_\pi\partial_\mu\pi^-+i\sqrt{2}\,(\pi^-\partial_\mu\pi^0-\pi^0\partial_\mu\pi^-)+\bar\psi_p\gamma_\mu(1-ig_A\gamma_5)\psi_n+ie\sqrt{2}\,F_\pi\pi^-A_\mu]:,$$

where $L_\mu^{(+)}=(G/\sqrt{2})\cos\theta\,\bar\mu(\bar e)\gamma_\mu(1-i\gamma_5)\nu$, G is the weak coupling constant, θ is the Cabibbo angle, and μ, e, and ν are the muon, electron, and neutrino fields.

We shall adopt the usual definition of the transition amplitudes T; for example, for the process $\pi^\pm\to\mu^\pm\nu\gamma$, we have

$$\langle\mu\nu(l),\ \gamma_\lambda(q)\,|\,S\,|\,\pi(p)\rangle=i\pi\frac{\delta^{(4)}(p-q-l)}{\sqrt{p^0q^0}}\,\varepsilon_\lambda^\mu T_\mu,$$

where ε_λ^μ is the polarization of the photon, and p, q, and l are the momenta of the pion, photon, and lepton pair, respectively. Since the contributions from baryon loops are much larger than those from pion loops, as can easily be seen from the preceding calculations, we shall take into account only the baryon contributions here.

1) We begin with a discussion of the fundamental *pion decay* $\pi^\pm\to\mu^\pm\nu(e^\pm\nu)$. (We first consider only the πN interaction. At the end of this section we indicate what corrections arise when allowance is made for all the baryons of the octet.) This process can be used to fix the only parameter F_π of the chiral theory. It turns out that the order of perturbation theory after the Born term gives only a small correction to F_π, and there is again a complete cancellation of the divergences in the loop diagrams b and c of Fig. 8.

Thus, in the single-loop approximation, we obtain

$$T_{(\pi\to\mu\nu)}=i\sqrt{2}\,F_\pi\left[1-\frac{1}{6}\left(\frac{g_A m_\pi}{2\pi F_\pi}\right)^2\right]p_\mu l_\mu^{(+)},\qquad(6.1)$$

where p_μ is the pion momentum, and $l_\mu^{(+)}=(G/\sqrt{2})\times\cos\theta\,\bar u_{(\mu)}\gamma_\mu(1-i\gamma_5)u_{(\nu)}$ is the leptonic current. The second term in the brackets is much less than unity. Comparing (6.1) with the experimental data, we have $F_\pi\approx 93$ MeV.

2) We now consider *the process* $\pi^\pm\to\mu^\pm\nu\gamma$. A detailed discussion of this process can be found in[45,50]. The Born approximation is determined by the diagrams of Fig. 9a:

$$T_\mu^{(b)}=ieF_\pi\left\{\sqrt{2}\left[g_{\mu\nu}+\frac{p_\mu(p-q)_\nu}{pq}\right]l_\nu^{(+)}-G\cos\theta\,\bar u_{(\mu)}\gamma_\mu(\hat k+\hat q-m_{(\mu)})^{-1}\hat p\,(1-\gamma_5)\,u_{(\nu)}\right\}.$$

The single-loop approximation consists essentially of the diagrams b and c of Fig. 9. These diagrams give a contribution of the form

$$T_\mu^{(0)}=-\frac{1}{6}\left(\frac{g_A m_\pi}{2\pi F_\pi}\right)^2 T_\mu^{(b)}-ie\sqrt{2}\,[ih_V\varepsilon_{\mu\nu\alpha\beta}p_\alpha q_\beta-h_A(g_{\mu\nu}pq-p_\mu q_\nu)]\,l_\nu^{(+)},$$

where

$$h_V=\frac{g_A}{8\pi^2F_\pi},\qquad h_A=\frac{g_A^2}{6\,(2\pi)^2F_\pi},\qquad(6.2)$$

and $\varepsilon_{\mu\nu\alpha\beta}$ is the completely antisymmetric tensor. Thus allowance for nucleon loops leads to 1) a renormalization of the constant F_π (see (6.1)); 2) terms describing structure radiation.

We find that the ratio $h_A/h_V=\gamma$ is given by

$$\gamma=\frac{g_A}{3}\approx 0.41,\qquad(6.3)$$

while experiment gives two possible values $\gamma=\{0.4, -2\}$.[51]

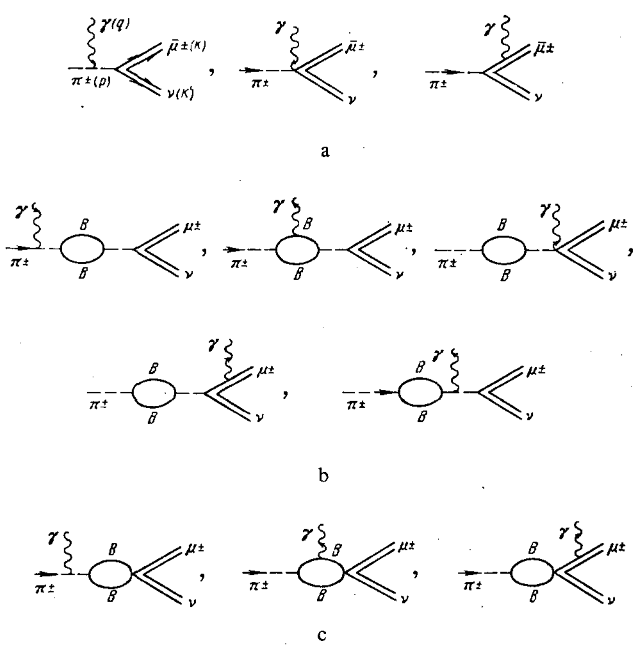

FIG. 9. Diagrams corresponding to the "tree" approximation (a) and the single-loop approximation (order eG/F_π) (b, c) for the decay $\pi^+\to\bar\mu\nu\gamma$.

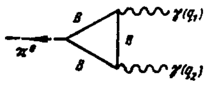

FIG. 10. Diagram corresponding to the single-loop approximation (order e^2/F_π) for the decay $\pi^0 \to \gamma\gamma$.

We note that our approach automatically satisfies the current-algebra relations[45] between the constant h_V and the constant f for the decay $\pi^0 \to \gamma\gamma$ and between the constant h_A and the pion polarizability β_π.

The amplitude for the process $\pi^0 \to \gamma\gamma$ was first calculated by Steinberger[52] in the single-loop approximation (Fig. 10):

$$T_{\mu\nu} = f\varepsilon_{\mu\nu\alpha\beta}q_1^\alpha q_2^\beta, \quad f = -\frac{e^2 g_A}{(2\pi)^2 F_\pi} = 0.59\frac{\alpha}{m_\pi}. \tag{6.4}$$

Here q_i are the photon momenta. The experimental values of f are as follows:

$$|f| = (0.45^{[53]},\ 0.57^{[54]})\frac{\alpha}{m_\pi}.$$

The current-algebra relation has the form

$$h_V = -\frac{f}{2e^2}. \tag{6.5}$$

Comparing (6.4) with (6.2), it is easy to see that this relation is satisfied.

The polarizability of the pion at energies $q_1 q_2 = 0$ is determined mainly by the baryon contributions (see Eq. (5.8)). Comparing (5.8) with (6.2), we obtain

$$h_A = F_\pi \beta_\pi^{(N)}. \tag{6.6}$$

This is in fact the relation which follows from current algebra.[45] Allowance for the contributions from the remaining members of the baryon octet leads to factors 1.7 in the coefficients with g_A^2 and 1.2 in the coefficients with g_A. Thus the relations (6.5) and (6.6) are still satisfied.

3) Finally, we consider *the process* $\pi^+ \to \pi^0 e^+ \nu$ (Fig. 11). The calculation of the amplitude for this process is very similar to the calculation of the pion form factor and gives the result

$$T_{(\pi^+\to\pi^0 e\nu)} = T^{(b)}\left[1 + \frac{1}{6}\left(\frac{g_A}{2\pi F_\pi}\right)^2 q^2\right] - \frac{\sqrt{2}}{6}\left(\frac{g_A}{2\pi F_\pi}\right)^2 (m_{\pi^0}^2 - m_{\pi^+}^2)\, q^\nu l_\nu^{(+)},$$

where

$$T^{(b)} = -\sqrt{2}(p_{\pi^+} + p_{\pi^0})^\nu l_\nu^{(+)}, \quad q = p_{\pi^+} - p_{\pi^0}.$$

This concludes our discussion of pion interactions.

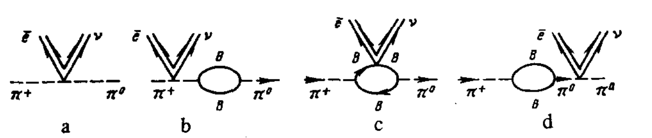

FIG. 11. Diagrams corresponding to the "tree" approximation (a) and the single-loop approximation (order G/F_π^2) (b–d) for the decay $\pi^+ \to \pi^0 \bar{e}\nu$.

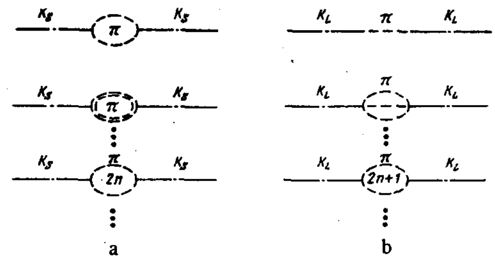

FIG. 12. Diagrams corresponding to the intermediate states of the K_S and K_L mesons. The dot-dash lines represent kaons.

B. The K_L-K_S mass difference[19]

To conclude our survey of the low-energy interactions of mesons, we calculate the mass difference of the neutral kaons.

We introduce one further Lagrangian, which describes the πK interaction and corresponds to the $\Delta T = 1/2$ rule. The simplest such chiral Lagrangian which contains no derivative couplings has the form[19]

$$L^{(1/2)}_{\pi K^0} = a : \left[K_S\left(\cos\sqrt{\frac{\pi^2}{F_\pi^2}} - 1\right) + K_L \frac{\pi^0}{F_\pi}\frac{\sin\sqrt{\pi^2/F_\pi^2}}{\sqrt{\pi^2/F_\pi^2}}\right]:$$

where $K_S = (\bar{K}_0 + K_0)/\sqrt{2}$ and $K_L = i(\bar{K}_0 - K_0)/\sqrt{2}$. The Born approximation for this Lagrangian accurately reproduces the low-energy theorems of current algebra concerning the non-leptonic decays of K_0 mesons into two or three pions.

The coupling constant a can be fixed by the probability for the decay $K_S \to 2\pi$ ($w(2\pi)$). This gives

$$a^2 = \frac{2\pi(2F_\pi)^4 m_K}{3\sqrt{1-(2m_\pi/m_K)^2}}\, w^{(2\pi)}. \tag{6.7}$$

We now proceed to the calculation of the mass difference between the K_L and K_S mesons. This mass difference is due to the different virtual states into which each of these mesons can be transformed, taking into account their combined parity (Fig. 12). Bearing this in mind, the mass difference Δm_{K_0} can be written in the form

$$\Delta m_{K_0} = m_{K_L} - m_{K_S} = 2(f_S - f_L), \tag{6.8}$$

where f_S is the sum of the matrix elements corresponding to the infinite set of diagrams containing an even number of virtual pions (Fig. 12a), and f_L is the same for the diagrams containing an odd number of pions (Fig. 12b).

The quantities f_S and f_L can easily be calculated by using the SP method.[25] It is found in this way that the value of Δm_{K_0} is determined almost entirely by the two-pion diagram. The diagrams containing three or more virtual pions contribute less than 1% to the value of Δm_{K_0}. The contribution from the single-pion diagram must be calculated together with the contribution from the diagram containing a single virtual η meson, and these contributions completely cancel according to exact $SU(3)$ symmetry. It is not essential to take into account the η meson in the loop diagrams.

The expression for the matrix element corresponding to the two-pion diagram takes the form

$$f_S^{(2\pi)} = \frac{3(2\pi a)^2}{m_K (4\pi F_\pi)^4}\left[\ln\left(\frac{4\pi F_\pi}{m_\pi}\right) - \frac{3}{2}C + \frac{13}{12} - J\left(\frac{m_K^2}{4m_\pi^2}\right)\right], \quad (6.9)$$

where C is the Euler constant, and $J(m_K^2/4m_\pi^2)$ is given in (4.3). Substituting (6.7) and (6.9) in (6.8), we obtain

$$\mathrm{Re}\,\Delta m_{K_0} = 0.52 w^{(2\pi)},$$

while the experimental value of Δm_{K_0} is $0.48 w^{(2\pi)}$ (see[55]).

7. CONCLUSIONS

The foregoing examples by no means exhaust all the problems which can be studied in the single-loop approximation of quantum chiral theory. We mention, as an example, the (leptonic, semi-leptonic, and non-leptonic) decays of kaons. Allowance for the single-loop diagrams in calculations of these decays can help to elucidate the group structure of chiral symmetry breaking.

To summarize the examples which we have given of the utilization of quantum chiral theory in describing low-energy mesonic processes, we can say that the results that have been obtained at least reproduce the actual qualitative features of the various physical processes, leading to good quantitative agreement with the experimental data in the majority of cases.

These results show that the universality of the strong, weak, and electromagnetic interactions of hadrons may account for the successful application of perturbation theory, not only to first order, but in the next order in the strong coupling constant, as confirmed by direct estimates of the two-loop approximation.[14,29] We recall that quantum chiral theory in the form in which it has been formulated here can be used successfully only at low energies much less than $4\pi F_\pi \approx 1.2$ GeV, which is the energy scale that occurs naturally in this theory. At higher energies, the approximation of point-like hadrons may be inapplicable, in view of the structure of the hadrons.

It is continually becoming more certain that such a structure does exist, thanks to the successes of the quark model of current algebra on the light cone and pure quark models in explaining electroproduction processes and neutrino reactions at high energies and in describing an enormous number of resonance decays and data on hadron spectroscopy.

At the same time, these successes definitely indicate that chiral symmetry is an approximate symmetry of the strong interactions for all currently accessible energies. However, this symmetry is realized in different ways at different energies. In this connection, it seems to us that one of the most interesting problems is that of understanding the transition from a dynamical realization to an algebraic realization of chiral symmetry in the region of moderate energies.

Dual resonance models which describe the interaction of extended objects (strings) provide certain possibilities of studying this problem. On the one hand, these models reproduce the spectrum of hadronic states (the classification by Regge trajectories) and lead to the Veneziano amplitudes; on the other hand, they reduce to field-theoretic models of point-like particles in the low-energy limit. In particular, the most realistic dual resonance model of Neveu and Schwarz[56] has as its point-like limit the chiral theory involving the non-linear phenomenological Lagrangian considered in the present review.

In conclusion, the authors express their gratitude to D. I. Blokhintsev for providing the initiative for writing this review and for valuable remarks, and to D. V. Volkov, V. A. Meshcheryakov, V. V. Serebryakov, and D. V. Shirkov for useful discussions.

[1]a) D. V. Shirkov, V. V. Serebryakov, and V. A. Meshcheryakov, Dispersionnye teorii sil'nykh vzaimodeĭstviĭ pri nizkikh énergiyakh (Dispersion Theories of Strong Interactions at Low Energy), Nauka, Moscow, 1967 (North-Holland, 1969). b) V. V. Serebryakov and D. V. Shirkov, Fortschr. Phys. 18, 527 (1970).
[2]a) R. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958). b) M. Gell-Mann, *ibid.* **125**, 1067 (1962); Physics **1**, 63 (1964).
[3]R. P. Feynman, Photon-Hadron Interactions, Benjamin, 1972 (Russ. Transl., Mir, Moscow, 1975). J. D. Bjorken and B. L. Ioffe, Usp. Fiz. Nauk **116**, 115 (1975) [Sov. Phys. Usp. 18, 361 (1975)].
[4]Y. Nambu, Phys. Rev. Lett. **4**, 380 (1960). F. Gursey, Nuovo Cimento **16**, 230 (1960). M. Gell-Mann and M. Levy, *ibid.*, p. 705.
[5]M. Goldberger and S. Treiman, Phys. Rev. **110**, 1478 (1958).
[6]A. N. Vaĭnshteĭn and V. I. Zakharov, Usp. Fiz. Nauk **100**, 225 (1970) [Sov. Phys. Usp. 13, 73 (1970)]. S. L. Adler and R. F. Dashen, Current Algebras, Benjamin, 1968 (Russ. Transl., Mir, Moscow, 1970). H. Pagels, Phys. Reports **16**, 219 (1975).
[7]S. Adler, Phys. Rev. **B139**, 1638 (1965).
[8]F. Low, *ibid.* **96**, 1428 (1954). M. Gell-Mann and M. L. Goldberger, *ibid.*, p. 1433.
[9]B. W. Lee, Nucl. Phys. **B9**, 649 (1969). J. L. Gervais and B. W. Lee, *ibid.* **B12**, 627 (1968). H. W. Grater, Phys. Rev. **D1**, 3313 (1970). A. A. Khelashvili and V. Yu. K Khmaladze, Teor. Matem. Fiz. **15**, 78 (1973).
[10]A. A. Slavnov and L. D. Faddeev, Teor. Matem. Fiz. **8**, 297 (1971). A. A. Slavnov, Nucl. Phys. **B31**, 301 (1971). S. Gerstein, R. Jackiw, B. W. Lee, and S. Weinberg, Phys. Rev. **D3**, 2486 (1971).
[11]J. Honerkamp, Nucl. Phys. **B36**, 130 (1972). V. N. Pervushin, JINR Preprint E2-7540, Dubna, 1973.
[12]V. N. Pervushin, Teor. Matem. Fiz. 22, 291 (1975).
[13]J. A. Mignaco and E. Remiddi, Nuovo Cimento **A1**, 376, 395 (1971). D. Bessis, S. Graff, V. Grecci, and G. Turchetti, Phys. Rev. **D1**, 2064 (1970). L. H. Chan and R. W. Haymaker, *ibid.* **D10**, 4170 (1974).
[14]K. S. Jhung and R. S. Willey, *ibid.* D9, 3132 (1974). J. L. Basdevant and B. W. Lee, *ibid.* D2, 1680 (1970).
[15]H. Lehmann and H. Trute, Nucl. Phys. **B52**, 280 (1973). H. Lehmann, Phys. Lett. **B41**, 529 (1972).
[16]G. Ecker and J. Honerkamp, Nucl. Phys. **B62**, 509 (1973).
[17]M. K. Volkov and V. N. Pervushin, Yad. Fiz. **20**, 762 (1974) [Sov. J. Nucl. Phys. 20, 408 (1975)].
[18]M. K. Volkov and V. N. Pervushin, Yad. Fiz. **19**, 652 (1974) [Sov. J. Nucl. Phys. 19, 330 (1974)]; Phys. Lett.

B51, 356 (1974).
[19]M. K. Volkov and V. N. Pervushin, Yad. Fiz. 21, 214 (1975) [Sov. J. Nucl. Phys. 21, 113 (1975)]; Phys. Lett. B51, 499 (1974).
[20]V. N. Pervushin and M. K. Volkov, *ibid.* B58, 177 (1975).
[21]M. K. Volkov and V. N. Pervushin, Yad. Fiz. 22, 366 (1975) [Sov. J. Nucl. Phys. 22, 189 (1975)].
[22]M. K. Volkov and V. N. Pervushin, Nuovo Cimento A27, 277 (1975).
[23]V. N. Pervushin and M. K. Volkov, Phys. Lett. B55, 405 (1975); Yad. Fiz. 22, 346 (1975) [Sov. J. Nucl. Phys. 22, 179 (1975)].
[24]V. I. Lend'el and M. I. Gaĭsak, Preprint 74-87E, Inst. of Theor. Physics, Ukrainian Academy of Sciences, Kiev, 1974.
[25]M. K. Volkov, Comm. Math. Phys. 7, 289 (1968); Ann. Phys. 49, 202 (1968); Teor. Matem. Fiz. 6, 21 (1971); Fortschr. Phys. 22, 499 (1974).
[26]H. Lehmann and K. Pohlmeyer, Comm. Math. Phys. 20, 101 (1971).
[27]A. Salam and I. Strathdee, Phys. Rev. D1, 3296 (1970); D3, 1805 (1971).
[28]M. Gell-Mann, R. J. Oakes, and B. Renner, *ibid.* 175, 2195 (1968).
[29]M. K. Volkov, D. I. Kazakov, and V. N. Pervushin, JINR Preprint E2-9170, Dubna, 1975.
[30]G. P. Pron'ko, Yad. Fiz. 22, 141 (1975) [Sov. J. Nucl. Phys. 22, 68 (1975)].
[31]S. B. Gerasimov and L. D. Soloviev, Nucl. Phys. 74, 589 (1965).
[32]a) J. P. Baton *et al.*, Phys. Lett. B33, 525, 528 (1970). b) S. D. Protopopescu *et al.*, L. A. L. Preprint 787, 1972. c) P. Baillon *et al.*, Phys. Lett. B38, 555 (1972). Y. Batusov, S. Buniatov *et al.*, JINR Preprint E1-7969, Dubna, 1974; Yad. Fiz. 21, 308 (1975) [Sov. J. Nucl. Phys. 21, 162 (1975)].
[33]F. P. Palou and F. J. Yndurain, Nuovo Cimento A19, 245 (1974).
[34]a) S. F. Berezhnev *et al.*, Yad. Fiz. 18, 102 (1973) [Sov. J. Nucl. Phys. 18, 53 (1974)]. b) G. Adylov *et al.*, Phys. Lett. B51, 402 (1974).
[35]D. V. Volkov, Fiz. Elem. Chastits At. Yadra 4, 3 (1973) [Sov. J. Part. Nucl. 4, 1 (1973)]; Preprint 69-75, Inst. of Theor. Physics, Ukrainian Academy of Sciences, Kiev, 1969.
[36]S. Coleman, G. Wess, and B. Zumino, Phys. Rev. 177, 2239 (1969). G. C. Callan *et al.*, *ibid.* 177, 2447 (1969).
[37]A. B. Borisov and V. I. Ogievetskiĭ, Teor. Matem. Fiz. 21, 329 (1974).
[38]V. B. Berestetskiĭ, Usp. Fiz. Nauk 85, 393 (1965) [Sov. Phys. Usp. 8, 147 (1965)]. N. N. Bogoliubov, in: Fizika vysokikh énergiĭ i teoriya élementarnýkh chastits (High Energy Physics and Elementary Particle Theory), Naukova dumka, Kiev, 1967, p. 5.
[39]G. V. Efimov, Zh. Eksp. Teor. Fiz. 44, 2107 (1963) [Sov. Phys. JETP 17, 1417 (1963)]. E. S. Fradkin, Nucl. Phys. B49, 624 (1963). B. A. Arbuzov and A. T. Filippov, JINR Preprint E2-3610, Dubna, 1967. M. K. Volkov, Fiz. Elem. Chastits At. Yadra 2, 33 (1971) [Sov. J. Part. Nucl. 2, 22 (1971)].
[40]N. N. Bogolyubov and D. V. Shirkov, Vvedenie v teoriyu kvantovannykh poleĭ (Introduction to the Theory of Quantized Fields), Nauka, Moscow, 1973 (Wiley, 1959).
[41]G. Ecker and J. Honerkamp, Nucl. Phys. B52, 211 (1973).
[42]V. N. Pervushin, JINR Preprint R2-7644, Dubna, 1973; JINR Preprint E2-8009, Dubna, 1974.
[43]A. Martin, Nuovo Cimento 47, 265 (1967).
[44]A. Martin, *ibid.* A58, 303 (1968).
[45]M. V. Terent'ev, Yad. Fiz. 16, 162 (1972) [Sov. J. Nucl. Phys. 16, 87 (1973)].
[46]T. E. O. Ericson and J. Hüfner, Nucl. Phys. B47, 205 (1972).
[47]N. M. Kroll, T. D. Lee, and B. Zumino, Phys. Rev. 157, 1376 (1967).
[48]M. V. Terent'ev, Yad. Fiz. 19, 1298 (1974) [Sov. J. Nucl. Phys. 19, 664 (1974)].
[49]G. Backenstoss *et al.*, Phys. Lett. B43, 431 (1973).
[50]D. Yu. Bardin and S. M. Bilen'kiĭ, Yad. Fiz. 16, 557 (1972) [Sov. J. Nucl. Phys. 16, 311 (1973)].
[51]D. Depommier *et al.*, Phys. Lett. 7, 885 (1963).
[52]I. Steinberger, Phys. Rev. 76, 1180 (1949).
[53]A. Rittenberg *et al.*, Rev. Mod. Phys. 43, 2 (1971).
[54]G. Belletini, C. Bemporad, and P. Braccini, Nuovo Cimento A66, 243 (1970).
[55]Rev. Mod. Phys. 45, Part II (1973).
[56]A. Neveu and J. H. Schwarz, Nucl. Phys. B31, 86 (1971). K. Kawarabayashi and S. Kitakado, Nuovo Cimento A14, 190 (1973).

Translated by N. M. Queen