Superfluidity of helium II near the λ point

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The properties (and particularly the superfluidity) of liquid helium near the λ point have long been and still are objects of numerous investigations. Nonetheless, much of the problem remains unsolved from both the theoretical and the experimental points of view. The main reason is the small correlation length in helium II even when the distance from the λ point is as small as hundredths of a degree, so that to reveal a number of specific effects one must work quite close to the λ point. This article treats in detail the phenomenological theory of superfluidity of helium near the λ point, a theory whose development dates back to 1958, and which is based on the use of the complex order parameter $\Psi = \eta e^{i\varphi}$ (the density of the superfluid part of helium is here $\rho_s = m\eta^2$, and the velocity of this helium component is $v = (\hbar/m)\nabla\varphi$, where m is the mass of the helium atom). In addition to formulating the general equations of the theory and discussing the regions where they are valid, we consider a number of concrete effects. The predictions of the theory are quite rich in content and can be verified in experiment. The main purpose of the article is just to contribute to such a verification.

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CONTENTS

I. INTRODUCTION

Helium was not observed on earth before 1895 (helium lines were observed earlier, in 1868, in the sun's spectrum), and was first liquefied in 1908 (the boiling temperature at atmospheric pressure is $T_b = 4.215$ °K; the critical point of the helium liquid-vapor system corresponds to a temperature $T_c = 5.20$ °K and a pressure $p_c = 2.26$ atm). It is curious that the first indication of the existence of a λ transition in liquid helium from helium I to helium II ($T_\lambda = 2.172$ °K at a saturated-vapor pressure $p_\lambda = 0.05$ atm), manifesting itself in an anomalous temperature dependence of the density ρ, was obtained by Kammerling-Onnes in the same year, 1911, in which he discovered superconductivity. It is possible that it is just this complete natural preoccupation with superconductivity which delayed further research on helium II; furthermore, the work was interrupted by the first world war. Be it as it may, not until 1924 was the λ transition in helium clearly noted on the $\rho(T)$ curve, and in 1928 this transition manifested itself even more pronouncedly in measurements of the dependence of the dielectric constant of liquid helium on the temperature (it was immediately afterwards that the constants of helium I and helium II were introduced). If

we speak of helium at equilibrium, the λ transition affects most strongly the course of its specific heat (1932), but a similar behavior of the specific heat is observed also for other transitions in other substances. Therefore the true distinguishing feature of helium II, for which the inclusive term "superfluidity" is presently used, manifests itself in the course of flow, heat transfer, etc. To be specific, the behavior of helium-II films, which at that time was not understood at all (and which we now know to be due to superfluidity), was observed back in 1922; the superthermal conductivity of helium II attracted attention in 1935, and finally, its superfluidity was discovered in 1938. This completed the first stage of the study of liquid helium (for more details see[1,2]).[1]

It took thus all of thirty years (and in fact, even somewhat more) to explain the most fundamental properties of helium II. It is therefore not surprising that the investigation of superfluidity has remained far from complete during the succeeding thirty years, in fact to this

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[1]For questions only indirectly connected with the topic of the present article we refer the reader, when possible, only to the reviews (see also the collection of original papers in[3]).

very day. The scale of low-temperature research, to be sure, became incomparably larger than during the first period, when liquid helium was produced in only about a dozen laboratories in the whole world (and for the first fifteen years, up to 1923, only in Leiden). On the other hand, hitherto undreamt of experimental capabilities came into being, and consequently the requirements imposed on the measurements have grown immeasurably and new problems have arisen. Furthermore, besides ^{4}He, studies of solutions of ^{3}He in ^{4}He were initiated, and then also of pure liquid ^{3}He (first produced in 1948). Phase transitions in liquid ^{3}He, due to the appearance of the superfluid phases of ^{3}He have been observed quite recently (1972–1974) at temperatures T_c 10^{-3}°K, and are now under intensive study (see[6]). The feasibility of producing superfluid molecular hydrogen,[7–9] or a superfluid excitonic phase in semiconductors has been suggested.[10,11] The question of superfluidity in cosmic conditions, especially in the case of a neutron liquid and neutron stars,[12,13] has also attracted much attention.

The study of superfluidity (more accurately speaking of superfluid substances) thus occupies at present a rather prominent place in physics, and there is no talk whatever of this field being "saturated," even if we restrict it only to helium II (i.e., to superfluid ^{4}He).

We regard it as quite probable that the unflagging interest in helium II is due, in particular, to the lack of an anywhere near complete microscopic theory of helium II, similar to the modern microscopic theory of superconductivity in metals. In brief, the Landau superfluidity theory[14] (see[2,4,5,15,16] concerning its subsequent development) can be divided into two parts: the quasi-microscopic theory, which operates with a gas of elementary excitations (quasiparticles), and the fluid hydrodynamics of superfluid liquids. Neither approach, however, is suitable in the vicinity of the λ point, i.e., in a region that is in any case very typical and important from the point of view of both theory and experiment. A consistent microscopic theory of liquid helium near the λ point would provide simultaneously a solution, "from first principles," of the problem of second-order phase transitions for a three-dimensional system, a task not yet completed even for simplified models. One such model, the most interesting from the viewpoint of liquid helium, is that of a non-ideal Bose gas without the assumption that the density is low (the gas approximation) or that the bond is weak.[17] However, even a perfectly successful further investigation of this model can hardly provide more than a check on the presently known phenomenological theory of second-order phase transitions,[17–19] supplemented by an estimate or a calculation of the critical exponents. Yet we expect much more from a theory that describes completely the properties and the behavior of helium II near the λ point—not only a determination of the temperature dependence of all the thermodynamic quantities for an unbounded liquid, but also an assessment of the influence of the boundaries (for example, the dependence of the λ-point temperature $T_\lambda(d)$ on the thickness d of the helium film), and finally, a dynamic theory of the flow of helium with a general allowance for the dissipation and for the relaxation of the order parameter.

It should be recalled in this connection that the order parameter does not enter in Landau's superfluidity theory at all, since this theory was constructed only for the region far from the λ transition. For this reason, a quantity connected with the order parameter, namely the density ρ_s of the superfluid part of the liquid, is regarded in the Landau theory as a given function of other quantities, for example ρ and T. Yet, when speaking of equilibrium, the density ρ_s near the transition point (the λ point) should be determined from the condition that the thermodynamic potential be a minimum. The quantity ρ_s was introduced as an order parameter in connection with the problem of critical velocities as early as in[20]; in this formulation, however, this question became of real interest much later (see Sec. 3.4 below). The introduction of the order parameter is important in the analysis of sound absorption in helium II near the λ point.[21] In this case, however, there was no need to spell out exactly which of the quantities is the order parameter. What was actually used was only a formula for the first-sound absorption coefficient, containing only the speed of sound on both sides of the transition point and the relaxation time (this is why neither ρ_s nor any other order parameter whatever is used for helium II in[21]).

The pressing need of generalizing Landau's theory of superfluidity, especially to include the region of the λ transition, was brought about by the problem of the discontinuity of the velocity v_s of the superfluid part of the liquid at the boundary with a solid wall. It is well known that the helium atoms stick to the wall, so that on the wall itself not only should the normal flux of the liquid vanish, $j_n = \rho_n v_n = 0$, but the flux of its superfluid part $j_s = \rho_s v_s$ should also be equal to zero. The equality $j_n = 0$ on the wall follows immediately from the usual condition for a viscous liquid, v (on the wall) $= 0$, which pertains also to the velocity v_n. For v_s, to the contrary, the Landau theory makes use of the condition for any ideal liquid on a wall, i.e., the component of v_s parallel to the boundary is assumed to be different from zero on the wall, in spite of the sticking of the helium atoms to the wall. An impression is thus gained that the velocity v_s should have a hydrodynamic discontinuity near the wall. For some reason, no one seemed to worry much about the question of this discontinuity (see, however,[22]), although it is impossible to get around it. In fact, the discontinuity at the wall, like a discontinuity of the velocity v_s in the interior of the helium (a vortex layer), should be connected with a surface energy $\sigma_s = \hbar^2/ma^4 \sim 10^{-2}$ erg/cm^2 (here $m = m_{He}$ is the mass of the helium atom and a is the characteristic thickness of the discontinuity, set equal to approximately 10^{-7} cm in the numerical estimates). The foregoing estimate of σ_s follows from different considerations[23]; for example, it is obtained from calculations of the zero-point (kinetic) energy following the change of the wave function of helium atoms in a layer of thickness a. The energy σ_s is always appreciable, and were a discontinuity with such a surface energy

actually to appear at a velocity $v_s \neq 0$ along the wall, then this would be clearly discerned in an experiment, in particular, an effect of "dry friction" would occur. Actually, however, there is no such effect, and the surface energy σ_s due to superfluid motion along the wall is smaller by at least 7–8 orders of magnitude than in the foregoing estimate, and can therefore be regarded as equal to zero. [24]

How is this contradiction to be explained? A way out can be found by assuming that $\rho_s = 0$ on the wall itself, and consequently, the density of the superfluid part of the liquid changes near the wall over a certain characteristic distance ξ, so that $\rho_s(0) = 0$ and $\rho_s(z \gg \xi) = \rho_{se}$, where z is the distance from the wall and ρ_{se} is the value of ρ_s in the interior of the helium II. Therefore, even at $v_s \neq 0$, the flux on the wall is $j_s(0) = \rho_s(0) v_s = 0$, and consequently the discontinuity of the velocity v_s on the wall itself becomes "innocent." In other words, a certain specific surface energy σ_s is always connected with the wall in the helium II, and is due to the presence of a density gradient $\rho_s(z)$ that has the same value at $v_s = 0$ as when $v_s \to 0$ (this means also that there is no "dry friction").

The publication of [25] was stimulated precisely by the desire to construct a theory in which the density ρ_s is not only not specified in the interior of the helium, but is defined by a certain equation compatible with the boundary condition $\rho_s(0) = 0$. At the same time, with the theory of superconductivity as an example, [26] it was already explained that the role of the order parameter for a superfluid (superconducting) system is played by a certain complex "effective wave function" $\Psi = \eta e^{i\varphi}$. Therefore, as applied to helium II, a similar complex order parameter was introduced in [25], with

$$\rho_s = m \mid \Psi \mid^2 = m\eta^2, \qquad (1.1)$$

$$v_s = \frac{\hbar}{m}\nabla\varphi, \qquad (1.2)$$

$$j_s = \rho_s v_s = -\frac{i\hbar}{2}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*) = \hbar\eta^2\nabla\varphi, \qquad (1.3)$$

where $m = m_{He}$ is the mass of the helium atom, and the normalization of Ψ was chosen from considerations of future convenience.

The introduced function Ψ is macroscopic and can be used only at scales greatly exceeding the atomic scale $a \sim 3 \times 10^{-8}$ cm. At the same time the density ρ_s in helium II far from the λ point changes near the wall over distances on the order of atomic, and in general the correlation length ξ characterizing the change of the order parameter is of atomic dimensions. It is clear therefore that a superfluidity theory based on the macroscopic order parameter Ψ (we shall henceforth call this the Ψ-theory of superfluidity) can be developed, if at all, only near the λ point, where the length ξ increases. We have specifically in mind the region of temperatures T satisfying the condition

$$\varepsilon = \frac{T_\lambda - T}{T_\lambda} \equiv \frac{t}{T_\lambda} \ll 1. \qquad (1.4)$$

We shall therefore assume henceforth, somewhat arbitrarily, that we are considering the region $\varepsilon \lesssim 0.1$, i.e., not farther than 0.1–0.2 °K from the λ point.

In the case of superconductivity, the correlation length (the coherence length) ξ is large (in a typical pure type-I superconductor, the length $\xi \sim \xi_0 \sim 10^{-4}$ cm is of the order of the Cooper-pair dimension even at $T \ll T_c$), so that near the superconducting-transition point (that is, under the condition (1.4) with T_λ replaced by the critical temperature T_c), the Ψ-theory of superconductivity [26] is applicable over a wide range if we use a self-consistent theory of second-order phase transitions, in which the fluctuations of the order parameter near the transition point are neglected. [2] For helium II, as already mentioned, the correlation length ξ is not anomalously large, and although on approaching the λ point we have $\xi \gg a \sim 3 \times 10^{-8}$ cm, which makes possible a rather extensive use of the macroscopic function Ψ, the self-consistent approximation (neglect of fluctuations) has, strictly speaking, no region of applicability. A self-consistent phase-transition theory was nevertheless used in [25], albeit with some stipulation (it was impossible to proceed otherwise at that time, in 1958). Therefore some of the final results obtained in [25], concerning the dependence of the temperature T_λ on the thickness d of the helium-II film, concerning the surface energy $\sigma_s(T)$ on the boundary of the helium II with a solid, concerning the heat capacity $C(T, d)$ of the films, and concerning the structure of the vortex filament, are all only of approximate character. The situation is considerably improved if we dispense with the expansion of the coefficients A, B, C...of $\mid\Psi\mid^2$, $\mid\Psi\mid^4$, $\mid\Psi\mid^6$, ...etc. in powers of $(T_c - T)$, and use instead [27] the experimentally-based dependence of these coefficients on $(T_c - T)$. It is precisely along this path, with allowance for experiment and the already existing general theory of phase transitions, [17–19] that the Ψ-theory of superfluidity is presently developing [28–40] (we are citing here also articles devoted to the more general question of extending the self-consistent theory of phase transitions, an extension based on the use of an arbitrary, in principle, temperature

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[2] A self-consistent phase-transition theory will be defined as one in which the thermodynamic potential is expanded near the transition point in powers of the order parameter, and the coefficients of this expansion are represented in the form of a series in $(T_c - T)$. It is precisely in such a scheme that many models are considered in the self-consistent (average or molecular) field approximation, i.e., neglecting fluctuations. The phase-transition theory called here "self-consistent" (this is not the most suitable term, but we do not know of a better one), is frequently called the Landau phase-transition theory. We have decided not to use this designation, first, to avoid confusion with the Landau theory of superfluidity, which is frequently mentioned in the present article. Second, the principal features of Landau's phase-transition theory [19] are the expansion in the order parameter, the evaluation of the role of the symmetry, etc., whereas an additional expansion in powers of $(T_c - T)$ is a much farther-reaching and not obligatory assumption. Third, the self-consistent theory of phase transitions was actually used in relatively simple concrete cases (ferromagnetism, liquid-vapor critical point) long before Landau's work.

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dependence of the coefficients A, B, $C\ldots$).

The development of the theory is not yet complete, and there are a number of unanswered questions in this field. This being the situation, it is quite natural to encounter also in the literature rather negative and contradictory assessments of the potentialities of the Ψ-theory of superfluidity of helium II. This is all the more a reason for casting light on the present status of the theory, including a comparison with experiment, and this is precisely the subject of this article. We note, however, that no attempt has been made at all to present a complete exposition (particularly when it comes to the references), since this would call for much more space and, principally, at the present state of the problem it would hardly be justified and only would make it difficult to see the forest for the trees.

Chapter II of the article is aimed at constructing the Ψ-theory of superfluidity for the stationary case at $v_n = 0$, and also at discussing the region of its applicability. A solution of a number of problems, such as the calculation of the temperature $T_\lambda(d)$ for films, is given in Chap. III. In Chap. IV we then discuss the Ψ-theory of superfluidity near the λ point for the general case ($v_n \neq 0$, nonstationarity). Finally, Chap. V is devoted to a certain summary and to a list of future research projects.

II. HELIUM II IN STATES OF COMPLETE AND INCOMPLETE THERMODYNAMIC EQUILIBRIUM (QUIESCENCE, STATIONARY SUPERFLUID FLOW)

2.1. Fundamental equation for the Ψ-function (the order parameter)

It is meaningful, or in any case useful, to single out the order parameter as one of the variables in the expression for the thermodynamic potential of the considered system (medium) only under certain conditions. It is necessary that the selected order parameter be a macroscopic quantity, i.e., a quantity already averaged over certain small scales. Another aspect of the problem is the question of the character of the relaxation of the order parameter. Generally speaking, the relaxation time of the order parameter (or parameters) is much longer than the relaxation times of the other variables (pressure, temperature, etc.). It is just under such conditions that we can consider states of incomplete thermodynamic equilibrium in which all the independent variables (quantities) except the order parameter are at equilibrium. Since the relaxation time $\tau(T)$ of the order parameter increases near the second-order phase transition point T_λ, with $\tau(T \to T_\lambda) \to \infty$, the consideration of states of incomplete thermodynamic equilibrium in this region is particularly justified.

In the case of helium II, the order parameter is chosen to be the already mentioned effective or macroscopic wave function $\Psi = \eta e^{i\varphi}$, which is connected with ρ_s and v_s by relations (1.1)–(1.3). To be sure, there is another quantity that can claim the role of the order parameter of helium II, namely the wave function $\bar{\Psi}$, the square of the modulus of which is equal to the concentration $n_0(T)$ of the helium atoms that are in a mac-

roscopic quantum state with zero momentum. It is clear even from this that the choice of the "correct" order parameter is not obvious. We shall return to this question later (Sec. 2.4), and note for the time being that the parameters n_0 and $\rho_s = m\eta^2$ may turn out to be proportional to each other, especially near the λ point, i.e., they are on a par or, more accurately, one reduces to the other. Be that as it may, we shall use below only the parameter Ψ.

Assuming the existence of states of incomplete thermodynamic equilibrium, having different values of Ψ, we shall consider the thermodynamic potential[3] $\Phi(p, T, \Psi)$ of helium II in such states. As the variables we chose here, besides Ψ, also the pressure and the temperature. It is possible, of course, to consider other thermodynamic potentials, such as free energy $F(\rho, T, \Psi)$, the thermodynamic potential Ω (μ, T, Ψ), and the internal energy $E(\rho, S, \Psi)$, where p is the pressure, ρ is the density, μ is the chemical potential, and S is the entropy. When solving spatially-inhomogeneous problems it is necessary, strictly speaking, to use the potential Ω (μ, T, Ψ), since constancy of μ and T corresponds to thermal and mechanical equilibrium, whereas the variables ρ, p, and S are generally speaking coordinate-dependent. When general equations are considered, however, particularly when it comes to deriving the fundamental equation for Ψ, it is more convenient (or at any rate more customary) to use the free energy or the potential Φ (p, T, Ψ). This is what we shall do.

We start with a homogeneous system (helium) and with absence of flow. Then Φ depends only on $\eta^2 = |\Psi|^2$, i.e., $\Phi = \Phi_{II\,0}$ (p, T, $|\psi|^2$). We now assume that the potential $\Phi_{II\,0}$ can be expanded near the λ-transition point in powers of $|\Psi|^2$, i.e.,

$$\Phi_{II\,0}(p, T \,|\, \Psi|^2) = \Phi_I(p, T) + A \,|\, \Psi\,|^2 + \frac{B}{2}\,|\,\Psi\,|^4 + \frac{C}{3}\,|\,\Psi\,|^6 + \ldots, \quad (2.1)$$

where Φ_I is the thermodynamic potential of helium I near the λ point, and A, B, C, $\ldots$, are functions of p and T.

The state of complete thermodynamic equilibrium corresponds to the minimum of $\Phi_{II\,0}$ with respect to Ψ^*, i.e., to the condition

$$\left(\frac{\partial \Phi_{II\,0}}{\partial\,|\,\Psi\,|^2} \right)_{p,\,T} \Psi = 0.$$

Below the transition point we have $\Psi \neq 0$ and the equilibrium condition can be written in the form

$$\left(\frac{\partial \Phi_{II\,0}}{\partial\,|\,\Psi\,|^2} \right)_{p,\,T} \equiv \left(\frac{\partial \Phi_{II\,0}}{\partial \eta^2} \right)_{p,\,T} = 0. \quad (2.2)$$

Confining ourselves to the terms written out in (2.1), we obtain the condition (2.2) in the form $A + B\eta^2 + C\eta^4 = 0$, whence the equilibrium value

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[3]More accurately speaking, we mean the density of the thermodynamic potential. For the sake of brevity, where no misunderstanding can result, we shall use the same term for the quantity and for its density.

$$\frac{\rho_{se}}{m} = \eta_e^2 = -\frac{B}{2C} + \sqrt{\left(\frac{B}{2C}\right)^2 - \frac{A}{C}}. \qquad (2.3)$$

We chose the + sign in front of the square root, since this choice ensures satisfaction, below the λ point, of the stability condition (the minimum of the potential) in the equilibrium state

$$\frac{\partial^2 \Phi_{\mathrm{II}\,0}}{(\partial \eta^2)^2} = B + 2C\eta_e^2 = \sqrt{B^2 - 4AC} > 0. \qquad (2.4)$$

At the λ point we have $\rho_{se} = 0$, and it is thus obvious that $A(T\lambda) = 0$. Above the λ point we get $A > 0$ from the requirement that the state with $\eta_e = 0$ be stable. Consequently, $A < 0$ at $T < T_\lambda$. The coefficient C must be positive to ensure positiveness of $\Phi_{\mathrm{II}0}$ at large values of η^2.

In the simplest variants, when $C = 0$ or $B = 0$, we have

$$C = 0, \quad \frac{\rho_{se}}{m} = -\frac{A}{B}; \qquad (2.4a)$$

$$B = 0, \quad \frac{\rho_{se}}{m} = \sqrt{-\frac{A}{C}}. \qquad (2.4b)$$

In the case (2.4a), of course, $B > 0$.

In the self-consistent theory of second-order phase transitions, the coefficients A, B, C, ... are regarded as expandable in powers of $(T_\lambda - T)$ and, for example, in variant (2.4a), which is valid at a sufficiently large distance from the tricritical point (at which the coefficient $B = 0$), we have

$$A = -a(T_\lambda - T), \quad a = \left(\frac{dA}{dT}\right)_{T_\lambda}, \quad B(T_\lambda) = b,$$
$$\rho_{se} = \frac{ma(T_\lambda - T)}{b}. \qquad (2.5)$$

The entropy $S_{\mathrm{II}} - \partial \Phi_{\mathrm{II}\,0}/\partial T$, by virtue of the condition (2.2), is in this case equal to

$$S_{\mathrm{II}} = S_{\mathrm{I}} - \frac{\partial A}{\partial T}\eta_e^2 = S_{\mathrm{I}} - \frac{a^2}{b}(T_\lambda - T), \quad S_{\mathrm{I}} = -\frac{\partial \Phi_{\mathrm{I}}}{\partial T} \qquad (2.6)$$

and the discontinuity of the heat capacity $C_p = T(\partial S/\partial T)_p$ at the λ point is equal to

$$\Delta C_p = C_{p,\,\mathrm{II}} - C_{p,\,\mathrm{I}} = \frac{a^2 T_\lambda}{b}. \qquad (2.7)$$

In the temperature interval $10^{-2} \lesssim \varepsilon = (T_\lambda - T)/T_\lambda \lesssim 0.1$ we can assume as a rough approximation $\rho_{se} \propto (T_\lambda - T)$, i.e., that formula (2.5) can be used. Assuming, for example, $\rho_{se} = 0.075$ g/cm³ at $\varepsilon = 0.1$ and $\Delta C_p = 5.2 \times 10^7$ erg-g⁻¹ deg⁻¹ = 0.76×10^7 erg-cm⁻³ deg⁻¹, we get[4]

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[4] The quantities cited here and below pertain to the λ point of pure ⁴He at saturated-vapor pressure. Thorough investigations of the λ transition in helium at all pressures have been initiated in recent years, from $p = p_{sat}(T_\lambda) = 0.05$ atm to the solidification pressure $p_m(T_\lambda) = 29.74$ atm, and also in ³He–⁴He mixtures (see, in particular,[45,46]). With increasing pressure, and also in mixtures, the region in which $\rho_{se} \propto (T_\lambda - T)^{2/3}$ with good accuracy becomes narrower, and the region where the linear relation $\rho_{se} \propto (T_\lambda - T)$ can be approximately used increases correspondingly. However, no detailed analysis has been made so far of the behavior of the coefficients $A, B, C, \ldots$ as functions of the pressure and of the ³He concentration. We shall therefore refer throughout, unless otherwise stipulated, only to the region of the λ point of pure ⁴He at saturated-vapor pressure.

$$a = 6.7 \cdot 10^{-17} \text{ erg/deg}, \quad b = 1.3 \cdot 10^{-39} \text{ erg-cm}^{-3} \qquad (2.8)$$

Actually, however, the density ρ_{se} is described sufficiently accurately in the entire interval $10^{-6} \lesssim \varepsilon \lesssim 0.1$ by the law[41-44]

$$\rho_{se} = m\eta_e^2 = m\,|\,\Psi\,|^2 = 1.43\rho_\lambda (T_\lambda - T)^\zeta, \qquad (2.9)$$
$$\rho_\lambda \equiv \rho(T_\lambda) = 0.146 \text{ g-cm}^{-3} \quad \rho_{00} = 1.43\rho_\lambda = 0.21 \text{ g-cm}^{-3}\text{deg}^{-\zeta}$$
$$\zeta = 0.67 \pm 0.01.$$

Assuming, as is quite probable, that $\zeta = 2/3$, we can put in (2.1)

$$A = -A_0(T_\lambda - T)\,|\,T_\lambda - T\,|^{1/3}, B = B_0\,|\,T_\lambda - T\,|^{2/3}, C = C_0, \qquad (2.10)$$

for in this case, according to (2.3) we have $\rho_{se} \propto (T_\lambda - T)^{2/3}$.

If $C_0 = 0$, then

$$A_0 = 1.11 \cdot 10^{-16} \text{ erg/deg}^{4/3}, \quad B_0 = 3.54 \cdot 10^{-39} \text{ erg-cm}^3/\text{deg}^{2/3}$$

Under conditions when coefficients of the type (2.10) are used (in contrast to the self-consistent variant of the type (2.5)), all the terms proportional to $|\Psi|^2$, $|\Psi|^4$ and $|\Psi|^6$ in (2.1) depend at equilibrium in the same manner on $(T_\lambda - T)$, and there are therefore no general reasons for discarding the term with $|\Psi|^6$ as $T \to T_\lambda$, as well as the higher terms in $|\Psi|^2$. By the same token, the use of variant (2.10) is an approximation. As will be shown below (Sec. 2.4), under the condition $\rho_s \lesssim \rho_{se}$, and especially if

$$\rho_s \ll \rho_{se}, \qquad (2.11)$$

it is already justifiable to discard the terms with $|\Psi|^8$ etc., and possibly also the term with $|\Psi|^6$. Yet in narrow channels and gaps, near a vortex axis, etc., the equilibrium value of the density ρ_s is precisely less than the corresponding equilibrium value of ρ_{se} for bulky helium II at rest. Thus, the approximation (2.10) is justified in a rather wide range.

We note that since the transition point is a singular point of the thermodynamic potential, the numerical values of the coefficients A_0, B_0, and C_0 can in general be different at $T > T_\lambda$ and $T < T_\lambda$. In what follows, however, we shall disregard this fact (for the sake of simplicity, and bearing in mind that we are applying the theory mainly in the temperature region below the λ point) (see also the remarks in Secs. 2.4 and 3.2).

We introduce the dimensionless variable (see (2.9))

$$\psi = \frac{\Psi}{\Psi_{00}}, \quad \Psi_{00} = \eta_{00} = \sqrt{\frac{1.43\rho_\lambda}{m}} = \sqrt{\frac{\rho_{00}}{m}}. \qquad (2.12)$$

In addition, we can express two of the coefficients A_0, B_0, and C_0 in (2.10) in terms of ρ_{se} and ΔC_p, using formula (2.3) and calculating $C_p = -T(\partial^2 \Phi/\partial T^2)_p$ on the basis of expressions (2.1), (2.9), and (2.10). As a result we can write

$$\Phi_{\mathrm{II}\,0} - \Phi_{\mathrm{I}} = \frac{3\Delta C_p}{(3+M)T_\lambda}\left(-t\,|\,t\,|^{1/3}\,|\,\Psi\,|^2 + \frac{(1-M)\,|\,t\,|^{2/3}}{2}\,|\,\Psi\,|^4 + \frac{M}{3}\,|\,\Psi\,|^6\right), \qquad (2.13)$$

where

$$t = T_\lambda - T = T_\lambda \varepsilon. \quad M = \frac{C_0}{A_0 \eta_{00}^3}. \qquad (2.14)$$

In the simplest variants of the theory[27,36] the dimensionless parameter is respectively $M = 0$ or $M = 1$.

As applied to helium II at equilibrium, the use of the thermodynamic potential (2.13) contributes, of course, nothing new. But this potential is needed in the Ψ theory of superfluidity as one of the blocks used to write down equations suitable for the solution of the inhomogeneous and nonstationary problems. In the latter case, however, it is necessary to take into account, generally speaking, the relaxation processes and dissipation (see Chap. IV). We shall therefore discuss first stationary inhomogeneous problems, in which it is necessary to take into account only the derivatives of Ψ with respect to the coordinates. In addition. we assume that $v_n = 0$, and consequently the Ψ-function (the order parameter) describes the state of the helium II completely.

Under the indicated conditions, we choose for the thermodynamic potential the expression

$$\Phi_{II}(p, T, \Psi) = \frac{\hbar^2}{2m} |\nabla\Psi|^2 + \Phi_{II\,0} = \frac{\hbar^2}{8m^2} \frac{(\nabla\rho_s)^2}{\rho_s} + \frac{\rho_s v_s^2}{2} + \Phi_{II\,0}, \quad (2.15)$$

where $\Phi_{II\,0}$ is the potential of the homogeneous helium II at rest (see (2.1) and (2.13)), and expressions (1.1) and (1.2) have been taken into account.

The choice of the form of the gradient term in (2.15) is dictated by the requirement of invariance (the potential Φ is a scalar), by the quantum-mechanical analogy, by the analogy with the Ψ-theory of superconductivity,[26] and by the fact that at $\nabla\rho_s = 0$ we arrive at an expression for the kinetic-energy density $\rho_s v_s^2/2$ (of course, these arguments are far from independent of one another). This is still not enough justification, but it pertains also to the choice of the expression for $\Phi_{II\,0}$ and, in general, to practically any phenomenological theory before it is deduced from the microscopic theory and before the region of its applicability is defined. The latter will be discussed further later on, and for the time being we shall assume, without further discussion, expression (2.15) for Φ_{II}. Then the complete thermodynamic potential is

$$\tilde{\Phi}_{II} = \int \Phi_{II}(p, T, \Psi) \, dV,$$

and the equation for Ψ is obtained as the extremum condition when $\tilde{\Phi}_{II}$ is varied with respect to Ψ^*. Hence

$$\frac{\hbar^2}{2m} \Delta\Psi = \left(\frac{\partial\Phi_{II\,0}}{\partial |\Psi|^2}\right)_{p, T} \Psi. \qquad (2.16)$$

If other thermodynamic potentials are used, then $(\partial\Phi_{II\,0}/\partial|\Psi|^2)_{p,T}$ in (2.16) is replaced by other derivatives, but all can be made equal to one another by a suitable change of variables

$$\left(\frac{\partial\Phi_{II\,0}}{\partial|\Psi|^2}\right)_{p,T} = \left(\frac{\partial F_{II\,0}}{\partial|\Psi|^2}\right)_{v,T} = \left(\frac{\partial\Omega_{II\,0}}{\partial|\Psi|^2}\right)_{\mu,T} = \left(\frac{\partial E_{II\,0}}{\partial|\Psi|^2}\right)_{\rho,S} = m\mu_s. \qquad (2.16a)$$

These relations follow from the identity $dE = TdS + \mu d\rho + \mu_s d\rho_s$ and from the definitions of the densities of the thermodynamic potentials Φ, F, and Ω: $\Phi = E - TS + p = \mu\rho$, $F = E - TS$ and $\Omega = E - TS - \mu\rho$.

Thus, if the symbol μ_s is used, Eq. (2.16) assumes the same form regardless of the choice of the thermodynamic variables.

We introduce now, besides the ψ-function (2.12), also the new coordinates

$$r_{**} = \frac{r}{\xi_{00}}, \quad \xi_{00} = \sqrt{\frac{\hbar^2\Psi_{00}^2 T_\lambda}{2m\Delta C_p}} = 2{,}74\cdot10^{-8} \text{ cm-deg}^{2/3} \qquad (2.17)$$

Then, taking (2.13) into account, Eq. (2.16) takes the form (the asterisks denote that the derivatives are taken with respect to $r_{**,i}$)

$$\Delta_{**}\psi = \frac{3}{3+M}\left(-t|t|^{1/3} + (1-M)|t|^{2/3}|\psi|^2 + M|\psi|^4\right)\psi. \qquad (2.18)$$

This is precisely the basic equation for what follows. Of course, in the region below the λ point, in which we are mainly interested, we have $t > 0$ and the absolute-value symbol for t in (2.18) can be omitted. We shall usually do so. The equilibrium value $\psi = \psi_e$, obtained from (2.18) at constant ψ, is

$$\psi_e = \frac{\Psi_e}{\Psi_{00}} = t^{1/3}, \qquad (2.12a)$$

as it should (see (2.9) and (2.12); the quantity ψ_e can be regarded as real). The physical meaning of the quantity ξ_{00} becomes particularly clear if one introduces the dimensionless coordinates

$$r_* = \frac{r}{\xi_M}, \quad \xi_M = \frac{\xi_{00}[(3-M)/3]^{1/2}}{t^{2/3}} \equiv \xi_0\left(\frac{3+M}{3}\right)^{1/2}$$
$$= \frac{2.74\cdot10^{-8}[(3+M)/3]^{1/2}}{(T_\lambda - T)^{2/3}} \text{ cm.} \qquad (2.19)$$

In terms of these coordinates we have for the function $\psi_0 = \psi/\psi_e$ (at equilibrium $\psi_0 = 1$ and $\psi = \psi_e = t^{1/3}$)

$$\Delta_*\psi_0 = [-1 + (1-M)|\psi_0|^2 + M|\psi_0|^4]\psi_0,$$
$$\psi_0 = \frac{\psi}{\psi_e} = \frac{\Psi}{(\rho_{00}/m)^{1/2}t^{1/3}}, \qquad (2.20)$$

and, at least as $|\psi_0|^2 \to 0$, the length ξ_M defines in obvious fashion the characteristic distance (the correlation length or the "coherence length") over which the order parameter ψ_0 changes. This will be discussed further in Sec. 2.3. At $M = 0$, the length $\xi_M = \xi_0$ coincides with that used in[27]. Were we to confine ourselves to the self-consistent variants (2.5) and (2.8), then the role of ξ_M would be assumed by the length[25] $\xi = \hbar/\sqrt{2ma\,(T_\lambda - T)} = 3.5\cdot10^{-8}/\sqrt{T_\lambda - T}$ (cm). However, such an approximation is not suitable near T_λ, since it corresponds to the relation $\rho_{se} \propto (T_\lambda - T)$ instead of the experimentally verified relation (2.9). We note that for superfluid ^{3}He the situation is already different,[6] inasmuch as near the λ point we can in fact put precisely $\rho_s \propto (T_\lambda - T)$.

The developed Ψ-theory can quantitatively be valid only under the condition

$$\xi_M \gg a \sim 3 \cdot 10^{-8} \text{ cm.} \tag{2.21}$$

At $M \lesssim 1$ this condition, according to (2.19), can be regarded as practically satisfied if $(T_\lambda - T) < 0.1 \,^\circ K$.

By varying the functional $\bar{\Phi}(\Psi)$ with respect to Ψ, we obtain for Ψ^* an equation that does not differ from (2.16). Multiplying now Eq. (2.16) by Ψ^*, multiplying the analogous equation for Ψ^* by Ψ, and subtracting one equation from the other, we obtain the continuity equation

$$\text{div } \mathbf{j}_s \equiv \text{div } (\rho_s \mathbf{v}_s) = 0, \tag{2.22}$$

where, of course, expressions (1.1)–(1.3) are used. Equation (2.2) can be derived also by somewhat different (but of course equivalent) method in which the real and imaginary parts of (2.16) are separated.

We note incidentally that in (2.15) and in the succeeding equations, if the definition (1.1) is used for ρ_s, the mass m of the helium cannot be replaced by some effective mass m_{eff}, nor to replace (1.2) by $\mathbf{v}_s = (\hbar/\sqrt{m m_{eff}}) \nabla \varphi$ (in order to retain the expression $\rho_s v_s^2/2$ for the kinetic energy). The point is that the effective mass m_{eff} should, generally speaking, depend on the temperature, pressure, etc. But this means that under spatially inhomogeneous conditions the mass m_{eff} would depend on the coordinates. The continuity equation (2.22) then no longer follows from the corresponding equations for Ψ and Ψ^*. That m cannot be replaced by m_{eff} is evidenced also by the requirement that other parameters of the circulation $\oint \mathbf{v}_s \cdot d\mathbf{l}$ be independent of the temperature (for details see[25] and Sec. 3.4).

2.2. Boundary conditions

When solving Eqs. (2.18) and (2.20) for ψ it is necessary to use definite boundary conditions. At the boundary with a solid wall, at least in the absence of heat exchange with the wall, one condition is obvious—the velocity v_s must be parallel to the wall, i.e.,

$$\mathbf{n}\nabla\varphi = v_s \mathbf{n} = 0. \tag{2.23}$$

where n is the normal to the wall.

Starting from the considerations discussed in the Introduction, we assume further that on the wall we have

$$\rho_s(0) = m \,|\, \Psi(0) \,|^2 = 0. \tag{2.24}$$

Actually this condition, even by virtue of the requirement (2.21), is indistinguishable from the condition that the density ρ_s vanish not on the wall itself, but at a certain distance from it, on the order of atomic dimensions (see Sec. 3.2). On the axis of a vortex filament in helium II we also assume condition (2.24), which arises naturally when the corresponding vortex problem is solved (see[25] and Sec. 3.4 below). The condition on the free boundary of helium II is not so reliably determined. Actual problems were solved both with condition (2.24) and with the condition

$$\mathbf{n}\nabla\Psi = 0, \text{ free boundary.} \tag{2.25}$$

We can introduce also a more general boundary condition; for example, for a real function Ψ we can put

$$\frac{\partial \Psi}{\partial z} + \lambda^{-1}\Psi = 0, \tag{2.26}$$

where z is the direction along the normal to the boundary and λ is a certain coefficient that can, in principle, depend on $t = T - T$ and on other variables.

We note that condition (2.25), which is valid in the Ψ-theory of superconductivity[26] for a boundary with vacuum, is obtained if we stipulate, when varying the functional $\Phi(\Psi)$, that it have an extremal value, without imposing any additional limitations on the boundaries.[5] The condition (2.26) can be arrived at, on the other hand (see, for example,[47]), by seeking, without imposing any other requirements on the boundary, the extremum of the functional

$$\tilde{\Phi}(\Psi) + \frac{\hbar^2 \lambda^{-1}}{2m} \int |\Psi(0)|^2 dS,$$

where $\Psi(0)$ is the value of Ψ on the boundary.

The need for using the condition (2.24) on a free surface as well is brought about by the analysis, presented in Sec. 3.3 below, of the situation on the boundary between helium II and its surface vapor.[37] Since the physical results (for example, the temperature $T_\lambda(d)$ of the λ point as a function of the film thickness d) depend on the employed boundary conditions, the validity of the employed boundary conditions should be checked, in final analysis, by comparing the theory with experiment. Finally, there is one more possibility, in principle, of refining the boundary conditions by resorting to the microscopic theory. But whereas in the case of superconductivity this procedure is quite reliable (and has already been employed[48,49]), it can hardly be regarded as particularly promising at the present state of the theory of helium II.

2.3. Accuracy of the Ψ-theory (allowance for fluctuations)

It is natural to raise the question of the region of applicability and of the accuracy of the Ψ-theory, based on the use of expressions (2.15) and (2.13) for the thermodynamic potential. Unfortunately we encounter here, besides the condition that we stay close to the λ point (see (1.4)), a number of other restrictions, which will be discussed, together with the meaning of the em-

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[5] In such cases we speak sometimes of natural boundary conditions. We arrive at these conditions by considering the variation

$$\int (\nabla\Psi\nabla\delta\Psi^*)dV = \int \nabla(\nabla\Psi\delta\Psi^*)dV - \int \delta\Psi^*\Delta\Psi\, dV.$$

Transforming

$$\int \nabla(\nabla\Psi\delta\Psi^*)dV = \oint (\mathbf{n}\nabla\Psi)\delta\Psi^*\, dS,$$

we see that the functional is minimal not only if (2.16) is valid, but also if condition (2.25) is satisfied.

ployed Ψ-function, in the following Sec. 2.4. Here we consider a very important aspect of the same problem, namely, we assess the role of fluctuations of the order parameter Ψ. In this case, however, if these fluctuations are small enough, we can use values of Ψ that are solutions of Eq. (2.16), or, specifically, (2.18) and (2.20).[6]

The incomplete thermodynamic potential

$$\tilde{\Phi}\,[p,\,T,\,\Psi\,(\mathbf{r})] = \int \Phi\,(p,\,T,\,\Psi\,(\mathbf{r}))\,dV, \qquad (2.27)$$

which depends, besides on p and T, also on the order parameter $\Psi(\mathbf{r})$, determines the probability density of a configuration with a given function $\Psi(\mathbf{r})$:

$$w\,(\Psi) = \exp \frac{1}{k_B T}\,[\tilde{\Phi}_e\,(p,\,T,\,\Psi_e\,(p,\,T)) - \tilde{\Phi}\,(p,\,T,\,\Psi\,(\mathbf{r}))]; \qquad (2.27a)$$

here $\tilde{\Phi}_e(p,\,T)$ is the complete thermodynamic equilibrium potential. (We note once more that it would be somewhat more consistent to use here the potential $\Omega(\mu,\,T)$.) From this, by virtue of the normalization $\int w(\Psi)D\Psi = 1$, follows an expression for the partition function

$$Z = \exp\left(-\frac{\tilde{\Phi}_e}{k_B T}\right) = \int \exp\left[-\frac{\tilde{\Phi}\,(\Psi\,(\mathbf{r}))}{k_B T}\right]D\Psi\,(\mathbf{r}), \qquad (2.27b)$$

with functional (configuration) integration with respect to $D\Psi(\mathbf{r})$.

In the self-consistent theory of phase transitions it is assumed, in fact, that

$$Z = Z_0 = \exp\left[-\frac{\tilde{\Phi}\,(\Psi_e\,(\mathbf{r}))}{k_B T}\right], \qquad (2.28)$$

i.e., it is assumed that

$$\tilde{\Phi}_e = \tilde{\Phi}\,(p,\,T,\,\Psi_e\,(p,\,T)),$$

where $\Psi_e(\mathbf{r})$ is the value of Ψ corresponding to the extremum (minimum) of $\tilde{\Phi}$, i.e., satisfying in our case Eq. (2.16). If the second-order phase transition point were not a singular point of the thermodynamic potential, then the fluctuations about the extremal value would play no noticeable role in the calculation of Z and $\tilde{\Phi}_e$ and, consequently, the approximation (2.28) would be perfectly satisfactory. In fact, however, the transition point is generally speaking a singular point, as manifest by the increase of the fluctuations when this point is approached. In the case of helium II (in contrast to superconductors and superfluid ^{3}He), the region of applicability of the self-consistent theory, understood here as the region of relative smallness of the fluctuations, turns out to be quite narrow or, strictly speaking, generally nonexistent (see Sec. 2.1 ff). The use, just

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[6] These are the only equations (and their generalizations to include nonstationary problems in the presence of normal flow) which will be used later on (see, however, the remarks made in Secs. 2.4 and 3.2). Therefore a detailed study of Secs. 2.3 and 2.4 is not needed before reading Chaps. III and IV of this article.

as in the self-consistent theory, of the expansion (2.1) but with nonanalytic coefficients A, B, $\ldots$ that depend on $t = (T_\lambda - T)$ is precisely a stratagem of the renormalization type, which changes the order parameter and makes it possible to take into account to a considerable degree the role of fluctuations even in a preliminary approximation such as (2.10). The necessary criterion for the success of this stratagem is that the part of the long-wave fluctuations (about the employed equilibrium solution) which is not taken into account in (2.13) must be small. We shall return to this question later.

We assume that the fluctuations $\delta\eta$ and $\delta\varphi$ of the modulus and of the phase of the function $\Psi = \eta e^{i\varphi}$, about a certain "equilibrium" value $\Psi_e = \eta_e e^{i\varphi_e}$ satisfying Eq. (2.16), are small quantities. This means that

$$\Psi = \eta e^{i\varphi} \approx \eta_e e^{i\varphi_e} + e^{i\varphi_e}\delta\eta + i\eta_e e^{i\varphi_e}\delta\varphi - \eta_e e^{i\varphi_e}\frac{(\delta\varphi)^2}{2}$$
$$\eta = \eta_e + \delta\eta, \quad \varphi = \varphi_e + \delta\varphi.$$

The corresponding fluctuation change of the potential Φ_{II}, taking into account terms of order not higher than quadratic in $\delta\eta$ and $\delta\varphi$, is equal to (see (2.15))

$$\delta\Phi = \frac{\hbar^2}{2m}\,(\nabla\varphi_e)^2\,(\delta\eta)^2 + \delta\Phi_0,$$
$$\delta\Phi_0 = \frac{\hbar^2}{2m}\,(\nabla\delta\eta)^2 + \frac{\hbar^2}{2m}\,\eta_e^2\,(\nabla\delta\varphi)^2 + \frac{1}{2}\,\Phi_{II\,0}''\,(\eta_e)\,(\delta\eta)^2, \qquad (2.29)$$

where $\Phi_{II\,0}''(\eta_e) \equiv (\partial^2\Phi_{II\,0}/\partial\eta^2)_{\eta=\eta_e}$ and we have discarded all the terms that are linear in $\delta\eta$ and $\delta\varphi$ (some of these terms vanish immediately by virtue of Eq. (2.16), while others vanish as a result of the volume integration of interest to us below, and specifically when $\delta\tilde{\Phi}\int = \delta\Phi dV$ is calculated).

In the absence of fluctuations, the gradient term is

$$\Phi_{II} - \Phi_{II\,0} = \frac{\hbar^2}{2m}\,|\nabla\Psi_e|^2 = \frac{\hbar^2}{2m}\,[(\nabla\eta_e)^2 + \eta_e^2\,(\nabla\varphi_e)^2],$$

and it is clear that $(\delta\Phi - \delta\Phi_0) \ll (\Phi_{II} - \Phi_{II\,0})$, provided that the conditions

$$\frac{\langle(\delta\eta)^2\rangle}{\eta_e^2} \ll 1, \quad \langle(\delta\varphi)^2\rangle \ll 1 \qquad (2.30)$$

are satisfied. Here and below the symbol $\langle\,\rangle$ denotes statistical averaging, since we are interested only in the contribution made by the fluctuations after such an averaging.

We shall show below that under the same conditions (2.30) the fluctuation increment $\delta\Phi_0$ or, more accurately, the fluctuation increment to the thermodynamic potential $\delta\Phi_{fl}$ which is defined below and is of interest to us, is also small. To be sure, the value of $\delta\Phi_{fl}$ will be calculated only neglecting the inhomogeneity of Ψ in the equilibrium state, i.e., assuming the quantities η_e and φ_e to be constant, but it seems to us that in all the cases of interest this assumption can not lead to significant changes in the situation (see, however, the end of Sec. 3.1).

From the general expression (2.27) it follows immediately that

$$\delta\tilde{\Phi}_{fl}=\tilde{\Phi}_e\,(\Psi_e)-\tilde{\Phi}\,(\Psi_e)$$

$$=-k_BT\ln\int\exp\left\{-\int\frac{\delta\Phi\,[\delta\eta,\,\delta\varphi]\,dV}{k_BT}\right\}D\eta\,D\varphi\equiv-k_BT\ln Z_{fl},$$

$$(2.31)$$

where $\tilde{\Phi}(\Psi_e)$ is defined in (2.28). The mean value of any physical quantity X is

$$\langle X\rangle=Z_{fl}^{-1}\int X\,[\delta\eta,\,\delta\varphi]\exp\left\{-\frac{\int\delta\Phi\,[\delta\eta,\,\delta\varphi]\,dV}{k_BT}\right\}D\eta\,D\varphi. \qquad (2.32)$$

We shall calculate the configuration integrals in a quadratic approximation (that is, including only terms of order $(\delta\eta)^2$ and $(\delta\varphi)^2$) and only for a system that is homogeneous in the equilibrium state. Concretely, we choose $\delta\Phi$ as given by expression (2.29) for $\delta\Phi_0$, the functions η_e and φ_e being assumed constant (properly speaking, the phase φ_e is now immaterial).

In this homogeneous case it is convenient to use a Fourier expansion, putting (V is the volume of the system, i.e., in our case, the volume of the liquid helium)

$$\delta\eta=\frac{1}{\sqrt{V}}\sum_q\eta_q e^{iqr},\quad \eta_e\delta\varphi=\frac{\eta_e}{\sqrt{V}}\sum_q\varphi_q e^{iqr},$$

$$\eta_{-q}=\eta_q^*,\quad \varphi_{-q}=\varphi_q^*,\quad \int(\delta\eta)^2\,dV=\sum_q|\eta_q|^2,$$

$$\int\eta_e^2\,(\delta\varphi)^2\,dV=\eta_e^2\sum_q|\varphi_q|^2. \qquad (2.33)$$

Therefore (see also (2.29))

$$\delta\tilde{\Phi}_0=\int\delta\Phi_0\,dV=\sum_q\left\{\left[\frac{\hbar^2}{2m}q^2+\frac{1}{2}\Phi_{II\,0}''(\eta_e)\right]|\eta_q|^2+\frac{\hbar^2}{2m}q^2\eta_e^2|\varphi_q|^2\right\}. \qquad (2.34)$$

Some of the calculations that follow are more conveniently carried out not on the basis of the general formulas (2.31) and (2.32), but directly on the basis of expression (2.27a). Namely, the probability of the fluctuation for which the thermodynamic potential changes by an amount $\delta\tilde{\Phi}$ is equal to $w=\text{const}\cdot\exp\{-\delta\tilde{\Phi}_0/k_BT\}$. Assuming that $\delta\tilde{\Phi}_0/k_BT=\lambda x^2/2$, we obtain for the probability of the fluctuation of the quantity x

$$w\,(x)\,dx=\sqrt{\frac{\lambda}{2\pi}}\,e^{-(\lambda/2)x^2}\,dx,\quad \langle x^2\rangle=\int x^2w\,(x)\,dx=\frac{1}{\lambda}, \qquad (2.35)$$

since

$$\int_{-\infty}^{+\infty}e^{-(\lambda/2)x^2}\,dx=\sqrt{\frac{2\pi}{\lambda}}.$$

We now use expression (2.34) for $\delta\tilde{\Phi}_0/k_BT=\lambda x^2/2$ and obtain directly with the aid of (2.35)[7]

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[7] Since η_q is complex, the square $|\eta_q|^2$ is the sum of the squares of two independent quantities (the same pertains also to $\eta_e^2|\varphi_q|^2$). This circumstance was already taken into account in (2.34) in the summation over all q, which is equivalent to taking the factor $|\eta_q|^2$ into account twice. We note also that we are confining ourselves to a classical (non-quantum) approximation, which is permissible under the condition $\langle|\eta_q|^2\rangle\gg1$ (see[40]). The classical approach is valid near the λ point in the case of long-wave fluctuations with $q\lesssim1/\xi_M$.

$$\langle|\eta_q|^2\rangle\equiv\langle\eta_q\eta_{-q}\rangle=\frac{k_BT}{\Phi_{II\,0}''(\eta_e)+(\hbar^2/m)\,q^2},$$

$$\langle|\varphi_q|^2\rangle=\frac{k_BT}{(\hbar^2/m)\,\eta_e^2q^2}. \qquad (2.36)$$

We have next

$$\langle(\delta\eta)^2\rangle\equiv\frac{1}{V}\left\langle\int(\delta\eta)^2\,dV\right\rangle=\frac{1}{V}\sum_q\langle|\eta_q|^2\rangle=\frac{k_BT}{2\pi^2}\int_0^{q_{max}}\frac{q^2\,dq}{\Phi_{II\,0}''(\eta_e)+(\hbar^2/m)\,q^2}$$

$$=\frac{mk_BT}{2\pi^2\hbar^2}\left[q_{max}-\sqrt{\frac{m\Phi_{II\,0}''(\eta_e)}{\hbar^2}}\,\text{arctg}\,\frac{q_{max}}{\sqrt{m\Phi_{II\,0}''(\eta_e)/\hbar^2}}\right],$$

$$(2.37)$$

$$\langle(\delta\varphi)^2\rangle=\frac{mk_BTq_{max}}{2\pi^2\hbar^2\eta_e^2}. \qquad (2.38)$$

It is important to recall the expression for the correlation function (we put henceforth $r=r_1-r_2$)

$$G_\eta\,(r)=\langle\delta\eta\,(r_1)\,\delta\eta\,(r_2)\rangle=\frac{1}{V}\sum_q\langle|\eta_q|^2\rangle\,e^{iqr}$$

$$=\int\langle|\eta_q|^2\rangle\,e^{iqr}\,\frac{d^3q}{(2\pi)^3}=\frac{k_BT}{4\pi\,(\hbar^2/m)\,r}\exp\left(-\frac{r}{\xi\,(t)}\right), \qquad (2.39)$$

where the correlation length for the fluctuations is

$$\xi\,(t)=\sqrt{\frac{\hbar^2/m}{\Phi_{II_0}''(\eta_e)}} \qquad (2.40)$$

and, of course, expression (2.36) is used.[8] Analogously,

$$G_\varphi\,(r)=\langle\delta\varphi\,(r_1)\,\delta\varphi\,(r_2)\rangle=\frac{k_BT}{4\pi\,(\hbar^2/m)\,\eta_e^2\,(t)\,r}. \qquad (2.41)$$

The quantity

$$\xi_\varphi=\frac{k_BTm^2}{4\pi\hbar^2\rho_{se}\,(t)} \qquad (2.41a)$$

can be naturally called the correlation length for the phase of the order parameter. In the microscopic approach[17] it is precisely the length ξ_φ which is regarded as fundamental.

In the self-consistent theory of phase transitions, the fluctuations are assumed to be of no significance at all and we can choose the maximum value q_{max} to be of the order of $\pi/\xi(0)$, where $\xi(0)$ is the short-range radius, with $\xi(0)\sim a\sim3\times10^{-8}$ cm for helium. We can here always assume that $\tan^{-1}(q_{max}/\sqrt{m\Phi_{II\,0}''/\hbar^2})=\pi/2$. Most importantly, the expression $(mk_BT/2\pi^2\hbar^2)q_{max}$, which enters in (2.37) and (2.38), is practically independent of the temperature near the λ point and plays no role if a fluctuation-dependent renormalization of the thermodynamic potential is used.[9] The condition for the

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[8] As is well known, the function (2.39) satisfies the equation

$$\left(\Delta-\frac{1}{\xi^2}\right)G_\eta=-4\pi\,\frac{k_BT}{4\pi\,(\hbar^2/m)}\,\delta\,(r).$$

Thus, apart from a factor, G_η is the Green's function for the equation written out above.

[9] This is particularly easy to verify above the transition point, by writing down the potential in the form $\Phi_{II\,0}=\Phi_I+\eta^2[\alpha(T_\lambda-T)+(b/2)\eta^2]$ and then replacing $b\eta^2/2$ by $3b\langle\eta^2\rangle/2\equiv3b\langle(\delta\eta)^2\rangle/2$. Obviously, the constant term $b^2(3b/2)\langle(\delta\eta)^2\rangle$ changes here the value of T_λ, i.e., the temperature, at which the coefficient of η^2 vanishes. Using the experimental (observable) value of T_λ, as is always the normal procedure, this change of T_λ (renormalization) does not play any role whatever.

smallness of the fluctuations (2.30) takes therefore the form

$$\langle(\delta\eta)^2\rangle = \frac{mk_BT}{4\pi\hbar^2}\sqrt{\frac{m\Phi_{\mathrm{II}\,0}''(\eta_e)}{\hbar^2}} \ll \eta_e^2,$$

or

$$t = (T_\lambda - T) \gg \frac{[(k_BT_\lambda)^2\,m^3}{4\pi^2\hbar^6}\frac{b^2}{a} \sim 1\cdot10^{-3}\ {}^\circ\mathrm{K}, \tag{2.42}$$

where we used the expressions (2.4a), (2.5), and (2.8), and took into account the fact that in this case

$$\Phi_{\mathrm{II}\,0}''(\eta_e) = -4A = 4at, \qquad \eta_e^2 = -\frac{A}{B} = \frac{at}{b}.$$

The correlation length (see (2.40) and (2.8)) below the λ-transition point is then

$$\xi^-(t) = \sqrt{\frac{\hbar^2}{4mat}} = \frac{3.5\cdot10^{-8}}{\sqrt{2}\sqrt{t}}\ \mathrm{cm}. \tag{2.43}$$

Above the transition point we have

$$\Phi_{\mathrm{II}\,0}''(\eta_e) = 2A = -2at, \qquad \eta_e^2 = 0$$

and the correlation length is

$$\xi^+(t) \equiv \xi(t) = \sqrt{\frac{\hbar^2}{|2ma|t|}} = \frac{3.5\cdot10^{-8}}{\sqrt{|t|}}\ \mathrm{cm}. \tag{2.44}$$

If the coefficient of the gradient term is expressed for an arbitrary second-order transition described by a certain order parameter η in the form δ (instead of $\hbar^2/2m$, as in the case of helium II), then the condition (2.42) for the smallness of the fluctuations takes the form[50-54] (see also[17,19]) [10]

$$t \gg \frac{(k_BT_\lambda)^2\,b^2}{32\pi^2\delta^3a}. \tag{2.45}$$

By virtue of the condition (2.42) and the requirement of sufficient proximity to the λ point (the condition $t \ll T_\lambda$), in the case of helium II at saturated-vapor pressure the self-consistent theory can be used with some reasonable accuracy only in the region

$$10^{-3} \ll t \leqslant 0.1\ {}^\circ\mathrm{K}. \tag{2.46}$$

We are furthermore interested also in much lower values of t, since in the experiments measurements are already feasible even at $t \lesssim 10^{-6}\ {}^\circ\mathrm{K}$. The scheme described in Sec. 2.1, in which the coefficients A, B, ... in the expansion (2.1) depend on t in non-analytic fashion (see (2.10), (2.13), (2.18), is resorted to in fact in practically the entire temperature interval near the λ point. In this scheme, even the initial expres-

sion for $\Phi_{\mathrm{II}\,0}(t)$ agrees with the observed relation (2.9) for $\rho_{se}(t)$, and leads for the correlation length below the λ point to the expression (see (2.13) and (2.17))

$$\xi_M^-(t) = \left[\frac{\hbar^2/m}{\Phi_{\mathrm{II}\,0}''(\eta_e)}\right]^{1/2} = \frac{[(3+M)/6\,(1+M)]^{1/2}\xi_{00}}{t^{2/3}}$$
$$= \frac{2.74\cdot10^{-8}\,[(3+M)/6\,(1+M)]^{1/2}}{t^{2/3}}\ \mathrm{cm}.$$

$$\Phi_{\mathrm{II}\,0}''(\eta_e) = \frac{12\Delta C_p\,(1+M)}{1.43\,(\rho_\lambda/m)\,(3+M)\,T_\lambda}t^{4/3}. \tag{2.47}$$

Above the λ point we have

$$\Phi_{\mathrm{II}\,0}''(\eta_e) = \frac{6\Delta C_p\,|t|^{4/3}}{(1.43\rho_\lambda/m)\,(3+M)\,T_\lambda},$$

and for the correlation length $\xi_M^+ \equiv \xi_M$ we obtain the expression

$$\xi_M^+ \equiv \xi_M = \xi_0\left(\frac{3+M}{3}\right)^{1/2} = \frac{2.74\cdot10^{-8}\,[(3+M)/3]^{1/2}}{t^{2/3}}, \tag{2.19a}$$

the meaning of which thus becomes finally clear. It is the lengths ξ_M and ξ_0 which we shall find the most convenient for use henceforth.

Since the self-consistent theory of phase transitions does not take temperature-dependent fluctuations into account, it is obvious that the theory with altered coefficients A, B, C... already takes these fluctuations into account to some degree. Namely, account is taken in this case of all the fluctuations with wavelengths smaller than and of the order of the correlation length $\xi(t)$, i.e., with wave vectors $q \gtrsim 1/\xi(t)$. The point is that the short-wave fluctuations, while small, are quite numerous (see (2.36); the integral $\int^{q_{max}}\langle|\eta_q|^2\rangle q^2 dq$ at large q_{max} increases in proportion to q_{max}). With increasing q, the fluctuations $\langle|\eta_q|^2\rangle$ increase up to values $q \sim \sqrt{m\Phi_{\mathrm{II}\,0}''/\hbar^2} \sim 1/\xi(t)$ (see (2.36)). In the calculation of the thermodynamic potential in the critical region it is therefore necessary to take into account all the fluctuations in the interval $1/a \sim q_{max} \gtrsim q \gtrsim q_c \sim 1/\xi$ (see also[56]) [11]. As to the longest-wave fluctuations with $q < 1/\xi(t)$, it is clear from (2.36) that they no longer increase with decreasing q and need not be taken into account in the initial expression for $\Phi_{\mathrm{II}\,0} - \Phi_{\mathrm{I}}$. Thus, the requirement that the fluctuations be small when expression (2.13) is used for the potential $\Phi_{\mathrm{II}\,0}$ pertains only to the long-wave fluctuations with $q \leqslant q_c$, with

$$q_c = \frac{Q}{\xi_M(t)} = 3.65\left[\frac{6\,(1+M)}{3+M}\right]^{1/2}\cdot10^7 t^{2/3}Q\ \mathrm{cm}^{-1}, \tag{2.48}$$

where $Q \equiv Q^-$ a numerical coefficient that serves as a parameter of the theory and ξ_M^- is the correlation length (2.47); we note that we have in mind here directly the region below the transition point, and above this point the value of Q can already be different.

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[10] The coefficient $(32\pi^2)^{-1}$ of (2.45) was left out in[50], with an appropriate stipulation, since only the relative values of the fluctuations for different transitions were compared there. Incidentally, it is clear even from (2.45) that near the tricritical point at which $b = 0$ the region of applicability of the self-consistent approximation is particularly large (for details see[55]).

[11] It must be stipulated, to be sure, that the short-wave fluctuations with $q \sim 1/a$ actually are independent of t and make no contribution to the singularity of the thermodynamic potential (from this point of view, only fluctuations with $q \ll 1/a$ are important). We emphasize in the text only the fact that all the fluctuations with $q \gtrsim 1/\xi$ can be regarded as taken into account in expression (2.13) for $\Phi_{\mathrm{II}\,0}$.

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TABLE I.

M	$Q_m = 1.1 \sqrt{\frac{3+M}{6(1+M)}}$	$\frac{\langle(\delta\eta)^2\rangle}{\eta_e^2}$	$\langle(\delta\varphi)^2\rangle$	Δ_2	Δ
0	0.776	0.0174	0.116	$-3.3 \cdot 10^{-3}$	$4.4 \cdot 10^{-3}$
1	0.635	0.0126	0.116	$-3.4 \cdot 10^{-2}$	$-1.5 \cdot 10^{-2}$
∞	0.449	0.0070	0.116	$-5.9 \cdot 10^{-2}$	$-3.7 \cdot 10^{-2}$

We can be sure that the fluctuations not included in expression (2.13) for $\Phi_{II\,0}$ are small if the conditions (2,30) are satisfied after we substitute $q_{max} = q_c$ in (2.37) and (2.38). We note that by proceeding in the same manner as in the self-consistent theory we obtain, in essence, the previous condition (2.42), for in this case[12]

$$q_{max} = \frac{Q}{\xi(t)} \sim \sqrt{\frac{m\Phi_{II\,0}''(\eta_e)}{\hbar^2}}. \qquad (2.49)$$

On the other hand, if we use the value $q_{max} = q_c = Q/\xi_M(t)$, in accordance with (2.48), then we arrive at the conditions (see (2.37), (2.38), and (2.19), (2.48))

$$\langle(\delta\eta)^2\rangle = \frac{mk_BT}{2\pi^2\hbar^2\xi_M}(Q - \mathrm{arctg}\,Q) \ll \eta_e^2 = \eta_{00}^2 t^{2/3},$$

$$\langle(\delta\varphi)^2\rangle = \frac{mk_BTQ}{2\pi^2\hbar^2\eta_e^2(t)\,\xi_M(t)} \ll 1. \qquad (2.49a)$$

At $T < T_\lambda$, the second and more stringent of these conditions reduces to (see (2.47))

$$0.1 \sqrt{\frac{6(1+M)}{3+M}}\, Q \ll 1. \qquad (2.49b)$$

It is important that the temperature t does not enter in the criterion (2.49a)–(2.49b) at all, and thus if it holds it does so in the entire temperature interval near the λ point.

To estimate the parameter Q and to obtain a quantitative idea of the accuracy of the theory based on the use of (2.13) and the expressions that follow, we turn to a calculation of the fluctuation-induced increment $\delta\Phi_{fl}$ to the thermodynamic potential.

To this end it is necessary to use the general formula (2.31). Substituting (2.34) in (2.31) we obtain for the density of the fluctuating part of the potential in the equilibrium state

$$\delta\Phi_{fl}(\eta_e, t) = \frac{1}{V}\,\delta\widetilde{\Phi}_{fl}(\eta_e, t) = \frac{k_BT}{2V}\sum_{q \leq q_{max}} \ln\frac{[(\hbar^2/m)q^2 + \Phi_{II\,0}''(\eta_e)](\hbar^2/m)q^2}{(2\pi k_BT)^2}$$

$$= \frac{k_BT}{2}\int_0^{q_{max}} \ln\frac{[(\hbar^2/m)q^2 + \Phi_{II\,0}''(\eta_e)](\hbar^2/m)q^2}{(2\pi k_BT)^2}\,\frac{4\pi q^2}{(2\pi)^3}\,dq$$

$$= \frac{k_BT}{6\pi^2\xi_M^3(t)}\left\{Q^3 \ln\left[\frac{Q\Phi_{II\,0}''(\eta_e)}{2\eta e^{2/3}k_BT}(1+Q^2)^{1/2}\right] + Q - \mathrm{arctg}\,Q\right\}. \qquad (2.50)$$

[12] The "cutoff" introduced in (2.37) and (2.38) at $q = q_{max} \sim 1/\xi$ when the criteria (2.30) are used can be interpreted in the following manner: we stipulate that the fluctuations $\langle\int(\delta\eta)^2 dV\rangle$ and $\langle\int\eta_e^2(\delta\varphi)^2 dV\rangle$ in the volume $V \sim \xi^3$ be small in comparison with $\eta_e^2 V \sim \eta_e^2\xi^3$. It is clear that only fluctuations with wavelength $\Lambda = 2\pi/q \lesssim \xi$ are significant in the integration over a region with dimensions on the order of ξ.

Substituting here $\xi_M(t)$ and $\Phi_{II\,0}''(\eta_e(t))$ in accordance with (2.19) and (2.47), we see that

$$\delta\Phi_{fl} = C_1 Q^3 t^3 (\ln|t| + C_2), \qquad (2.51)$$

where C_1 and C_2 are quantities independent of t. Allowance for the potential $\delta\Phi_{fl}$ leads thus to a renormalization of the specific-heat discontinuity at the λ point (owing to the term proportional to C_2) and to the appearance of a logarithmic singularity in the heat capacity. It is possible, in principle, to assume that the entire observed logarithmic (or near-logarithmic)[13] variation of the heat capacity is due to long-wave fluctuations (see, in particular,[31]), i.e., it is included in $\delta\Phi_{fl}$. We then obtain the upper bound Q_m of Q from (2.50) and (2.51) as well as from experimental data,[57,58] according to which

$$C_p = \begin{cases} 4.55 - 3.00\,\lg(T_\lambda - T) \;\;(\text{J/g-deg}), & T < T_\lambda, \\ -0.65 - 3.00\,\lg|T_\lambda - T| \;\;(\text{J/g-deg}), & T > T_\lambda, \end{cases} \qquad (2.52)$$

Without dwelling on the details, we present in Table I the values of Q_m at different values of the parameter M (see[61]).

It is clear from the table that the conditions (2.49) are satisfied. The role of the fluctuations becomes even clearer if we calculate first $\delta\Phi_{fl}(\eta)$ for some nonequilibrium value of η and then find $(\partial^2\delta\Phi_{fl}/\partial\eta^2)_{\eta=\eta e} \equiv (\delta\Phi_{fl})_{\eta_e}''$. It is next necessary to find the ratio $\Delta = (\delta\Phi_{fl})_{\eta_e}''/\Phi_{II\,0}''(\eta_e)$, which characterizes the change of the factor preceding $(\eta - \eta_e)^2$ in the expansion of $\Phi_{II\,0}(\eta)$ about the equilibrium value η_e, due to the long-wave fluctuations. The renormalized value is $\Phi_{II\,0,\,ren}''(\eta_e) = \Phi_{II\,0}''(\eta_e)(1 + \Delta)$. The calculation of Δ in the harmonic (quadratic) approximation leads to two values Δ_2 indicated in Table I. In general, however, $\Delta = \Delta_2 + \Delta_4 + \Delta_6 + \Delta\varphi$, where Δ_4, Δ_6, and $\Delta\varphi$ are corrections needed respectively to allow for the fluctuations of $(\delta\eta)^4$, $(\delta\eta)^6$, and $(\delta\eta)^2 \cdot (\nabla\delta\varphi)^2$. Calculation shows[61] that the corrections Δ_4, Δ_6, and $\Delta\varphi$ are positive (in contrast to Δ_2), and are of the order of $|\Delta_2|$ in the sum. As a result we obtain the values of Δ given in the last column of Table I, which do not exceed several percent. Higher accuracy can hardly be expected of our entire scheme, which involves the selection of the function $\Phi_{II\,0}(\eta)$, or the coefficients of the expansion of this function in powers of η, on the basis of empirical data. We can thus state that allowance for the long-wave fluctuations (wave numbers $q \lesssim 1/\xi_M(t)$) does not affect the results obtained without their allowance (this does not pertain, of course, to phenomena connected with the fluctuations themselves, for example, with the scattering of light by fluctuations or with calculations of the fluctuating part of the heat capacity).

By the same token, in view of the foregoing, we prove at least that the scheme based on the use of coefficients

[13] According to[58,59] $C_p \sim t^{-\alpha}$ with $\alpha = -0.026$ (see also[60]). If $\alpha \neq 0$, the Ψ-theory developed by us requires certain modifications, but these seem still premature to us, (see Chap. V).

A, B, ..., which are not analytic in t, in the expansion of Φ_{II0} in $|\Psi|^2$ (we consider specifically expression (2.13)) is not contradictory (or is self-consistent to some degree). Incidentally, in connection with the discussion of the role of fluctuations from the point of view of the accuracy of the Ψ-theory of superfluidity mention should also be made of[62] (see also[17] Chap. IV, §7), according to which the Ψ-theory of superfluidity of helium II has no region of applicability at all. To the extent that we are dealing with the initial self-consistent variant[25] (see also Sec. 2.1 and, specifically, expression (2.5)), this statement is correct to a certain degree (more accurately, see the condition (2.46)) and has already been noted in[50]. However, in the generalized theory (see (2.13) ff.), the role of long-wave fluctuations is insignificant, and the conclusion drawn above remains in force also when account is taken of divergence of the longitudinal susceptibility $\chi_{\shortparallel} \propto h^{-1/2}$ for a "field" $h \to 0$.[14] The point is that the nonanalytic dependence of $\chi_{\shortparallel}$ on h as $h \to 0$, as well as the logarithmic singularity of the heat capacity, can be completely taken into account in the zeroth term of the expansion of the potential $\Phi_{II0}(\eta, h)$ in powers of $(\eta - \eta_e)$, and has no bearing on the other terms of the corresponding series, which are the only ones of interest to us from the point of view of deriving (2.18) or the initial equations (2.16), (2.13).[61]

We can thus forget about the fluctuations of the parameter Ψ (or, equivalently, the fluctuations of the quantities η and φ) when solving any problem that does not deal with the fluctuations themselves.

2.4. Conditions and region of applicability of the Ψ theory

We have indicated above certain restrictions under which the Ψ-theory of superfluidity can be used. Thus, an important condition is (2.21), $\xi_M \gg a$, which makes possible a phenomenological description, say, of the distribution of $\rho_s(z)$ near a wall only at $t = (T_\lambda - T) < 0.1°$K. The same condition is necessary if the density $\rho_{se}(t)$ is to be small in comparison with $\rho_{se}(T = 0) = \rho$ (only in this case can we count on being able to use the expansion of the thermodynamic potential in $|\Psi|^2$ at $|\Psi|^2 \simeq |\Psi_e|^2 = \rho_{se}/m$). Finally, the same condition (2.21), in all probability, is necessary also when only

the gradient term $(\hbar^2/2m)|\nabla\Psi|^2$ is used in (2.15). Indeed, terms with higher derivatives or powers of the derivative such as $a_1|\nabla\Psi|^4$, $a_2|\Delta\Psi|^2$, etc., are usually of the same order in the self-consistent variant of phase-transition theory when Ψ is varied over the coherence length or the correlation length at $T = 0$, i.e., over the length $\xi(0)$, which has the meaning of the short-range order or, as is sometimes said, of the molecular-action radius. For helium II we have $\xi(0) \sim a \sim 3 \times 10^{-8}$ cm, so that for the order parameter $\Psi(r)$, which varies over a length L, we have the ratio

$$\frac{a_2|\Delta\Psi|^2}{(\hbar^2/2m)|\nabla\Psi|^2} \sim \left(\frac{a}{L}\right)^2.$$

It is clear therefore that a Ψ-theory that takes into account only the term $(\hbar^2/2m)|\nabla\psi|^2$ (this approximation is sometimes called hydrodynamic) cannot be used if the Ψ function varies over a characteristic length $L \lesssim a$. It must be emphasized at the same time that when Eqs. (2.18) and (2.20) are used for Ψ this function turns out to be variable, generally speaking, over distances $L \gtrsim \xi_M(t)$ or, in the case of the self-consistent theory, over distances $L \gtrsim \xi(t) = \hbar/\sqrt{2mat}$, i.e., satisfaction of a condition such as (2.21) near the λ point ensures in the self-consistent variant of the theory that the term $(\hbar^2/2m)|\nabla\Psi|^2$ predominates over the other terms with the derivatives.[15] Unfortunately, when Ψ-theory variants with coefficients that do not depend analytically on t are used, it is possible to work only under the condition $L > \xi_M(t)/Q$ (see (2.48)), and smallness of terms of the type $a_1|\nabla\Psi|^4$ and $a_2|\Delta\Psi|^2$ is ensured reliably only under the condition

$$L \gg \xi_M(t), \tag{2.53}$$

inasmuch as $Q \sim 1$. In practice, however, conditions of the type (2.53) contain the parameter $(\xi_M/L)^2$, and we may expect to be able to retain only the term with $(\hbar^2/2m)|\nabla\Psi|^2$ even at $L \gtrsim \xi_M(t)$.

We must dwell specially on the very possibility of expanding the potential Φ_{II0} in powers of $|\Psi|^2$ (see (2.1) ff.). In modern theory of phase transitions[17,18,30,32,35] it is assumed in fact that after the regular part is subtracted the thermodynamic potential $\Phi_0(\Psi, \varepsilon)$ takes the form

$$\Phi_0(\Psi, \varepsilon) = |\varepsilon|^{2-\alpha} f\left(\frac{|\Psi|}{\Psi_e(0)|\varepsilon|^\beta}\right), \tag{2.54}$$

where Ψ is the order parameter, $\Psi_e(0) \equiv \Psi_e(T = 0)$,

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[14] By "field" h in the case of a system having a complex order parameter Ψ, we mean a quantity the introduction of which requires addition of a term $-(1/2) \cdot (h\Psi^* + h^*\Psi)$ to the thermodynamic potential. The longitudinal susceptibility is

$$\chi_{\shortparallel} = \frac{\partial\langle\Psi_{\shortparallel}\rangle}{\partial h}, \quad \Psi_{\shortparallel} = \mathrm{Re}\,\Psi = \eta\cos\varphi,$$

and in the quadratic approximation we have $\langle\Psi_{\shortparallel}\rangle = \eta_e[1 - (\langle\delta\varphi\rangle^2/2)] \approx \eta_e(1 - C_1 + C_2\sqrt{h})$, where C_1 and C_2 are constants and η_e is the equilibrium value of η in the field h.

Obviously, for a ferromagnet, in which the order parameter is the magnetization m, the field h is the magnetic field and the increment to the thermodynamic potential is $-$ m · h. Although the field h cannot be realized physically for a superfluid liquid or for a superconductor, introduction of this field is a convenient device in the investigation of a number of questions.

[15] For superconductors, for superfluid ^{3}He (see[6,63]), and for a superfluid neutron fluid (as well as for a superconducting proton fluid), the coherence length $\xi(0) \equiv \xi(T = 0)$ is large in comparison with distance a between particles even at $T \ll T_\lambda$ (the role of $\xi(0)$ is played by the dimension of the Cooper pairs). Under similar conditions the term $(\hbar^2/2m)|\nabla\Psi|^2$ dominates at temperatures at which $\xi(T) \gg \xi(0)$. At the same time, in these cases the region of applicability of the self-consistent variant of the Ψ-theory turns out to be appreciable, since the fluctuations decrease rapidly with increasing ratio $\xi(0)/a$ (see (2.45)), where the denominator contains the cube of the coefficient δ, which is precisely the measure of this ratio.

$\varepsilon = (T_\lambda - T)/T_\lambda$, while α and β are exponents that determine the temperature dependences of the heat capacity $C_p \propto |\varepsilon|^{-\alpha}$ and of the equilibrium order parameter $|\Psi_e| = \eta_e \propto \varepsilon^\beta$; in this case the function f takes different forms at $\varepsilon > 0$ (the function f_-) and $\varepsilon < 0$ (the function f_+).

The function f_- can be expanded in powers of $x = |\Psi| / \Psi_e(0)| \varepsilon|^\beta$ only at small values of x, and, more concretely, this expansion is rigorously valid only so long as $\eta/\eta_e \ll 1$ or $\rho_s/\rho_{se} \ll 1$ (see (2.11)).

Actually, however, at $\varepsilon > 0$ we can confine ourselves to the condition[16]

$$\frac{\rho_s}{\rho_{se}} \lesssim 1. \tag{2.11a}$$

Fortunately, this condition is satisfied in the overwhelming majority of cases of interest to us (see Chap. III), and only when we consider the boundary between helium I and II in a gravitational field (Sec. 3.2) will we need to use the function f as $x \to \infty$, when it cannot be expanded in powers of x^2 (see also[32]). In that case, however, the potential $\Phi_0(\Psi, \varepsilon)$ can be represented in the form

$$\Phi_0(\Psi, \varepsilon) = |\Psi|^{(2-\alpha)/\beta} \, \bar{f} \left(\varepsilon \left(\frac{\Psi_e(0)}{|\Psi|} \right)^{1/\beta} \right),$$

while the function $\bar{f}$ can be expanded in powers of $y = \varepsilon(\Psi_e(0)/|\Psi|)^{1/\beta}$. At $\alpha = 0$ and $\beta = 1/3$, the expansion of Φ_0 begins with a term proportional to $|\Psi|^6$ (we assume that $\bar{f}(0) \neq 0$). Thus, it is possible to treat all the problems by using the initial trinomial (2.13) as the interpolation equation.

We emphasize that we have immediately chosen above the critical exponents $\alpha = 0$ and $\beta = 1/3$, since they agree at the presently attainable accuracy with the experimental data (all the critical exponents above and below the transition point are assumed to be the same, as is also confirmed by the available experiments; this remark holds also for the exponents we shall use somewhat later on). It would be possible, of course, to develop the Ψ-theory without fixing the critical exponents beforehand, but, at least at the present stage, this would lead only to still-unjustifiable complications. This pertains, in essence, to the gradient term $(\hbar^2/2m)|\nabla\Psi|^2$, which could be generalized by making the substitution[32,61]

$$\frac{\hbar^2}{2m} \to \frac{\hbar^2}{2m^2} m^*(\eta, \varepsilon) = \varepsilon^{-\sigma} f_1 \left(\frac{\eta}{\eta_e(0)|\varepsilon|^\beta} \right),$$

with $\sigma = \hat{\eta}\nu$, while the exponents ν and $\hat{\eta}$ enter in the expressions for the coherence length $\xi \sim \varepsilon^{-\nu}$ and the Green's function $G_\eta(r \to 0) \sim r^{-(1+\hat{\eta})}$. We have assumed

above that $\nu = 2/3 = 2\beta$ and that $\hat{\eta} = 0$ (it is known from a number of examples[17] that the exponent $\hat{\eta}$ is quite small—usually of the order of several percent, and can perhaps also be equal to zero). In addition, the function f_1 was replaced by a constant.

The question of the value of the exponent $\hat{\eta}$ is connected with the problem of choosing the order parameter for helium II. The point is that besides the order parameter Ψ, which determines the observed quantities v_s and ρ_s (or j_s) in accordance with (1.1)–(1.3), in the case of helium II we deal also with a quantity such as the concentration n_0 of the helium atoms that have zero momentum (the Bose condensate). It is precisely the quantity n_0 (and the complex function $\tilde{\Psi}$ related with it by $n_0 = |\tilde{\Psi}|^2$) which has a clear-cut microscopic meaning and can be regarded as an order parameter. According to[28], if we use this order parameter $\tilde{\Psi}$ (we note that the notation in[28] is different, so that $\tilde{\Psi}$ plays the role of Ψ and vice versa) and put $|\tilde{\Psi}| = \sqrt{n_0} \propto \varepsilon^\beta$ and $C_p \propto \varepsilon^{-\alpha}$ at $T < T_\lambda$, then $\rho_s \propto |\Psi|^2 \propto \varepsilon^{2\beta-\hat{\eta}\nu}$. Taking into account one of the relations employed in similarity theory, namely $2\beta - \hat{\eta}\nu = \frac{1}{3}(2-\alpha)$, we obtain at $\alpha = 0$ (logarithmic variation of the heat capacity) the value $2\beta - \hat{\eta}\nu = 2/3$. Thus, the temperature dependence of ρ_s turns out to agree with experiment also at $\hat{\eta} \neq 0$. Conversely, if $\hat{\eta} = 0$, then the order parameters Ψ and $\tilde{\Psi}$, at least near the λ point, coincide (or, more accurately, are proportional to each other). Let us make a few more remarks in this connection. The single-particle density matrix of a system of N particles is by definition (the system is assumed here to be in a pure state, i.e., at any rate if $T = 0$)

$$\rho(r, r') = \int \psi^*(r, r_i) \psi(r', r_i) \, dr_i, \tag{2.55}$$

where $\psi(r, r_i)$ is the true wave function of the system and depends on the coordinates of all N particles ($i = 2, \ldots, N$; the coordinates of one of the particles will be designated by r and at another point by r'; of course, $dr_i = dr_2, dr_3, \ldots, dr_N$).

For ordinary liquids or non-superconductors we have $\rho(r, r') \to 0$ as $|r - r'| \to \infty$, but for a superfluid it can be assumed that $\rho(r, r') \to \rho_0 \neq 0$ as $|r - r'| \to \infty$. In any case, this property means that the system particles have a non-zero probability of having a zero momentum, and this precisely corresponds to Bose condensation (the concentration n_0 of particles with zero momentum differs from zero or, as is customarily stated, is finite).[17] Using the operators $\hat{\psi}(r)$ and $\psi(r)$ for the

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[16]Favoring this conclusion are, in particular, the measured values of the magnetization m as a function of the field h in magnetic phase transitions (see, e.g.,[64]). The connection between m and h is in this case similar to (2.54): $h = |\varepsilon|^{2-\alpha-\beta} f'(m/m_e(0)|\varepsilon|^\beta)$. It is known from experiment[64] that expansion of the function $f' \equiv x(df/dx)$ in powers of $x^2 = m/m_e(0)|\varepsilon|^\beta$, with the first three terms retained, can be used up to values $x \approx 1.8$.

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[17]Insofar as we know, this property of the single-particle density matrix ($\rho_0 \neq 0$), as a characteristic of the superfluid state, was first formulated by L. D. Landau (see[65]). This property was named later "off-diagonal long-range order" (ODLRO), since it referred to the off-diagonal elements of the ρ matrix ($r = r'$ for its diagonal elements). Brief mention of ODLRO or of the interpretation of the macroscopic Ψ function (the order parameter) associated with this concept was made already in[26]. This question was subsequently considered in a large number of articles (see[66,67] and the bibliography in[67]).

creation and annihilation of a particle at the point $\mathbf{r}$ we can write down for any temperature $\rho(\mathbf{r}, \mathbf{r}')$ $= \langle\langle \hat{\psi}^*(\mathbf{r})\hat{\psi}(\mathbf{r})\rangle\rangle$, where the double brackets $\langle\langle \; \rangle\rangle$ denote both quantum-mechanical and statistical averaging. The macroscopic wave function characterizing the superfluid (superconducting) system can be naturally introduced in the form[26]

$$\rho(\mathbf{r}, \mathbf{r}')_{|\mathbf{r}-\mathbf{r}'|\to\infty} = \widetilde{\Psi}^*(\mathbf{r})\,\widetilde{\Psi}(\mathbf{r}'). \qquad (2.56)$$

In (2.56) we write $\widetilde{\Psi}$ instead of Ψ, in order to emphasize the connection, which is clear from the foregoing, that exists between just the quantity $\widetilde{\Psi}$ and the particle concentration $n_0(t)$ in the condensate. If the averaging $\langle\langle \; \rangle\rangle$ is over an ensemble with violated symmetry, for which the order parameter has a definite (specified) phase, then we can write $\widetilde{\Psi} = \langle\langle \hat{\psi}(r)\rangle\rangle$ (see[28,67]).

In the case of helium II, both $\rho_s(t)$ and $n_0(t)$ are observable; however, measurement of $n_0(t)$, say by neutron scattering, is in fact still very difficult and has not been carried out at all near the λ point. The question remains whether it is more correct to relate ρ_s or n_0 with the modulus of the order parameter and, accordingly to choose Ψ or $\widetilde{\Psi}$ as this parameter. In the phenomenological approach to the problem we see no arguments of fundamental character for such a choice, and we assume that the choice should be dictated only by considerations of convenience and maximum simplicity of the equations (and, or course, by the results of comparison with experiment, since even the most convenient and simple equations that do not agree well with experiment are of no value). From this point of view there is no doubt (at least at the present stage) that the order parameter in the Ψ-theory of superfluidity should be chosen to be Ψ, which is connected with ρ_s and $\mathbf{v}_s$ by expressions (1.1) and (1.2). This is how we shall proceed.

Once the microscopic theory is developed, the question of the roles of Ψ and $\widetilde{\Psi}$ should be resolved automatically, since both quantities will be expressed in terms of microscopic parameters. Here, as we have seen, the function $\widetilde{\Psi}$ is expressed directly in terms of the density matrix $\rho(\mathbf{r}, \mathbf{r}')$ and in this sense it is "closer" to the microscopic theory. But from this it does not follow at all that the equation for $\widetilde{\Psi}$ near the λ point is simpler and more lucid than the equation for the function Ψ, which will be expressed somehow in terms of $\rho(\mathbf{r}, \mathbf{r}')$ and possibly other quantities. [18]

It remains to add that the definitions of the order parameter in terms of Ψ and $\widetilde{\Psi}$ near the λ point are not

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[18] An analogous problem can arise, in fact, in the case of superconductivity. Phenomenologically, however, only the function Ψ, which is connected with the depth of penetration of the magnetic field into the superconductor, was introduced. [26] The same equations for Ψ were derived later on[68] from the microscopic theory of superconductivity. The fluctuations, which are particularly small for superconductors, were not taken into account, and the functions Ψ and $\widetilde{\Psi}$ were in fact assumed to be identical (see the remarks connected with relation (2.57) below).

mutually exclusive and may turn out to be identical. The point is that the quantity $\sqrt{\rho_s/m} \equiv \sqrt{n_s}$ is set in correspondence, in fact (apart from a possible mass renormalization $m \to m^n \propto t^{-\tilde{n}\nu}$, which depends little on the temperature), with the modulus of the local value of a certain macroscopic wave function Ψ, which already incorporates averaging over regions of the order of the correlation length ξ_M. At the same time, it is natural to set the square root of the particle density in the condensate $\sqrt{n_0}$ in correspondence with the modulus of the total average value of the same function Ψ, with allowance for arbitrary long-wave fluctuations, particularly phase fluctuations. These two types of averages, generally speaking, do not coincide, for when account is taken of the long-wave phase fluctuations[89,61] we have

$$|\widetilde{\Psi}| = |\langle\Psi\rangle| \equiv |\langle \eta e^{i\varphi}\rangle| \approx |\Psi||\langle e^{i\varphi}\rangle| = |\Psi| \exp\left(-\frac{\langle\varphi^2\rangle}{2}\right), \qquad (2.57)$$

where the angle brackets $\langle \; \rangle$ correspond to additional averaging over the fluctuations with wavelengths $\Lambda = 2\pi/q \gtrsim \xi_M$. In[70], starting from this difference, a simple explanation was obtained for the temperature dependence of $n_0(T)$ at low temperatures, which differs from the relation $n_s(T) \equiv \rho_{se}(T)/m$. It was also made clear, in principle, that n_0 differs from the total particle density n. Near the λ point one can in principle not exclude the possibility that $|\Psi|$ differs from $|\widetilde{\Psi}| = |\langle\Psi\rangle|$, but, as we have seen (see (2.49a)), in this region the quantity $\langle(\delta\varphi)^2\rangle$, at least in the employed approximation, is small and does not depend on $t = T_\lambda - T$. This is precisely why it is quite probable that near the λ point the parameters Ψ and $\widetilde{\Psi}$ differ only by a numerical factor (in the spatially-homogeneous case) or practically coincide (at $\hat{\eta} = 0$). Incidentally, it can be concluded from the foregoing analysis that for our purposes it is natural to use in fact the local order parameter Ψ.

Thus, the Ψ-theory of superfluidity, formulated in Sec. 2.1, is actually subject to a number of restrictions and assumptions. However, for reasons that are clear from the foregoing and are still discussed in the concluding Chap. V, we regard this situation as perfectly normal and not precluding the extensive use of the Ψ-theory of superfluidity to solve actual problems, which is the subject of the next Chap. III.

III. CONSIDERATION OF VARIOUS PROBLEMS AND EFFECTS IN HELIUM NEAR THE λ POINT

3.1. Size effects in films, capillaries, and pores (shift of λ point, decrease of density ρ_s, and change of heat capacity C_p)

The vanishing of the modulus of the order parameter $|\Psi| = \eta = \sqrt{\rho_s/m}$ on a solid wall (boundary condition (2.24), and probably also on the free surface of helium II (see Sec. 3.3 below) leads to the appearance of size effects in "samples" (films, capillaries, etc.) with characteristic dimensions L comparable with the correlation length $\xi_M(t)$. In fact, if $|\Psi| = 0$ "on the ends" (on the walls or on the free surface), then the mean value of $|\Psi|$ over the entire volume of the film or of the capillary will be less than for helium II in a large volume. Thus, it is

directly clear that the mean value $\bar{\rho}_s$ is decreased, and this leads, e. g., to a decrease in the velocity of fourth sound in porous materials. With increasing dimension L (film thickness, etc.) the λ-point temperature $T_\lambda(L)$ at which $\rho_s(T_\lambda) = 0$ is also naturally lowered. A change takes place also in the equilibrium value of the thermodynamic potential, and consequently also in the heat capacity and other thermodynamic quantities. These effects have been discussed on the basis of the Ψ-theory of superfluidity in[25,27,29,61-75], and there are also a number of experimental data[44,76-80] (see also references to earlier work in the cited articles).[74,76,77] But the question of a quantitative comparison of theory with experiment is in fact still open. We shall therefore dwell on size effects in considerable detail.

In this Chap. III (with the exception of 3.4), the helium is assumed to be at rest, so that the parameter Ψ can be regarded as positive, real, and satisfying Eq. (2.20). For convenience, we write down this equation for the case when ψ_0 varies only in the z direction

$$\frac{d^2\psi_0}{dz_*^2} = [-1 + (1-M)\psi_0^2 + M\psi_0^4]\psi_0,$$

$$z_* = \frac{z}{\xi_M(t)}, \quad \xi_M = \xi_0\left(\frac{3+M}{3}\right)^{1/2} = \xi_{00}\left(\frac{3+M}{3}\right)^{1/2} t^{-2/3}$$

$$= 2.74 \cdot 10^{-8}\left(\frac{3+M}{3}\right)^{1/2} t^{-2/3} \text{ cm,}$$

$$\psi_0 = \frac{\Psi}{(1.43\rho_\lambda/m)^{1/2} t^{1/3}}, \quad t = T_\lambda - T. \quad (3.1)$$

Equation (3.1) has as its first integral

$$\left(\frac{d\psi_0}{dz_*}\right)^2 + \psi_0^2 - \frac{1-M}{2}\psi_0^4 - \frac{M}{3}\psi_0^6 = \text{const} \equiv I, \quad (3.2)$$

and any problem for a plane layer can be solved in quadratures. We begin, however, with the simplest case, when a plane layer (film) of thickness d goes over into the superfluid state at a certain temperature $T_\lambda(d)$ via a second-order transition, i.e., $\psi_0 = 0$ at the transition point. Then the value of ψ_0 close enough to the transition point tends to zero, and we can confine ourselves in the right-hand side of (3.1) to the first term, meaning that $\psi_0(z) = C_1 \sin z_* + C_2 \cos z_*$. For a layer, taking the boundary conditions (2.24) into account, we have

$$\psi_0(0) = 0, \quad \psi_0(d) = 0 \quad (3.3)$$

or, in a form that is frequently more convenient

$$\psi_0(0) = 0, \quad \left(\frac{d\psi_0}{dz}\right)_{z=d/2} = 0. \quad (3.3a)$$

Obviously, under these conditions we have $\psi_0'(z) \simeq C_1 \sin z_*$, and a nontrivial solution is obtained only at $d \geq d_c$, where $\sin(d_c/\xi_M) = 0$, i. e.,

$$d_c = \pi\xi_M(t) = 8.61 \cdot 10^{-8}\left(\frac{3+M}{3}\right)^{1/2} t^{-2/3} \text{ cm.} \quad (3.4)$$

Hence, by virtue of the definition of $\xi_M(t)$, and recognizing that in the foregoing we had everywhere $T_\lambda \equiv T_\lambda(d \to \infty)$, we get

$$\Delta T_\lambda \equiv T_\lambda - T_\lambda(d) = \frac{[(3+M)/3]^{3/4} (\pi\xi_{00})^{3/2}}{d^{3/2}} = \frac{2.53 \cdot 10^{-11} [(3+M)/3]^{3/4}}{d^{3/2}} \text{ }^\circ\text{K},$$

$$(3.4a)$$

where d is the thickness (in centimeters) of the layer in question (in this formulation of the problem, when we determine the temperature $T_\lambda(d)$, it is natural to omit the subscript "c" of d). For a round capillary of radius r_0 and for a spherical cavity (pore or drop) of radius R_0 it is more effective to use in (2.20) cylindrical or spherical coordinates, respectively, and in (3.1) the derivative $d^2\psi_0/dz_*^2$ is replaced by

$$\frac{1}{r_*}\frac{d}{dr_*}\left(r_*\frac{d\psi_0}{dr_*}\right) \text{ and } \frac{1}{R_*^2}\frac{d}{dR_*}\left(R_*^2\frac{d\psi_0}{dR_*}\right).$$

In place of the boundary condition (3.3a) we now have

$$\psi_0(r_0) = 0, \quad \left(\frac{d\psi_0}{dr}\right)_{r=0} = 0; \quad \psi_0(R_0) = 0, \quad \left(\frac{d\psi_0}{dR}\right)_{R=0} = 0. \quad (3.5)$$

Proceeding as before, we obtain (2.4048 is the first root of the Bessel function $I_0(r)$)

$$r_{0,c} = 2.4048\xi_M, \quad \Delta T_\lambda = T_\lambda - T_\lambda(r_0) = \frac{1.69 \cdot 10^{-11} [(3+M)/3]^{3/4}}{r_0^{3/2}}, \quad (3.6)$$

$$R_{0,c} = \pi\xi_M, \quad \Delta T_\lambda = T_\lambda - T_\lambda(R_0) = \frac{2.53 \cdot 10^{-11} [(3+M)/3]^{3/4}}{R_0^{3/2}}. \quad (3.7)$$

A check on the validity of the theory would be satisfaction of the relation[19] $\Delta T_\lambda \propto d^{-3/2}$, and measurement of the coefficient will make it possible, in principle, to determine the only remaining free parameter M. Unfortunately, at $M \leq 1$, when a second-order phase transition takes place and the formulas written above are valid, the dependence of d_c or of ΔT_λ on M is quite weak (when M varies from 0 to 1 the coefficients in formulas (3.4a), (3.6), and (3.7) change by only a factor $(4/3)^{4/3}$, i. e., by 24%). At $M > 1$ the transition of a film from the normal state (helium I) to the superfluid state (helium II) is of first order, as will be shown below, and the critical value of the thickness d_{tr} corresponding to the thermodynamic transition point (the point where the thermodynamic potentials are equal), as well as the corresponding "shift" $\Delta T_\lambda = T_\lambda - T_{tr}(d)$, depend on M very weakly, as before. Thus, for example, for a plane-parallel gap we have $d_{tr} = 4.4\xi_0$ as $M \to \infty$, i. e., it differs from the value of d_c at $M = 0$ by only a factor of 1.4. Thus, even very large values of M cannot be regarded as excluded from the known experimental data. On the other hand, the characteristic relation $\Delta T_\lambda \propto d^{3/2}$, which does not depend on M, is confirmed by experiment.[44,74,78,81,82,88]

One of the most pressing problems of future research is measurement of $\Delta T_\lambda(d)$ in rather wide gaps, i. e., close to $T_\lambda \equiv T_\lambda(d \to \infty)$. Only in this case is the condition $\xi_M \gg a \sim 3 \times 10^{-8}$ cm, which is needed for the theory to hold quantitatively, satisfied (see (1.4) and (2.21)), and at the same time it is possible to measure the thickness d with sufficient accuracy. On the other hand, such measurements appear to be quite realistic,

[19]More accurately, failure to satisfy this relation would contradict the theory, whereas the relation $\Delta T_\lambda \propto d^{-3/2}$ should be satisfied even under simpler assumptions, provided only that $\xi(t) \propto t^{-2/3}$. We shall no longer repeat below the similar stipulations connected with the universally known "asymmetry" in the interpretation of experiments that agree or do not agree with the conclusions of some particular theory.

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since it is possible to work with helium II very close to the λ point (see, e.g., [58,91] where values $t \lesssim 10^{-6}$ °K were attained). By way of example we note that at $d = 1\mu = 10^{-4}$ cm the length is $\xi_M(t = \Delta T_\lambda) = d_c/\pi = 3.2 \times 10^{-5}$ cm and the difference is

$$\Delta T_\lambda(d) = 2.5 \cdot 10^{-5} \left[\frac{3+M}{3}\right]^{3/4} \text{°K};$$

it is clear that even gaps with $d = 10^{-5}$ to 10^{-6} cm, if they can be produced and their dimensions controlled, are still perfectly suitable from the point of view of satisfying the condition $\xi_M \gg a$.

We turn now to the other limiting case of very thick layers, when it suffices to solve the problem for a half-space. In this case it is convenient to measure the distances in units of $\xi_0(t) = \xi_M\sqrt{3/(3+M)} = 2.74 \cdot 10^{-8} t^{-2/3}$ cm. Thus, we put $y = z/\xi_0(t)$, after which Eq. (3.1) and its first integral (3.2) take the form

$$2\frac{d^2\psi_0}{dy^2} = [-(2-v_0^2) + 2(1-2v_0^2)\psi_0^2 + 3v_0^2\psi_0^4]\psi_0, \tag{3.8}$$

$$2\left(\frac{d\psi_0}{dy}\right)^2 + (2-v_0^2)\psi_0^2 - (1-2v_0^2)\psi_0^4 - v_0^2\psi_0^6 = E, \tag{3.9}$$

where $v_0^2 = 2M/(M+3)$ and $E = (M+3)I/6$ is a certain new constant. Equation (3.9) can also be rewritten in the form

$$2\left(\frac{d\psi_0}{dy}\right)^2 - (1-\psi_0^2)^2(1+v_0^2\psi_0^2) + 1 = E, \tag{3.9a}$$

from which it is clear why it is convenient to measure the distances of units of ξ_0 and to replace the parameter M by v_0.

Let us find the solution of Eq. (3.8) under the boundary conditions (the helium fills the region $0 \leq y \leq \infty$)

$$\psi_0(0) = 0, \quad \psi_0(\infty) = 1.$$

Of course, as $y \to \infty$ the derivative $d\psi_0/dy \to 0$. In this case, obviously, the constant E in (3.9a) is equal to 1, and the equation can be integrated in terms of elementary functions

$$\psi_0(y) = \frac{\text{th}(y\sqrt{(1+v_0^2)/2})}{\sqrt{1+v_0^2 \text{ ch}^{-2}(y\sqrt{(1+v_0^2)/2})}}. \tag{3.10}$$

In the simplest variant, when $M = 0$, we have

$$\psi_0(z) = \text{th}\frac{z}{\xi_0\sqrt{2}}, \quad \rho_s(z) = \rho_{se}\text{th}^2\frac{z}{\xi_0\sqrt{2}}, \quad \rho_{se} = 1.43\rho_\lambda t^{2/3}. \tag{3.11}$$

We note that at large distances from the wall we have

$$\psi_0(z) \approx 1 - \exp\left(-\frac{z\sqrt{2(1+v_0^2)}}{\xi_0}\right).$$

The length $\xi_M = \xi_0/\sqrt{2(1+v_0^2)} = \xi_0\sqrt{(3+M)/6(1+M)}$, which has the meaning of the correlation function of the modulus of the parameter Ψ below the λ point, changes only by $\sqrt{3}$ times when M is varied from 0 to ∞. This is indeed the cause, as we shall show below, of the relatively weak dependence of various surface effects in thick films on the value of M.

Besides the distribution $\rho_s(z)$ itself near the wall, interest attaches also to the surface "deficit" of the superfluid mass

$$\Delta = \int_0^\infty [\rho_{se} - \rho_s(z)]\,dz, \tag{3.12}$$

to the surface energy

$$\sigma = \int_0^\infty [\Phi_{II}(\rho_s(z)) - \Phi_{II}(\rho_{se})]\,dz = \int_0^\infty [\Phi_{II-I}(\rho_s) - \Phi_{II-I}(\rho_{se})]\,dz, \tag{3.13}$$

and also to certain derivatives of the surface energy,[20] viz., the surface entropy $S_\sigma = -(\partial\sigma/\partial T)_\mu$, the surface heat capacity $C_\sigma = -T(\partial^2\sigma/\partial T^2)_\mu$, and the excess surface mass $m_\sigma = -(\partial\sigma/\partial\mu)_T$.

In addition, in the case of solutions of helium-3 in helium-4, interest can also attach to the excess surface mass of the He[3]: $m_{3,\sigma} = -(\partial\sigma/\partial\mu_3)_{\mu,T}$, where μ_3 is the chemical potential of He[3].

The thermodynamic potential $\Phi_{II-I}(\rho_s)$ per unit volume at $v_s = 0$ is, according to (2.13) and (2.15),

$$\Phi_{II-I}(\rho_s) \equiv \Phi_{II}(p, T, \rho_s) - \Phi_I(p, T)$$
$$= \Phi_{I-II}(\rho_{se})[-(2-v_0^2)\psi_0^2 + (1-2v_0^2)\psi_0^4 + v_0^2\psi_0^6 + 2\left(\frac{d\psi_0}{dy}\right)^2],$$
$$\Phi_{I-II}(\rho_{se}) \equiv \Phi_I - \Phi_{II}(\rho_{se}) = \frac{\Delta C_p}{2T_\lambda}t^2, \quad t = T_\lambda - T,$$
$$y = \frac{z}{\xi_0(t)}, \quad v_0^2 = \frac{2M}{M+3}. \tag{3.14}$$

Substituting the solution (3.10) in (3.12) and (3.13) and taking (3.14) into account, it being convenient to change from integration with respect to the coordinate to integration with respect to ψ_0 and to use (3.9a), we obtain

$$\Delta(t) = \xi_0(t)\rho_{se}(t)\frac{\sqrt{2}}{v_0}\ln(v_0 + \sqrt{1+v_0^2}), \tag{3.15}$$

$$\sigma(t) = \xi_0(t)\Phi_{I-II}(\rho_{se})\frac{1}{2\sqrt{2}v_0^3}[(1+4v_0^2)\ln(v_0+\sqrt{1+v_0^2})$$
$$-v_0\sqrt{1+v_0^2}(1-2v_0^2)]. \tag{3.16}$$

The derivatives of σ with respect to T and with respect to μ, in view of the power-law dependence of σ on $t = T_\lambda(\mu) - T$, are obviously expressed in terms of σ and t themselves:

$$S_\sigma = \sigma_t'(t) = (2-\alpha-v)t^{-1}\sigma(t) = \frac{4}{3}t^{-1}\sigma(t), \tag{3.17}$$

$$C_\sigma = -T_\lambda\sigma_{tt}''(t) = (2-\alpha-v)(1-\alpha-v)T_\lambda t^{-2}\sigma(t) = -\frac{4}{9}\frac{T_\lambda}{t^2}\sigma(t), \tag{3.18}$$

$$m_\sigma = \sigma_\mu'(t = T_\lambda(\mu) - T) = -\sigma_t'\frac{dT_\lambda}{d\mu} = \left|\frac{dT_\lambda}{d\mu}\right|S_\sigma, \tag{3.19}$$

$$m_{3,\sigma} = -\sigma_{\mu_3}''(t = T_\lambda(\mu, \mu_3) - T) = -\frac{dT_\lambda}{d\mu_3}S_\sigma. \tag{3.20}$$

In the scheme considered by us, the critical exponents are $\alpha = 0$ and $v = 2/3$. If $M = 0$, when the solution (3.11) is valid, the calculation is particularly simple and we obtain directly, or else from (3.14) as $v_0 \to 0$,

$$\Delta_0 = \sqrt{2}\xi_0(t)\rho_{se}(t) = 0.81 \cdot 10^{-8} \text{ g/cm}^2 \tag{3.15a}$$

$$\sigma_0 = \frac{4\sqrt{2}}{3}\xi_0(t)\Phi_{I-II}(\rho_{se}) = 9.0 \cdot 10^{-2}t^{4/3} \text{ erg/cm}^2 \tag{3.16a}$$

In this case

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[20] We have already emphasized (see Sec. 2.1) that to solve static spatially-inhomogeneous problems we must use not the potential $\Phi(p, T)$ but the potential $\Omega(\mu, T)$.

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TABLE II.

M	$v_0^2 = 2M/(3+M)$	$\Delta/\xi_0\rho_{se}$	$\sigma/\xi_0\Phi_{I-II}(\rho_{se})$
0	0	1.414	1.886
1	1/2	1.317	1.976
2	4/5	1.272	2.026
∞	2	1.146	2.208

$$S_\sigma = 0.12 t^{1/3}\,\text{erg/cm}^2\,\text{deg,}, \tag{3.17a}$$

$$C_\sigma = -8.7\cdot10^{-2}t^{-2/3}\,\text{erg/cm}^2\,\text{deg,} \tag{3.18a}$$

$$m_\sigma = 1.54\cdot10^{-10}t^{1/3}\,\text{g/cm}^2 \tag{3.19a}$$

To measure the profile $\rho_s(z)$ we can use a procedure, albeit not an easy one, of probing with second sound; this will be discussed in Sec. (3.2). The quantities Δ and σ (as well as their derivatives) can be measured by using "thick" films of thickness[21]

$$d \gg \xi_0 = 2.74\cdot10^{-8}t^{-2/3}\,\text{cm} .$$

In this case the mean values $\bar\rho_s$ and $\bar\Phi_{II} \equiv \bar\Phi_{\perp} - \bar\Phi_{I-II}$ (averaged over the cross section) in the film are equal to

$$\bar\rho_s = \rho_{se} - \frac{2\Delta}{d} = \rho_{se}\left(1 - \frac{2l(t)}{d}\right), \tag{3.20a}$$

$$\bar\Phi_{II} = \Phi_{II}(\rho_{se}) + \frac{2\sigma}{d}. \tag{3.20b}$$

The quantity $l = \Delta(t, m)/\rho_{se}(t)$ is frequently called the healing length. Analogous formulas hold also for other quantities. For example,

$$\bar S_{II} = S_{II}(\rho_{se}) + \frac{2S_\sigma}{d} = S_{II}(\rho_{se}) + \frac{8}{3}t^{-1}\frac{\sigma}{d}, \tag{3.20c}$$

$$\bar C_p = C_p(\rho_{se}) + \frac{2C_\sigma}{d} = C_p(\rho_{se}) - \frac{8}{9}\frac{T_\lambda}{t^2}\frac{\sigma}{d}, \tag{3.20d}$$

$$\rho = \rho(\rho_{se}) + \frac{2m_\sigma}{d} = \rho(\rho_{se}) + \frac{8}{3}t^{-1}\left|\frac{dT_\lambda}{d\mu}\right|\frac{\sigma}{d}. \tag{3.20e}$$

Measurements of the foregoing quantities, and in the case of solutions also of $m_{3,\sigma}$, can be used in principle to determine the parameter M, or to check the theory if M is known from other experiments. Unfortunately, the dependence of Δ and σ on M (it enters in (3.15) and (3.16) via $v_0^2 = 2M/(3+M)$ is quite weak, as is clear from Table II.

For films of arbitrary thickness, the results of $\psi_0(z)$ and other quantities cannot be expressed in terms of elementary functions. In a qualitative investigation of the problem it is convenient to use a mechanical analogy, for when y is replaced by the time t, and Ψ_0 is replaced by the coordinate of the point x, Eq. (3.8) describes the motion of a point having a mass 4 in the field of a potential (Fig. 1).

$$V(x) = (2 - v_0^2)x^2 - (1 - 2v_0^2)x^4 - v_0^2x^6 = 1 - (1 - x^2)^2(1 + v_0^2x^2). \tag{3.21}$$

On the other hand, Eq. (3.9) has the meaning of an

energy conservation law, the total energy $(V + 2\dot x^2)$ being equal to E. The distribution of $\psi_0(y)$ at half the film thickness $(0 \leqslant y \leqslant d/2\xi_0)$ corresponds obviously to the trajectory of the particle on the section from $x = 0$ to the turning point x_m, where x_m is determined from the solution of the equation

$$E - V(x_m) = 0. \tag{3.22}$$

Using this circumstance, the expression for the distribution $\psi_0(y)$ can be immediately written in parametric form (y varied in the interval from 0 to $d/2\xi_0$)

$$y = \sqrt{2}\int_0^{x\equiv\psi_0} \frac{dx}{\sqrt{E - V(x)}}.$$

Substituting x_m as the upper limit in this expression, we obtain the connection between the amplitude of the distribution $x_m \equiv \psi_0(d/2)$ and the film thickness

$$\frac{d}{\xi_0} \equiv D = 2\sqrt{2}\int_0^{x_m} \frac{dx}{\sqrt{E - V(x)}}. \tag{3.23}$$

The quantity D has obviously, in the language of the mechanical analogy, the meaning of the half-period of the oscillations. Let us analyze (3.23) in somewhat greater detail. To this end we substitute in it $V(x)$ and E in accord with (3.21) and (3.22), and make the substitution $x/x_m = \sin\varphi$ under the integral sign.

After some transformations we obtain

$$D \equiv \frac{d}{\xi_0} = 2\sqrt{2}\int_0^{\pi/2} \frac{d\varphi}{R(x_m^2, \varphi)}, \tag{3.24}$$

$$R(x_m^2, \varphi) = [(2 - v_0^2) - (1 - 2v_0^2)(1 + \sin^2\varphi)x_m^2$$
$$- v_0^2(1 + \sin^2\varphi + \sin^4\varphi)x_m^4]^{1/2}. \tag{3.25}$$

As $x_m^2 \to 0$, the integrand becomes $(2 - v_0^2)^{-1/2}$, and we obtain, as we should,

$$d_c = D_c\xi_0 = \pi\xi_0\sqrt{2/(2 - v_0^2)} = \pi\xi_0\sqrt{(3 + M)/3} = \pi\xi_M(t).$$

We now obtain the next (first) term of the expansion of D in x_m^2

$$D = \frac{d}{\xi_0} = 2\sqrt{\frac{2}{2 - v_0^2}}\int_0^{\pi/2}\left[1 + \frac{1 - 2v_0^2}{2(2 - v_0^2)}x_m^2(1 + \sin^2\varphi) + \ldots\right]d\varphi$$
$$= D_c\left(1 + \frac{3}{8}(1 - M)x_m^2 + \ldots\right), \quad D_c = \pi\sqrt{\frac{3 + M}{3}}. \tag{3.26}$$

It is seen from (3.26) that at $M > 1$ the dimensionless length D first decreases with increasing x_m. At the same time, as $x_m \to 1$ the "period" D (the film thickness) increases without limit. Thus D, as a function of x_m, has a minimum $D = D_{c1}$ at a certain value $x_m = x_{m,c}$. If now, conversely, we regard x_m as a function of D, then in the interval $D_{c1} < D < D_c$ this function is double-val-
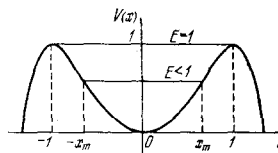
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[21]More accurately, as we shall see below, it suffices to have d larger than the "critical" thickness d_c or d_{tr} (in the case when the phase transition in the film is of first order) by an amount on the order of $2\xi_M^- = 2\xi_0\sqrt{(3 + M)/6(1 + M)}$.
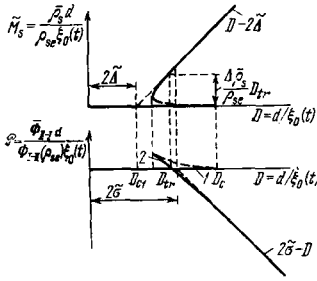

FIG. 1. Plot of the function $V(x)$.

FIG. 2. Dependence of $\tilde{M}_s$ and of the dimensionless thermodynamic potential $\mathcal{P}$ on the film thickness for the case $M > 1$.

ued, i.e., one and the same value of the film thickness $d = D\xi_0$ corresponds here to two different distributions of $\psi_0(y)$ with different amplitudes x_m.

To determine the cause of this ambiguity, we turn to an analysis of the behavior of the total excess thermodynamic potential of the film referred to a unit surface, $\tilde{\Phi}_{II-I} \equiv \int \Phi_{II-I}(\rho_s) dz = \int [\Phi_{II}(\rho_s) - \Phi_I] dz$. The mechanical analog of $\tilde{\Phi}_{II-I}$ (more accurately, of the ratio $\mathcal{P} = \tilde{\Phi}_{II-I}/\Phi_{I-II}(\rho_{se})\xi_0(t))$ is the action integral of the particle over a half-period of the oscillations

$$\frac{\tilde{\Phi}_{II-I}}{\xi_0 \Phi_{I-II}(\rho_{se})} \equiv \mathcal{P} = 2 \int_0^{D/2} [2\dot{x}^2 - V(x)] dt$$

$$= -ED + 4\sqrt{2} \int_0^{x_m} \sqrt{E - V(x)}\, dx.$$

$$(3.27)$$

The derivative $\mathcal{P}'(D)$ of the action with respect to the half-period of the oscillation is, as is well known, the negative total energy of the particle (this can be verified also by direct differentiation of (3.25), after first making the substitution $\bar{x} = x/x_m$ under the integral sign). Since $E = V(x_m) > 0$ by its very meaning, this means that $\mathcal{P}$ decreases monotonically with increasing D. It is easy also to determine the sign of the second derivative of the function $\mathcal{P}(D)$. We have

$$\mathcal{P}''(D) = -E'(D) = -\frac{E'(x_m^2)}{D'(x_m^2)}.$$

But $E'(x_m^2) = dV/d(x_m^2) > 0$. Therefore $\mathcal{P}''(D) < 0$ if $D''(x_m^2) > 0$, and conversely $\mathcal{P}''(D) > 0$ if $D''(x_m^2) < 0$. Using the obtained information concerning the signs of the derivatives, we can easily construct the entire plot of $\mathcal{P}(D)$. It must be borne in mind here that $\mathcal{P}(D = D_c) = \mathcal{P}'(D = D_c) = 0$, and that at large D the $\mathcal{P}(D)$ curve approaches asymptotically the straight line $2\bar{\sigma} - D$, where $\bar{\sigma} = \sigma/(\xi_0 \Phi_{I-II}(\rho_{se}))$ (see formula (3.20b)). The approximate form of the plot of $\mathcal{P}(D)$ is shown in the lower half of Fig. 2. This plot is typical of first-order phase transitions. The point D_{tr} where the lower curve crosses the abscissa axis corresponds here to the thermodynamic transition point (the point where the thermodynamic potentials are equal), and sections 1 and 2 correspond to the superheated and supercooled states. The upper branch of the curve (shown dashed in Fig. 2) corresponds to unstable states. In addition to the plot of $\mathcal{P}(D)$, the upper half of Fig. 2 shows a plot of the function $\tilde{M}_s(D) = \rho_{se}^{-1} \int \rho_s(z) dz$. The fact that this plot is S-shaped also indicates a first-order transition.

Approximate plots of the functions $\mathcal{P}(D)$ and $\tilde{M}_s(D)$ in the case $M \lesssim 1$, which is simpler to analyze, are shown in Fig. 3.

Thus, depending on the value of the parameter M, the phase transition in the film can be of first order $(M > 1)$ or of second order $(M < 1)$.[29,61] The value $M = 1$ corresponds to the tricritical point. The parameter M can, in principle be affected by pressure or by introduction of He^3 as an impurity. In fact, if the He^3 concentration in the solution is $x_3 > x_{3,t}$ $(x_{3,t} \simeq 67\%$ at saturated-vapor pressure), then the transition to the superfluid state is of first order also in an unbounded liquid. In this case certainly $M > 1$. The situation is less clear at $x_3 < x_{3,t}$ and even in pure He^4, since systematic investigations of the λ transition in sufficiently thick films or gaps (with $d \gtrsim (0.1$ to $1)\mu)$, with well-controlled geometry, have not yet been carried out, and it is difficult in practice to distinguish between a second-order phase transition and a weak first-order one (according to[74], the transition in thin films is more likely to be of second order).

In a quantitative study of the behavior of various thermodynamic functions of a film, the integral in the right-hand side of (3.27) is best written in a form similar to (3.24) and (3.25):

$$I_1 = 4\sqrt{2} \int_0^{x_m} \sqrt{E - V(x)}\, dx = 4\sqrt{2}\, x_m^3 \int_0^{\pi/2} \cos^2\varphi R(x_m^2,\varphi)\, d\varphi. \quad (3.28)$$

The average thermodynamic potential of the film is in this case:

$$\tilde{\Phi}_{II}(d) = \Phi_I + \Phi_{I-II}(\rho_{se})\left(-E + \frac{I_1}{D}\right), \quad (3.29)$$

$$\Phi_{I-II}(\rho_{se}) = \frac{\Delta C_p}{2T_\lambda} t^2, \quad D = \frac{d}{\xi_0(t)},$$

where according to (3.20) and (3.22),

$$E \equiv E(x_m^2) = x_m^2 [(2 - v_0^2) - (1 - 2v_0^2) x_m^2 - v_0^2 x_m^4]$$
$$= 1 - (1 - x_m^2)^2 (1 + v_0^2 x_m^2). \quad (3.30)$$

It is convenient to reduce also the expression for $\bar{\rho}_s(d)$ to a similar form:

$$\bar{\rho}_s(d) = \rho_{se}(t)\left(x_m^2 - \frac{I_2}{D}\right), \quad \rho_{se} = 1.43\rho_\lambda t^{2/3}, \quad (3.31)$$

$$I_2 = 2\sqrt{2}\, x_m^2 \int_0^{\pi/2} \frac{\cos^2\varphi\, d\varphi}{R(x_m^2, \varphi)}.$$

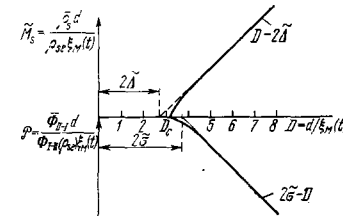Expressions (3.29) and (3.31) in conjunction with

FIG. 3. Dependence of $\tilde{M}_s$ and of the dimensionless thermodynamic potential $\mathcal{P}$ on the film thickness for the case $M < 1$.

(3.28), (3.30) and (3.24), (3.25) define $\overline{\Phi}_{\text{II}}(d)$ and $\overline{\rho}_s(d)$ in parametric form as functions of the parameter x_m, which has the meaning of the amplitude of the distribution of $\psi_0(z)$, i.e., of the maximum value reached by the function $\psi_0(z)$ at half the film thickness.

Differentiating (3.29) with respect to $t = T_\lambda - T$ and recognizing at the same time, as already mentioned, that $\mathcal{F}'(D) = d/dD(-ED + I_2) = -E(D)$, we obtain an expression for the average entropy density and for the average heat capacity of the film

$$\overline{S}_{\text{II}}(d) = S_{\text{I}} + S_{\text{I-II}}(\rho_{se})\left(-E + \frac{2}{3}\frac{I_1}{D}\right), \quad S_{\text{I-II}}(\rho_{se}) = \frac{\Delta C_p}{T_\lambda}t,$$

(3.32)

$$\overline{C}_{\text{II}}(d) = C_{\text{I}} + \Delta C_p\left[E - \frac{2}{9}\frac{I_1}{D} + \frac{2}{9}DE'(D)\right], \quad (3.33)$$

$$E'(D) = \frac{E'(x^2_m)}{D'(x^2_m)} = \frac{(2-v_0^2)-2(1-2v_0^2)x_m^2-3v_0^2x_m^4}{I_3(x_m^2)}, \quad (3.34)$$

$$I_3(x_m^2) = \sqrt{2}\int_0^{\pi/2}\frac{(1-2v_0^2)(1+\sin^2\varphi)+2v_0^2x_m^2(1+\sin^2\varphi+\sin^4\varphi)}{R^3(x_m^2,\varphi)}d\varphi.$$

Let us analyze the derived relations. We begin with thick films, of thickness

$$d \gg \overline{\xi}_M(t) = \xi_0\left[\frac{3+M}{6(1+M)}\right]^{1/2} = 2.74 \cdot 10^{-8}t^{-2/3}\left[\frac{3+M}{6(1+M)}\right]^{1/2}\text{ cm.}$$

In this case we can put $x_m^2 = E = 1$ and $E'(D) = 0$ in (3.29), (3.31), (3.32), and (3.33). The corresponding integrals I_1 and I_2 can be easily evaluated at $x_m^2 = 1$, and from (3.29), (3.31)–(3.33) we return, as we should, to the asymptotic formulas (3.20). We note that the asymptotic values given by (30) are rapidly (exponentially) reached in the case of thick films, with corrections on the order of $\exp\{-(d-d_0)/\xi_M^-\}$, where $d_0 \sim \min(d_{\text{tr}}, d_0)$ (d_{tr} is the critical value of the thickness, corresponding at $M > 1$ to the equilibrium transition temperature). Thus, with accuracy better than 5%, these formulas can be used already at

$$d > d_{\text{tr}} + 2\sqrt{2}\,\xi_M.$$

A few remarks are in order concerning the temperature dependences of $\overline{\rho}_s$ and $\overline{C}_p$ in thick films. In conjunction with (3.15), the result (3.20a) means that

$$\overline{\rho}_s = \rho_{se}(t) - \frac{\xi_0(t)\rho_{se}(t)}{d}\cdot\frac{2\sqrt{2}}{v_0}\ln(v_0+\sqrt{1+v_0^2})$$
$$= \rho_{se}(t) - K_M\rho_{se}(T_\lambda(d)), \quad v_0^2 = \frac{2M}{M+3},$$

(3.35)

i.e., for a thick film the average density $\overline{\rho}_s$ is simply "shifted" by an amount $K_M\rho_{se}(T_\lambda(d))$ in comparison with the density $\rho_{se}(T)$ in a large volume. The constant K_M ranges from 0.90 (at $M = 0$) to 0.52 (as $M \to \infty$). It is precisely this kind of "shift" of the function $\overline{\rho}_s(T)$ which is observed in experiment (see, e.g., [88-90]), but it cannot be used to estimate the numerical value of the parameter M, since the data obtained in the corresponding experiments refer to channels that are not wide enough ($\Delta T_\lambda \gtrsim 10^{-2}\,°\text{K}$) and are as a rule irregular, with a great uncertainty in the pore dimensions.

On the other hand, the difference between the heat capacity $\overline{C}_p$ of a thick film from the heat capacity $C_p(\rho_{se})$

in a large volume no longer reduces to a simple shift, and according to (3.20d) it increases in proportion to $t^{-2/3}$ with increasing temperature. The maximum value of the difference $C_p(\rho_{se}) - \overline{C}_p(d)$, reached at the limit of applicability of the asymptotic formulas, i.e., at $d \simeq d_{\text{tr}} + 2\sqrt{2}\xi_M$, is approximately $(0.1-0.2)\Delta C_p$. Unfortunately, we know of no exact measurements of the heat capacity C_p in "thick helium films."

We turn now to the case of thin films with $d \lesssim d_{\text{tr}} + 2\sqrt{2}\xi_M^-$. The analysis is simple enough here only if $M \lesssim 1$, i.e., when the phase transition in the film is of second order. Assume first that $M < 1$. In the limit when $x_m^2 \ll 1$, we obtain from (3.26)

$$x_m^2 = \frac{8}{3(1-M)}\left(\frac{D}{D_c} - 1\right) = \frac{8}{3(1-M)}\left[\frac{\xi_0(T_\lambda(d))}{\xi_0(T)} - 1\right]$$
$$\approx \frac{16}{9(1-M)}\frac{T_\lambda(d)-T}{T_\lambda-T_\lambda(d)}.$$

(3.36)

Using this expression, we obtain from (3.31)[22]

$$\overline{\rho}_s \approx \rho_{se}(T)\frac{x_m^2}{2} = \frac{8}{9(1-M)}\rho_{se}(T)\frac{T_\lambda(d)-T}{T_\lambda-T_\lambda(d)}$$
$$\approx \frac{1.27}{1-M}[T_\lambda-T_\lambda(d)]^{-1/3}[T_\lambda(d)-T],$$

(3.37)

where the last expression is obviously valid only if

$$T_\lambda(d) - T \lesssim \Delta T_\lambda \equiv T_\lambda - T_\lambda(d). \quad (3.38)$$

It is clear from (3.37) and (3.38) that a linear dependence of $\overline{\rho}_s$ on the temperature should be observed only in a narrow region near the temperature $T_\lambda(d)$ at which superfluidity appears in the film. One can hardly hope to reveal this effect distinctly by measuring the velocity of fourth sound in porous materials filled with helium II. The uncertainty in the pore dimensions and the need for taking into account the variation of these dimensions over distances on the order of $\xi_0(t)$ can lead to substantial changes of the picture. The available data for porous materials merely confirm the presence of the shift $\Delta T_\lambda(d)$, but cannot be used for a quantitative verification of the formulas given above. It must furthermore be added that the pores must in any case not be small enough to make $\Delta T_\lambda(d) \gtrsim 0.1\,°\text{K}$, for otherwise the Ψ-theory is quantitatively not valid at all (condition (1.4)). The last remark refers, in particular, to[90], which cites $\Delta T_\lambda = 0.22\,°\text{K}$. Both for this reason and on the basis of Fig. 1 of[90], we still see no justification for stating that the theory contradicts the observations. We note furthermore that in[89], in contrast to[90], a region (3.38) in which $\overline{\rho}_s$ depends linearly on $T_\lambda(d) - T$ has been observed. It is undoubtedly desirable to measure $\overline{\rho}_s(t)$ for sufficiently wide gaps (or capillaries, pores, etc.) with distinctly determined dimensions.

We now calculate the jump of the heat capacity $\Delta\overline{C}_p \equiv \overline{C}_{p,\text{II}} - C_{p,\text{I}}$ in a film at $T = T_\lambda(d)$. From (3.33) and

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[22]The result (3.37) at $M = 0$ coincides with that given in[73].

TABLE III.

M	$v_0^2 = \frac{2M}{3+M}$	$D_{tr} = \frac{d_{tr}}{\xi_0}$	$D_c = \pi\sqrt{\frac{3+M}{3}}$	$D_{c1} = \frac{d_{c1}}{\xi_0}$	$x_m^2, c = \psi_0^2\left(\frac{d_{tr}}{2}\right)$	$\frac{\Delta\bar\rho_s}{\rho_{se}}$	$\frac{\Delta\bar S}{S_{I-II}(\rho_{se})}$	$\frac{\Delta\bar C_p}{\Delta C_p}$	E_{tr}
9/7	0.6	3.735	3.755	3.728	0.20	0.100	0.094	6.11	0.28
2	0.8	3.895	4.056	3.845	0.45	0.226	0.196	2.04	0.59
3	1	4.017	4.443	3.898	0.60	0.304	0.248	1.36	0.74
4,5	1.2	4.101	4.967	3.914	0.69	0.352	0.280	1.10	0.82
7	1.4	4.186	5.736	3.910	0.76	0.391	0.293	1.00	0.88
17	1.7	4.291	8.112	3.882	0.83	0.434	0.308	0.91	0.93
∞	2	4.404	∞	3.841	0.88	0.468	0.320	0.85	0.96

(3.34), using also (3.30) and (3.36), we obtain[23]

$$\Delta\bar C_p(d) \equiv \bar C_{p,II}(T_\lambda(d)) - C_{p,I}(T_\lambda(d))$$
$$= \Delta C_p\left(E - \frac{2}{9}\frac{I_2}{D} + \frac{2}{9}DE_D'\right)\Big|_{x_m^2=0}$$
$$= \frac{2}{9}\Delta C_p D_c\left(\frac{E'(x_m^2)}{D'(x_m^2)}\right)_{x_m^2=0} = \frac{32\Delta C_p}{9(3+M)(1-M)}.$$

$$(3.39)$$

We note that the "jump" of the heat capacity in the film is somewhat larger, even at $M=0$, than the "jump" ΔC_p of the heat capacity in "bulky" helium. In addition, the quantity $\Delta C_p(T_\lambda(d))$ does not depend on the thickness of the film, in contradiction to the conclusion drawn in[72]. The interpretation of the experimental data given there seems therefore doubtful to us. The general picture of the behavior of $\Delta\bar C_p(T)$, which follows from the analysis presented here, is obviously the following: at $d > d_c + 2\sqrt{2}\,\xi_M$, when the film can be regarded as "thick," $\Delta\bar C_p$ first decreases with increasing temperature, in accordance with formula (3.20d): next, at $d \sim d_c + 2\sqrt{2}\,\xi_M$ it reaches a minimum $\Delta\bar C_{p,min} \simeq (0.8-0.9)\Delta C_p$, after which it begins to grow and tends as $T \to T_\lambda(d)$ to the value determined by (3.39).

As $M \to 1$, the heat capacity $\bar C_p$ diverges at $T = T_\lambda(d)$ because the value $M = 1$ corresponds to the tricritical point. It can be shown that as $T \to T_\lambda(d)$ the character of the singularity of $\bar C_p(T)$ is in this case of the square-root type

$$\bar C_{p,II} = C_{p,I} + 0.37\Delta C_p\sqrt{\frac{T_\lambda - T_\lambda(d)}{T_\lambda(d) - T}}, \quad M = 1, \quad \frac{T_\lambda - T_\lambda(d)}{T_\lambda} \ll 1.$$

$$(3.39b)$$

The density of the superfluid component also vanishes in this case in accordance with a square-root law

$$\bar\rho_s = 0.73\rho_{se}(T)\sqrt{\frac{T_\lambda(d) - T}{T_\lambda - T_\lambda(d)}}, \quad M = 1, \quad \frac{T_\lambda - T_\lambda(d)}{T_\lambda} \ll 1. \quad (3.37a)$$

At $M > 1$, to assess the positions of the transition point (the temperature $T_{tr}(d)$ or the "critical" thickness of the layer d_{tr}) and of the discontinuities that take place at $T = T_{tr}(d)$ in the density $(\Delta\bar\rho_s)$, in the entropy $(\Delta\bar S)$,

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[23]If the critical exponents α for the heat capacity and ν for the correlation length are not fixed, the result (3.39) takes the form

$$\Delta\bar C_p = \frac{v^2}{(2-\alpha)(1-\alpha)}\frac{16}{(3+M)(1-M)}\Delta C_p. \quad (3.39a)$$

In particular, if the self-consistent variant (2.5)-(2.8) were used, the relative discontinuity of the heat capacity in the film would be $\Delta\bar C_p/\Delta C_p = 2/3$.

and in the heat capacity (ΔC_p), as well as to determine the limits of the temperature hysteresis, numerical calculations are needed. Some results of calculations of this kind are listed in Table III.

The last column of Table III gives the values of the relative pressure jump $E_{tr} \equiv [p_{II}(T_{tr}(d)) - p_I]/\Omega_{I-II}(\rho_{se})$ which is realized in the film if the thickness of the film (layer) is fixed.[24] Actually, however, in the case of films the given quantity is the vapor pressure over the film, and it is the thickness which changes jumpwise at $T = T_{tr}(p)$. This circumstance must be taken into account when the predictions of the theory are compared with experiment. A rapid change in thickness or vapor pressure over the film takes place at $T \sim T_\lambda(d)$ also if the phase transition in the film is of second order. For details see[74,75].

Unfortunately, it is difficult to compare the thermodynamic parameters of the films as given by the developed theory and by experiment in the immediate vicinity of the transition temperature $T_\lambda(d)$ or $T_{tr}(d)$. The point is that, as already mentioned in Sec. 2.4, the heat capacity and other analogous quantities (entropy, thermodynamic potential, etc.) depend on the long-wave fluctuations. The corresponding contribution was attributed by us to terms of the type $\Phi_I(p, T)$, $S_I(p, T)$, etc., and was assumed to be symmetrical with respect to replacement of t by $-t$, i.e., to have no influence on the difference $\Phi_{II} - \Phi_I$. In the case of helium in large volumes this, in general, is confirmed by experiment. In the case of films, however, the situation can be different, and the contribution of the fluctuations at $t < 0$ and $t > 0$ may turn out to be different (see, e.g., [92]). In this case the considered Ψ-theory must definitely be modified. We note that in the calculation of $\bar\rho_s(d)$ this difficulty is not so crucial, because the long-wave fluctuations smear out the transition only slightly (so that there is a certain "para-superfluidity" also at $T > t_\lambda(d)$, see, e.g. [40,83]). We emphasize that the noted difficulties encountered when it comes to compare theory with experiment (even for films whose thickness can be accurately controlled) make the corresponding experiments no less interesting and important.

3.2. Density distribution of superfluid component in external fields

The effect of the boundaries is not the only factor that leads to inhomogeneity of the modulus of the order parameter $|\Psi| = \eta = \sqrt{\rho_s/m}$ in immobile helium. A similar result is produced also by the action of various external fields on the liquid, e.g., the gravitational field, striction forces of electric and magnetic fields, or the fields of the van der Waals forces. The field increases the density ρ of the liquid, and with it the temperature T_λ

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[24]The additional pressure produced in the film as a result of its transition to the superfluid state is given by the formula

$$p_{II} - p_I = \left(\frac{\partial\tilde\Omega_{I-II}}{\partial d}\right)_{\mu, T} = \Omega_{I-II}(\rho_{se})E\left(\frac{d}{\xi_0(t)}\right),$$

where $\Omega_{I-II}(\rho_{se}) = (\Delta C_\mu/2T_\lambda)t^2$, $\Delta C_\mu \approx \Delta C_p = 0.76$ J-cm^{-3} deg, and E is defined in (3.30).

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$\equiv T_\lambda(\rho)$ of the λ transition. As a result, if the potential $V(\mathbf{r})$ of the field forces is not constant in space, then the phase transition into the superfluid state occurs not simultaneously over the entire "sample," and the first to become superfluid (with decreasing temperature) are those liquid regions in which the density ρ is lower. The regions that have already gone over into the superfluid state are separated from the neighboring "normal" regions by diffuse phase boundaries, and the width and shape of the boundary (the character of the distribution of $\rho_s(\mathbf{r})$ in the transition layer) depend on the field gradient, on the correlation effects that are taken into account in (2.15) with the aid of the term containing the gradient of Ψ, and also on the form of the density of the thermodynamic potential. It is clear even from this that a study of the distribution of $\rho_s(\mathbf{r})$ in the transition region is of considerable interest.

The question of the character of the inhomogeneous distributions of $\rho_s(\mathbf{r})$ in external fields and of the possibility of experimentally studying distributions of this kind were considered on the basis of the Ψ-theory of superfluidity in[38,94]. In addition, for the particular case of the He I–He II boundary in a gravitational field, a number of preliminary estimates were made also in[95,36] and in the experimental paper[91]. For lack of space, and in view of the rather complete treatment in[38,94], we omit many details in our review of the cited papers.

When solving spatially-inhomogeneous problems in which the changes of the density ρ are significant, it is necessary to consider the thermodynamic potential

$$\tilde{\Omega}_{II}(\mu, T; \Psi, \rho) = \int \left[F_{II\,0}(\rho, T, |\Psi|^2) + \frac{\hbar^2}{2m}|\nabla\Psi|^2 - \mu\rho \right] dV \quad (3.40)$$

and to minimize this potential with respect to $\Psi^*(\mathbf{r})$ and $\rho(\mathbf{r})$ simultaneously.

In the presence of an external field $\mathbf{G} = \nabla V(\mathbf{r})$ (per unit mass) it is necessary to add to the integrand of (3.40) the term $\rho V(\mathbf{r})$, representing the potential energy of the particles in the field, and also, generally speaking (if the density changes noticeably over distances on the order of interatomic ones), the term $(\delta/2)(\nabla\rho)^2$, which takes into account the correlations of ρ. As a result, by minimizing (3.40), we obtain the following system of coupled differential equations for the equilibrium values of $\Psi_s(\mathbf{r})$ and $\rho_s(\mathbf{r})$

$$\frac{\hbar^2}{2m}\Delta\Psi = \left(\frac{\partial F_{II\,0}}{\partial|\Psi|^2}\right)_{\rho,\,T}\Psi,$$

$$\delta\Delta\rho = \left(\frac{\partial F_{II\,0}}{\partial\rho}\right)_{|\Psi|^2,\,T} - \mu + V(\mathbf{r}). \quad (3.41)$$

If however, the gradients of the density ρ are small, as will be assumed from now on and is assured if the condition (2.21) holds, then the terms with the spatial derivatives of ρ can be discarded. Furthermore, it is convenient to change over from the density of the free energy $F_{II\,0}(\rho, T, |\Psi|^2)$ to the density of the thermodynamic potential $\Omega_{II\,0}(\mu_0, T, |\Psi|^2)$ in terms of the variables μ_0, T, and $|\Psi|^2$, where μ_0 is the chemical potential of helium in the absence of an external field

$$\mu_0 = \left(\frac{\partial F_{II\,0}}{\partial\rho}\right)_{|\Psi|^2,\,T} = \mu - V(\mathbf{r}). \quad (3.42)$$

Equations (3.41) are then separable and take the form

$$\frac{\hbar^2}{2m}\Delta\Psi = \left(\frac{\partial\Omega_{II\,0}}{\partial|\Psi|^2}\right)_{\mu_0,\,T}\Psi, \quad (3.43)$$

$$\rho = -\left(\frac{\partial\Omega_{II\,0}}{\partial\mu_0}\right)_{|\Psi|^2,\,T}. \quad (3.43a)$$

We change over now to the reduced ψ function (3.12) and to the dimensionless coordinates $\mathbf{r}_{**}$ (see (2.17)), and use for $\Omega_{II\,0} - \Omega_I$ a formula similar to (2.13). Equation (3.43a) then takes the form

$$\Delta_{**}\psi = \frac{3}{3+M}\left(-t\,|t|^{1/3} + (1-M)\,|t|^{2/3}\,|\psi|^2 + M\,|\psi|^4\right)\psi, \quad (3.44)$$

which is analogous to (2.18), except that the distance to the λ transition, $t = T_\lambda - T$, is itself now, generally speaking, a function of the coordinates[25], inasmuch as in accord with (3.42) we have

$$t = T_\lambda(\mu_0) - T = T_\lambda(\mu - V(\mathbf{r})) - T \approx t_0 - \frac{dT_\lambda}{d\mu}V(\mathbf{r}), \quad (3.45)$$

where $t_0 = T_{\lambda,0} - T + (dT_\lambda/d\mu)(\mu - \mu_{\lambda,0})$ is the initial distance to a certain point $(T_{\lambda 0}, \mu_{\lambda 0})$ on the λ curve in the absence of an external field, and $dT_\lambda/d\mu$ is the slope of the λ curve at the indicated point (at saturated-vapor pressure we have $dT_\lambda/d\mu = -1.27\times10^{-9}$ deg-g-erg^{-1}).[26]

We note that if we were to use for $\Omega_{II\,0} - \Omega_I$ in (2.13) the more general formula (see[38] and Sec. 2.4)

$$\Omega_{II\,0} - \Omega_I = \frac{\Delta C_\mu}{2T_\lambda}t^2 f\left(\frac{\psi}{|t|^{1/3}}\right),$$

then Eq. (3.44) would take the form

$$\Delta_{**}\psi = |t|^{5/3}f'\left(\frac{\psi}{|t|^{1/3}}\right).$$

Thus, were we to know from experiment the function $\psi(z)$ in the region of the transition layer, then by plotting the dependence of $y \equiv (d^2\psi/dz_{**}^2)\,|t(z_{**})|^{-4/3}$ on $x \equiv \psi/|t|^{1/3}$ it would be possible to determine the form of the function $f(x)$, and hence the form of the density of the thermodynamic potential, and furthermore for all x and $t \lessgtr 0$, and not only for $x \lesssim 1$ and $t > 0$, as is the case for films. We shall return to the question of the

[25]In principle, a changeover to other thermodynamic variables can lead also to a change in the coefficients of (2.18) (the numerical value of the parameter M and the numerical values of the scales ξ_{00} and Ψ_{00} indicated in (2.12) and (2.17); see also (2.9)). Actually, however, the corresponding renormalization of the coefficients turns out to be very small and, in particular, on going from the variables (p, T, Ψ) to (μ_0, T, Ψ) it does not exceed 2% (for more details see[38]).

[26]The equations used in[95] and[36] contain instead of the derivative $dT_\lambda/d\mu$ the derivative $\rho_\lambda dT_\lambda/dp$. That these two derivatives are equivalent is implied also in the experimental paper.[91] We note that although the difference between $dT_\lambda/d\mu$ and $\rho_\lambda dT_\lambda/dp$ is quite small ($\sim 2\%$), allowance for it brings the slope of the λ curve $dp_\lambda/dT = -113.9$ atm/°K, obtained in[91], closer to the best experimental value $dp_\lambda/dT = -111.05$ atm/°K[96] (the corrected value that follows from[91] for dp_λ/dT is -11.6 atm/°K).
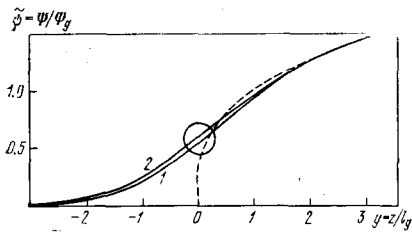
FIG. 4. Distribution of the order parameter $\tilde{\psi}(y)$ near the He I–He II interface in a gravitational field. The solid curves were obtained by numerically solving Eq. (3.50) for two values of the parameter M, namely $M = 0$ (curve 1) and $M = 1$ (curve 2). The dashed curve shows the distribution $\tilde{\psi}(y) = y^{1/3}$ that would be obtained without taking correlation effects into account.

methods used to measure the density profile $\rho_s(z)$, but for the time being, having no information on the complete form of the function $f(x)$, we assume Eq. (3.44) and consider a number of concrete problems on its basis.

We begin with the case of slowly varying fields and assume first that the field $\mathbf{G}$ is entirely homogeneous. In this case (the z axis is assumed directed along the field)

$$V_G = Gz. \qquad (3.46)$$

In the particular case of a gravitational field the constant G has obviously the meaning of the acceleration g due to gravity. It is convenient to measure the coordinate z from the plane $t = t_0 - (dT_\lambda/d\mu) Gz = 0$, which corresponds to the He I–He II phase separation boundary in a field (without allowance for correlation effects), and to change over to a new relative ψ function and to a new dimensionless coordinate y

$$\tilde{\psi} = \frac{\Psi}{\Psi_G}, \quad y = \frac{z}{l_G}, \qquad (3.47)$$

where the characteristic scales Ψ_G and l_G are equal to

$$\Psi_G = \left(\frac{\rho_{sG}}{m}\right)^{1/2}, \quad \rho_{sG} = 1.43\rho_\lambda \left(\frac{\xi_{00}}{l_G}\right) = 0.21 \left(\frac{\xi_{00}}{l_G}\right) \text{g/cm}^3$$

$$l_G = \xi_{00}^{3/5} \left(\left|\frac{dT_\lambda}{d\mu}\right| G\right)^{-2/5}. \qquad (3.48)$$

In particular, for the gravitational field ($G = g = 981$ cm/sec²) we have

$$\rho_{sg} = 1.43\rho_\lambda \frac{\xi_{00}}{l_g} = 8.6 \cdot 10^{-7} \text{ g/cm}^3,$$

$$l_g = \xi_{00}^{3/5} \left(\left|\frac{dT_\lambda}{d\mu}\right| g\right)^{-2/5} = 6.7 \cdot 10^{-3} \text{ cm}. \qquad (3.49)$$

In terms of the variables (3.47), Eq. (3.44) takes the form

$$\frac{d^2\tilde{\psi}}{dy^2} = \frac{3}{3+M} [-y |y|^{1/3} + (1-M)|y|^{2/3} \tilde{\psi}^2 + M\tilde{\psi}^4] \tilde{\psi}. \qquad (3.50)$$

Solutions of (3.50) corresponding to the values $M = 0$ and $M = 1$ are shown in Fig. 4, where the dashed curve shows also the $\tilde{\psi}(y)$ distribution that would be obtained without allowance for the correlation effect. Attention is called to the fact that the curves with $M = 0$ and $M = 1$ are very close to each other, despite the essentially different character of the nonlinear term in (3.50). The

difference in the nonlinearity is most important in the region where $\tilde{\psi} \gtrsim |y|^{1/3}$, i.e., exactly where the expansion (2.13) ceases to be valid, and where it would be necessary to expand in powers of $t/|\psi|^3$ (see [35,38] and Sec. 2.4)—the corresponding region is circled in Fig. 4. The close character of the curves with $M = 0$ and $M = 1$ indicates, however, that the exact form of the balance equation in this region is of no great significance, and that the use of (3.44) is in all probability an adequate approximation of the behavior of $\rho_s(z)$ in the transition region.

In the case of an inhomogeneous but weakly varying field, the potential $V(z)$ can be expanded in a series about the point $z = z_0$ at which $t = T(z) - T = 0$

$$V(z) = V(z_0) + G(z-z_0) + \frac{1}{2}\left(\frac{d^2V}{dz^2}\right)_{z=z_0}(z-z_0)^2 + \cdots \qquad (3.51)$$

In this case, if

$$\left(\frac{d^2V}{dz^2}\right)_{z=z_0}\frac{l_G}{2G} \ll 1, \qquad (3.51a)$$

then the field in the vicinity of the point $z = z_0$ can be regarded as homogeneous, as before, and the entire preceding analysis is valid. The foregoing pertains obviously also to non-planar interfaces, provided that the curvature radius of the boundary is much larger than its thickness l_G.

From the point of view of experiment, apart from the force of gravity, interest attaches also to the striction forces that arise in electric and magnetic fields. In an electric field, the additional chemical potential of the liquid is equal to

$$\mu_E = V_E(\mathbf{r}) = -\left(\frac{\partial \varepsilon}{\partial \rho}\right)_T \frac{E^2(\mathbf{r})}{8\pi} \approx -\frac{\alpha_0 E^2(\mathbf{r})}{2}, \qquad (3.52)$$

where E is the field intensity, ε is the dielectric constant, and $\alpha_0 = 3.1 \times 10^{-2}$ cm³/g is the polarizability of the helium per gram. A similar formula, but with α_0 replaced by the diamagnetic susceptibility $\alpha_H = -0.47 \times 10^{-6}$ cm³/g of helium, holds also in the case of a magnetic field (E replaced by H). The shifts of the λ point under the influence of a homogeneous electric or homogeneous magnetic field, due to the change of ρ, are quite small. Thus, at $E \sim E_{breakdown} \sim 2 \times 10^6$ V/cm the λ-point shift is $\Delta T_\lambda(E) = T_\lambda(0) - T_\lambda(E) \sim 1 \times 10^{-3}$ °K, and for a magnetic field the corresponding values are even smaller (at $H \sim 10^5$ Oe the shift is $\Delta T_\lambda(H) \sim -3 \times 10^{-6}$ °K). Naturally, even in an inhomogeneous field the distribution $\rho_s(\mathbf{r})$ changes (with a transition into He II) only in a relatively narrow vicinity of the λ transition. Nonetheless, inside the indicated region the effects of the inhomogeneous electric or magnetic field are fully comparable and can even exceed the action due the weight of, say, a helium column 1 cm high (the λ-point shift due to the pressure of the helium column is 1.25×10^{-6} °K/cm). For this reason, an investigation of the action of electric and magnetic fields is also desirable, for by varying the intensity and configuration of the field it is possible to change (contract or extend) the width of the transition region between the helium I and helium II, produce local regions with decreased or increased density concentration of the superfluid part and thus acting as converg-

ing or dispersing lenses for the second-sound waves, [94] etc. Nor can we exclude the use of local condensations of ρ in He II in an electric field to investigate effects such as the proximity effect or an analog of the Josephson effect. [34]27) To illustrate the possibilities of using electric and magnetic fields, let us estimate, for example, the width of the phase-separation boundary between He I and He II in the field of a charged filament of radius R. In this case $E \sim 1/r$ (r is the distance to the filament axis) and as follows from (3.48), (3.50), and (3.52),

$$l_E = \xi_0^{3/5}(t_0)\left[\frac{r_0(t_0)}{2}\right]^{2/5}, \qquad (3.53)$$

where $r_0(t_0) = \left[\,|dT_\lambda/d\mu|(\alpha_0 E_R^2/2t_0)\right]^{1/2}R$, E_R is the field intensity on the surface of the filament, and t_0 is the temperature difference $T_{\lambda,0} - T$ far from the filament (for details see[38]). At $E_R \sim 10^6$ V/cm, $t_0 \sim 10^{-6}$ °K, and $R \sim 0.1$ cm, the length $l_E \sim 8.8 \times 10^{-3}$ cm. In the case of a current-carrying conductor at a field intensity $H \sim 10^5$ Oe at the conductor surface and at the same values of t_0 and R, an analogous estimate yields $l_H \sim 2.7 \times 10^{-3}$ cm.

We turn now to the case of fields that vary rapidly in space, for which the condition (3.51) no longer holds. Real fields of this type are those of the van der Waals forces near a solid wall, or the field of microscopically charged bodies (ions in helium). We consider first the former case.

The potential acting on the helium because of the solid is equal to[97]

$$V(z) = -\frac{\theta}{z^3}, \qquad (3.54)$$

where z is the distance to the wall and θ is a constant that depends on the wall material (a typical value is $\theta \sim (1 \text{ to } 10) \times 10^{-14}$ erg-cm^{3}g^{-1}). Without allowance for the correlation effects, the action of the potential (3.54) should lead to the production near the wall of an He I layer of thickness z_0 that increases as $T \to T_{\lambda 0}$ in accordance with the law

$$z_0 = \left(\frac{\theta}{t_0}\left|\frac{dT_\lambda}{d\mu}\right|\right)^{1/3} = z_{00}t_0^{-1/3} = (2.3-5.0)\cdot 10^{-8}t_0^{-1/3} \text{ cm}. \qquad (3.55)$$

Actually, however, no such distinct layer is produced, and to determine the behavior of $\rho_s(z)$ in the region of the action of the "wall potential" (3.54) it is necessary to solve Eq. (3.44) with

$$t \equiv t(z) = t_0 - \left|\frac{dT_\lambda}{d\mu}\right|\frac{\theta}{z^3} \qquad (3.56)$$

and with the boundary conditions

$$\psi(0) = 0, \quad \psi(\infty) = \psi_e = t_0^{1/3}. \qquad (3.57)$$

We note that when account is taken of the van der Waals forces, one can dispense with the condition $\psi(0) = 0$ on the wall, and this condition can be replaced (just as in the vortex problem, see Sec. 3.4) by the physically obvious requirement that there be no exponentially growing solutions.

Inasmuch as in the region $z \simeq z_0$ the potential (3.54) varies quite rapidly, for a qualitative investigation of the problem the $t(z)$ dependence can be replaced approximately by the expression

$$t(z) = \begin{cases} t_0 & \text{at } z > z_0, \\ -|dT_\lambda/d\mu|\,\theta/z^3 & \text{at } z < z_0. \end{cases} \qquad (3.56a)$$

The asymptotic form of $\psi(z)$ at $z < z_0$, with allowance for (3.57), takes the form

$$\psi(z < z_0) = cz\exp\left(-\frac{l_0}{z}\right), \qquad (3.58a)$$

where c is the integration constant and

$$l_0 = \frac{z_{00}^2}{\xi_{00}}\sqrt{\frac{3}{3+M}}, \quad z_{00} = \left(\theta\left|\frac{dT_\lambda}{d\mu}\right|\right)^{1/3} = (2.3-5.0)\cdot 10^{-8} \text{ cm-deg}^{-1/3}. \qquad (3.59)$$

On the other hand, in the region $z > z_0$, a "free" solution without a field is valid (see (3.10))

$$\psi(z > z_0) = \psi_e(t_0)\frac{\text{th}\,[(z-b)/2\xi_M^-]}{\sqrt{1+[2M/(3+M)]\,\text{ch}^{-2}[(z-b)/2\xi_M^-]}}, \qquad (3.58b)$$

which is "shifted" away from the wall a certain distance b. By matching expressions (3.58a) and (3.58b) at $z = z_0$ (we require equality of the functions as well as of their derivatives), we obtain for the constants c and b the values

$$b = \frac{l_0}{1+(l_0/z_0)}, \quad c = \frac{\psi_e(t_0)}{\sqrt{2}\,\xi_0[1+(l_0/z_0)]}e^{l_0/z_0}, \qquad (3.60)$$

and the condition of the applicability of these formulas is the requirement

$$\psi_0(z_0) = \psi(z_0)t_0^{-1/3} = \frac{\psi(z_0)}{\psi_e(t_0)} \approx \frac{cz_0 e^{-l_0/z_0}}{\psi_e(t_0)} \approx \frac{z_0}{\sqrt{2}\xi_0} = \frac{z_{00}}{\sqrt{2}\,\xi_{00}}t_0^{1/3} \ll 1. \qquad (3.61)$$

It follows from (3.60) and (3.59) that at small values of the difference $t_0 \equiv T_{\lambda 0} - T$, when $l_0/z_0 = (l_0/z_{00})t_0^{1/3} \ll 1$, the distance b over which ψ effectively vanishes is practically independent of temperature and does not exceed several interatomic distances. This justifies the use, near the λ point, of the formulas of Sec. 3.2, which were obtained without allowance for the van der Waals forces. At the same time, with increasing distance from the λ point (e.g., for a material with $\theta \sim 10^{-13}$ erg-cm^{3}g^{-1} at $t_0 \sim 2 \times 10^{-3}$ °K), the effects connected with the influence of the van der Waals forces become generally speaking significant, as is clear from Fig. 5 (curves 2), and as was in fact observed in experiment. [83]

In the case of microscopic charged bodies (ions in helium), the presence of a short-range electrostriction potential (3.52) also leads to a certain shift of the point at which we have effectively $\psi = 0$, away from the boundary of the ion in the interior of the helium. This shift, however, is quite small and even as $T \to T_{\lambda 0}$ it does not exceed the dimension of the ion (i.e., the radius of the

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27)It would also be of interest to use electric and magnetic fields to eliminate the effect of the inhomogeneity of a liquid column in a gravitational field near the ordinary "liquid–vapor" critical points or near the mixture-stratification critical points, particularly for solutions of ^{3}He in ^{4}He.

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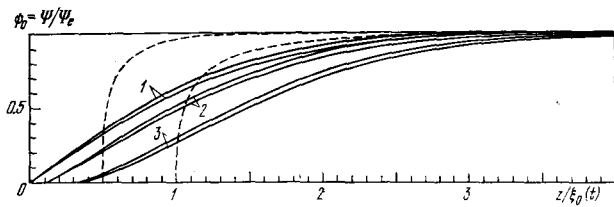V. L. Ginzburg and A. A. Sobaynin   795

FIG. 5. Distribution of the order parameter $\psi_0(y)$ near a solid wall with allowance for the field of the van der Waals forces. The solid curves were obtained by numerically solving Eq. (3.44) with potential (3.54) for two values of the parameter M, namely $M = 0$ (lower curves) and $M = 1$ (upper curves). Curves 1 were obtained without allowance for the influence of the field, but with the boundary condition (2.26). Curves 2 and 3 correspond to the ratio z_0/ξ_0 equal to 1/4 and 1/2, respectively. The dashed lines show the $\psi_0(y)$ distribution that would be obtained with allowance for the van der Waals forces but in the absence of correlation effects.

"hard core" $R_0^+ \approx 7$ Å in the case of a positive ion, or the radius of the electron "bubble" $R_0^- \approx 17$ Å in the case of a negative ion; see[15], Sec. 81). Nonetheless, the presence of this very condition means the existence, near the ion, of a region with decreased concentration of the superfluid component, the effective width R_{eff} of which, defined by the integral

$$\frac{4\pi}{3}(R_{eff}^\pm)^3 = 4\pi \int_{R_0^\pm}^{\infty} \left[1 - \frac{\rho_s(r)}{\rho_{se}}\right] r^2 \, dr \sim R_0^\pm \xi_M^2(t),$$

increases without limit as $T \to T_{\lambda 0}$. This can lead to a number of effects. Consider, for example, the question of ion mobility.[98] Above the λ point, the ion mobility can be estimated by Stokes's formula for the resistance to the flow of a viscous liquid around a solid sphere (for positive ions)

$$F^+ = 6\pi\eta R_0^+ v_\infty \qquad (3.62)$$

or from the Rybchinskii–Hadamard formula for the resistance to flow around an empty cavity (negative charges)

$$F^- = 4\pi\eta R_0^- v_\infty. \qquad (3.62a)$$

Here η is the viscosity of helium I and v_∞ is the velocity of the liquid flow at infinity. Below the λ point we can use the same formulas (with η replaced by η_n and v by v_n, the subscript n designating the viscosity and velocity of the normal component), but it is then necessary to take account of the fact that the density of the normal component is not uniformly distributed around the ion, as a result of which the velocity $v_{n,i}$ near the ion differs from the velocity $v_{n,\infty}$ far from the ion. As a rough approximation we can assume that

$$\rho_n(r) = \begin{cases} \rho & \text{at} \quad r < R_{eff}, \\ \rho_{ne} = \rho - \rho_{se} & \text{at} \quad r > R_{eff}, \end{cases}$$

whence, using the continuity equation $(\rho_n(r)v_n(r) = \text{const})$, we get

$$v_{ni} = \frac{\rho_{ne}}{\rho} v_{n\infty}.$$

But it is precisely the velocity v_{ni} near the ion that should obviously enter in expressions (3.62) for the resistance forces. Thus, immediately below the λ point we have

$$F^+ = 6\pi\eta_n R_0^+ \left(1 - \frac{\rho_{se}}{\rho}\right) v_{n\infty}, \quad F^- = 4\pi\eta_n R_0^- \left(1 - \frac{\rho_{se}}{\rho}\right) v_{n\infty}$$

or, since $F^\pm = (e/\mu_\pm) v_{n\infty}$,

$$\frac{1}{\mu_\pm} = \frac{1}{\mu_{\pm,\lambda}} \left(1 - \frac{\rho_{se}}{\rho}\right), \qquad (3.63)$$

where $\mu_{\pm,\lambda}$ is the ion mobility at the λ point. The foregoing arguments are only qualitative. Nonetheless, Eq. (3.63) agrees well with experiment.[99]

In conclusion, we dwell briefly on the possibility of experimentally studying the inhomogeneous distributions of the density of the superfluid part of helium II near the λ point by "probing" them with second-sound waves.[94] For simplicity we confine ourselves to the case of slowly varying density, when geometric optics suffices for the analysis of the wave propagation, and furthermore we can neglect the effects of absorption and dispersion of the sound.

Specifically, we consider the normal incidence of a plane second-sound wave on an He I–He II interface in a gravitational field. Then, as shown in[94], at the optimal choice of frequency $\omega \sim 200$ Hz (when, on the one hand, the geometric-optics approximation is still applicable, and at the same time the effects of sound absorption and dispersion can be regarded as small), penetration of the second-sound wave into the region of the "normal" phase ($z < 0$, see Fig. 4) is possible up to distances

$$|z| \lesssim z_c \sim \frac{5}{6} \left\{\ln\left[\left(\frac{3+M}{3}\right)^{1/2} \frac{0.84}{\gamma}\right]\right\}^{3/5} l_g,$$

where γ is a certain coefficient determined by the character of the increase of second-sound absorption at high frequencies, while the length l_g is given by formula (3.48). If $\gamma \lesssim 0.1$, as follows from the experimental data,[100] then $z_c \gtrsim 1.5 l_g$, and the phase shift in the vicinity of the "tail" of the distribution of $\Psi(z)$ (Fig. 4)

$$\delta\varphi = \omega \int_0^{-z_c} \frac{dz}{u_2(z)} \qquad (3.64)$$

is large enough ($\delta\varphi \gtrsim 2\pi$). Here $u_2(z) = [TS^2 \rho_s(z)/\rho_n c_p]^{1/2} \propto \Psi(z)$ is the velocity of the second sound.

To measure the phase of the wave (or the delay time of the sound pulse $\tau = \delta\varphi/\omega$) it is apparently necessary to place some wall at a distance $|z| < z_c$ and to observe the reflection of the second sound from it. By varying, furthermore, the position of the wall or by shifting (via a temperature change) the phase boundary, we can find $\delta\varphi(z)$, and consequently, according to (3.64), reconstruct also the function $\Psi(z)$ of interest to us.

All the foregoing applies to an equal degree to an electric (or magnetic) field, except that the characteristic scale l_g must be replaced here by l_E (or l_H). This offers an additional opportunity of varying the width of

the separation boundary and the depth of sound pene-
tration, by varying the field intensity.

Thus, the procedure discussed above can be counted
upon to yield information on the function $\rho_s(\mathbf{r})$ in a weak-
ly varying field. We note, however, that the appro-
priate experiments are not easy to perform, for in the
case, say, of a gravitational field, in order to regulate
the relative position of the phase separation boundary
and of the "mirror" accurate to $(1-0.1)l_e \sim 10^{-2}$ to 10^{-3}
cm, it is necessary to regulate the temperature with
accuracy to $(0.1-1)l_e \, |dT_\lambda/d\mu| g \sim 10^{-8} - 10^{-9}$ °K. None-
theless, such a degree of stability is apparently fea-
sible (see[91]).

In the study of the reflection of second sound from a
wall without a field, and in some other cases, when the
geometric-optics approximation cannot be used, the
complete system of equations of helium II near the λ
point can be verified (see Sec. 4.1), and the behavior
of $\rho_s(\mathbf{r})$ investigated, near a wall, near the boundary of
the helium with its vapor, etc. The use of the methods of
inverse scattering theory for the analysis of a sound
pulse from an inhomogeneous layer can be of definite
interest from this point of view.[101]

There are grounds for hoping that study of second-
sound propagation near the λ point in inhomogeneous
helium will become an important method for the study
of superfluidity.

3.3. Surface tension and boundary condition on a free surface

When evaluating the character of the boundary con-
dition for the macroscopic order parameter (the den-
sity of the superfluid component) on a free surface of
helium II, it is impossible to advance as direct and as
convincing considerations in favor of the condition $\Psi = 0$
as in the case of the boundary between helium II and a
solid (see the Introduction and Sec. 2.2). Nonetheless,
a number of conclusions based on the analysis of the
energy of the boundary as a function of the proposed
value of the parameter Ψ on it (i. e., the value $\Psi(0)$
$\equiv \Psi(z = 0)$ can be drawn for this case, too. We begin
with the following simple estimate[37] (the helium will
henceforth be assumed to be at rest, so that the func-
tion $\Psi(z)$ is assumed to be real and positive).

The boundary between helium II and its vapor can be
regarded in analogy between the boundary between a
superconductor and a normal metal. It is known in the
latter case[48] that the macroscopic wave function pene-
trates into the normal metal, but this penetration is
connected with a loss of free energy (per unit area of
the boundary)

$$\sigma_N = \frac{\hbar^2}{m} \int_{(N)} \left(\frac{d\Psi}{dz}\right)^2 dz \approx \frac{\hbar^2}{2m} \frac{\Psi^2(0)}{\lambda}. \tag{3.65}$$

Here $\Psi(0) \equiv \Psi(z=0)$ is the already mentioned value of

$\Psi(z)$ on the boundary (for simplicity, the function Ψ is
assumed to be continuous on the boundary), and λ is a
certain characteristic depth of "penetration" of the Ψ
function into the normal ("N") region. In the case of
contact between the superconductor and the normal
metal[48] we have $\lambda \sim \xi_N = \hbar v_F/k_B T$, where v_F is the elec-
tron velocity on the Fermi surface in the normal metal;
in the case of vapor above a helium surface, on the
other hand, for lack of any other information, we can
assume the length λ to be of the order of the thermal
wavelength of the helium atom in the vapor, i.e.,

$$\lambda \sim \lambda_T = \sqrt{2\pi\hbar^2/mk_BT} \approx 5.9 \text{ Å}.$$

On the other hand, in the superconducting (super-
fluid) phase ("S"), a change of Ψ near the boundary is
connected with a surface energy

$$\sigma_S = \frac{\hbar^2}{m} \int_{(S)} \left(\frac{d\Psi}{dz}\right)^2 dz \sim \frac{\hbar^2}{m} \frac{(\Psi_\infty - \Psi(0))^2}{2\sqrt{2}\,\xi_0(t)}, \tag{3.66}$$

where $\Psi_\infty \equiv \Psi_e(t)$ is the equilibrium value of the order
parameter far from the boundary and $\xi_0(t)$ is the tem-
perature-dependent correlation radius (in the variant
of the theory with $M=0$).

Adding these two contributions (3.65) and (3.66) and
minimizing with respect to $\Psi(0)$, we see that the mini-
mum of the total surface energy $\tilde{\Phi}_S = \sigma_S + \sigma_N$ is reached
at[28]

$$\psi_0(0) \equiv \frac{\Psi(0)}{\Psi_\infty} \approx \frac{\lambda}{\sqrt{2}\,\xi_0(t)} \propto t^{2/3}. \tag{3.67}$$

The corresponding minimal value of the surface energy
is equal in this case to

$$\sigma = \sigma_S + \sigma_N \approx \sigma_S \sim \frac{\hbar^2 \Psi_e^2(t)}{2\sqrt{2}\,m\xi_0(t)} \sim 10^{-2}t^{4/3} \text{ erg/cm}^2. \tag{3.68}$$

We see thus that if the parameter λ is finite at $T = T_\lambda$,
then it can be assumed, with a good degree of accuracy,
that near the λ point $\Psi = 0$ on the free boundary. We
have also found in passing that under these conditions
the contribution σ_N can be disregarded in the calcula-

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[28] In the case of superconductors, besides a decrease, there
is also a "jump" of the value of the parameter Ψ on the
boundary. This circumstance can be taken into account
phenomenologically by considering, besides expressions
(3.65) and (3.66), also the correlation contribution made to
the surface energy by the difference between the values
$\Psi_S(0)$ and $\Psi_N(0)$ of the order parameter on both sides of the
boundary

$$\sigma_{SN} \sim \frac{\hbar^2}{m\lambda_1} [\Psi_S(0) - \Psi_N(0)]^2,$$

where λ_1 is a certain new length. Minimization of the total
surface energy $\sigma = \sigma_N + \sigma_S + \sigma_{SN}$ with respect to $\Psi_S(0)$ and
$\Psi_N(0)$ would indeed lead to the appearance of a "jump" in the
parameter Ψ, with

$$\Delta\Psi \equiv \Psi_S(0) - \Psi_N(0) \approx \frac{\lambda_1}{\lambda_1 + \lambda}\Psi_S(0).$$

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tion of the thermodynamic quantities.[29]

The problem of deriving the boundary conditions can be approached also in a somewhat different but essentially equivalent manner by considering, without any concrete assumptions concerning the meaning of the parameter λ, the functional already mentioned in Sec. 2.2[47,48,103–107]

$$\delta\tilde{\Phi}_S = \int \left\{ [\Phi_{\Pi 0}(\Psi) - \Phi_{\Pi 0}(\Psi_e)] + \frac{\hbar^2}{2m}|\nabla\Psi|^2 \right\} dV + \frac{\hbar^2}{2m\lambda} \int |\Psi(0)|^2 \, dS. \quad (3.69)$$

The second integral takes into account here the entire genuine "surface" contribution to the energy of the boundary, i.e., the sum of the contributions $\sigma_N + \sigma_{SN}$, already minimized with respect to $\Psi_N(0)$ (see footnote 28).

Varying (3.69) with respect to $\Psi(0)$, we obtain for the determination of $\Psi(0)$ the equation

$$\frac{d\Psi}{dz}\Big|_{z=0} + \lambda^{-1}\Psi(0) = 0, \quad (3.70)$$

or, changing over to the relative ψ_0 function (2.20) and using Eq. (3.9b) with $E=1$ (half-space),

$$\psi_0(0) = \frac{\lambda}{\xi_0(t)\sqrt{2}}[1 - \psi_0^2(0)]\sqrt{1 + v_0^2\psi_0^2(0)}.$$
$$\psi_0(0) \equiv \frac{\Psi(z=0)}{\Psi_e(t)}, \quad \xi_0(t) = 2.74 \cdot 10^{-8} t^{-2/3} \text{ cm.} \quad (3.71)$$

The corresponding total value of the surface energy is then

$$\delta\Phi_S \equiv \frac{\delta\tilde{\Phi}_S}{S}\sigma(t) - \frac{3}{4}\sigma_0(t)\left\{ 2\int_0^{\psi_0(0)}(1 - \psi_0^2)\sqrt{1 + v_0^2\psi_0^2}\, d\psi_0 \right.$$
$$\left. -\psi_0(0)[1 - \psi_0^2(0)]\sqrt{1 + v_0^2\psi_0^2(0)} \right\}, \quad (3.72)$$

_____

[29] If the values of the critical exponents were not fixed, the result (3.67) would take the form[37]

$$\Psi_0(0) \propto t^{\nu-\nu'}, \quad (3.67a)$$

whereas the total surface energy would be determined, as before, by the contribution made by the change of Ψ to the ordered (superfluid) phase

$$\sigma \approx \sigma_S \sim t^{2-\alpha-\nu} = t^{2(2-\alpha)/3}. \quad (3.68a)$$

Relation (3.67a) was criticized in[102] because it is violated[103] for a semi-infinite two-dimensional Ising model, and apparently also a three-dimensional model, so that λ can no longer be regarded as temperature-independent in the critical region. We note, however, that in the case of the Ising model we deal in fact with a contact between two media, of which only one can have magnetic order (i.e., in fact we are speaking of a contact between a ferromagnetic system and a vacuum). In the case of vapor above the surface of helium, however, both adjacent subsystems could become superfluid (although, to be sure, in a vapor with given density ρ_{vap} this transition would occur at a much lower temperature), and thus correlations can exist between the values of Ψ on both sides of the boundary (or, as is also said, a proximity effect is possible). It is obvious that it is meaningless to speak of such correlations in the case of a boundary with a vacuum, and in general in the case of the interface between different materials.

where S is the area of the boundary and the values of $\sigma(t)$ and $\sigma_0(t)$ are given by (3.16) and (3.16a). At $\lambda/\sqrt{2}\,\xi_0(t) \ll 1$ we obtain from (3.71) the result (3.67), and it follows from (3.72) that

$$\delta\Phi_S \approx \sigma(t) \propto t^{4/3}, \quad (3.72a)$$

in agreement with the expression (3.68) above.

From (3.71) and (3.72) we can determine also the correction terms for (3.67) and (3.68)

$$\psi_0(0) = \frac{\lambda}{\sqrt{2}\,\xi_0(t)}\left[1 - \frac{1}{2(M+1)}\left(\frac{\lambda}{\xi_0}\right)^2 + O\left(\frac{\lambda}{\xi_0}\right)^4\right], \quad (3.71a)$$

$$\delta\Phi_S = \sigma(t) - \frac{3}{4}\frac{\sigma_0(t)\lambda}{\xi_0(t)\sqrt{2}}\left[1 + O\left(\frac{\lambda}{\xi_0}\right)^2\right]. \quad (3.72b)$$

Analogous correction terms, which can be of interest at distances away from the λ point, can be easily obtained also for the other quantities used in Sec. 3.1. We shall not dwell on them, however.

The functional (3.69) and the expressions (3.71) and (3.72) that follow from it can be used also in the analysis of more general cases, when, e.g., $\lambda \gg \xi_0(t)$ or when λ is negative. Such situations are possible in the case of certain other (e.g., magnetic) phase transitions.[48,103–107][30] As applied to our problem, however, we assume that $\lambda > 0$ and is a quantity on the order of atomic dimensions (there are simply no other likely parameters in the problem, since at $T \approx T_\lambda$ all the lengths a, λ_T, and $a_{vap} \approx 10a$ are quantities of the same order).

It is possible to verify the boundary conditions for the function Ψ by using the measured values of the displacement of the λ point, of the average density of the superfluid part, and of the thermodynamic quantities in thin slits, channels, and films of He II, as already discussed in Secs. 2.2 and 3.1. However, the condition on the free boundary can manifest itself perhaps in the most direct manner in the behavior of the surface-tension coefficient near the λ point.[37] The point is that the inhomogeneity of Ψ near the boundary gives rise to an additional surface energy, by virtue of which the surface tension of helium II should also contain an additional contribution $\delta\sigma \equiv \sigma_{\Pi} - \sigma_I = \delta\Phi_S$, which is missing from the normal phase. Since the exponent in the expression for $\delta\sigma(t)$ is close to unity (see (3.68) and (3.72)), the presence of this contribution should obviously reveal itself in rough measurements as a certain change of slope on the plot of $\sigma(T)$ at $T \simeq T_\lambda$. The

_____

[30] The situation $\lambda \gg \xi_0(t)$ occurs, in particular, on the boundary between a superconductor and vacuum (or a superconductor and a dielectric). In this case[48] $\lambda \sim \xi^2(0)/a \sim 1$ cm, and condition (3.70) is close to the condition $d\Psi/dz = 0$ to within a very small vicinity of the superconducting transition temperature $T_c - T \lesssim 10^{-8}\,°K$. The foregoing, of course, does not contradict the remark in footnote 29, since the parameter λ can be introduced as a characteristic of the boundary energy also for those media between which proximity effects are impossible. We note that it is precisely in the latter case that one can expect the parameter λ to be temperature-dependent in the critical region.

existence of such a change of slope was noted already in[108] and was confirmed by subsequent more accurate measurements[109,110]: The experimental value of the change of slope $(d\delta\sigma/dT)_{exp} = 0.0226$ erg-cm^{-2}K^{-1} (estimated in[110] at $\varepsilon \equiv t/T_\lambda = 0.01$) is close to that calculated from formula (3.16), according to which $(d\delta\sigma/dT)_{theor} \simeq \delta\sigma/t = 0.0252-0.0295$ erg-cm^{-2}°K^{-1} at $t = 0.0217$°K. Some difference between the theoretical and experimental values, as well as the poor functional agreement, observed in[109,110], between the temperature dependence $\delta\sigma(t)$ in the region $t \gtrsim 0.01$ °K and formula (3.16), can be fully attributed to corrections of the type given in (3.72b). The corresponding value of the parameter λ, required to explain the experimental data, is approximately $\lambda \simeq (1-3)\lambda_T \simeq 6-18$ Å.

In connection with the foregoing it must be noted, however, that a behavior analogous to (3.68) can result also[102] in some cases from allowance for the long-wave fluctuations of the order parameter. The point is that for a "sample" of finite dimensions (say a helium layer of thickness L), the fluctuating part $\delta\Phi_{f1}(\eta_e, t, L)$ of the free-energy density contains corrections of the type $\delta\Phi_{f1}(\eta_e, t) \xi_0(t)/L$, compared with the corresponding fluctuating part $\delta\Phi_{f1}(\eta_e, t)$ of the free-energy density of a bulk sample (see (2.50), where in the case of a finite system the summation must be carried out over the discrete spectrum of the fluctuations). When multiplied by the volume $V = SL$ of the layer, these corrections obviously yield the corrections $\delta\Phi_{s,f1}$ to the surface energy.

We emphasize that the existence of fluctuation corrections $\delta\Phi_{f1}$ is also connected with a boundary condition of a definite type,[111,112] and is therefore also of considerable interest; to be sure, the boundary condition is imposed in this case not on the equilibrium order parameter itself, but on its fluctuations. Unfortunately, a detailed quantitative calculation of the corresponding fluctuation contribution has not been carried out as yet. From general considerations and by analogy with the behavior of the fluctuating part of the heat capacity we can only state that this contribution should be approximately the same above and below the λ point, but it can have different signs at $t > 0$ and $t < 0$[102] (see also[103] and[111]).

When account is taken of the possible fluctuation corrections to the temperature dependence of the surface-tension coefficient, the interpretation of the experimental data still remains ambiguous, in view of the difficulty of separating the regular part of the relation $\sigma(t) = \sigma_{reg} + \sigma_{f1} + \delta\sigma$, with $\sigma_{f1} = 0$ at the λ point and $\delta\sigma = 0$ at the λ point. In principle, however, the regular part σ_{reg} (by definition, the part having no singularity at the λ point) can be separated. Further, above the λ point we have $\delta\sigma = 0$, and this determines thus the contribution $\sigma_{f1}(t < 0) \equiv \sigma_{f1}(T > T_\lambda)$. If, finally, $\sigma_{f1}(t) = |\sigma_{f1}(-t)|$, then we can separate the contribution $\delta\sigma(t > 0)$ calculated above. It is perfectly possible, at the same time, that $\sigma_{f1} \ll \delta\sigma$ and then the already available measurement data will yield just the surface energy $\delta\sigma$; as already mentioned, in this case $\lambda = 6-18$ Å.

Besides the obvious desirability of determining $\sigma(t)$

with maximum accuracy in the case of pure helium II at saturated-vapor pressure, it is also desirable to determine the tension $\sigma(t)$ for solutions of ^{3}He in ^{4}He (in this case it is necessary also to measure the concentration of the ^{3}He near the surface).

3.4. Critical velocity of superfluid flow. Vortex filament

In addition to the problems considered above under the assumption that there is no flow, there are many others in which $\mathbf{v}_s \neq 0$. It must be pointed out, first, that the onset of flow changes the equilibrium value of ρ_s. As a result, the superfluidity vanishes, or else the superfluid flow loses stability, at a certain velocity v_{sc}. Second, within the framework of the Ψ-theory, the vortex filament in helium II has a structure; in particular, $\rho_s(0) = 0$ on the filament axis itself. The same pertains to a vortex ring and to some other types of flow.

We begin with a uniform and homogeneous flow in infinite helium II, i.e., in practice in a sufficiently broad channel. In this case, according to (2.15) and (2.13), the thermodynamic potential of helium II is equal to

$$\Phi_{II}(p, T, \Psi) = \Phi_{II0} + \frac{\rho_s v_s^2}{2}$$
$$= \Phi_I + \frac{3\Delta C_p}{(3+M)T_\lambda}\left[-t^{4/3}\frac{\rho_s}{\rho_{00}} + \frac{1-M}{2}t^{2/3}\frac{\rho_s^2}{\rho_{00}} + \frac{M}{3}\frac{\rho_s^3}{\rho_{00}^2}\right] + \frac{\rho_s v_s^2}{2},$$
$$\rho_{00} = 1.43\rho_\lambda \approx 0.21 \text{ g-cm}^{-3}\text{deg}^{-2/3}. \quad (3.73)$$

For a second-order phase transition we have $\rho_s = 0$ at the transition point, and the transition point $T_\lambda(v_{sc2})$ itself is determined from the condition that the coefficient of ρ_s vanish. Thus, the transition takes place at a velocity

$$v_{sc2} = \left(\frac{6\Delta C_p}{(3+M)T_\lambda\rho_{00}}\right)^{1/2}t^{2/3} = 5.8 \cdot 10^3\left(\frac{3}{3+M}\right)^{1/2}t^{2/3} \text{ cm-sec}^{-1}.$$
$$(3.74)$$

We recall that according to (2.12), (2.17), and (2.19), the correlation length is

$$\xi_M = \left(\frac{3+M}{3}\right)^{1/2}\left(\frac{\hbar^2\rho_{00}T_\lambda}{2m^2\Delta C_p}\right)^{1/2}t^{-2/3}$$

and consequently

$$v_{sc2} = \frac{\hbar}{m\xi_M}.$$

We note that in the self-consistent variant (2.5), (2.8)

$$v_{sc2} = \left(\frac{2a}{m}\right)^{1/2}t^{1/2} = \left(\frac{2\Delta C_p}{T_\lambda|d\rho_{se}/dT|}\right)^{1/2}t^{1/2} = 4.5 \cdot 10^3t^{1/2} \text{ cm-sec}^{-1}.$$
$$(3.75)$$

This expression was derived already in[20] prior to the development of the Ψ-theory; since the relation $\rho_{se} \propto t$ is in fact no longer valid in a sufficiently wide temperature interval, we shall not use (3.75) below. For superfluid ^{3}He, however, and probably also for solutions of ^{3}He in ^{4}He near the tricritical point, it is perfectly reasonable to use just an expression such as (3.75).

Relation (3.74) can, just as in the case of (3.4a), be "inverted," i.e., it is possible to find the shift of the λ-point temperature occurring in the course of the flow

$$\Delta T_\lambda (v_s) \equiv T_\lambda (v_s = 0) - T_\lambda (v_s)$$

$$= \frac{v_s^{3/2}}{[6\Delta C_p/(3+M)T_\lambda \rho_{00}]^{3/4}} = 2.3 \cdot 10^{-6} \left(\frac{3+M}{3}\right)^{3/4} v_s^{3/2} \, °K.$$

(3.76)

In principle it is possible to measure not only v_{sc} or ΔT_λ, but also the change of the density ρ_s with changing v_s. Thus, using (3.73) and the condition $\partial \Phi_{II}/\partial \rho_s = 0$, we have

$$\rho_{ss}(v_s) = \rho_{00} t^{2/3} \left(1 - \frac{T_\lambda \rho_{00}}{2\Delta C_p t^{4/3}} v_s^2\right),$$

(3.77)

where we put $M = 0$ for simplicity.

It was assumed above that the velocity of the normal part of the liquid is $v_n = 0$. But if $v_n \neq 0$, then it follows both from general considerations and from the actual expressions given in Sec. 4.1 that it is necessary to make in formulas (3.74)–(3.77) the substitution

$$v_s \rightarrow |\mathbf{v}_s - \mathbf{v}_n| \equiv w.$$

(3.78)

One of the possibilities of observing the shift ΔT_λ or the change of $\rho_{ss}(w)$ due to the presence of the velocity $w = |\mathbf{v}_s - \mathbf{v}_n|$ is to measure the transition temperature T_λ in the presence of heat flow. A lowering of T_λ in the presence of heat flow is actually observed (see, e.g., [113]), but the interpretation of the corresponding measurements is not clear enough. In any case, we are not dealing with uniform flow in the absence of vortices, so that the formulas given above cannot be used. On the other hand, it is clear that near the λ point the dependence of different effects and quantities on the velocity v_n or on the heat flow can be considered on the basis of the equations discussed in Sec. 4.1 (see also Sec. 4.2).

The experimentally observed[5,114] critical velocities for broad channels and gaps are much lower than the velocity (3.74). This is quite understandable, inasmuch as in broad channels vortices are produced already at velocities $v_s \ll v_{sc2}$. For thin channels and gaps, vortex production is difficult, and the critical velocity can be determined by the effect discussed here.

Assuming that $v_n = 0$ for very narrow channels and gaps, we shall use Eq. (2.20). We direct the z axis perpendicular to the gap, and the y axis along the gap and along the direction of the velocity $\mathbf{v}_s$. The velocity $\mathbf{v}_s$ can be regarded as constant along the gap (it is assumed that the gap width is constant). We put $\Psi(y, z) = \exp[i(m/\hbar)v_s y]\Psi(z)$, which corresponds exactly to flow with velocity v_s (see (1.2)).

We then obtain from (2.20) for $\psi_0(z) = \Psi(z)/[(\rho_{00}/m)^{1/2} t^{1/3}]$ the equation

$$\frac{d^2\psi_0}{dz_*^2} = \left[-1 + \frac{3+M}{6}\frac{\rho_{00}T_\lambda}{\Delta C_p}\frac{v_s^2}{t^{4/3}} + (1-M)\psi_0^2 + M\psi_0^4\right]\psi_0.$$

(3.79)

It is obvious that this equation differs from (3.1) only in the term $(y_s/v_{sc2})^2\psi_0$, which is proportional to v_s^2. Therefore all the results obtained in Sec. 3.1 at $v_s = 0$ can be readily generalized to include the case $v_s \neq 0$. Thus, for a gap of width d we obtain as the condition for the appearance of a solution with $\rho_s \neq 0$, instead of

(3.4), the equation (see also 3.74))

$$d\left[1 - \left(\frac{v_s}{v_{sc2}}\right)^2\right]^{1/2} = \pi \xi_M(t) = \frac{8.6 \cdot 10^{-8}[(3-M)/3]^{1/2}}{t^{2/3}} \, cm,$$

$$v_{sc2} = 5.8 \cdot 10^3 \left(\frac{3}{3+M}\right)^{1/2} t^{2/3} \, cm\text{-sec}^{-1}.$$

(3.80)

Hence

$$\Delta T_\lambda = T_\lambda - T_\lambda(d, v_s) = \left(\frac{3+M}{3}\right)^{3/4} \left(\frac{\pi^2 \xi_{00}^2}{d^2} + \frac{\rho_{00}T_\lambda v_s^2}{2\Delta C_p}\right)^{3/4}$$

$$= \left[\left(\frac{3+M}{M}\right)\left(\frac{7.4 \cdot 10^{-15}}{d^2} + 3 \cdot 10^{-8}v_s^2\right)\right]^{3/4} \, °K,$$

(3.81)

where $T_\lambda \equiv T_\lambda(d \to \infty, \, v_s = 0)$, the thickness d is in centimeters, and v_s is in cm/sec; of course, Eq. (3.81) goes over into (3.76) or (3.4a), respectively, as $d \to \infty$ or at $v_s = 0$.

Given d and t, the critical velocity v_{sc2} at which $\rho_s = 0$ is, according to (3.81),

$$v_{sc2}(d) = \left[1 - 7.4 \cdot 10^{-15}\frac{(3+M)/3}{d^2 t^{4/3}}\right]^{1/2} v_{sc2},$$

(3.82)

where, of course, $v_{sc2} \equiv v_{sc2}(d \to \infty)$ is given by (3.74).

We have labeled the critical velocity v_{sc2} with a subscript 2, since this velocity can be said to be the highest critical velocity—the density $\rho_s \to 0$ as $v_s \to v_{sc2}$. However, even if we disregard the possibility of formation of vortices or of some other "excitations," the superfluid flow becomes unstable at a certain critical velocity $v_{sc1} < v_{sc2}$. The point is that the flux of the superfluid part of helium II, $j_s = \rho_s v_s$, has a maximum as a function of v_s. In fact, for uniform flow (wide gaps) we have, according to (3.77)

$$j_s = \rho_s(v_s) v_s = \rho_{00} t^{2/3} \left(1 - \frac{T_\lambda \rho_{00}}{2\Delta C_p t^{4/3}} v_s^2\right) v_s,$$

(3.83)

and $dj_s/dv_s = 0$ for the velocity

$$v_{sc1} = \frac{1}{\sqrt{3}} v_{sc2} = 3.3 \cdot 10^3 t^{2/3}$$

(3.84)

(see (3.74); just as in (3.77), we confine ourselves in (3.84) to the case $M = 0$).

At $v_s > v_{sc1}$ the flux j_s decreases with increasing v_s, and this is indeed an indication of the onset of instability. The appearance of instability can be verified by considering the law governing the variation of small perturbations with time. In our case, however, there is no need for this, since the superfluid flow can be assumed to be in thermodynamic quasi-equilibrium. It suffices therefore to calculate the second derivatives of the thermodynamic potential and to ascertain whether the potential has a minimum. It is such a simple calculation (see, e.g., [391]) which leads to the conclusion that flow with $dj_s/dv_s < 0$ is unstable.

As applied to narrow gaps and channels, the calculation of v_{sc1} makes it necessary to use for $\rho_s(z)$ solutions of the type obtained in Sec. 3.1. The flux j_s, of course is here not homogeneous in z ($j_s = 0$ on the wall and is maximal at the midpoint of the gap). It is therefore not clear whether the instability condition $dj_s/dv_s < 0$ should
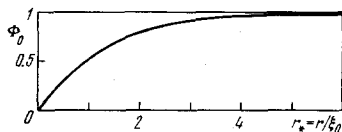
FIG. 6. Solution of Eq. (3.86) for the function $\Phi_0(z_*)$.

be used locally (at each z) or averaged over the gap. We shall therefore not present here the results of some published calculations. [39] It is still necessary to generalize these calculations (particularly to include the case $M \neq 0$) and, principally, test the flow for stability by the method of small perturbations. It is necessary to compare next the obtained values of v_{scl} with estimates based on the concept wherein the critical velocity of superfluid flow is regarded as a threshold of vortex formation, etc. (it must be borne in mind here that, as will be made clear below, in helium II vortices near the λ point are in general substantially different from the vortices far from the λ point). The problem of the critical velocity near the λ point has thus not yet been completely solved even under the simplest conditions (uniform gap). The same can be said of effects in porous media, of channels that are inhomogeneous in cross section, of the "proximity effect," etc.[71,115,116] (the statements made above pertain to the presentation of the sufficiently complete picture, and are of course not meant to negate the value of the qualitative or even quantitative considerations advanced in the cited papers and in a number of others).

The second most important class of superfluid flows, besides one-dimensional linear flow, which will be considered below, are flows connected with rotation. The simplest and at the same time most important object is evidently an isolated vortex filament. We now proceed to consider the corresponding problem. [25]

On the basis of Eq. (2.20) with $M = 0$, we represent the function

$$\Psi = \psi_0 \left[\left(\frac{\rho_{00}}{m} \right)^{1/2} t^{1/3} \right]$$

in the form

$$\Psi(r, \varphi) = \Psi(r) e^{in\varphi},$$

where $n = 1, 2, \ldots$, and r and φ are cylindrical coordinates with z axis coinciding with the axis of the vortex filament. The velocity $\mathbf{v}_s$ has only one component $v_{s\varphi} = \hbar n/mr$, and the circulation is

$$\oint \mathbf{v}_s \, dl = 2\pi r v_{s\varphi} = \frac{2\pi n \hbar}{m}. \tag{3.85}$$

We note that were the helium-atom mass m in the initial expressions (2.15)–(2.16) to be replaced by a certain effective mass $m_{eff}(p, T)$, then the circulation would be equal to $2\pi n \hbar/\sqrt{m m_{eff}}$. But the circulation cannot change with changing temperature. At the same time, formula (3.85) is also valid at $T = 0$. [117] In addition, under inhomogeneous external conditions (say in a gravitational field) the circulation, were m to be replaced by $m_{eff}(p, T)$, would depend on the choice of

the contour, i.e., it would not be constant. Hence, and also from the arguments advanced at the end of Sec. 2.1, it follows that $m_{eff} = m$ in (2.15) and (2.16).

For a vortex filament, Eq. (2.20) takes the form $(r_* = r/\xi_0, \psi_0 = \Phi_0 e^{in\varphi})$

$$r_*^2 \frac{d^2 \Phi_0}{dr_*^2} + r_* \frac{d\Phi_0}{dr_*} + (r_*^2 - n^2)\Phi_0 - r_*^2 \Phi_0^3 = 0. \tag{3.86}$$

As $r_* \to 0$, the function Φ_0 takes the form $C_1 r_*^{\pm |n|}$. Solutions with $\Phi_0 \sim C_1 r_*^{-|n|}$ have no physical meaning, since $\psi_0 \to \infty$ as $r \to 0$. We must thus choose the solution $\Phi_0(r_* \to 0) = C r_*^{|n|}$, and consequently $\rho_s = 0$ on the filament axis. As $r_* \to \infty$, the density $\rho_s = m|\Psi|^2 = m\Phi^2$ should equal the unperturbed density ρ_{se}, i.e., $\Phi_0(r_* \to \infty) = 1$. It is easily seen that the sought solution of Eq. (3.86) at large r_* takes the form

$$\Phi_0^2 = 1 - \frac{n^2}{r_*^2}, \quad r_* \gg 1. \tag{3.87}$$

In the entire interval $0 \leqslant r_* \leqslant \infty$, Eq. (3.86) cannot be integrated in quadratures. The results of its numerical solution for $n = 1$ are shown in Fig. 6.

The thermodynamic potential per unit filament length (we have in mind the difference between the thermodynamic potentials in the presence and in the absence of velocity circulation) is equal to

$$
\begin{aligned}
f &= \left(\frac{\pi \hbar^2}{m^2} \right) \rho_{se} N, \quad N = N_1 + N_2, \\
N_1 &= \frac{1}{2} \int_0^\infty \left[\Phi_0^4 - 2\Phi_0^2 + 2\left(\frac{d\Phi_0}{dr_*} \right)^2 + 1 \right] r_* \, dr_*, \\
N_2 &= n^2 \int_0^{R/\xi_0} \Phi_0^2 \frac{dr_*}{r_*}.
\end{aligned}
\tag{3.88}
$$

Here R is a certain maximal integration radius (the diameter of the vessel, the distance between the vortex filaments), which must be introduced if an infinitely long filament is considered. If the density $\rho_s = \rho_{se}$ is constant, as would be the case if the gradient term were not taken into account, then $N = N_e = n^2 \ln(R/a)$, where a is a certain distance of atomic scale. In our problem, however, numerical calculation yields

$$
\begin{aligned}
n = 1: &\quad N = \ln\left(1.46 \frac{R}{\xi_0} \right), \\
n = 2: &\quad N = 4 \ln\left(0.59 \frac{R}{\xi_0} \right), \\
n = 3: &\quad N = 9 \ln\left(0.38 \frac{R}{\xi_0} \right).
\end{aligned}
\tag{3.89}
$$

It is clear therefore that at sufficiently large R (isolated filament) the difference between N_e and N is relatively small (the reason is that the main contribution to f is made by the far, "classical" region). The energy is here approximately proportional to n^2 and states with $n > 1$ need not be taken into account (it is more efficient to form two filaments with $n = 1$ than one filament with $n > 1$).

Differentiating an expression of the type (3.88) with respect to the temperature, the chemical potential, or the chemical potential of the ^{3}He impurity, we can obtain the contributions made to the density by the vortex filament, and also find the corresponding change of the

Since $\xi_0 = 2.74 \times 10^{-8} t^{-2/3}$, it is clear that as the λ point is approached the filament becomes thicker and, for example at $t = T_\lambda - T = 10^{-3}$ °K, the density ρ_s near the filament axis differs significantly from the equilibrium value $\rho_{s\infty}$, up to distances $r \sim (3-4)\xi_0 \sim 10^{-5}$ cm from the filament axis (see Fig. 6). At still lower values of t, the filament thickness may turn out already to be commensurate with the widths of even macroscopic channels and gaps. It is precisely under these conditions that the Ψ-theory of superfluidity must be used. Unfortunately, the problems encountered here (vortex ring; short vortex filament between walls perpendicular to its axis; vortex filament in gap with walls parallel to the filament axis; rotation of the drop,[118] and others) have not yet been solved (for some results, incidentally, see[119]). W won't even speak of the various nonstationary problems connected with the formation and oscillations of vortex filaments and vortex rings (to solve problems of this type it is necessary to use the equations given in Sec. 4.1, since we have considered above only stationary flow of the helium II).

It is known that rotating helium II is one of the basic objects in the study of superfluidity (see the reviews[16,120,121]). As the λ point is approached, some problems involving rotation require, naturally, the use of the Ψ-theory of superfluidity (see[120], Sec. 12). We confine ourselves here to such an example: a vessel with helium II is rotated with angular velocity ω, and the radius of the vessel is $R \gg \xi_0$. With increasing ω, the distance between the vortex filaments becomes of the order of ξ_0, i.e., of the order of the thickness of the filament, and the average density ρ_s in the vessel is greatly decreased. With further increase of ω, the density ρ_s vanishes completely, i.e., the superfluidity vanishes. According to[27], the corresponding critical value is $\omega_{c2} \simeq 10^{11} t^{4/3}$ sec^{-1}, meaning that the rotation has lowered the λ-point temperature by an amount

$$\Delta T_\lambda = T_\lambda(\omega = 0) - T_\lambda(\omega = \omega_{c2}) = 5.4 \cdot 10^{-9} \omega^{3/4}.$$

Unfortunately, this effect is quite small, for even at $\omega \sim 10^4$ the shift is $\Delta T_\lambda \sim 5 \times 10^{-6}$ °K. On the other hand, in this problem we are dealing with a "gross" effect — complete suppression of superfluidity. More delicate changes in rotating helium II, due to the non-constancy of ρ_s in vortex filaments, may turn out to be significant also at much smaller values of ω, but at $t \lesssim 10^{-3}$ °K.

IV. GENERAL Ψ-THEORY OF SUPERFLUIDITY NEAR THE λ POINT (MOTION OF SUPERFLUID AND NORMAL COMPONENTS, NONSTATIONARY STATE)

4.1. Fundamental equations

We have considered above either helium II at rest or its stationary superfluid motion. It is obvious that the Ψ-theory of superfluidity near the λ point should apply in general also to cases when not only the superfluid but also the normal component of helium II are in motion (the normal component has a density $\rho_n = \rho - \rho_s$ and a velocity $\mathbf{v}_n$). The motion can be also nonstationary, and dissipation and relaxation of the order parameter must be taken into account. The corresponding theory has been the subject of only relatively few studies,[122–125,93] many questions have not yet been clarified, and the encountered concrete problems have not been solved. The present section will of necessity be rather short.

We start with stationary motion, but with $\mathbf{v}_n \neq 0$. Moreover, we assume that the velocity $\mathbf{v}_n$ is homogeneous in space, so that

$$\text{div } \mathbf{v}_n = 0, \quad \frac{\partial \mathbf{v}_n}{\partial t} = 0. \tag{4.1}$$

In this case, in a reference frame K' that moves with velocity $\mathbf{v}_n$ relative to the laboratory frame K, the motion of the helium II is purely superfluid and cannot differ in any way from that considered above. This means that the thermodynamic potential for the macroscopic wave function Ψ' of helium II in the system K' takes the form (2.15). We now carry out a Galilean transformation in the system K, whereby, as is well known (see, e.g.,[126] Sec. 17):

$$\mathbf{r} = \mathbf{r}' + \mathbf{v}_n t, \quad \mathbf{p} = \mathbf{p}' + m\mathbf{v}_n, \quad E = E' + \mathbf{p}\mathbf{v}_n + \frac{mv_n^2}{2},$$
$$\Psi(\mathbf{r}) = \Psi'(\mathbf{r} - \mathbf{v}_n t) \exp\left(\frac{i}{\hbar} m\mathbf{v}_n\mathbf{r}\right), \tag{4.2}$$

where we consider a certain particle with mass m and, naturally, the primed quantities pertain to the system K' and the unprimed ones to K.

Transforming in (2.15) from Ψ' to Ψ, in accord with (4.2), we obtain directly

$$\Phi_{II} = \frac{m}{2}\left|\left(-\frac{i\hbar}{m}\nabla - \mathbf{v}_n\right)\Psi\right|^2 + \Phi_{II\,0}(p, T, |\Psi|^2). \tag{4.3}$$

The same result can also be arrived in the following way: The energy density of the helium II in the laboratory frame K can be naturally expressed in the form

$$E = \frac{\hbar^2}{2m}|\nabla\Psi|^2 + \frac{\rho_n v_n^2}{2} + E_{II\,0}(\rho, S, |\Psi|^2), \tag{4.4}$$

where $\rho_n = \rho - m|\psi|^2 = \rho - \rho_s$ is obviously the density of the normal part of helium II. We find now the energy density E' in the system K', in which $v_n' = 0$. It is known that, in accord with (4.2) if m is replaced by ρ, that

$$E = E' + \mathbf{p}'\mathbf{v}_n + \frac{\rho v_n^2}{2},$$
$$\mathbf{p}' = \mathbf{j} - \rho\mathbf{v}_n = \rho_s(\mathbf{v}_s - \mathbf{v}_n), \tag{4.5}$$

where $\mathbf{j} = \rho_s\mathbf{v}_s + \rho_n\mathbf{v}_n \equiv \mathbf{j}_s + \mathbf{j}_n$ is the total momentum density in the K system.

From (4.4) and (4.5), on the other hand, we obtain directly

$$E' = \frac{m}{2}\left|\left(-\frac{i\hbar}{m}\nabla - \mathbf{v}_n\right)\Psi\right|^2 + E_{II\,0}(\rho, S, |\Psi|^2). \tag{4.6}$$

The transition to (4.3) is a transition to other variables and to another thermodynamic potential, and is subject to no doubt, at least if ρ is constant.

Varying the integral $\int \Phi_{II} dV$ with respect to Ψ^* by using expressions (4.3), at a specified velocity $\mathbf{v}_n$, we obtain the sought equation for Ψ:

$$\frac{m}{2}\left(-\frac{i\hbar}{m}\nabla - \mathbf{v}_n\right)^2 \Psi + \frac{\partial \Phi_{II\,0}}{\partial |\Psi|^2}\Psi = 0, \qquad (4.7)$$

which goes over, of course, into (2.06) at $\mathbf{v}_n = 0$.

An equation equivalent to (4.7) was derived in[122] by variation (minimization) of (4.4), but under the condition that $\mathbf{j}$ be constant, i.e., at a specified total momentum of the liquid. This requirement is equivalent to an energy release E'; on the other hand, minimization of the energy E without the condition $\mathbf{j} = \text{const}$ would mean minimization also of the kinetic energy of the liquid as a whole, having no bearing on the thermodynamic properties of the system.

Separating the real and imaginary parts of (4.7), we obtain two equations, which we express in terms of $\rho_s = m\eta^2$ and $\mathbf{v}_s = (\hbar/m)\nabla\varphi$,

$$\Psi = \eta e^{i\varphi}, \quad \mu_s = \left(\frac{\partial \Phi_{II\,0}}{\partial |\Psi|^2}\right)_{p,T} = m\left(\frac{\partial \Phi_{II\,0}}{\partial \rho_s}\right)_{p,T}:$$

$$-\frac{\hbar^2}{4m^2}\Delta\rho_s + \frac{\hbar^2(\nabla\rho_s)^2}{8m^2\rho_s} + \frac{\rho_s(\mathbf{v}_s - \mathbf{v}_n)^2}{2} + \mu_s\rho_s = 0, \qquad (4.8)$$

$$\text{div}\,[\rho_s(\mathbf{v}_s - \mathbf{v}_n)] = 0. \qquad (4.9)$$

Of course, Eq. (4.9) can be obtained also by an equivalent method—by multiplying (4.7) by Ψ^*, multiplying the conjugate equation by Ψ, and subtracting the equations from each other.

We note that on going from (4.7) to (4.8) and (4.9), we did not use the condition $\text{div}\,\mathbf{v}_n = 0$. On the other hand, if the conditions $\text{div}\,\mathbf{v}_n = 0$ and $\nabla\varphi = 0$, under which Eq. (4.7) was derived above, then (4.9) becomes identical with the continuity equation

$$\text{div}\,\mathbf{j} = \text{div}\,(\rho_s\mathbf{v}_s + \rho_n\mathbf{v}_n) = 0.$$

We have gone into such detail in the derivation (by three methods!) of Eq. (4.7), because this equation is regarded as incorrect in[123] (see also[15], Sec. 33), and another equation is proposed, differing by the term

$$\frac{i\hbar\Psi}{2\rho_s}\,\text{div}\,(\mathbf{j} - \rho\mathbf{v}_n) = \frac{i\hbar\Psi}{2\rho_s}\,\text{div}\,\rho_s(\mathbf{v}_s - \mathbf{v}_n)$$

and obtaining by varying $\int \Phi_{II} d\xi V$ with Φ_{II} according to (4.3) at a fixed difference $\mathbf{v}_s - \mathbf{v}_n$. But it is precisely the last requirement that seems to us completely unfounded. Besides the arguments above, the inconsistency follows also from the fact that the equations proposed in[123] do not lead to the continuity equation.

This pertains, in particular, to the case when $\mathbf{v}_n = 0$ and, consequently $\text{div}\,(\mathbf{j} - \rho\mathbf{v}_n) = \text{div}\,\rho_s\mathbf{v}_s$; then the equation given in[123] in place of our Eq. (3.16) is

$$-\frac{\hbar^2}{2m}\Delta\Psi + \frac{\partial \Phi_{II\,0}}{\partial |\Psi|^2}\Psi + \frac{i\hbar\,\text{div}\,(\rho_s\mathbf{v}_s)}{2\rho_s}\Psi = 0. \qquad (4.10)$$

This equation and its conjugate do not lead to any conservation law; in other words, the imaginary part of (4.10) vanishes identically. There are therefore no grounds for assuming that $\text{div}\,\rho_s\mathbf{v}_s = 0$. On the other hand, postulation of this law from independent consid-

erations is both contrary to nature and returns us to Eq. (2.16). Since the Ψ-theory of superconductivity[26] follows from the microscopic theory, it can also be noted that it corresponds exactly to Eq. (2.16) and not to (4.10).

An analogous situation obtains also at $\mathbf{v}_n \neq 0$: the equation proposed in[123] does not lead to the momentum conservation law, whereas such a law, $\text{div}\,\mathbf{j} = \text{div}\,(\mathbf{j} - \rho\mathbf{v}_n) = 0$, follows directly from (4.7) at $\text{div}\,\mathbf{v}_n = 0$ and $\nabla\rho = 0$, as we have already seen (incidentally, it is stated in[123] that the condition $\text{div}\,(\mathbf{j} - \rho\mathbf{v}_n) = 0$ is superfluous and there are no grounds for it whatever).

In what follows, however, we must have an equation of the type (4.7), suitable at both $\text{div}\,\mathbf{v}_n \neq 0$ and $\nabla\rho \neq 0$. If we add to the left-hand side of (4.7) the term

$$-\left(\frac{i\hbar}{2m|\Psi|^2}\,\text{div}\,\rho\mathbf{v}_n\right)\Psi, \qquad (4.11)$$

then the imaginary part of such an equation takes the form $\text{div}\,\mathbf{j} = 0$ both for $\text{div}\,\mathbf{v}_n \neq 0$ and $\nabla\rho \neq 0$. On the other hand, the term (4.11) contains Ψ^* in the denominator, which is not likely (this of course, pertains also to (4.10) and to the entire variant proposed in[123]). Further, at $\text{div}\,\mathbf{v}_n \neq 0$, effects such as second viscosity appear, and there are no grounds for the Ψ function to be defined by an equation such as (4.7), which contains no dissipative terms. Taking all the foregoing into consideration, we shall use Eq. (4.7) to change over to a general system of equations suitable both in the nonstationary case and when $\text{div}\,\mathbf{v}_n \neq 0$, just as was done in[122].

Referring the reader for details to[122], we write down directly the corresponding complete system of equations for the function $\Psi = \eta\,e^{i\varphi}$, for the normal-component velocity $\mathbf{v}_n$, and also for the density ρ and the entropy per unit volume S:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi + (\mu + \mu_s)m\Psi - i\Lambda\left[\frac{1}{2}\left(\frac{i\hbar}{m}\nabla + \mathbf{v}_n\right)^2 + \mu_s\right]m\Psi,$$

$$\mu_s = \left(\frac{\partial E_{II\,0}}{\partial\rho_s}\right)_{\rho,S}, \quad \mu = \left(\frac{\partial E_{II\,0}}{\partial\rho}\right)_{\rho_s,S}; \qquad (4.12)$$

$$\frac{\partial\rho}{\partial t} + \text{div}\,\mathbf{j} = 0,$$
$$\mathbf{j} = \rho_s\mathbf{v}_s + \rho_n\mathbf{v}_n, \quad \rho_n = \rho - m|\Psi|^2,$$
$$\rho_s\mathbf{v}_s = -\frac{i\hbar}{2}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*), \qquad (4.13)$$
$$\rho_s = m|\Psi|^2;$$

$$\frac{\partial j_i}{\partial t} + \frac{\partial\Pi_{ik}}{\partial x_k} = 0,$$

$$\Pi_{ik} = \rho_n v_{ni}v_{nk} + p\delta_{ik} + \frac{\hbar^2}{4m}\left(\frac{\partial\Psi}{\partial x_i}\frac{\partial\Psi^*}{\partial x_k} - \Psi\frac{\partial^2\Psi^*}{\partial x_i\partial x_k} + \text{c.c.}\right) \qquad (4.14)$$
$$-\eta_n\left(\frac{\partial v_{ni}}{\partial x_k} + \frac{\partial v_{nk}}{\partial x_i} - \frac{2}{3}\delta_{ik}\,\text{div}\,\mathbf{v}_n\right);$$

$$\frac{\partial S}{\partial t} + \text{div}\left(S\mathbf{v}_n - \frac{\varkappa}{T}\nabla T\right) = \frac{R}{T},$$

$$R = \varkappa\left(\frac{\nabla T}{T}\right)^2 + \frac{\eta_n}{2}\left(\frac{\partial v_{ni}}{\partial x_k} + \frac{\partial v_{nk}}{\partial x_i} - \frac{2}{3}\delta_{ik}\,\text{div}\,\mathbf{v}_n\right)^2 \qquad (4.15)$$
$$+ \frac{2\Lambda m\rho_s}{\hbar}\left|\left[\frac{1}{2}\left(\frac{i\hbar}{m}\nabla + \mathbf{v}_n\right)^2 + \mu_s\right]m\Psi\right|^2,$$

where $p = -E_{II\,0} + TS + \mu\rho + \mu_s\rho_s$ is the pressure, η_n is the first viscosity coefficient for the normal component of helium II, and the dimensionless real coefficient Λ determines the value of the second viscosity in the considered vicinity of the λ point. It will be made clear later that the order-parameter relaxation time τ is given by

$$\frac{1}{\tau} = \frac{2\Lambda m}{\hbar}\rho_s\frac{\partial \mu_s}{\partial \rho_s} \equiv \frac{2\Lambda m}{\hbar}\rho_s\left(\frac{\partial^2 E_{\Pi 0}}{\partial \rho_s^2}\right)_{\rho,\,S}. \qquad (4.16)$$

We note that in the expression for Π_{ik} in (4.14) we have discarded terms connected with second viscosity, which is independent of the relaxation of ρ_s (the point is that the second viscosity due to the relaxation of ρ_s turns out to be predominant near the λ point).

The real part of (4.12), after dividing by $m\eta$ and taking the gradient, takes the form

$$\frac{\partial v_s}{\partial t} + \nabla\left\{\mu + \mu_s - \frac{\hbar^2}{2m}\frac{\Delta\eta}{\eta^2} + \frac{v_s^2}{2} - \frac{\hbar\Lambda}{2m\rho_s}\operatorname{div}[\rho_s(v_s - v_n)]\right\} = 0. \qquad (4.17)$$

On the other hand, the imaginary part of (4.12), after multiplication by $2m\eta/\hbar$, is given by

$$\frac{\partial \rho_s}{\partial t} + \operatorname{div}\rho_s v_s = -\frac{2\Lambda m}{\hbar}\left[\mu_s - \frac{\hbar^2}{2m^2}\frac{\Delta\eta}{\eta} + \frac{(v_s - v_n)^2}{2}\right]\rho_s. \qquad (4.18)$$

In spatially homogeneous helium II, and at $v_s = v_n$, we get from (4.18) for small deviations from equilibrium $\rho_s' = \rho_s - \rho_{se}$,

$$\frac{\partial \rho_s'}{\partial t} = -\frac{2\Lambda m}{\hbar}\left(\frac{\partial \mu_s}{\partial \rho_s}\right)_{\rho_{se}}\rho_{se}\rho_s', \qquad (4.19)$$

inasmuch as in this case

$$\mu_s(\rho_{se}) = \left[\left(\frac{\partial E_{\Pi 0}}{\partial \rho_s}\right)_{\rho,\,S}\right]_{\rho_s = \rho_{se}} = 0;$$

(4.19) leads precisely to the relation (4.16).

Far enough from the λ point we can assume that the relaxation sets in instantaneously, i.e., $\tau \to 0$, or formally, $\Lambda \to \infty$. It is just this limit which corresponds to the Landau hydrodynamic theory of superfluidity,[14-16] in which we put, in terms of our variables, $\rho_s = \rho_s(\rho, S)$ and $\mu_s + (v_s - v_n)^2/2 = 0$ (see, e.g., Eq. (4.8) with the gradient terms neglected). In addition, Landau's theory does not take into account, of course, the gradient terms proportional to $\hbar^2$. Taking the foregoing into account it is clear that as $\Lambda \to \infty$ Eq. (4.18) remains indeterminate, i.e., it imposes no restrictions whatever on ρ_s, as should indeed be the case under similar conditions. On the other hand, Eq. (4.17) goes over into the corresponding equation of two-fluid hydrodynamics

$$\frac{\partial v_s}{\partial t} + \nabla\left(\mu + \frac{v_s^2}{2}\right) = \nabla[\zeta_3\operatorname{div}(j - \rho v_n)] \equiv \nabla[\zeta_3\operatorname{div}\rho_s(v_s - v_n)], \qquad (4.20)$$

where

$$\zeta_3 = \frac{\hbar\Lambda}{2m\rho_s} = \frac{\hbar^2}{4m^2\rho_s^2(\partial\mu_s/\partial\rho_s)\tau}$$

is one of the second-viscosity coefficients (see[15], Sec. 18); we cannot put $\Lambda \to \infty$ in the expression for ξ_3, for then the second viscosity would become infinite. If Λ is constant (is independent of $t = T_\lambda - T$), then the relaxation time is $\tau \sim t^{-4/3}$ (see (4.16) with $\partial\mu_s/\partial\rho_s = \text{const} \cdot t^{2/3}$). On the other hand if $\tau \sim t^{-1}$, an assump-

tion for which there are some grounds,[70,122a,123] then $\Lambda \sim t^{-1/3}$.[31]

Under the assumptions mentioned above, the entire system of equations (4.12)–(4.15) goes over into the set of equations of two-fluid hydrodynamics,[32] as is to be naturally expected (we see none of the difficulties mentioned concerning this question at the end of article[121]).

Beyond the scope of the Landau theory, but under the assumption that the gradients of ρ_s are small, as can be the case in the case of sound propagation in the interior of helium II, the system (4.12)–(4.15) simplifies to

$$\frac{\partial v_s}{\partial t} + \nabla\left[\mu + \mu_s + \frac{v_s^2}{2} - \frac{\hbar\Lambda}{2m\rho_s}\operatorname{div}\rho_s(v_s - v_n)\right] = 0, \qquad (4.21)$$

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho_s v_s + \rho_n v_n) = 0, \qquad (4.22)$$

$$\frac{\partial}{\partial t}(\rho_n v_{ni} + \rho_s v_{si}) + \frac{\partial}{\partial x_k}\left[\rho_n v_{ni}v_{nk} + \rho_s v_{si}v_{sk} + p\delta_{ik}\right.$$
$$\left. - \eta_n\left(\frac{\partial v_{ni}}{\partial x_k} + \frac{\partial v_{nk}}{\partial x_i} - \frac{2}{3}\delta_{ik}\operatorname{div}v_n\right)\right] = 0, \qquad (4.23)$$

$$\frac{\partial S}{\partial t} + \operatorname{div}(Sv_n) = \frac{1}{T}\operatorname{div}(\varkappa\nabla T) + \frac{1}{T}\left\{\frac{\hbar\Lambda}{2m\rho_s}[\operatorname{div}\rho_s(v_s - v_n)]^2\right.$$
$$\left. + \frac{2\Lambda m}{\hbar}\left[\frac{1}{2}(v_s - v_n)^2 + \mu_s\right]^2\rho_s + \frac{\eta_n}{2}\left(\frac{\partial v_{ni}}{\partial x_k} + \frac{\partial v_{nk}}{\partial x_i} - \frac{2}{3}\delta_{ik}\operatorname{div}v_n\right)^2\right\}, \qquad (4.24)$$

$$\frac{\partial\rho_s}{\partial t} + \operatorname{div}\rho_s v_s = -\frac{2\Lambda m}{\hbar}\left[\mu_s + \frac{(v_s - v_n)^2}{2}\right]\rho_s. \qquad (4.25)$$

The system (4.21)–(4.25) is of the same form as the equations of two-fluid hydrodynamics, with the chemical potential μ replaced by

$$\mu + \mu_s = \left(\frac{\partial E_{\Pi 0}}{\partial\rho}\right)_{\rho_s,\,S} + \left(\frac{\partial E_{\Pi 0}}{\partial\rho_s}\right)_{\rho,\,S}$$

and with the expression for the second-viscosity coefficient ξ_3 spelled out concretely. Besides, of course, the equation (4.25) is added and determines the relaxation of the density ρ_s, which is no longer regarded as a specified function of ρ and S.

4.2. Sound propagation. Thermal resistance near a solid wall

The complete system of hydrodynamic equations of helium near the λ point, which was obtained in Sec. 4.1, has been used relatively little so far. Its most important application is in an investigation, based on (4.21)–(4.25), of the character of sound propagation in homogeneous immobile helium II (and furthermore in the simplest situation, when the sound wavelength $\lambda = 2\pi/k$ is much larger than the coherence length $\xi_M(t)$, and consequently the gradient terms with respect to ρ_s can

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[31] We present an estimate of the parameter Λ in (4.12), (4.16), and elsewhere, based on the absorption of first sound. It is assumed that in the considered region $\omega\tau \ll 1$ the absorption is determined entirely by the relaxation mechanism (see Sec. 4.2). The data of M. Barmatz and I. Rudnick (Phys. Rev. 170, 224, 1968) yield $\Lambda \approx 2.4$ (at $T_\lambda - T = 10^{-4}\,°K$), while according to C. Buchal and V. Pobell (Phys. Rev. B14, 1103, 1976) $\Lambda \approx 5$ (likewise at $T_\lambda - T = 10^{-4}\,°K$).

[32] One of the fundamental equations of the hydrodynamic theory of superfluidity, the condition $\operatorname{curl}v_s = 0$, follows obviously from the very definition $v_s = (\hbar/m)\nabla\varphi$ (see Eq. (1.2)).

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be disregarded). However, even in this case the question cannot be regarded as fully answered. The point is that[123,15] contain only estimates aimed at establishing the temperature dependences of the dispersion and absorption of the sound. What is of interest, however, is also a quantitative comparison of the predictions of the theory with experiment.[127] Furthermore, the system of equations used differs in[123,15] from (4.21)–(4.25) in the absence of a term of the second-viscosity type, such as is contained in the left-hand side of (4.21) (see Sec. 4.1). Yet allowance for this term, at least in the problem of second-sound absorption, may turn out to be essential. Finally (and perhaps most importantly), only the relaxation mechanism of the absorption of first and second sound was considered in[122] and[123], and no account is taken at all of the interaction between the sound and the fluctuations of the parameter Ψ (see[128–130]). Yet the "fluctuation mechanism" (which estimates show to have the same temperature dependence as the contribution from the relaxation of Ψ), can play a noticeable role even in the hydrodynamic frequency region $\omega\tau \ll 1$; in the region $\omega\tau \gg 1$, and also at $T > T_\lambda$, this mechanism becomes in fact decisive.

An attempt to examine the effect of flucutations on the absorption (dispersion) of first sound and on the thermal conductivity of helium was made in[93,125] for the temperature region above the λ point. In[93], however, a highly simplified variant of the theory was used, and while the results of the calculations in[93] are probably correct from the qualitative point of view, their connection with the approach developed in Sec. 4.1 still calls for an evaluation. In[125], to the contrary, the initial equations were precisely of the type (4.12)–(4.15), but in a variant[124] in which a critical exponent $\hat{\eta} = 1/2$ was used so that the connection between the Ψ function and the density ρ_s was given by the expression $\rho_s = (m^2/m^*)|\Psi|^2$, with $m^* \propto t^{1/3}$. In[125], furthermore, just as in[123], the second-viscosity term was left out of the equation for $\mathbf{v}_s$, and a number of other assumptions, the validity of which is still to be verified, have been made.

By virtue of the foregoing remarks, and also for lack of space, we shall not dwell on the results of the cited and other papers. We indicate only that from the qualitative point of view, for the hydrodynamic frequency region $\omega\tau \ll 1$, the expressions obtained in[93,123,125] agree in the main with experiment[127] and with other more rigorous calculation methods (see[70,130] and the references in[130]), in which the ideas of the dynamic similarity theory and the ε-expansion method are used.

We note that a more detailed study of sound propagation (primarily first and second sound, and also the thermal conductivity above the λ point) in homogeneous helium can cast more light on the question discussed in Sec. 4.1, whether the additional terms, which are clear from (4.10) and (4.11) are present or absent in Eq. (4.7).

In light of the foregoing it becomes clear that the study of the singularities of sound propagation in homogeneous helium, and all the more in inhomogeneous

helium (particularly an investigation of the reflection and transformation of sound waves on the boundary of helium II with a solid or with vapor) is, generally speaking, a very complicated problem. Some idea of the character of the singularities that are possible here can be gained from a solution of the particular problem of the "interaction" of a static heat flux ($\omega = 0$) with an inhomogeneous distribution $\rho_s(z)$ near a solid wall. This problem is of independent interest, since it leads to the conclusion that a helium II layer next to a wall has an internal boundary-layer thermal resistance that increases sharply near the λ point.

Consider[61,131] a helium-II half-space bounded by a solid, and assume a stationary small heat flux q into the helium from the solid. At large distances z from the wall, this flux causes opposing convective flows of the normal and superfluid components. At short distances, however, if the condition $\rho_s(0) = 0$ is satisfied on the wall, the convective component of the heat flux is absent. Indeed, from the condition $\rho_s = 0$ and from the condition that the perpendicular component of the total mass flux vanish ($j_\perp = \rho_s v_{s\perp} + \rho_n v_{n\perp} = 0$) it follows that $v_{n\perp}$ also vanishes on the boundary. The condition $v_{n\perp} = 0$ remains in force, obviously, also in the case of arbitrary flow. Thus, at small distances from the wall, the heat exchange between the solid and the helium II proceeds, as in ordinary liquids, only as a result of the normal thermal conductivity.

To estimate the distance l_T over which the thermal-conductivity component of the heat flux is attenuated and the convective counterflow is produced, we turn to Eqs. (4.7)–(4.12). The linearized form of these equations for the stationary case of interest to us is

$$j = \rho_s^{(0)} v_s + \rho_n^{(0)} v_n, \tag{4.26}$$

$$T^{(0)} S^{(0)} v_n - \varkappa \frac{dT'}{dz} = q, \tag{4.27}$$

$$\frac{dp'}{dz} = \rho^{(0)} \frac{d\mu'}{dz} + S^{(0)} \frac{dT'}{dz} + \rho_s^{(0)} \frac{d\mu_s'}{dz} = \frac{4}{3} \eta_n \frac{d^2 v_n}{dz^2} + \rho_s^{(0)} \frac{d}{dz} \left(\frac{\hbar^2}{2m^2 \eta^{(0)}} \frac{d^2 \eta'}{dz^2} \right), \tag{4.28}$$

$$\frac{d\mu'}{dz} = \frac{d}{dz} \left(\frac{\hbar^2}{2m^2 \eta^{(0)}} \frac{d^2 \eta'}{dz^2} - \mu_s' - \frac{\hbar\Lambda}{2m\rho_s^{(0)}} \frac{dp^{(0)} v_n}{dz} \right), \tag{4.29}$$

$$\frac{d}{dz} (\rho_n^{(0)} v_n) = \frac{2\Lambda m \rho_s^{(0)}}{\hbar} \left(\mu_s' - \frac{\hbar^2}{2m^2 \eta^{(0)}} \frac{d^2 \eta'}{dz^2} \right). \tag{4.30}$$

The primes denote here small increments, and the superscript 0 labels the unperturbed quantities. We note that in all these equations the perturbations μ_s' and η' enter in the form of the combination, $\mu_s' - (\hbar^2/2m^2 \eta^{(0)}) \times d^2 \eta'/dz^2$. This makes it possible to eliminate, with the aid of (4.30), the variable η from the first four equations. Eliminating furthermore with the aid of (4.27) and (4.29) the derivatives of the thermodynamic quantities $d\mu'/dz$ and dT'/dz from the third equation, we obtain for $v_n(z)$ the equation

$$\frac{4}{3} \eta_n \frac{d^2 v_n}{dz^2} + \rho_n^{(0)} \frac{d}{dz} \left(\frac{\hbar}{2\Lambda m \rho_s^{(0)}} \frac{dp_n^{(0)} v_n}{dz} \right)$$
$$+ \rho^{(0)} \frac{d}{dz} \left(\frac{\hbar\Lambda}{2m\rho_s^{(0)}} \frac{dp^{(0)} v_n}{dz} \right) - \frac{S^{(0)}}{\varkappa} (S^{(0)} T^{(0)} v_n - q) = 0.$$

$$\tag{4.31}$$

Near the λ point we can put $\rho_n^{(0)} \simeq \rho^{(0)} \simeq \rho_\lambda$ and $S^{(0)} \simeq S_\lambda$. Further, we can neglect in (4.31) the term containing the first viscosity, since the coefficient η_n has no

anomaly at the λ point. Finally, it is convenient to change over from the velocity v_n to the thermal-conductivity component of the heat flux $q_T = q - S^{(0)} T^{(0)} v_n$. With all these simplifications, Eq. (4.31) takes the form

$$l^2 \frac{d}{dz} \left[\frac{\rho_{se}}{\rho_s^{(0)}(z)} \frac{dq_T}{dz} \right] - q_T(z) = 0, \tag{4.32}$$

where

$$l^2 = \frac{\kappa}{S_\lambda^2 T_\lambda} \frac{\hbar \rho_\lambda^2 (1 + \Lambda^2)}{2 \Lambda m \rho_{se}}. \tag{4.33}$$

In the case of temperatures relatively distant from the λ point, the change of the density $\rho_s(z)$ near the wall can be neglected, and it is then evident that it is precisely the length l which has the meaning of the characteristic distance over which the thermal-conductivity component of the heat flux attenuates and the convective counterflow is produced (see also[16,132]). The temperature T has in this case the following distribution near the wall:

$$T(z) = T(0) + \frac{ql}{\kappa} (1 - e^{-z/l}). \tag{4.34}$$

We see that heat exchange between helium II and a solid (or vapor) is accompanied by the onset of a temperature gradient localized in a layer (of thickness l) at the wall inside the liquid. The total temperature drop is $\Delta T \equiv T(z = \infty) - T(z = 0) = lq/\kappa$. The quantity $R_{\text{int}} = \Delta T/q = \kappa/l$ can be naturally called the internal boundary-layer thermal resistance.[33] In the critical region, even if the parameter Λ varies with the distance to the λ point like $\Lambda = \Lambda_0 t^{-1/3}$ (see Sec. 4.1), the length l increases like $l = l_0 t^{-1/2}$, i.e., in accord with a power law weaker than that of the correlation radius, and thus the change of $\rho_s^{(0)}(z)$ near the boundary becomes appreciable. Various forms of the $\rho_s^{(0)}(z)$ distribution near the wall are given in Sec. 3.1. We shall use the simplest of them, corresponding to retaining only the first two terms in the phenomenological expansion of the thermodynamic potential density

$$\rho_s^{(0)}(z) = \rho_{se} \,\text{th}^2 \left[\frac{z}{\sqrt{2} \, \xi_0(t)} \right].$$

For this profile, it is possible to solve (4.2) in explicit form. The solution is expressed in terms of a hypergeometric function

$$q_T(z) = q \frac{\Gamma(c-a)\Gamma(c-b)}{\Gamma(c)\Gamma(c-a-b)} \,\text{ch}^{-B} \left(\frac{z}{\sqrt{2}\,\xi_0} \right) {}_2F_1 \left(a, b; c; \text{ch}^{-2} \frac{z}{\sqrt{2}\,\xi_0} \right), \tag{4.35}$$

---

[33]R_{int} must be distinguished from the "acoustic" thermal resistance R_a, which is also present on the boundary and is connected with singularities of phonon exchange between the media (see[15], Chap. 12). The latter mechanism is not peculiar to superfluid liquids, but is the predominant one far from the λ transition. Consequently at low temperatures R_{int}, in the case of an He II–solid boundary, yields only a small correction to the total thermal resistance $R_K = R_a + R_{\text{int}}$. In the immediate vicinity of the λ point, to the contrary, and also in the case of heat exchange between the He II and the vapor, the main contribution may be that of R_{int}.

where

$$a = \frac{1}{2} \left(B - \frac{1}{2} + \sqrt{B^2 + \frac{1}{4}} \right), \qquad b = \frac{1}{2} \left(B - \frac{1}{2} - \sqrt{B^2 + \frac{1}{4}} \right), \qquad c = B + 1$$

and B is defined by the relation $B^2 = 2\xi_0^2(t)/l^2(t)$. Integrating (4.35), we obtain also a general expression for the thermal resistance

$$R_{\text{int}} \equiv \frac{\Delta T}{q} = \frac{\xi_0}{\sqrt{2}\,\kappa} \frac{\Gamma(c-a)\Gamma(c-b)\,\Gamma(B/2)\,\Gamma(1/2)}{\Gamma(c)\,\Gamma(c-b-a)\,\Gamma((B+1)/2)} {}_3F_2 \left(a, b, \frac{B}{2}; c, \frac{B+1}{2}; 1 \right). \tag{4.36}$$

We consider two limiting cases:

1) $B \ll 1$ (i.e., $\xi_0 \sqrt{2} \ll l$). In this limit ${}_3F_2(a, b, B/2; c, (B+1)/2; 1) \to 1$, $\Gamma(B/2) \to 2/B$, and all the remaining Γ functions cancel out. As a result we get $R_{\text{int}} = l/\kappa$, in agreement with (4.34) and[16,132].

2) $B \gg 1$. At large values of B, we have ${}_3F_2(a, b, B/2; c, (B+1)/2; 1) \to (9/4B)^{1/4}$, and the factor containing the Γ functions tends to the value $2\Gamma(5/4)B^{1/4}(2/B)^{1/2}$. As a result, in this limit, which corresponds to $T \to T_\lambda$, we obtain for R_{int} the asymptotic formula

$$R_{\text{int}} = \sqrt[4]{18} \, \Gamma \left(\frac{5}{4} \right) \frac{\sqrt{\xi_0(t)\, l(t)}}{\kappa} \approx 1.84 \frac{\sqrt{\xi_0(t)\, l(t)}}{\kappa}.$$

Thus, at $l \ll \xi_0$, the convective flux is effectively formed not over a length l, but over the geometric mean of the lengths l and ξ_0. The temperature dependence of R_{int} as $T \to T_\lambda$ is given by

$$R_{\text{int}} \propto \begin{cases} t^{-1/2}, & \Lambda = \text{const}, \\ t^{-7/12}, & \Lambda = \Lambda_0 t^{-1/3}. \end{cases}$$

We note that the character of the temperature dependence of R_{int} depends also on the form of the employed equations. In particular, when the system of equations of[123] is used we have

$$R_{\text{int}} \propto \begin{cases} t^{-1/2}, & \Lambda = \text{const}, \\ t^{-5/12}, & \Lambda = \Lambda_0 t^{-1/3}. \end{cases}$$

Substituting the numerical values, we find that R_{int} at $t \sim 0.1\,°K$ is smaller by 3–4 orders of magnitude than typical values of the "acoustic" thermal resistance on the boundary between He II and a solid (at $T \sim 2\,°K$). This makes it difficult to observe the anomalies of the thermal resistance in the case of heat exchange between the helium II and the solid. Nonetheless, when heat passes through the boundary between helium and the vapor, and possibly also in the case of "soft" crystals with low Debye temperatures, observation of the indicated anomaly is feasible in principle even at $t \sim 0.1\,°K$.

The presence of exponentially damped solutions of the type (4.34), (4.35) can manifest itself also in a number of other effects. In particular, they must be taken into account in the reflection, transmission, and transformation of second-sound waves at helium-II boundaries, in the calculation of the mobility of small particles (ions) in He II, and in other problems. It is also possible that the experimentally observed[133] abrupt decrease of the coefficient of second-sound reflection from a free helium II surface near the λ point is due precisely to this circumstance. This group of problems is of

great interest and it would be appropriate to consider them in the future.

One of the interesting applications of the complete system of time-dependent equations (4.12)–(4.15) is also the calculation, near the λ point, of the mutual-friction forces acting on vortices in rotating helium II.[122a] Unfortunately, we learned of[122a] only after completing this paper, and are unable therefore to discuss it here in any detail.

V. CONCLUDING REMARKS

The Ψ-theory developed above for the superfluidity of helium II is, first, a generalization of two-fluid hydrodynamics of a superfluid liquid to include the region of the λ point. Second, the Ψ-theory describes phenomenologically the λ transition itself in liquid helium, with definite boundary conditions taken into account (we have in mind in fact the condition $\rho_s = 0$ on the wall and presumably on a free surface). As a consequence, various dimensional and hydrodynamics effects are predicted and can be quantitatively considered. The Ψ-theory of superfluidity has much in common with the Ψ theory of superconductivity.[26,48,68,134,135] Related schemes have been developed and are discussed as applied to superfluid ^{3}He (see [63,136,137]), in the case of superfluidity of neutron and proton liquids in neutron stars,[13] for the description of an aggregate of excitons,[11] in laser theory,[138] as well as in the analysis of domain walls[139] and surface effects in ferromagnets.[103–107] Mention must be made of the known close relation between the Ψ-theory and investigations of the quasi-hydrodynamics of a non-ideal Bose gas[140,141] and of a "dense" superfluid liquid.[142] We shall not refer here to the numerous recent articles in which analogous concepts and formalisms are used in quantum field theory.

Like any phenomenological theory (apart from those that are derived more or less consistently from the microscopic theory), the Ψ-theory of superfluidity is based on a number of assumptions that are justified either by using the experimental data, or by striving to make the initial equations and the calculations with them as simple as possible. Accordingly, we specify beforehand the critical exponents, putting $\beta = 1/3$, $\nu = 2/3$, $\alpha = 0$ and $\hat{\eta} = 0$, i.e., we use the relations

$$\rho_s \propto t^{2/3}, \ \xi \propto |t|^{-2/3}, \ C_p \propto \ln |t|, \ G_n(r \to 0) \propto r^{-1}$$

(see in particular (2.9)–(2.10), (2.19), (2.39)). All the other assumptions (i.e., discarding terms of order higher than $|\Psi|^6$ in the expansion of the thermodynamic potential) likewise seem quite natural, at least at the present stage, and are apparently not connected with any essential restrictions or conditions that limit the accuracy or even the very possibility of comparing the theory with experiment. Incidentally, it would be useful and desirable to investigate in all the actual problems (see Chaps. III and IV) the possible influence of various conceivable modifications of the theory (change of the critical exponents, allowance for terms of order $|\Psi|^8$, etc.). It is clear, however, that at present the most justified procedure is to employ just the simplest scheme that does not contradict the experimental data, and

such is precisely the Ψ-theory of superfluidity of helium II near the λ point, which is discussed in the present article.

We emphasize that this theory, as well as its possible extensions (as applied to helium II under pressures and to ^{3}He-^{4}He solutions) and generalizations (among which the most interesting is the analysis of the superfluidity of ^{3}He), is much farther reaching than the existing theory of second-order phase transitions, in which one establishes for a homogeneous system only the limiting relations (as $t = (T_\lambda - T) \to 0$) and connections between the critical exponents. Indeed, the general system of equations given in Sec. 4.1 is complete and makes it possible, in principle, to find the dependence of all the quantities (ρ_s, v_s, v_n, as well as ρ and S or any other pair of variables) on the coordinates and on the time. By the same token, the predictions of the theory are also rich in content and can be verified in experiment. It is precisely the last problem which is at present, of course, the central one, and the main purpose of the present article is to help in its solution. Those concrete experiments that should be performed are clear in part from the statements made in Chap. III and Sec. 4.3. Here we wish to emphasize particularly the need for working sufficiently close to the λ point. Thus, for example, the correlation length is $\xi_0 = (2.73 \times 10^{-8}) t^{-2/3}$ ~3×10^{-7} cm ~$10a$ at $t \sim (1/30)$ °K. Thus, the main condition of the applicability of the theory, $\xi \gg a \sim 3 \times 10^{-8}$ cm (see (2.21)), is satisfied even formally only at $t \lesssim 1/10$ to $1/30$ °K. Actually, however, it is desirable to have a larger "margin" and to carry out the measurements at $t < 10^{-2}$ °K.

On the other hand, according to (3.4b) with $M = 0$, for a gap of width, say, 10^{-4} cm, which can be reliably regulated, the shift of the λ-point temperature is ΔT_λ ~3×10^{-5} °K, i.e., very small. Briefly speaking, in order for both the theory to be easily applicable (if, of course, it is valid at all!) and for the employed gaps, channels, etc. to be of macroscopic size, it is actually necessary to concentrate on the immediate vicinity of the λ point. It is just the failure to satisfy this condition which has led, in the main, to the uncertainty that must always be mentioned when theory is compared with experiment. What are principally lacking are measurements made close enough to the λ point and at the same time under strictly controllable conditions (we have in mind an accurate determination of the gap or film thickness, etc.). Only after the appropriate data are obtained will it be possible to ascertain to what degree and with which accuracy the discussed Ψ-theory of superfluidity corresponds to reality. If it is reliably established that theory and experiment do not agree then, depending on the character of the discrepancies, it will be possible to consider concretely ways of extending the theory. We assume, however, that the Ψ-theory of superfluidity described here will retain a certain value even in this case. Incidentally, it is quite possible that the accuracy of this theory is high enough, and requires no significant extension whatever.

SYMBOLS AND SOME NUMERICAL PARAMETERS

1. Atomic and other constants

$m \equiv m_4 = 6.6464 \times 10^{-24}$ g—mass of He^4 atom,

$M_4 = 4.0026$ g/atom—atomic weight of He^4,

$2\pi\hbar/m = h/m = 0.997 \times 10^{-3}$ cm/sec—circulation quantum in superfluid He II,

$\alpha_0 = 3.1 \times 10^{-2}$ cm^3/g—dielectric susceptibility of He^4,

$\alpha_H = -0.47 \times 10^{-6}$ cm^3/g—diamagnetic susceptibility of He^4,

$g = 980.665$ cm/sec^2—acceleration due to gravity.

2. Parameters of λ point of pure He^4 at saturated-vapor pressure

$T_\lambda = 2.172\,^\circ$K—temperature of λ point ($^\circ$K $\equiv$ g deg),

$p_\lambda = 0.05$ atm—pressure

$\rho_\lambda = 0.14617$ g/cm^3—density $\Big\}$ at λ point,

$S_\lambda = 1.56 \cdot 10^7$ erg-g$^{-1}\,^\circ$K^{-1} ($^\circ$K$^{-1} \equiv$ g deg^{-1})—entropy

$dp_\lambda/dT = -111.05$ atm $^\circ$K^{-1}

$d\rho_\lambda/dT = -0.2425$ g-cm$^{-3}\,^\circ$K^{-1}

$dS_\lambda/dT = 2.558 \times 10^7$ erg-g$^{-1}\,^\circ$K^{-1} $\Big\}$ slope of λ curve in terms of different variables,

$d\mu_\lambda/dT = -78.54 \cdot 10^7$ erg-g$^{-1}\,^\circ$K^{-1}

$\Delta C_p = 5.2 \cdot 10^7$ erg-g$^{-1}\,^\circ$K^{-1} = 0.76 $\cdot$ 10^7 erg-cm$^{-3}\,^\circ$K^{-1}—"jump" of heat capacity at λ point,

$C_0 = 1.30 \cdot 10^7$ erg-g$^{-1}\,^\circ$K^{-1} = 0.190 $\cdot$ 10^7 erg-cm$^{-3}\,^\circ$K^{-1}—coefficient of the logarithmic part of the heat capacity $(C_p = \text{const} - C_0 \ln |T - T_\lambda|)$,

$\rho_{se} = \rho_{00}(T_\lambda - T)^{2/3}$, $\rho_{00} = 1.43\rho_\lambda = 0.209$ g-cm$^{-3}\,^\circ$K$^{-2/3}$—temperature dependence of the density of the superfluid part in "bulky" He II near the λ point.

3. Some characteristic lengths

$\lambda_T = \sqrt{2\pi\hbar^2/mk_B T_\lambda} = 5.92 \cdot 10^{-8}$ cm—thermal wavelength of He^4 atom at $T = T_\lambda$,

$a = (m/\rho_\lambda)^{1/3} = 3.57 \cdot 10^{-8}$ cm—average interatomic distance in liquid helium at $T = T_\lambda$,

$\xi_M \equiv \xi_M^+ = \xi_0(3 + M/3)^{1/2} = 2.74 \cdot 10^{-8}(3 + M/3)^{1/2} |T_\lambda - T|^{-2/3}$ cm—temperature-dependent correlation length for region above the λ point (see (2.17) and (2.19)),

$\xi_M^- = \xi_0 \sqrt{3 + M/6(1 + M)}$—the same length for the region below the λ point (see (2.47)),

$\xi_0 = \xi_{00}(T_\lambda - T)^{-2/3}$, $\xi_{00} = (\hbar^2 \rho_{00} T_\lambda/2m^2 \Delta C_p)^{1/2} = 2.74 \cdot 10^{-8}$ cm $^\circ$K$^{2/3}$—correlation length in the theory variant with $M = 0$ (see (2.17) and (2.19)),

$\xi_\varphi = k_B T_\lambda m^2/4\pi\hbar^2 \rho_{se} = (a^3 \rho_\lambda/2\lambda_T^2 \rho_{00})(T_\lambda - T)^{-2/3} = 0.453 \cdot 10^{-8}(T_\lambda - T)^{-2/3}$ cm—correlation length for phase fluctuations (see (2.41a)),

$l_g = \xi_{00}^{3/5}(1/g|d\mu_\lambda/dT|)^{2/5} = 6.67 \times 10^{-3}$ cm—characteristic width of He I–He II phase separation boundary in a gravitational field,

λ—length parameter in the boundary condition (2.26).

4. Most important among the remaining symbols

$\Psi = \eta e^{i\varphi}$—complex order parameter ("effective wave function"),

$\rho_s = m|\Psi|^2 = m\eta^2$—density of superfluid part of helium II,

$\mathbf{v}_s = (\hbar/m)\nabla\varphi$—velocity of superfluid part of helium II,

$\mathbf{j}_s = \rho_s \mathbf{v}_s = \hbar\eta^2 \nabla\varphi = -i\hbar/2(\Psi^* \nabla\Psi - \Psi\nabla\Psi^*)$—flux of superfluid part of liquid,

$\rho = \rho_s + \rho_n$—total density of helium,

ρ_n—density of normal part of the liquid,

$\mathbf{v}_n$—velocity of the normal part of the liquid,

p—pressure,

S—entropy per unit volume,

μ—chemical potential (usual in erg/g),

x_3, μ_3—concentration and chemical potential He^3 (in the case of He^3–He^4 solutions),

C—heat capacity, C_p—heat capacity at constant pressure, C_μ—heat capacity at constant μ,

$\Phi(p, T, \Psi)$—density of thermodynamic potential,

$\Omega(\mu, T, \Psi)$—density of thermodynamic potential for variable μ of the chemical potential,

$F(\rho, T, \Psi)$—free-energy density,

$E(\rho, S, \Psi)$—internal-energy density,

$\Phi_{II0}(p, T, |\Psi|^2)$—density of thermodynamic potential for homogeneous helium II at rest,

$\Phi_I(p, T)$—the same for helium I,

$\mu_s = (\partial \Phi_{II0}/\partial \rho_s)_{p,T}$—chemical potential of superfluid part of liquid (see (2.16a)),

$\Phi_{II}(\rho_{se})$—equilibrium value of the density of the thermodynamic potential Φ_{II0},

$\Phi_{I-II}(\rho_{se}) \equiv \Phi_I - \Phi_{II}(\rho_{se}) = (\Delta C_p/2T_\lambda)t^2$—difference between the equilibrium values of the densities of the thermodynamic potential for HeI and $HeII$ (after subtracting the regular part),

$\Omega_{I-II}(\rho_{se}) \equiv \Omega_I - \Omega_{II}(\rho_{se}) = (\Delta C_\mu/2T_\lambda)t^2 \approx (\Delta C_p/2T_\lambda)t^2$—the same for the variables μ and T,

$\tilde{\Phi}_{II} = \int \Phi_{II}\, dV$—thermodynamic potential of helium II,

$M = C_0/A_0 \eta_{00}^2$—parameter of the theory (see (2.13), (2.14), (2.18), (2.20)),

$\nu_0^2 = 2M/M + 3$—different form of the same parameter,

$\Psi_e = \eta_e = \sqrt{\rho_{se}/m} = \Psi_{00}\, t^{1/3}$, where $\Psi_{00} = \sqrt{\rho_{00}/m}$, —equilibrium value of the Ψ function in homogeneous helium at rest,

$\psi = \Psi/\Psi_{00}$—reduced Ψ function,

$\psi_e = t^{1/3}$—its equilibrium value (at constant Ψ) in helium II,

$\psi_0 = \Psi/\Psi_e$—reduced Ψ function used in the solution of boundary-value problem $\psi_{0e} = 1$ (at equilibrium for homogeneous helium II); a reduced function is used also in Sec. 2 of Chap. III (see (3.47) and (3.48)),

$\mathbf{r}_* = \mathbf{r}/\xi_M$, $r_* = r/\xi_M$—dimensionless radius vector and its modulus,

$\mathbf{r}_{**} = \mathbf{r}/\xi_{00}$—relative coordinates; z, z_*, z_{**} —distances in one-dimensional problems,

$y = z/\xi_0(t)$—distances measured in units of the length ξ_0,

$\mathbf{q}$—wave vector of fluctuations,

$q_{max} \equiv q_c = Q/\xi_M$—cutoff wave vector, Q is a parameter of the theory,

$\alpha, \beta, \gamma, \delta, \zeta, \hat{\eta}, \nu$—critical exponents (see, e.g., [17,19]),

d—thickness of film or gap,

$T_\lambda(d)$—λ-transition temperature of helium in film, gap, or capillary,

$T_{tr}(d)$—the same, but for first-order transition,

$\Delta T_\lambda = T_\lambda - T_\lambda(d)$—shift of λ-transition temperature of helium in a film,

$d_c = \pi \xi_M(t)$—minimum film thickness for which superfluidity is still possible,

d_{tr}—film thickness at the point of the first-order equilibrium phase transition,

$D = d/\xi_0(t)$—dimensionless film thickness,

Δ—surface "deficit" of superfluid mass (see (3.12)),

σ—"excess" surface energy of helium II (see (3.13)),

$S_\sigma, C_\sigma, m_\sigma, m_{3\sigma}$—derivatives of this energy; surface entropy and heat capacity, excess surface mass, excess He^3 concentration,

$\bar{\rho}_s \equiv \bar{\rho}_s(d)$—averaged (over the cross section) density of the superfluid part in a film, gap, or capillary,

$\Phi_{\text{I-II}} \equiv \bar{\Phi}_{\text{I-II}}(d)$, $\bar{S}_{\text{I-II}} \equiv \bar{S}_{\text{I-II}}(d)$, $\bar{C}_{\text{II-I}} \equiv \bar{C}_{\text{II-I}}(d)$—the same for the differences between the densities of the thermodynamic functions of He I and He II (after subtracting the regular part),

$E = \bar{\Omega}_{\text{I-II}}/\Omega_{\text{I-II}}(\rho_{se})$,

$\mathscr{F} = \bar{\Phi}_{\text{I-II}}\, d / \Phi_{\text{I-II}}(\rho_{se})\xi_0(t)$,

$\bar{M}_s = \rho_s \cdot d/\rho_{se}\xi_0(t)$,

V_G—potential of forces of the field G acting on a unit mass of helium,

$v_{sc2} = \hbar/m\xi_M = 5.8 \cdot 10^3(3/3+M)^{1/2}t^{2/3}$ cm/sec—critical velocity of superfluid flow, corresponding to the vanishing of ρ_s,

$v_{sc1} = v_{sc2}/\sqrt{3}$ (at $M = 0$)—velocity at which the superfluid flow becomes unstable (see (3.74), (3.84)),

Λ—dimensionless parameter of the theory, used in the time-dependent equations,

$\tau = [(2\Lambda m/\hbar)\rho_s(\partial\mu_s/\partial\rho_s)]^{-1}$—relaxation time of the parameter Ψ,

η_n—first viscosity,

$\zeta_i (i = 1, 2, 3, 4)$—second viscosities,

$\varkappa$—thermal conductivity,

$u_2 = \sqrt{T(S^2\rho_s/\rho\rho_n C_p)}$—velocity of second sound.

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