# Remarks on forces and the energy-momentum tensor in macroscopic electrodynamics

V. L. Ginzburg and V. A. Ugarov

*P.N. Lebedev Physics Institute, USSR Academy of Sciences and Moscow Pedagogical Institute*
Ups. Fiz. Nauk. **118**, 175–188 (January 1976)

The conservation laws which follow from the field equations and their relation to the energy and momentum conservation laws are discussed. On the basis of the electrodynamics of slowly moving bodies, an expression is derived for the density of the force which acts on an isotropic inhomogeneous medium in an electromagnetic field. Attention is concentrated on elucidating the difference between the energy-momentum tensors of Minkowski and Abraham. It is emphasized either of these can be used in practice to consider the exchange of energy and momentum between an emitter and a medium in which the emitter is placed. However, to analyze the processes in the medium itself, Abraham's should be used because it takes into account Abraham's volume force, which acts even on a homogeneous medium (whereas according to Minkowski no force acts at all on a transparent, homogeneous medium with density-independent permittivity in an electromagnetic field).

PACS numbers: 03.50.Cd

## METHODOLOGICAL NOTES

The problem of the energy–momentum tensor in macroscopic electrodynamics has been discussed comparatively recently in this journal.[1,2] Nevertheless, for a number of reasons we believe it is appropriate to return to this problem. First, Abraham's force has finally been measured experimentally[3] almost seventy years after the theoretical expression for it was obtained.[4–6] Second, in recent years (in particular, after the publication of [1,2]) much material relating to these questions has appeared (apart from the literature cited in[1,2] see, for example, [7–22]). Third, we wish, besides taking into account this material, to make some additional remarks in connection with [2], which may help to remove some misunderstandings. In view of the methodological nature of the note, we cannot strive for too great brevity, and we shall therefore give once more expressions that occurred earlier in[2].

1. In macroscopic electrodynamics (and, for that matter, in electrodynamics generally) the energy–momentum tensor is in a certain sense an auxiliary quantity. The fundamental quantities are the volume forces (or, as they were called earlier, the ponderomotive forces), and also the energy density and energy flux. It is the forces that occur in the equations of motion for a medium or individual charges and can, in principle, be measured. We shall consider a macroscopic medium characterized by real permittivity $\varepsilon$ and real magnetic permeability $\mu$ (in fact, some of the expressions will correspond to a more general case). We shall ignore both frequency and spatial dispersion. Nor shall we take into account anisotropy. Of course, this approximation has a very restricted field of application (essentially, to low-frequency fields), but it happens to be this situation in which we are interested. Besides the Lorentz force with density

$$\mathbf{f}^L=\rho_e\mathbf{E}+\frac{1}{c}\mathbf{j}\times\mathbf{B} \qquad (1)$$

an isotropic medium at rest (a fluid) in a static field is subject to a further force whose density $\mathbf{f}_m$ is equal to the sum of the densities of two forces:

$$\mathbf{f}_m=\mathbf{f}_{m1}+\mathbf{f}_{m2},\qquad \mathbf{f}_{m1}=-\frac{E^2}{8\pi}\nabla\varepsilon-\frac{H^2}{8\pi}\nabla\mu,$$

$$\mathbf{f}_{m2}\equiv\mathbf{f}_{\text{str}}=\nabla\left(\left(\frac{\partial\varepsilon}{\partial\rho}\rho\right)\frac{E^2}{8\pi}\right)+\nabla\left(\left(\frac{\partial\mu}{\partial\rho}\rho\right)\frac{H^2}{8\pi}\right). \qquad (2)$$

In Eqs. (1) and (2) and below, $\rho_e$ is the charge density, $\mathbf{j}$ is the current density, $\rho$ is the density of the medium, $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic field intensities and $\mathbf{D}$ and $\mathbf{B}$ are the electric displacement and magnetic induction. In Eqs. (2) it is assumed that

$$\mathbf{D}=\varepsilon\mathbf{E},\quad \mathbf{B}=\mu\mathbf{H}, \qquad (3)$$

and $\varepsilon$ and $\mu$ are assumed to be functions of the point $\mathbf{r}$ and the density $\rho$ of the medium.

In a fairly slowly varying electromagnetic field the total expression for the force density is obtained by adding to (2) the density of "Abraham's force"

$$\mathbf{f}_{m3}\equiv\mathbf{f}^A=\frac{\varepsilon\mu-1}{4\pi c}\frac{\partial}{\partial t}\mathbf{E}\times\mathbf{H} \qquad (4)$$

and, thus, the density $\mathbf{f}_m^{(t)}$ of the total force is[1)]

$$\mathbf{f}_m^{(t)}=\mathbf{f}_{m1}+\mathbf{f}_{m2}+\mathbf{f}^A. \qquad (5)$$

Both the static force (2) and the total force (5) are obtained usually in a somewhat inconsistent manner (here, and sometimes below we shall merely refer to the force and not the force density for brevity).

Thus, one first considers the variation of the energy

[1)]We omit here the force density $-\nabla p$, where $p$ is the pressure in the medium.

under displacements of the medium, which enables one to find $\mathbf{f}_m$ (see Secs. 15 and 34 of [23] and Secs. 32 and 66 of [24]); the force $\mathbf{f}^A$ is then added to $\mathbf{f}_m$ on the basis of the field equations and general arguments (Sec. 56 of[23]). However, it would be desirable to obtain the forces and other expressions (energy density, energy flux, momentum density) in a unified manner on the basis of the field equations. Let us consider such a derivation.

2. We write the field equations in the usual form

$$\operatorname{curl}\mathbf{H}=\frac{4\pi}{c}\mathbf{j}+\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t},\qquad(6)$$

$$\operatorname{curl}\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t},\qquad(7)$$

$$\operatorname{div}\mathbf{D}=4\pi\rho_e,\qquad(8)$$

$$\operatorname{div}\mathbf{B}=0.\qquad(9)$$

These equations can also be regarded as valid for moving bodies (media) by dispensing with constraints of the type (3). For our purposes, it is sufficient to consider slowly moving bodies (neglecting terms of the order $u^2/c^2$, where $\mathbf{u}$ is the velocity of a body or the medium with respect to the "laboratory" frame of reference that is used). This case is considered in detail in the course[24] (see Ch. VIII). In a discussion of the energy conservation law it is natural to consider moving media since the force $\mathbf{f}_m$ acting on a medium does "work" only if the velocity $\mathbf{u}$ of the medium is not zero. In addition, the striction force $\mathbf{f}_{m2}$ depends on $\partial\varepsilon/\partial\rho$ and is associated with the possibility of changing the density of the medium, which is controlled by the continuity equation $\partial\rho/\partial t+\operatorname{div}\rho\mathbf{u}=0$, which contains the velocity $\mathbf{u}$.

In a slowly moving medium (see Sec. 111 of [24])

$$\left.\begin{aligned}\mathbf{D}&=\varepsilon\mathbf{E}+\left(\varepsilon-\frac{1}{\mu}\right)\frac{\mathbf{u}}{c}\times\mathbf{B},\\ \mathbf{E}&=\frac{1}{\varepsilon}\mathbf{D}-\left(1-\frac{1}{\varepsilon\mu}\right)\frac{\mathbf{u}}{c}\times\mathbf{B},\end{aligned}\right\}\qquad(10)$$

$$\mathbf{H}=\frac{1}{\mu}\mathbf{B}+\left(\varepsilon-\frac{1}{\mu}\right)\frac{\mathbf{u}}{c}\times\mathbf{E}=\frac{1}{\mu}\mathbf{B}+\left(1-\frac{1}{\varepsilon\mu}\right)\frac{\mathbf{u}}{c}\times\mathbf{D},\qquad(11)$$

where it is assumed that in a medium at rest the constraints (3) hold; in addition, the velocity $\mathbf{u}$ of the medium is constant by assumption in space and time or, rather, the derivatives with respect to $\mathbf{r}$ and $t$ can be ignored everywhere (only the divergence $\operatorname{div}\mathbf{u}$ in the continuity equation will be retained).

Multiplying Eqs. (6) and (7) scalarly by $\mathbf{E}$ and $\mathbf{H}$, respectively, and then subtracting the expressions and using the identity $\mathbf{E}\operatorname{curl}\mathbf{H}-\mathbf{H}\operatorname{curl}\mathbf{E}=-\operatorname{div}(\mathbf{E}\times\mathbf{H})$, we arrive at Poynting's theorem[2)]:

$$\frac{1}{4\pi}\left(\frac{\partial\mathbf{D}}{\partial t}\cdot\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{H}\right)=-\mathbf{j}\cdot\mathbf{E}-\operatorname{div}\mathbf{S};\quad \mathbf{S}=\frac{c}{4\pi}\mathbf{E}\times\mathbf{H}.\qquad(12)$$

We now calculate the derivative

$$\frac{\partial w^M}{\partial t}=\frac{1}{8\pi}\frac{\partial}{\partial t}(\mathbf{D}\cdot\mathbf{E}+\mathbf{B}\cdot\mathbf{H}),$$

using Eqs. (10) and (11). To calculate $\partial\varepsilon/\partial t$ and $\partial\mu/\partial t$ we use the relation

$$\frac{d\varepsilon}{dt}\equiv\frac{\partial\varepsilon}{\partial t}+\mathbf{u}\nabla\varepsilon=\frac{\partial\varepsilon}{\partial\rho}\frac{d\rho}{dt}=-\frac{\partial\varepsilon}{\partial\rho}\rho\operatorname{div}\mathbf{u}$$

and the analogous one for $d\mu/dt$, where we have used the continuity equation $\partial\rho/\partial t+\operatorname{div}\rho\mathbf{u}=0$, by virtue of which

$$\frac{d\rho}{dt}\equiv\frac{\partial\rho}{\partial t}+\mathbf{u}\nabla\rho=-\rho\operatorname{div}\mathbf{u}.$$

In addition, it is obviously assumed that the variation of $\varepsilon$ (and also of $\mu$) for a given element of the medium is due solely to the change in its density $\rho$. Then

$$\begin{aligned}\frac{\partial\varepsilon}{\partial t}&=-\mathbf{u}\nabla\varepsilon-\frac{\partial\varepsilon}{\partial\rho}\rho\operatorname{div}\mathbf{u},\\ \frac{\partial\mu}{\partial t}&=-\mathbf{u}\nabla\mu-\frac{\partial\mu}{\partial\rho}\rho\operatorname{div}\mathbf{u},\end{aligned}\qquad(13)$$

and by virtue of the constraints (10) and (11),

$$\begin{aligned}\frac{\partial\mathbf{E}}{\partial t}&=\frac{1}{\varepsilon}\frac{\partial\mathbf{D}}{\partial t}-\left(1-\frac{1}{\varepsilon\mu}\right)\frac{\mathbf{u}}{c}\times\frac{\partial\mathbf{B}}{\partial t}-\mathbf{D}\left\{\mathbf{u}\nabla\left(\frac{1}{\varepsilon}\right)-\frac{1}{\varepsilon^2}\left(\frac{\partial\varepsilon}{\partial\rho}\right)\rho\operatorname{div}\mathbf{u}\right\},\\ \frac{\partial\mathbf{H}}{\partial t}&=\frac{1}{\mu}\frac{\partial\mathbf{B}}{\partial t}+\left(1-\frac{1}{\varepsilon\mu}\right)\frac{\mathbf{u}}{c}\times\frac{\partial\mathbf{D}}{\partial t}-\mathbf{B}\left\{\mathbf{u}\nabla\left(\frac{1}{\mu}\right)-\frac{1}{\mu^2}\left(\frac{\partial\mu}{\partial\rho}\right)\rho\operatorname{div}\mathbf{u}\right\}.\end{aligned}\qquad(14)$$

Using (14) and taking into account once more (10) and (11), and ignoring, as before terms of order $u^2/c^2$, we obtain

$$\frac{\partial w^M}{\partial t}=\frac{1}{4\pi}\left(\frac{\partial\mathbf{D}}{\partial t}\cdot\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{H}\right)+\frac{1}{8\pi}E^2(\mathbf{u}\nabla\varepsilon)+\frac{E^2}{8\pi}\left(\frac{\partial\varepsilon}{\partial\rho}\right)\rho\operatorname{div}\mathbf{u}+\frac{1}{8\pi}H^2(\mathbf{u}\nabla\mu)+\frac{H^2}{8\pi}\left(\frac{\partial\mu}{\partial\rho}\right)\rho\operatorname{div}\mathbf{u}.\qquad(15)$$

Finally, combining (15) and (12), we obtain

$$-\frac{\partial}{\partial t}\left(\frac{\mathbf{D}\cdot\mathbf{E}+\mathbf{B}\cdot\mathbf{H}}{8\pi}\right)=-\frac{\partial w^M}{\partial t}=\mathbf{j}\cdot\mathbf{E}+\mathbf{f}_m\mathbf{u}+\operatorname{div}\left\{\mathbf{S}-\mathbf{u}\frac{1}{8\pi}\left(\frac{\partial\varepsilon}{\partial\rho}\rho\right)E^2-\mathbf{u}\frac{1}{8\pi}\left(\frac{\partial\mu}{\partial\rho}\rho\right)H^2\right\},\qquad(16)$$

where $\mathbf{f}_m$ is the expression (2) and $\mathbf{f}_m\cdot\mathbf{u}$ is the work of the force on the medium.[3)] Obviously, Eq. (16) has the meaning of the energy conservation law, in which $w^M$ can be regarded as the energy density, and the total energy flux is

$$\mathbf{S}^t=\mathbf{S}-\frac{\mathbf{u}}{8\pi}\left\{\left(\frac{\partial\varepsilon}{\partial\rho}\right)\rho E^2+\left(\frac{\partial\mu}{\partial\rho}\right)\rho H^2\right\}.\qquad(17)$$

By itself, the appearance of the additional energy flux proportional to the velocity $\mathbf{u}$ of the medium need not provoke surprise. However, we are not yet sure that allowance for the correction to $\mathbf{S}$ proportional to $\mathbf{u}$ does not go beyond the accuracy of the calculation made here. However, this remark does not apply to the term $\mathbf{f}_m\cdot\mathbf{u}$ in which we are here interested. The expression for this term determines the force $\mathbf{f}_m$. Essentially, our derivation of the expression (2) for the force $\mathbf{f}_m$ is equivalent to the usually employed static calculations. But the derivation is suitable not only for the static case, and is therefore more general. We also feel that it is more consistent.

But why does not Abraham's force $\mathbf{f}^A$ appear in (16)? If this force really exists, it should do work on an equal

2)The relation obtained from Eq. (12) by integrating over the volume is also sometimes called Poynting's theorem.

3)The derivative here differs from that in Sec. 115 of the third edition (1946) of Tamm's book[24] only by allowance for the terms with $\partial\varepsilon/\partial\rho$ and $\partial\mu/\partial\rho$ (Tamm assumed that $\partial\varepsilon/\partial\rho=0$ and $\partial\mu/\partial\rho=0$; note that Secs. 115 and 116 of the 1946 edition were removed by Tamm from later publications in connection with his dissatisfaction of the treatment of the Minkowski and Abraham energy–momentum tensors).

footing with the force $\mathbf{f}_m$. It would therefore appear that (16) should contain the total force

$$\mathbf{f}_m^{(t)} = \mathbf{f}_m + \mathbf{f}^A$$

The answer to this perfectly natural question is that the conservation law (16), like other similar relations, by no means uniquely determines all quantities. For suppose that the medium is also subject to Abraham's force (4), which we write in the more general form

$$\mathbf{f}^A = \frac{1}{4\pi c}\frac{\partial}{\partial t}\{(\mathbf{D}\times\mathbf{B}) - (\mathbf{E}\times\mathbf{H})\}. \qquad (18)$$

Then nothing prevents us writing down the relation (16)—the energy conservation law—in which we replace $\mathbf{f}_m$ by the total force $\mathbf{f}_m^{(t)} = \mathbf{f}_m + \mathbf{f}^A$ and, simultaneously, the energy density $w^M$ by the density

$$w^A = \frac{\mathbf{D}\cdot\mathbf{E}+\mathbf{B}\cdot\mathbf{H}}{8\pi} - \frac{\mathbf{u}}{4\pi c}\{(\mathbf{D}\times\mathbf{B}) - (\mathbf{E}\times\mathbf{H})\}, \quad \frac{\partial w^A}{\partial t} = \frac{\partial w^M}{\partial t} - \mathbf{f}^A\mathbf{u}. \qquad (19)$$

Thus, if we take Minkowski's expression $w^M$ for the energy density, the force $\mathbf{f}^A$ should not be taken into account—one must assume that it is absent. Conversely, the assumption that the force $\mathbf{f}^A$ does exist entails a choice of Abraham's expression $w^A$ in (19) for the energy density in a slowly moving medium.

What we have said also applies to the momentum conservation law and, naturally, the energy–momentum conservation law. None of these conservation laws can uniquely determine the quantities they contain; concretely, they cannot determine the energy–momentum tensor $T_{ik}$. Even if the divergence of this tensor is zero, one can add to it an expression with vanishing divergence. But if, as in our case, we consider the energy density, the stress tensor, and the momentum density of the electromagnetic field in the medium, the divergence of the energy–momentum tensor $\partial T_{ik}/\partial x_k$ is not zero at all—all that must vanish is the divergence of the total energy–momentum tensor $T_{ik}^{(t)} = T_{ik} + T_{ik}^{(m)}$, where $T_{ik}^{(m)}$ is the energy–momentum tensor of the medium. It is clearly impossible to decompose $T_{ik}^{(t)}$ into $T_{ik}$ and $T_{ik}^{(m)}$ solely on the basis of the conservation law $\partial T_{ik}^{(t)}/\partial x_k = 0$.

Before we make some more remarks about this point—they refer to the choice of $T_{ik}$ in the Minkowski or Abraham form—let us consider the derivation of the conservation law for the momentum of the electromagnetic field on the basis of the field equations. More precisely, we consider the conservation law that acquires the meaning of the momentum conservation law after the meaning of the quantities contained in it has been settled.

3. To this end, we multiply Eq. (6) vectorially by $\mathbf{B}$, Eq. (7) by $\mathbf{D}$, and add the resulting expressions. We immediately obtain

$$\frac{1}{4\pi}\{(\mathbf{D}\times\operatorname{curl}\mathbf{E}) + (\mathbf{B}\times\operatorname{curl}\mathbf{H})\} = -\frac{1}{c}\,\mathbf{j}\times\mathbf{B} - \frac{1}{4\pi c}\frac{\partial}{\partial t}\,\mathbf{D}\times\mathbf{B}. \qquad (20)$$

This is the only relation connected with the momentum conservation law that can be obtained from the field equations. Because of the uncertainty discussed above in the derivation of the expression for the force on the basis of the conservation (12) alone, the relation (20) can be given the form of the conservation law

$$\frac{\partial\sigma_{\alpha\beta}}{\partial x_\beta} - \frac{\partial g_\alpha}{\partial t} = f_\alpha \qquad (\alpha = 1, 2, 3) \qquad (21)$$

with the individual terms particularized only if one invokes additional arguments relating to the form of the total force $\mathbf{f}$, the stress tensor $\sigma_{\alpha\beta}$, and the momentum density $\mathbf{g}$ of the field. We certainly have $\mathbf{f} = \mathbf{f}^L + \mathbf{f}_m'^{(t)}$, where the density $\mathbf{f}^L$ of the Lorentz force is known [see Eq. (1)], but $\mathbf{f}_m'^{(t)}$ must still be determined. We therefore add to the right- and left-hand sides of Eq. (20) the identical terms $-\rho_e\mathbf{E} = -\mathbf{E}\operatorname{div}\mathbf{D}/4\pi$ [see Eq. (8)]. Then Eq. (20) can be written in the form

$$\frac{1}{4\pi}\{(\mathbf{D}\times\operatorname{curl}\mathbf{E}) + (\mathbf{B}\times\operatorname{curl}\mathbf{H}) - \mathbf{E}\operatorname{div}\mathbf{D}\} = -\mathbf{f}^L - \frac{1}{4\pi c}\frac{\partial}{\partial t}\mathbf{D}\times\mathbf{B}. \qquad (20a)$$

We now introduce the Minkowski stress tensor

$$\sigma_{\alpha\beta}^{(1)} = \frac{1}{4\pi}\{E_\alpha D_\beta + H_\alpha B_\beta\} - \delta_{\alpha\beta}\frac{\mathbf{D}\cdot\mathbf{E}+\mathbf{B}\cdot\mathbf{H}}{8\pi}. \qquad (22)$$

As a result of differentiation and simple transformations (given in detail in Sec. 105 of the book [24]), we obtain

$$\frac{\partial\sigma_{\alpha\beta}^{(1)}}{\partial x_\beta} = -\frac{1}{4\pi}\{(\mathbf{D}\times\operatorname{curl}\mathbf{E}) + (\mathbf{B}\times\operatorname{curl}\mathbf{H}) - \mathbf{E}\operatorname{div}\mathbf{D}\}_\alpha + \frac{1}{8\pi}\left\{\left(\frac{\partial E_\beta}{\partial x_\alpha}D_\beta - E_\beta\frac{\partial D_\beta}{\partial x_\alpha}\right) + \left(\frac{\partial H_\beta}{\partial x_\alpha}B_\beta - H_\beta\frac{\partial B_\beta}{\partial x_\alpha}\right)\right\}. \qquad (23)$$

Combining (23) and (20a), we can write the momentum conservation law in the form (21):

$$\frac{\partial\sigma_{\alpha\beta}^{(1)}}{\partial x_\beta} = f_\alpha^L + \frac{1}{4\pi c}\frac{\partial}{\partial t}\mathbf{D}\times\mathbf{B}_\alpha - \frac{E^2}{8\pi}\frac{\partial\varepsilon}{\partial x_\alpha} - \frac{H^2}{8\pi}\frac{\partial\mu}{\partial x_\alpha}, \qquad (24)$$

where the second expression in the curly brackets in (23) has been transformed with allowance for the constraints (3), which are valid for a medium at rest.

If we cast the expression for the force into the form (5), Eq. (24) must be modified further. We must introduce the stress tensor of the striction forces:

$$\sigma_{\alpha\beta}^{(2)} = \left\{\frac{1}{8\pi}\left(\frac{\partial\varepsilon}{\partial\rho}\rho\right)E^2 + \frac{1}{8\pi}\left(\frac{\partial\mu}{\partial\rho}\rho\right)H^2\right\}\delta_{\alpha\beta} \qquad (25)$$

and the momentum density of the electromagnetic field:

$$\mathbf{g}^A = \frac{1}{4\pi c}\,\mathbf{E}\times\mathbf{H}. \qquad (26)$$

Then the momentum conservation law (24) takes the final form

$$\frac{\partial\sigma_{\alpha\beta}}{\partial x_\beta} - \frac{\partial g_\alpha^A}{\partial t} = f_\alpha^L + f_{m,\alpha}^{(t)}, \qquad (27)$$

where $\sigma_{\alpha\beta} = \sigma_{\alpha\beta}^{(1)} + \sigma_{\alpha\beta}^{(2)}$, and $\mathbf{f}^L$ and $\mathbf{f}_m^{(t)}$ are the forces defined in (1) and (5).

It is obvious that if we were to take the momentum density of the field to be

$$\mathbf{g}^M = \frac{1}{4\pi c}\mathbf{D}\times\mathbf{B}, \qquad (28)$$

then the momentum conservation law would have the form

$$\frac{\partial\sigma_{\alpha\beta}}{\partial x_\beta} - \frac{\partial g_\alpha^M}{\partial t} = f_\alpha^L + f_{m,\alpha}, \qquad (29)$$

where $\mathbf{f}_m = \mathbf{f}_m^{(t)} - \mathbf{f}^A$ is the force (2) acting on the medium.

Thus, in agreement with what we have said earlier, we see that on the basis of a conservation law alone one cannot uniquely choose the expression for the momentum density of the field in the medium. Of course, all that we have said applies equally to the use of the energy–momentum tensor $T_{ik}$, whose conservation law simply combines the conservation laws for the energy and momentum.

For the convenience of the readers, we give here the expressions for the energy–momentum tensors of Minkowski, $T^M_{ik}$, and Abraham, $T^A_{ik}$ ($i, k = 1, 2, 3, 4$; $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = ict$). Their form for the case of a medium at rest—it is clear from (16), (27), and (29)—is[4)]:

$$T^M_{ik} = \begin{pmatrix} \sigma_{\alpha\beta} & -ic\mathbf{g}^M \\ -\frac{i}{c}\mathbf{S} & w \end{pmatrix}, \quad \mathbf{g}^M = \frac{\varepsilon\mu}{4\pi c}(\mathbf{E}\times\mathbf{H}) = \frac{\varepsilon\mu}{c^2}\mathbf{S}, \quad \mathbf{S} = \frac{c}{4\pi}(\mathbf{E}\times\mathbf{H}), \tag{30}$$

$$\frac{\partial T^M_{ik}}{\partial x_k} = f^L_i + f_{m,i}, \quad f^L_\alpha = \left\{\rho_e\mathbf{E} + \frac{1}{c}\mathbf{j}\times\mathbf{B}\right\}_\alpha, \quad f^L_4 = \frac{i}{c}(\mathbf{j}\cdot\mathbf{E}), \tag{31}$$

$$T^A_{ik} = \begin{pmatrix} \sigma_{\alpha\beta} & -ic\mathbf{g}^A \\ -\frac{i}{c}\mathbf{S} & w \end{pmatrix}, \quad \mathbf{g}^A = \frac{1}{4\pi c}\mathbf{E}\times\mathbf{H} = \frac{1}{c^2}\mathbf{S}, \quad \mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{H}, \tag{32}$$

$$\left.\begin{aligned} &\frac{\partial T^A_{ik}}{\partial x_k} = f^L_i + f_{m,i} + f^A_i, \\ &f^A_\alpha = \frac{\varepsilon\mu - 1}{4\pi c}\frac{\partial}{\partial t}\mathbf{E}\times\mathbf{H}_\alpha; \qquad f^A_4 = 0. \end{aligned}\right\} \tag{33}$$

The three-dimensional force $\mathbf{f}_m$ in (31) and (33) is determined by the expression (2). The fourth components of the force density $f_{m,i}$ and the force $f^A_i$ defined in the usual manner are equal to $(i/c)(\mathbf{f}\cdot\mathbf{u})$, and vanish for a medium at rest, i.e., $f_{m,4} = 0$, $f^A_4 = 0$.

In addition, in a medium at rest

$$w = w^M = w^A = \frac{\varepsilon E^2 + \mu H^2}{8\pi}. \tag{34}$$

The expressions (30) and (33) differ from those given in [2] only by the fact that in [2] it was assumed that $\mu = 1$ and only a homogeneous medium ($\varepsilon = \text{const}$) with $\partial\varepsilon/\partial\rho = 0$ was considered, as a result of which the force $\mathbf{f}_m = \mathbf{f}_{m1} + \mathbf{f}_{m2}$ was in fact ignored. If the constraints (3) are not used, then in (30) it is necessary to replace $\mathbf{g}^M$ by the expression (28) and in (33) to use the expression (18) for $\mathbf{f}^A$; in addition

$$w = (\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})/8\pi$$

and the tensor $\sigma_{\alpha\beta}$ must be chosen, respectively, in Minkowski's form or Abraham's form [$\sigma_{\alpha\beta}$ with neglect of the friction forces; see (22) and (26)].

We see that the entire difference between the Minkowski, $T^M_{ik}$, and Abraham, $T^A_{ik}$, tensors in this case of an isotropic medium at rest depends on whether in the equation $\partial T_{ik}/\partial x_k = f_i$ one sets the force of the type of Abraham's equal to zero, and then takes the momentum density of the field in the medium to be $\mathbf{g} = \mathbf{g}^M$ (Minkowski), or one takes the momentum density of the field in the medium equal to $\mathbf{g}^A = \mathbf{S}/c^2$, assuming that a force with density $\mathbf{f}^A$ also acts on the medium (Abraham).

4. The choice between the Minkowski and Abraham tensors, i.e., the problem of finding the "true" energy–momentum tensor of the field in the medium, thus reduces to the reality of Abraham's force. Since this force acts on a medium, to measure it one must consider motion of a medium and, generally speaking, processes in a medium. Theoretically, there can be no doubt about the existence of Abraham's force (or, rather, a force of this type). For if this force were absent, a homogeneous medium with $\partial\ /\partial\rho = 0$ (and for $\rho_e = 0$, $\mathbf{j} = 0$) would not be subject to any force; in particular, not even in the presence of the polarization current $\partial/\partial t[(\mathbf{D} - \mathbf{E})/4\pi]$.

But the Lorentz force $(1/c)(\mathbf{j}\times\mathbf{B})$ acts on the conduction current. Thus, the displacement current would not be on an equal footing with the conduction current, which contradicts the spirit of Maxwell's theory and electron theory. Essentially, it was for this reason that from the very start objection was made[4] to the choice of Minkowski's tensor and the corresponding expression for the forces acting on a medium. It is true that Abraham's force $\mathbf{f}^A$ is not equal to the Lorentz force acting on the polarization current:

$$\mathbf{f}^P = \frac{1}{4\pi c}\left(\frac{\partial(\mathbf{D}-\mathbf{E})}{\partial t}\times\mathbf{B}\right) \equiv \frac{1}{c}\left(\frac{\partial\mathbf{P}}{\partial t}\times\mathbf{B}\right),$$

but these forces are related to one another and the appearance of the expression $\mathbf{f}^A$ rather than $\mathbf{f}^P$ is due to the allowance for certain other terms in the expression for the forces. Let us demonstrate this for the example of a nonmagnetic medium of dipoles. In this case, the force acting on unit volume of a polarized medium in a field $\mathbf{E}$ is

$$\mathbf{f} = (\mathbf{P}\nabla)\mathbf{E} = \frac{\varepsilon-1}{4\pi}(\mathbf{E}\nabla)\mathbf{E} = \frac{\varepsilon-1}{8\pi}\nabla E^2 - \frac{\varepsilon-1}{4\pi}(\mathbf{E}\times\operatorname{curl}\mathbf{E}),$$

since the polarization is

$$\mathbf{P} = N\mathbf{p} = \frac{\varepsilon-1}{4\pi}\mathbf{E}.$$

The second term here is, by virtue of the equation $\operatorname{curl}\mathbf{E} = -(1/c)\partial\mathbf{H}/\partial t$, equal to the force

$$\mathbf{f}^H = \frac{\varepsilon-1}{4\pi c}\left(\frac{\partial\mathbf{H}}{\partial t}\times\mathbf{E}\right) = \frac{1}{c}\left(\frac{\partial\mathbf{H}}{\partial t}\times\mathbf{P}\right).$$

Obviously,

$$\mathbf{f}^P + \mathbf{f}^H = \mathbf{f}^A = \frac{\varepsilon-1}{4\pi c}\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{H}).$$

Of course, this example is not a proof; it merely illustrates the connection between Abraham's force and the Lorentz force acting on the polarization current.

The most general expression (18) for $\mathbf{f}^A$ follows from the "momentum conservation law" (20)–(20a), in which the term $(1/4\pi c)(\partial/\partial t)[\mathbf{D}\times\mathbf{B}]$ automatically appears, and from the fact that the force acting on the medium must of course vanish on the transition to vacuum. One can put this differently as follows: To obtain the force

---

4)The striction forces are frequently ignored, and therefore the tensors $T^M_{ik}$ and $T^A_{ik}$ are frequently taken equal to the expressions in which $\sigma_{\alpha\beta}$ is replaced by the tensor $\sigma^{(1)}_{\alpha\beta}$ [see (22)–(24)]. In addition, in a medium that is at rest but anisotropic Abraham's stress tensor differs slightly from the tensor (22), a point to which we shall return below [see Eq. (36)].

acting on the medium, subtract from the corresponding total expression for the force and the change in the momentum, $(1/4\pi c)(\partial/\partial t)[\mathbf{D}\times\mathbf{B}]$, the change in the momentum of the field itself: $(1/4\pi c)(\partial/\partial t)[\mathbf{E}\times\mathbf{H}]$. The fact that for this one must take the expression we have written down follows from the requirement of relativistic invariance and the law of conservation of the angular momentum, by virtue of which the energy flux density $\mathbf{S}=(c/4\pi)[\mathbf{E}\times\mathbf{H}]$ must (to within a factor $c^2$) be equal to the spatial density of the field momentum (see, for example, Sec. 56 in [23]). But it cannot be claimed that any of these arguments unambiguously single out the expression (18) for a force of the Abraham type. The whole point is that it is only the field and the medium which together form a closed system; division of the total quantities (momentum, energy, etc.) into parts corresponding to the subsystems (the field and the medium) cannot but contain an arbitrariness. Therefore, for example, it is not *a priori* clear why the energy-momentum tensor of the field should be symmetric, whereas such a requirement is unexceptionable when applied to the complete system because of the need to guarantee angular momentum conservation.

We shall not dwell on these questions in more detail, but in the light of what we have already said it is clear that an experimental measurement of Abraham's force is far from superfluous. Unfortunately, the force $\mathbf{f}^A$ is very small and, in addition, can be measured only with a rather special arrangement of the problem[1,25] (in the majority of cases, the force acting on a medium cannot be directly measured and the use of the Minkowski and Abraham tensors leads to the same results; see [25] for more details). Recently, however, it proved possible to make such measurements,[3] and, apparently, with complete success. In the experiment, a disk made of barium titanate, for which $\varepsilon$ was equal to about 4000, was used. In the center of the disk there was a small opening, while the edges were aluminized, so that a cylindrical condenser was obtained. The disk played the role of the mass in a torsion pendulum and was hung between the poles of an electromagnet (with field $B=H \approx 10$ kG). An alternating voltage of amplitude 150 V was applied to the disk condenser (the field frequency was not specified in the brief communication,[3] but it is clear that it was very low and could be tuned to the characteristic frequency of the torsional vibrations). The constant field $\mathbf{B}$ was perpendicular to the alternating polarization $\mathbf{P}=[(\varepsilon-1)/4\pi]\mathbf{E}$ and the density of Abraham's force was equal to $(1/c)[\partial\mathbf{P}/\partial t\times\mathbf{B}]$. The amplitude of the vibrations of the pendulum was very close to the value expected from a calculation using Abraham's force. There is hope that in the near future Abraham's force will be measured with high accuracy. But already one can apparently no longer doubt the existence of this force on the ground of experimental data, to say nothing of theoretical arguments. Thus, there is now no doubt that Minkowski's tensor cannot be regarded as the "true" one, and the choice of Abraham's tensor corresponds to reality for fairly simple cases.

5. It is, however, a fact that, in the theory of emission of charges and other sources moving in a medium, the use of Minkowski's tensor is particularly effective and, one may say, direct. More precisely, we are speaking of the use of the expression (28) for the field momentum density in a medium. Namely, assuming that the momentum of a wave train is equal to $\mathbf{G}^M=\int \mathbf{g}^M\,dV$, we arrive at the connection $\mathbf{G}^M=(Wn/c)\mathbf{k}/k$, where $n=\sqrt{\varepsilon\mu}$ is the refractive index and $W$ is the field energy in the train (here, averaging with respect to the high frequency is performed; for details see [2]). At the quantum level, this connection corresponds to choosing for the momentum of a "photon in a medium" the expression $\hbar\mathbf{k}=(\hbar\omega n/c)\mathbf{k}/k$, where $\hbar\omega$ is the photon energy and $\mathbf{k}$ is its wave vector. It is these expressions that must be used to obtain correct results on the basis of the use of the energy and momentum conservation laws; an example is the Vavilov–Čerenkov radiation condition. Naturally, the problem of why the use of the "false" Minkowski tensor should be successful was discussed very fully in the paper [2].

The problem reduces to this. The field in a medium, assumed to be nondispersive, obtains, not the momentum $\mathbf{G}^M$, but the momentum

$$\mathbf{G}^A=\frac{W}{nc}\frac{\mathbf{k}}{k}=\frac{W}{c^2}\frac{c}{n}\frac{\mathbf{k}}{k},$$

which corresponds to the use of Abraham's tensor.[5)] At the same time, in connection with the existence of Abraham's force

$$\mathbf{f}^A=\frac{n^2-1}{4\pi c}\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{H}),$$

the emission of a wave train by an emitter in the medium imparts to the medium the impulse

$$\mathbf{F}^A=\int \mathbf{f}^A dt\,dV=\frac{n^2-1}{cn}W\frac{\mathbf{k}}{k}=\mathbf{G}^M-\mathbf{G}^A.$$

Therefore, the emitter changes its momentum by $-\mathbf{G}^M$, and $\mathbf{G}^M=\mathbf{G}^A+\mathbf{F}^A$, which is exactly the same as if one has assumed that the Minkowski momentum is transferred solely to the field in the medium (and not to the field and the medium).

In[2] this result was derived and discussed only under a number of assumptions. The medium was assumed to be at rest, nonmagnetic, and homogeneous; the equation $\mathbf{D}=\varepsilon\mathbf{E}$ was used. However, we are concerned here with a very general relation, which must have universal nature. This is in reality the case.

The conservation laws (12) and (20) were directly obtained from the field equations (6)–(7), and are therefore valid without any further assumptions. Introducing further, the stress tensor $\sigma_{\alpha\beta}$, we can reduce Eq. (20) to the form

$$\frac{\partial\sigma_{\alpha\beta}}{\partial x_\beta}=f^L_\alpha+\frac{1}{4\pi c}\frac{\partial}{\partial t}(\mathbf{D}\times\mathbf{B})_\alpha+f'_{m,\alpha}, \qquad (35)$$

where $\mathbf{f}'_m$ is the force acting on the medium.

---

5)For a dispersive medium, it is clear from general considerations that the electromagnetic momentum of the wave train is $\mathbf{G}^A=(W/c^2)\mathbf{v}_{gr}=(W/c^2)d\omega/d\mathbf{k}$, where $\mathbf{v}_{gr}=d\omega/d\mathbf{k}$ is the group velocity. It is curious that for an isotropic plasma with $=n^2=1-(\omega_0^2/\omega^2)$, when $v_{gr}=cn$, the equation $\mathbf{G}^M=(Wn/c)\mathbf{k}/k=\mathbf{G}^A$ holds. Note that in all the above the total energy is $W=\int w^M dV=\int w^M AV$, since we are dealing with a medium at rest.

If Minkowski's expression (22) for $\sigma_{\alpha\beta}$ is used, the form of the force $\mathbf{f}'_m$ is clear from Eq. (23), and if we also use the constraints (3) we have $\mathbf{f}'_m = \mathbf{f}_{m1}$ [see (2) and (24)]. In addition, the force $\mathbf{f}_{m1}$ for a homogeneous medium is zero. If for $\sigma_{\alpha\beta}$ for a medium at rest we take, not Minkowski's expression (22), but Abraham's symmetric stress tensor, which differs from (22) only by the replacement of $E_\alpha D_\beta$ by $(E_\alpha D_\beta + E_\beta D_\alpha)/2$ and similarly for $H_\alpha B_\beta$:

$$\sigma_{\alpha\beta}^A = \frac{1}{8\pi}\{(E_\alpha D_\beta + E_\beta D_\alpha + H_\alpha B_\beta + H_\beta B_\alpha) - \delta_{\alpha\beta}(\mathbf{D}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{H})\}$$
$$= \sigma_{\alpha\beta}^M + \frac{1}{8\pi}(E_\beta D_\alpha - E_\alpha D_\beta + H_\beta B_\alpha - H_\alpha B_\beta), \tag{36}$$

then in an isotropic medium at rest [the constraints (3)] we nevertheless have $\sigma_{\alpha\beta}^A = \sigma_{\alpha\beta}^M$. The addition to the tensors $\sigma_{\alpha\beta}^A$ or $\sigma_{\alpha\beta}^M$ of the striction tensor $\sigma_{\alpha\beta}^{(2)}$ [see (25)], or any other equal terms, clearly does not affect the difference $\sigma_{\alpha\beta}^M - \sigma_{\alpha\beta}^A$.

We now consider a medium on which the force $\mathbf{f}^L + \mathbf{f}'_m$ acts in the presence of a field. Then the change in the momentum density $\mathbf{g}_{m,M}$ of the medium is described by the equation[6)]

$$\frac{\partial \mathbf{g}_{m,M}}{\partial t} = \mathbf{f}^L + \mathbf{f}'_m + \mathbf{h}, \tag{37}$$

where $\mathbf{h}$ is the density of the forces that are not directly related to the presence of the field (for example, $\mathbf{h} = -\nabla p + \rho\mathbf{a}$, where $p$ is the pressure and $\mathbf{a}$ is the acceleration due to gravity).

We subtract the expression (35) written in the vector form from (37) and we then integrate over the whole of space under the assumption that the field "at infinity" decreases fairly rapidly. The term with the divergence does not then play a role (it is transformed into a surface integral and vanishes), and we obtain

$$\frac{d}{dt}(\mathbf{G}_{m,M} + \mathbf{G}^M) = \int \mathbf{h}\, dV, \tag{38}$$

where

$$\mathbf{G}_{m,M} = \int \mathbf{g}_{m,M}\, dV, \qquad \mathbf{G}^M = \frac{1}{4\pi c}\int (\mathbf{D}\times\mathbf{B})\, dV = \int \mathbf{g}^M\, dV.$$

In Eq. (38), the partial derivatives with respect to the time are replaced by total derivatives, since the integrals over the whole of space depend only on $t$; of course, for a closed system, we have in addition $\int \mathbf{h}\, dV = 0$.

Equation (38) is the momentum conservation law for the system consisting of the field and the medium. But to assert, as seems natural on the first glance, that $\mathbf{G}_{m,M}$ is the momentum of the medium and $\mathbf{G}^M$ is that of the field, would be wrong. For suppose that the term $(1/4\pi c)(\partial/\partial t)[\mathbf{D}\times\mathbf{B}]$ in Eq. (35) is not the change in the field momentum, but is equal to

$$\frac{\partial \mathbf{g}_{m,A}}{\partial t} \equiv \frac{1}{4\pi c}\frac{\partial}{\partial t}(\mathbf{D}\times\mathbf{B}) = \mathbf{f}^A + \frac{\partial}{\partial t}\mathbf{g}^A, \tag{39}$$

where $\mathbf{f}^A$ is the density of some volume force which acts on the medium and $\mathbf{g}^A$ is the momentum density of the field (the superscript $A$ does not yet necessarily mean that we identify $\mathbf{f}^A$ and $\mathbf{g}^A$ with Abraham's expressions). Then on the right-hand side of Eq. (37) one must also add the force $\mathbf{f}^A$; of course, the momentum density $\mathbf{g}_{m,A}$ of the medium is then different—its variation is now determined by the equation

$$\frac{\partial \mathbf{g}_{m,A}}{\partial t} = \mathbf{f}^L + \mathbf{f}'_m + \mathbf{f}^A + \mathbf{h}. \tag{40}$$

Subtracting now the expression (35) from (40) and then integrating over the whole of space with allowance for (39), we obtain

$$\frac{d}{dt}(\mathbf{G}_{m,A} + \mathbf{G}^A) = \int \mathbf{h}\, dV, \tag{41}$$

where

$$\mathbf{G}_{m,A} = \int \mathbf{g}_{m,A}\, dV, \quad \mathbf{G}^A = \int \mathbf{g}^A\, dV.$$

The most natural thing, of course, would be to identify $\mathbf{g}^A$ with Abraham's expression (26), and then $\mathbf{f}^A$ is determined by Eq. (18). However, as is clear from what we have said, the choice of the expression for $\mathbf{g}^A$ or $\mathbf{f}^A$ must be based on experimental data or calculations outside the scope of the actual equations for the macroscopic field, from which there follows only the conservation law (20) or its direct consequences.

As is obvious from (39) and (40),

$$\frac{\partial \mathbf{g}_{m,M}}{\partial t} = \frac{\partial \mathbf{g}_{m,A}}{\partial t} - \mathbf{f}^A, \tag{42}$$

but this is true only under the assumption that two variants of the theory with the same forces $\mathbf{f}'_m$ and $\mathbf{h}$ are compared. In the case of the integral quantities, the equation

$$\frac{d\mathbf{G}_{m,M}}{dt} = \frac{d\mathbf{G}_{m,A}}{dt} + \int \mathbf{f}^A\, dV \tag{43}$$

does hold for variants of the theory with different $\mathbf{f}'_m$ and $\mathbf{h}$ provided

$$\int (\mathbf{f}'_{m,M} - \mathbf{f}'_{m,A})\, dV = 0 \text{ and } \int (\mathbf{h}_M - \mathbf{h}_A)\, dV = 0,$$

where the subscripts $M$ and $A$ correspond to the forces in the variants $M$ and $A$ of the theory. (Of course, if the volume forces $\mathbf{f}'_m$ and $\mathbf{h}$ are expressed in terms of $\partial\sigma_{\alpha\beta}/\partial x_\beta$, the corresponding volume integrals become surface integrals and, in general, vanish.)

It is clear, finally, from (38), (41), and (43) that the equation

$$\mathbf{G}^M = \mathbf{G}^A + \int \mathbf{f}^A\, dt\, dV \tag{44}$$

has a very general nature and does not entail particular assumptions.

In particular, it holds for a moving medium.

Of course, by virtue of the relativistic invariance of the field equations it is immediately clear that on the transition from a medium at rest to one moving uniformly no general relations can be affected. However, it is helpful to examine this for the example of the Minkowski and Abraham tensors.

According to Minkowski, for both a medium at rest and a moving medium the expressions for $\sigma_{\alpha\beta}^M$, $\mathbf{g}^M$, and $w_M$ are determined by (22), (28), and (34), and the stric-

[6)]We assume that the velocity of the medium is zero or so small small that $d\mathbf{g}_m/dt = (\partial\mathbf{g}_m/\partial t) + (\mathbf{u}\nabla)\mathbf{g}_m \approx \partial\mathbf{g}_m\ \partial t$, etc.

tion forces, solely for the sake of simplicity, are not considered here. According to Abraham, in a medium at rest the expressions for $\sigma^A_{\alpha\beta}$, $\mathbf{g}^A$, and $w^M$ are given by (36), (26), and (34). In a moving medium[7)]

$$w^A = \frac{1}{8\pi}(\mathbf{D}\cdot\mathbf{E}+\mathbf{B}\cdot\mathbf{H}) - \frac{\mathbf{u}/c}{4\pi[1-(u^2/c^2)]}((\mathbf{D}\times\mathbf{B})-(\mathbf{E}\times\mathbf{H})),$$
$$\mathbf{g}^A = \frac{\mathbf{S}}{c^2} = \frac{1}{4\pi c}\left\{(\mathbf{E}\times\mathbf{H}) - \frac{\mathbf{u}/c^2}{1-(u^2/c^2)}(\mathbf{u}\cdot(\mathbf{D}\times\mathbf{B})-\mathbf{u}\cdot(\mathbf{E}\times\mathbf{H}))\right\}. \quad (45)$$

Of course, if the expression for $T_{ik}$ (i.e., for $w$, $\mathbf{g}$, $\mathbf{S}$, and $\sigma_{\alpha\beta}$) in the medium at rest is given, the corresponding expressions for a medium moving with velocity $\mathbf{u}$ = const can be obtained uniquely by relativistic transformations.

For a moving medium, by what we have said earlier,

$$\mathbf{f}^A = \frac{\partial}{\partial t}(\mathbf{g}^M - \mathbf{g}^A) = \frac{1}{4\pi c}\frac{\partial}{\partial t}\left\{(\mathbf{D}\times\mathbf{B})-(\mathbf{E}\times\mathbf{H}) + \frac{\mathbf{u}(\mathbf{u}\cdot(\mathbf{D}\times\mathbf{B})-\mathbf{u}\cdot(\mathbf{E}\times\mathbf{H}))}{1-(u^2/c^2)}\right\}. \quad (46)$$

In a medium at rest (or one moving slowly when terms of order $u^2/c^2$ can be ignored) the expression (46) is identical to (18), in which Abraham's force was introduced for a medium at rest and it was only on the transition to (19) that the medium was assumed to move slowly.

Forming the balance of the energy and momentum when a wave train is emitted in a moving medium and using Abraham's tensor, we must bear in mind that the field energy is written in the form

$$W^A = \int w^A\, dV = W^M - \frac{\mathbf{u}/c}{4\pi[1-(u^2/c^2)]}\int\{(\mathbf{D}\times\mathbf{B})-(\mathbf{E}\times\mathbf{H})\}\, dV. \quad (47)$$

In addition, under the influence of the force $\mathbf{f}^A$ defined in accordance with (46) the medium receives the energy

$$R^A = \mathbf{u}\cdot\int \mathbf{f}^A\, dt\cdot dV = \mathbf{u}\,\frac{\int((\mathbf{D}\times\mathbf{B})-(\mathbf{E}\times\mathbf{H}))dV}{4\pi c(1-u^2/c^2)}. \quad (48)$$

Thus, the total change in the energy of the emitter is equal to $-(W^A+R^A) = -W^M$. Of course, this result is also clear without integration since $\partial w^M/\partial t = \partial w^A/\partial t + \mathbf{f}^A\cdot\mathbf{u}$, which for a slowly moving medium was already reflected in (19).

The change in the momentum of the emitter is also determined as before by the expression (44). [When applied to a wave train and with averaging with respect to the high frequency, this relation was used in [2]; see Eqs. (38) and (39).]

The difference between the tensors $\sigma^M_{\alpha\beta}$ and $\sigma^A_{\alpha\beta}$ for a moving medium does not here play a role, as we have seen [in expressions of the type (44) and (48) integration is performed over the whole space, and in the integration with respect to the time the period of time when the wave train is already sufficiently "separated" from the emitter is taken into account]. Therefore, in accordance with our general arguments, we arrive at the same results when considering emission processes in a moving medium as for a medium at rest. In particular, in the quantum treatment we can use the expressions $\hbar\omega$ and $\hbar\mathbf{k} = (\hbar\omega n/c)\mathbf{k}/k$ for an emitted and an absorbed "photon" in a moving medium as well, although we know that (as in a medium at rest) we are concerned, not with the energy and momentum of the field itself in the medium, but with the changes in the energy and the momentum of an emitter in the medium. For this reason, $\hbar\mathbf{k}$ is, in fact, sometimes called the photon quasimomentum. And if $\hbar\omega$ is the photon energy in a medium at rest, the energy $\hbar\omega$ in a moving medium is rather the quasienergy, in the sense that it is the sum of the field energy $W^A$ and the work $R^A$ of Abraham's force on the medium [see (47) and (48)].

6. Above and in [2], as in the earlier papers of one of the authors, [26] attention was entirely concentrated on the exchange of energy and momentum between the emitter and the medium. But of course it is also of interest to analyze processes taking place in the medium itself. Here, the differences between Minkowski's and Abraham's approach become very significant and lead to different results. Indeed, according to Minkowski no force at all (ignoring the striction force) acts on a transparent and homogeneous uncharged medium in an electromagnetic field (for example, when a wave train propagates). According to Abraham—and this corresponds to reality—the medium is subject to a force with volume density $\mathbf{f}^A$ [see (4), (18), and (46); Eqs. (18) and (4) follow from (46) under the appropriate special assumptions].

Under the influence of the force $\mathbf{f}^A$ the motion of the medium is of course changed, but no universal conclusions about this can be drawn apart from the conservation law of the total momentum (41). The point is that the momentum density $\mathbf{g}_m$ acquired by the medium depends on the equations that describe the motion of the medium and, concretely, on the force density $\mathbf{h}$ in the equation of motion (37) or (40) of the medium. In addition, besides Abraham's force $\mathbf{f}^A$, the striction force (2) also acts on the homogeneous medium in general:

$$\mathbf{f}_{m,2} \equiv \mathbf{f}_{str} = \frac{1}{8\pi}\nabla\left\{\left(\frac{\partial\varepsilon}{\partial\rho}\rho\right)E^2\right\}$$

(we ignore the force proportional to $\partial\mu/\partial\rho$, which in the majority of cases is considerably smaller).

The density of the momentum imparted to a medium through which a train of electromagnetic waves propagates was calculated for various models of a medium in [1,12,20,22]. For example, for a gas of heavy dust particles[1] the momentum density acquired by the gas is exactly equal to[8)]: $\mathbf{g}^M - \mathbf{g}^A = (n^2-1)\mathbf{g}^A$. Therefore, the total momentum density of the field and the medium is $\mathbf{g}^M$. For a very rigid solid, the effect of the force $\mathbf{f}^A$ at a not too high frequency in a quasistationary regime is almost completely compensated by forces of elas-

---

[7)]See[24] in the third edition (1946), Secs. 115 and 116; see also Appendix 4 in [1] or Sec. 36 in [6].

[8)]For an equilibrium gas

$$\frac{\partial\varepsilon}{\partial\rho} = \frac{\partial\varepsilon}{\partial p}\frac{\partial p}{\partial\rho} = \frac{\partial\rho}{\partial N}\frac{1}{M}, \quad \varepsilon = 1+4\pi\alpha N, \quad p = NkT,$$

$\rho = NM$, where $\alpha$ is the polarizability of the particle (molecule) $T$ is the temperature, $N$ is the concentration, and $M$ is the mass of the particles. Obviously, $\partial p/\partial\rho = kT/M \to 0$ as $M \to \infty$. Thus, in this case the striction force is equal to zero and the medium is subject to only the force with density $\mathbf{f}^A = (n^2-1)\partial\mathbf{g}^A/\partial t$, which imparts to unit volume of the gas the momentum $\int\mathbf{f}^A dt = (n^2-1)\mathbf{g}^A$ as a result of the passage of the front of the wave train.

ticity (but, of course, the medium as a whole receives the momentum $\int \mathbf{f}^A dt dV$). This assertion can be illustrated by the example of an oscillator whose equation of motion has the form $m(d^2x/dt^2) + kx = f = dg/dt$, where $f$ is the effective force. If the coefficient of elasticity $k$ is very large, the displacement is $x \approx f/k$ for the greater part of the time, and $dx/dt \approx (df/dt)/k$, so that the oscillator momentum tends to zero, $mdx/dt \to 0$, as $k \to \infty$; but in the limit $k \to 0$, we have $mdx/dt = \int f dt = g$.

For the model of a medium consisting of dipoles, the value obtained in [20] for the momentum density of the field and the medium—it is intermediate between $\mathbf{g}^M$ and $\mathbf{g}^A$—is also in no way universal, and characterizes in the first place the model of the medium that is used. A more detailed discussion of the question of the density of the momentum that arises in a medium under the influence of the electromagnetic field goes beyond the scope of the present paper (in particular, see [22]).


[1] L. V. Skobel'tsyn, Usp. Fiz. Nauk **110**, 253 (1973) [Sov. Phys. Uspekhi **16**, 381 (1973)].

[2] V. L. Ginzburg, *ibid.* p. 309, [**16**, 434 (1973)].

[3] G. B. Walker and D. G. Lahoz, Nature **253**, 339 (1975).

[4] A. Einstein and J. Laub, Ann. Phys. (Leipzig). **26**, 541 (1908).

[5] M. Abraham, Rc. Circ. mat. Palermo **28**, 1 (1909); **31**, 527 (1910).

[6] W. Pauli, The theory of Relativity (Pergamon, Oxford, 1968), (Russian translation, Gostekhizdat, Moscow-Leningrad (1947), §35).

[7] U. E. Schröder, Z. Naturforsch **24a**, 1356 (1969).

[8] J. Meixner, Z. Phys. **229**, 352 (1969).

[9] J. Vlieger and S. Emid, Physica **41**, 368 (1969).

[10] K. Furutsu, Phys. Rev. **185**, 257 (1969).

[11] M. G. Burt and R. Peierls, Proc. R. Soc. London **333**, 149 (1973).

[12] J. P. Gordon, Phys. Rev. **A8**, 14 (1973).

[13] K. K. Thornber, Phys. Lett. **A43**, 501 (1973).

[14] M. Pitteri, Nuovo Cimento **B18**, 144 (1973).

[15] O. Costa de Beauregard, Lett. Nuovo Cimento **10**, 852 (1974).

[16] R. Klima and V. A. Petržilka, Phys. Lett. A **43**, 151 (1973).

[17] V. A. Zhelnorovich, Dokl. Akad. Nauk SSSR **217**, 1029 (1974) [Sov. Phys. Doklady **19**, 499 (1975)].

[18] D. V. Skobeltzyne, C. R. Ac. Sci. B **280**, 251, 287 (1975).

[19] W. Israel, California Institute of Technology Preprint OAP-387 (1975).

[20] R. Peierls, The Momentum of Light in a Refracting Medium, Preprint (1975).

[21] N. P. Konopleva, Izv. Vyssh. Uchebn. Zaved. Radiofiz. (1976).

[22] F. N. H. Robinson, Phys. Rep. **C16**, 315 (1975).

[23] L. D. Landau and E. M. Lifshitz, Elektrodinamika Sploshnykh Sred [Electrodynamics of Continuous Media (Pergamon Press, Oxford, 1960)], Moscow (1957).

[24] I. E. Tamm, Osnovy Teorii Élektrichestva (Fundamentals of the Theory of Electricity), 8th edition, Nauka, Moscow (1966) (the 9th edition, which is almost identical to the 8th, is in preparation).

[25] I. Brevik, Mat. Fys. Medd. Dan. Vid. Selsk. **37**, No. 13 (1970).

[26] V. L. Ginzburg, Zh. Eksp. Teor. Fiz. **10**, No. 583 (1940) Usp. Fiz. Nauk **69**, 537 (1959) [Sov. Phys. Uspekhi, **2**, 874 (1960)].


Translated by J. B. Barbour