Multiple scattering of waves by ensembles of particles and the theory of radiation transport

Yu. N. Barabanenkov

All-Union Scientific Research Institute of the Metrological Service

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The problem is outlined of how to approach the description of partially coherent multiple scattering of waves by ensembles of particles in terms of the photometric theory of radiation transport in a scattering medium. The treatment is carried out with the example of a scalar monochromatic wavefield. The apparatus of equations of the Dyson and Bethe-Salpeter types is used. This apparatus is shown to be adequate for the fundamental concepts of transport theory. In particular, correspondence is established between an effective inhomogeneity of an ensemble of correlated particles and the volume element of transport theory. It is shown that neglect of the effect of mutal illumination of correlation groups of particles within a given inhomogeneity leads to spatial localization of the inhomogeneities, while the additional hypothesis that the localized inhomogeneities lie close to one another in the Fraunhofer zone gives the transport equation. The contribution is estimated of the effect of mutual illumination of correlation groups of particles within a given inhomogeneity to the partially coherent scattering of waves by the volume of the medium. The role of the detector of the scattered radiation in making this estimate is discussed.

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CONTENTS

1. INTRODUCTION

The set of phenomena that arise from multiple scattering of waves by an ensemble of particles is very broad. For example, it includes light scattering in dispersed substances [1,2] such as pigments, powders, colloids, polymers, aero- and hydrosols, emulsions, mineral textures, and snowlike and paperlike materials. Multiple scattering of light occurs in astrophysical [3,4] objects (stellar and planetary atmospheres, gaseous nebulas, and the interstellar medium), geophysical objects (soils, waters, and their deposits), and in gases and liquids near the critical point. [5,6] Multiple scattering of waves also includes processes such as the scattering of electromagnetic waves in a plasma, [7] scattering of electrons by impurities in a crystal structure, [8] neutron transport in objects of varied shapes and dimensions, [9] passage of charged particles through matter, [10,11] and the interaction of cosmic rays with matter. [12]

The phenomenological and statistical approaches have been applied for treating multiple-scattering processes.

The gist of the phenomenological approach is embodied in the theory of radiation transport in a scattering medium. Its apparatus is the transport equation, [1,13,14] which expresses the law of conservation of the radiation energy or the condition of balance of intensities of light beams with account taken of their polarization.

In the statistical treatment of multiple scattering of waves, one starts with the stochastic wave equation or with a system of such equations, for which one poses and studies the problem of diffraction of waves by a statistical ensemble of particles.

The problem of the statistical basis of the theory of radiation transport using the theory of multiple scattering of waves by an ensemble of particles belongs to the set of fundamentally important and as yet not fully solved problems of theoretical physics. The formulation of this problem arises from the following factors.

Transport theory has existed for about a hundred years. Nevertheless, it hasn't yet been tested experimentally, owing to difficulties [2] that involve the limited nature of the models of a scattering medium that are amenable to fully valid mathematical analysis, and the imperfection of the known methods of measuring the coefficients that enter into the transport equation. Yet the transport theory gives no hint of the method for calculating these coefficients, apart from determining them experimentally.

In connection with the rise laser technique of creating wavefields of a high degree of coherence, the topic has arisen of elucidating the conditions that should be imposed on the properties of the medium and of the incident wavefield, as well as the methods of detecting the scattered wavefield, such that transport theory proves to be applicable.

It has become possible in recent years to find exact solutions [15-19] of the steady-state problem of scattering of a monochromatic wave for a one-dimensional model of a scattering medium by starting with the stochastic wave equation of Helmholtz. The results obtained thereby considerably disagree [20] with those that stem from the solution [14] of the transport equation.

A review by Rozenberg [1] and the review [21] have been concerned with the problem of the statistical basis of the theory of transport of radiation from the standpoint of the theory of multiple scattering of waves. The first of these reviews systematically formulates the problem, and reveals the physical principles of its solution. However, the theory of multiple scattering of waves was not yet sufficiently developed when this review was written. The second of the cited reviews paid major attention to the methods of solving problems of wave propagation in a randomly-inhomogeneous medium having smooth fluctuations of the refractive index, and did not fully treat problems of transport theory.

This review pursues the aim of a generalizing presentation of how one solves the problem of the statistical basis of transport theory by using the modern theory of multiple scattering of waves by an ensemble of particles, and of how far people have as yet succeeded in solving this problem. In order to construct a theory of multiple scattering of waves, this review uses the method of Green's functions and Feynman diagram technique, which leads to equations of the Dyson and Bethe-Salpeter types. This method is marked by the advantage that it is physically pictorial and more adequate than transport theory. Moreover, it includes as special cases many other known methods. The studies [22-24] have recently made it possible to impart to the method of the Dyson and Bethe-Salpeter equations a mathematically rigorous corroboration in problems of non-steady-state multiple scattering of wave packets in a randomly-variable medium.

2. THE PROBLEM OF THE STATISTICAL BASIS OF TRANSPORT THEORY

The concept of the volume element [1,25-27] of a scattering medium plays an important role in the theory of radiation transport. This theory is constructed on the notions of geometrical optics [1,2,28] and it assumes total incoherence of scattering events. [27] The objects of transport theory are the photometric quantities [27,28] that describe the light beam, and which satisfy the transport equation.

The concept of the volume element of a scattering medium is rather complex. This volume attenuates and scatters the radiation incident on it; quantitatively, the attenuation and scattering are proportional to the size of the volume. [25] The optical properties of the volume element are characterized by the extinction and scattering coefficients, or when the polarization of the light beams is taken into account, by the extinction and scattering matrices. [29,30] Transport theory essentially assumes that the light beams scattered by different volume elements are mutually incoherent, i.e., their intensities add together. [1] The light beams propagate according to geometric optics between two successive scattering events by volume elements.

Transport theory deals with the photometric quantity of radiance, [26-28] which defines the flux of radiation energy through a unit area in a unit solid angle per unit time. When we account for the polarization of the light radiation, the four-component Stokes vector-parameter [13,29,30] replaces the radiance. Here the first component is identical with the radiance of the light beam, and the three others define its polarization. One draws up a radiation transport equation for the radiance in the scattering medium. [13,14] The source of this equation in physics in its original form is due to O. D. Khvol'son, and it dates back to the seventies of the past century, and is also due to Schwartzschild and Schuster. The physical content of the equation is that the change in radiance of the light beam per element of length is composed of attenuation due to absorption and scattering, and of intensification owing to scattering in the given direction of light that illuminates the corresponding volume element from all other directions.

The transport equation with account taken of polarization of the radiation was first formulated simultaneously and independently by Chandrasekhar [13] and Rozenberg [29] for an isotropic scattering medium, for which the extinction matrix is scalar, and somewhat earlier by Sobolev, [31] who treated the special case of Rayleigh scattering. Rozenberg [30,32] has derived a more general transport equation for polarized light radiation. It permits one to solve all problems of optics of anisotropic scattering media, including application of the theory to electro- and magnetooptic phenomena in colloids. Instead of the extinction coefficient, the dispersion matrix figures in this equation, and it is composed of the sum of the extinction and phase matrices. Of these, the extinction matrix describes the attenuation of a light beam of a given polarization owing to absorption and scattering, while the phase matrix describes the change in polarization of the light beam owing to the difference in velocities of propagation of the two oppositely polarized components.

The transport equation for light radiation as written with the Stokes vector-parameter proves to be a matrix equation, and it looks far more complex than the classical transport equation for the radiance. However, in studying multiple scattering of electromagnetic radiation, one must adopt such a complex transport equation, since the many terms of this equation, which account for polarization effects, are generally of the same order of magnitude as the terms that contain only the first component of the Stokes vector-parameter. Strictly speaking, the classical transport equation for the radiance is applicable only in the case of scalar radiation, e.g., in describing neutron transport. [9]

In the statistical approach, one deals directly with the wavefields and their multiple scattering by the ensemble of particles, and here the properties of this ensemble are assumed to be given. In other words, one assumes that one is dealing with weak fields in which the effect of the field on the state of the matter can be ignored or taken as a small perturbation. This permits us to treat the medium and the field as being independent systems, and to restrict the object of study to the action of the matter on the field. [2,27]

Several complex, fundamental problems stem from this on the pathway to the statistical basis of transport theory.

One of these consists in elucidating the photometric concept of the radiance from the standpoint of statistical wave theory. [26-28] This problem arises also even in the absence of a scattering medium when one tries to describe photometrically a given partially-coherent wavefield.

Perhaps the most complex problem is the possible introduction of the concept of the volume element of the scattering medium, starting with the statistical theory of multiple scattering of waves. [26,27] The proof that this can be done would give a theoretical solution to the problem of determining the optical parameters of the volume element (the extinction and scattering coefficients) by expressing them in terms of the result of solving the problem of diffraction of waves by a set of small number of particles and in terms of the correlation functions of the particles.

The problem of the relationship between this theory and the theory of cooperative effects [26,27,32] proves to be no less complex in the statistical basis of transport theory. Here the phases of the waves scattered by the particles of the medium and the correlations of the particles play a substantial role.

The statistical theory of multiple scattering of waves by an ensemble of particles (see the reviews [1,33,34])[1)] is currently far from being perfected. Mathematically, it boils down to studying a stochastic wave equation with a random effective scattering potential. As a rule, one solves this equation by asymptotic methods that use expansions in small parameters. Here one estimates only certain ones of the terms to be dropped in these expansions. Rigorous methods of solving the stochastic wave equation with special assumptions on the properties of the scattering medium have begun to appear only in recent years. [15,22-24,36]

The imperfection of the statistical theory of multiple scattering of waves hampers the final solution of the problem of the statistical basis of transport theory. Nevertheless, a number of interesting results that are rather convincing from the physical standpoint have been obtained on the way toward solving this problem, and they are presented in this review. Among them, we shall especially distinguish the physical principles that mark the path of approach to solving the studied problem, and also certain experimental studies involved with it.

Rozenberg [1,32] has made a fundamental contribution to the study of the problem of the statistical basis of transport theory. He has established from physical considerations that the mutual effect of particles in the scattering of waves from them is divided into two components: coherent and incoherent. Here the coherent component, for which only the nearest neighbors of a given particle are responsible, is manifested exclusively in two cooperative effects that are responsible for a set of dispersion phenomena, namely, in the variation of the effective complex refractive index (in the general case of a matrix), and in the difference of the scattering coefficient (or matrix) of the volume element from its value for an isolated particle. At the same time, the incoherent component of the interaction, which originates from the entire volume of the scattering medium, arises in the form of multiple scattering, and it becomes the object of transport theory.

Rozenberg [26-28] has studied the problem of the statistical-electrodynamics content of the photometric quantities and the rules in applying them to describe a partially coherent wavefield. He has shown that the introduction of photometric quantities as observables (in the quantum-mechanical sense of this word) assumes that one uses light detectors that perform square-law detection, and which have finite dimensions and a finite time constant.

One of the cooperative optical effects in scattering media that lies outside the limits of applicability of transport theory is coherent forward scattering. This effect has been discussed in a number of studies, [32,37-39] and it has been experimentally detected by Ivanov, Khaĭrullina, and Khar'kova. [40] Khaĭrullina and Ivanov [41] have reported that, when a scattering medium is illuminated with radiation having a high degree of spatial coherence, the light field formed by interference of the waves scattered by the particles will be inhomogeneous in space (the so-called grain structure) and fluctuating in time. The nature of the grain structure that is formed is due to the optical and geometric parameters and relative arrangement of the particles, while the frequency of the fluctuations of the light field is determined by the mobility of the particles, e.g., by their Brownian movement.

Rozenberg [27] has given a general physical analysis of the conditions for applicability of transport theory in describing multiple scattering of a partially coherent incident wavefield in a scattering medium whose properties vary randomly with time owing, e.g., to Brownian movement of particles. He has elucidated the criteria for choosing the time and space scales for averaging the quadratic functions of the field to give the photometric quantities. He has examined the limiting case in which the propagation of the field in the medium can be considered to be steady-state from the optical standpoint, and the case in which this process is non-steady-state.

The review to be presented contains the results of the theory of multiple scattering of waves by a statistical ensemble of particles, which directly bear upon the problem of the basis of the photometric theory of radiation transport. I shall pay major attention to the case in which one can consider the medium to be constant in time, and the incident field is purely coherent in space and in time. The results to be given were derived for a scalar wavefield. Their known generalizations to an electromagnetic field are noted in passing.

3. THE MODEL OF A DISCRETE SCATTERING MEDIUM

A discrete scattering medium comprises a set of particles. By analogy with quantum mechanics, we can conveniently characterize each particle with a certain scattering potential. The potential $V(\mathbf{r})$ of the discrete scattering medium equals the sum of the potentials of its particles:

$$V(\mathbf{r})=\sum_{j=1}^{N} V_0(\mathbf{r}-\mathbf{r}_j), \tag{3.1}$$

Here $V_0(\mathbf{r}-\mathbf{r}_j)$ is the potential of the j-th particle, $j = 1, \ldots, N$, having its center at the point $\mathbf{r}_j$, and N is the total number of particles.

The Helmholtz equation for the scalar wave monochromatic field $\psi(\mathbf{r})$ in a scattering medium having the potential $V(\mathbf{r})$ has the form

$$[\Delta+k_0^2-V(\mathbf{r})]\,\psi(\mathbf{r})=0, \tag{3.2}$$

where k_0 is the wavenumber in free space. If the wavefield and its normal derivative are continuous at the surfaces of the particles, then the solution of the Helmholtz equation with account taken of the radiation conditions at infinity reduces to the wave integral equation

$$\psi(\mathbf{r})=\psi_0(\mathbf{r})+\int G_0(\mathbf{r}-\mathbf{r}')\,V(\mathbf{r}')\,d^3\mathbf{r}'\,\psi(\mathbf{r}'); \tag{3.3}$$

[1)]The reviews [21,35] treat the statistical theory of propagation of waves in a randomly-inhomogeneous continuous medium.

Here the inhomogeneous term $\psi_0(\mathbf{r})$ constitutes the incident field created by the assigned distribution of sources, and $G_0(r) = -\exp(ik_0r)/4\pi r$ is the Green's function for free space. In the case in which the scattering medium occupies a limited volume, and the source lies at infinity, the incident field has the form of the plane wave $\psi_0(\mathbf{r}) = \exp(ik_0\mathbf{s}_0\cdot\mathbf{r})$, which propagates in the direction of the unit vector $\mathbf{s}_0$. For a point source concentrated at the point $\mathbf{r}'$, the incident field $\psi_0(\mathbf{r})$ is equal to $G_0(\mathbf{r}-\mathbf{r}')$, and the solution of the integral wave equation (3.3) gives the Green's function $G(\mathbf{r}, \mathbf{r}')$ of the scattering medium.

a) The Scattering Operator and the Optical Theorem

In solving the diffraction problem for some particle, one usually restricts the treatment to calculating the scattering amplitude that determines the scattered field at long range. However, in studying multiple scattering of waves, one must know the scattered field at any distance from the particle, since in the statistical ensemble the particles can approach one another considerably.

One can conveniently represent the complete solution of the diffraction problem by using the scattering operator T.[42] This concept is borrowed from the quantum theory of scattering. The solution of the integral wave equation (3.3) for the field $\psi(\mathbf{r})$ is expressed in terms of the kernel $T(\mathbf{r}, \mathbf{r}')$ of the scattering operator by the relationship

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) + \int G_0(\mathbf{r}-\mathbf{r}'')\, d^3r''\, T(\mathbf{r}'', \mathbf{r}')\, d^3r'\, \psi(\mathbf{r}'). \quad (3.4)$$

We shall designate the Fourier transform $\widetilde{T}(\mathbf{k}, \mathbf{k}')$ of the kernel of the scattering operator $T(\mathbf{r}, \mathbf{r}')$, when calculated on the spherical surface $k^2 = k'^2 = k_0^2$ of wavenumbers (k_0^2 being the surface), as $T(\mathbf{s}, \mathbf{s}_0)$, where the unit vectors $\mathbf{s}$ and $\mathbf{s}_0$ lie along the wave vectors $\mathbf{k}$ and $\mathbf{k}'$. Apart from a constant coefficient, it gives the scattering amplitude of the scatterer being studied; the latter could be an individual particle, a certain set of particles, or the entire volume of the scattering medium. This property of the scattering operator explains its physical meaning.

In studying electromagnetic waves, one writes the Helmholtz equation for the electric field intensity in vector form (see, e.g.[43,44]). The Green's function of free space and of the scattering medium, and also the scattering operator for the electric field have tensor dimensionality.

The law of conservation of energy in wave scattering gives rise to the optical theorem.

In the case of a non-absorbing scatterer having a real potential, the optical theorem is formulated as a relationship involving its scattering operator:

$$T(\mathbf{r}_2', \mathbf{r}_1') - T^*(\mathbf{r}_1', \mathbf{r}_2') = \int [G_0(\mathbf{r}_2-\mathbf{r}_1) - G_0^*(\mathbf{r}_1-\mathbf{r}_2)]\, d^3r_1 d^3r_2 T(\mathbf{r}_1, \mathbf{r}_1')\, T^*(\mathbf{r}_2, \mathbf{r}_2'), \quad (3.5)$$

Here the asterisk indicates that we take the complex con-conjugate. In the Fourier representation, the optical theorem (3.5) takes on its customary form[45]

$$-\frac{\operatorname{Im}\widetilde{T}(\mathbf{s}_0, \mathbf{s}_0)}{k_0} = \frac{1}{(4\pi)^2}\int_{4\pi} d^2s\, |\widetilde{T}(\mathbf{s}, \mathbf{s}_0)|^2. \quad (3.5a)$$

The quantities on the left- and right-hand sides of this relationship are called the extinction and total scattering cross sections.[46] These cross sections are equal to one another for a non-absorbing scatterer.

If the scatterer can absorb the energy of the wave incident on it because it has a complex potential, then the formulation of the optical theorem changes. Here the left-hand side of (3.5a) proves to be larger than the right-hand side. That is, the extinction cross-section is larger than the total scattering cross section. The difference between the extinction and the total scattering cross sections is called the absorption cross section.[46] The optical theorem for an absorbing scatterer states that the extinction cross section equals the sum of the total scattering and the absorption cross sections. In other words, the energy extracted from the incident wave goes into scattering and absorption.

Rozenberg[1,32] has studied the meaning of the optical theorem for an absorbing scatterer of electromagnetic waves.

b) The System of Multiple Wave-Scattering Equations

If a volume of a discrete scattering medium exists with a potential $V(\mathbf{r})$ equal to the summation of (3.1), then we can conveniently deal also with the scattering operators t_j of the isolated particles having the potentials $V_0(\mathbf{r}-\mathbf{r}_j)$, as well as with the scattering operator of this volume. The relationship between T and all the t_j ($j = 1, \ldots, N$) is established by the system of multiple wave-scattering equations:

$$T = \sum_{j=1}^{N} T_j, \qquad T_j = t_j + t_j G_0 \sum_{j'=1\,(j'\neq j)}^{N} T_{j'}; \quad (3.6)$$

Here each operator T_j describes the scattering by the j-th particle in the presence of the remaining $(N-1)$ particles. As we see, T_j equals the scattering operator t_j of the isolated particle having the given number j plus the effect of the rest of the particles. We can completely reveal the physical meaning of the system of equations (3.6) by writing its solution in the form of a series of successive approximations. Each term of this series describes a wave-scattering process such that the wave travels from one particle to another, yet can return to each particle, while undergoing repeated scattering by it.

Watson[42] has derived a system of multiple wave-scattering equations in the form (3.6). Rozenberg[32] had formulated this system earlier in the representation of inhomogeneous plane waves.

c) Distribution Functions and Correlation Functions of the Particles

Foldy[47] introduced into the multiple wave-scattering theory the concept of the configurational average over a statistical ensemble of particles randomly arranged in space. This concept is applied for calculating the averages of such quantities as the wavefield and its bilinear combination. Like the statistical theory of gases and liquids, it is based on the probability density of configurations of centers of particles in space, as normalized to unity and symmetrical with respect to interchange of its arguments.[2)] One constructs from the probability densities of the particles their class distribution functions $f_n(\mathbf{r}_1, \ldots, \mathbf{r}_n)$ of different orders $n = 1, 2, \ldots$ (see[48], p. 81). Here the class distribution function of

2)We restrict the treatment to identical, spherical particles.

the n-th order determines the probabilities of configurations of any n particles out of the total number of N. In addition to the distribution functions for describing the ensemble of particles, one also uses their correlation functions $g_n(\mathbf{r}_1, \ldots, \mathbf{r}_n)$, n = 1, 2, ... They are related to the class distribution functions by relationships of the type [33,49,50]

$$f_1(\mathbf{r}) = g_1(\mathbf{r}),$$
$$f_2(\mathbf{r}_1, \mathbf{r}_2) = g_2(\mathbf{r}_1, \mathbf{r}_2) + g_1(\mathbf{r}_1)\, g_1(\mathbf{r}_2), \qquad (3.7)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

The class distribution function and the correlation function of the first order coincide, and they give the density of particles. The correlation functions of order $n \geq 2$ have the property of attenuated correlation, according to which they rapidly (exponentially as a rule) approach zero as the distance between the points in even one pair of their arguments increases by an amount exceeding the scale of the correlations of the particles. The two-particle correlation function $g_2(\mathbf{r}_1, \mathbf{r}_2)$ is especially important in the statistical theory of gases and liquids. It has been applied by Lax [51] and by Fürth and Williams [52] (see also [48]) for studying molecular scattering of x-rays in liquids and of light in liquids near the critical point (the critical-opalescence phenomenon). The two-particle correlation function is used in treating the effect of an electrostatic interaction on scattering of electromagnetic waves by atmospheric aerosols. [53]

d) Discrete and Continuous Scattering Media

The system of multiple wave-scattering equations (3.6) and the set of distribution functions or correlation functions of the particles define a model of a discrete scattering medium. When a wave enters such a medium, then it is said to undergo multiple scattering by the ensemble of particles. In addition to the theory of multiple scattering of waves in a discrete medium, there is a theory of propagation of waves in a randomly-inhomogeneous medium that has been developed by Bourret, [54] Furutsu, [55] Tatarskiĭ, [56] Frisch, [35] Finkel'berg, [50] et al. (see the review [21]).

The model of a randomly-inhomogeneous medium, which is sometimes also called a continuous random or scattering medium, is defined by the law of the spatial fluctuations of its potential. This law is fully defined by the set of moment or cumulant functions of the potential.[35,49,50] In the simplest case of Gaussian potential fluctuations, it suffices to fix its first two cmulants. Generally one must know the entire set of cumulants of different orders of the potential of the medium.

Instead of the system of multiple wave-scattering equations (3.6), which pertains to the model of a discrete scattering medium, one writes for the model of a continuous random medium a Born series of perturbation theory for the field. The latter is obtained by solving the wave integral equation (3.3) by successive approximations in terms of the potential of the medium. The terms of this series describe the multiple scattering of waves by the elements of volume of the continuous medium.

Frisch [33] has discussed the relationship between the models of discrete and continuous random media. If one knows the distribution functions or correlation functions of the particles of a discrete medium, one can calculate by the formulas of [49] the moment and cumulant functions of its potential, which is given by (3.1). This indicates that the propagation of waves in a discrete scattering medium can be studied by the same method as for a continuous medium.

Yet the transition from the discrete to the continuous model of a medium assumed that scattering events by elements of volume of separate particles and by those of a certain given particle are equivalent to one another. This assumption is justified if the particles of the medium are weak or "soft" scatterers. For each of such isolated particles, the scattering operator t fits a Born series of perturbation theory in powers of its potential V_0. In this case, the model of a discrete medium has no advantages over the model of a continuous medium.

Yet if the particles have sufficiently highly marked scattering properties (strong or "hard" scatterers), for which a Born approximation for the scattering operator of the isolated particle is inapplicable, the transition from the model of a discrete to a continuous medium is not justified, either from the physical or the practical standpoint.

It is sometimes convenient to use a combined model of discrete and continuous scattering media. Ovchinnikov [57] has applied this approach for studying radiation transport in the visible range in a turbulent atmosphere containing an aerosol.

4. THE CONCEPT OF AN EFFECTIVE INHOMOGENEITY

In order to elucidate the conditions of applicability of the theory of radiation transport within the framework of the multiple-scattering theory, it suffices to restrict the treatment to the averages over the ensemble of field values $\langle \psi(\mathbf{r}) \rangle$ and the bilinear combination of the field $\langle \psi(\mathbf{r}_1)\psi^*(\mathbf{r}_2) \rangle$, which is also called the covariance, and which is the mutual spatial coherence function of the field.

There are several asymptotic methods for calculating the mean field and the covariance of the field. The most pictorial of them from the physical standpoint and most general from the standpoint of getting concrete results is the Green's function method and the diagram technique of Feynman, which lead to equations[3)] of the Dyson (D) and Bethe-Salpeter (BS) types.

In the case of a discrete scattering medium[4)], the D and BS equations have been formulated through the studies of Foldy, [47] Lax, [51] Gnedin and Dolginov, [59] Frisch, [33] and Finkel'berg. [50] Frisch [33] has derived exact D and BS equations. They contain unknown kernels, which are called the mass operator M and the intensity operator K. They are equal to the sums of all possible strongly connected one-row and two-row diagrams without external lines of propagation. Approximate D and BS equations having assigned approximate values of the kernels M and K are of great interest in applications. The most general equations of this type were derived by Finkel'berg [50] by the method of correlation groups with the kernels M and K in the single-group approximation.

The D and BS equations in symbolic-operator form are written respectively as (see the review [21]):

3)See the review of Apresyan [58] for the derivation of the D and BS equations without using the diagram technique.

4)See the review [21] for the D and BS equations for a continuous medium.

$$\langle\psi\rangle = \psi_0 + G_0 M\,\langle\psi\rangle,$$
$$\langle\psi\times\psi^*\rangle = \langle\psi\rangle\times\langle\psi^*\rangle + \langle G\rangle\times\langle G^*\rangle K\,\langle\psi\times\psi^*\rangle; \qquad (4.1)$$

In the BS equation, $\langle G(\mathbf{r}, \mathbf{r}')\rangle$ denotes the average of the Green's function of the scattering medium, and the multiplication sign $\times$ denotes the bilinear combination of the values of the field or of the kernels of the operators (of the average Green's function). These equations are macroscopic in nature in the sense that all of the information on the optical and statistical properties of the ensemble of particles is contained in the kernels M and K.

The D and BS equations are integral equations, and they describe coherent and partially coherent scattering of waves. The physical meaning of the kernels M and K is revealed by representing the solution of the D equation as a series of successive approximations in powers of the kernel M, and the solution of the BS equation in powers of the kernel K. We shall associate with the kernels M and K the concept of an effective inhomogeneity of the scattering medium. Then the terms of the series for the D and BS equations describe the consistently coherent and partially coherent scattering of waves by the effective inhomogeneities. The kernel $M(\mathbf{r}, \mathbf{r}')$ is a two-point kernel that serves to express the spatial dispersion of the waves of the mean field. It gives the optical properties of an inhomogeneity with respect to coherent scattering of waves. The kernel $K(\mathbf{r}_1, \mathbf{r}_1'; \mathbf{r}_2, \mathbf{r}_2')$ is a function of four points, and it shows that an inhomogeneity plays the role of a quadrupole-type converter of the mutual-coherence function. In order to find the mean intensity of the field scattered by an inhomogeneity, we must generally know the mutual-coherence function (rather than the mean intensity alone) of the incident field. This indicates that the inhomogeneities act as partially coherent (instead of incoherent) scatterers, and the kernel K gives the optical properties of the inhomogeneity with respect to partially coherent scattering of the waves.

a) The Optical Theorem for an Inhomogeneity and for the Entire Scattering Volume

In transport theory, the extinction and scattering coefficients of a volume element (or the extinction and scattering matrices for electromagnetic radiation) are connected by a relationship [30] that expresses the law of conservation of energy. An analogous relationship exists also in the theory of multiple scattering of waves. It is formulated as an optical theorem for the kernels M and K. For a medium having no true absorption, it is written as [60]:

$$M(\mathbf{r}_2', \mathbf{r}_1') - M^*(\mathbf{r}_1', \mathbf{r}_2') = \int [\langle G(\mathbf{r}_2, \mathbf{r}_1)\rangle - \langle G^*(\mathbf{r}_1, \mathbf{r}_2)\rangle]\, d^3r_1 d^3r_2 K(\mathbf{r}_1, \mathbf{r}_1'; \mathbf{r}_2, \mathbf{r}_2'). \qquad (4.2)$$

This relationship establishes the connection between the imaginary components of the kernel M, the mean Green's function $\langle G\rangle$, and the kernel K.[5)]

The optical theorem (4.2) expresses the law of conservation of energy for an effective inhomogeneity. Its right-hand side contains the mean Green's function, rather than the Green's function of free space, as in the optical theorem for the entire volume of the scattering medium, which one gets by averaging Eq. (3.5) over the ensemble. This distinction stems from the fact that each effective inhomogeneity lies within the scattering medium, and is surrounded by other inhomogeneities.

It is customary to characterize the wave scattering by the volume of the medium in terms of the amplitudes and cross-sections for coherent and partially coherent scattering. The quantities $(-1/4\pi)\langle\tilde{T}(\mathbf{s}, \mathbf{s}_0)\rangle$, $(-1/k_0)\,\mathrm{Im}\langle\tilde{T}(\mathbf{s}_0, \mathbf{s}_0)\rangle$, and $(4\pi)^{-2}|\langle\tilde{T}(\mathbf{s}, \mathbf{s}_0)\rangle|^2$ are the amplitude, the extinction cross section, and the differential coherent-scattering cross section. The difference

$$C = -\frac{1}{k_0}\,\mathrm{Im}\,\langle\tilde{T}(\mathbf{s}_0, \mathbf{s}_0)\rangle - \frac{1}{(4\pi)^2}\int_{4\pi} d^2s\,|\langle\tilde{T}(\mathbf{s}, \mathbf{s}_0)\rangle|^2 \qquad (4.3)$$

between the attenuation cross section and the total coherent-scattering cross section is called the absorption cross section for coherent radiation.[46] It serves to measure the fraction of the energy of the coherent radiation that goes into partially coherent scattering, or can also undergo true absorption.

Partially coherent scattering is characterized by its differential cross section $(4\pi)^{-2}\tilde{U}(\mathbf{s}, \mathbf{s}_0)$, where we have denoted

$$\tilde{U}(\mathbf{s}, \mathbf{s}_0) = \langle|\tilde{T}(\mathbf{s}, \mathbf{s}_0)|^2\rangle - |\langle\tilde{T}(\mathbf{s}, \mathbf{s}_0)\rangle|^2,$$

to be equal to the mean square of the scattering amplitude fluctuations of the medium. According to the optical theorem for the entire volume of a medium lacking true absorption, the absorption cross section for coherent radiation equals the total cross section for partially coherent scattering:

$$C = \frac{1}{(4\pi)^2}\int_{4\pi} d^2s\,\tilde{U}(\mathbf{s}, \mathbf{s}_0) \equiv H. \qquad (4.4)$$

This equality means that the energy taken from the wave incident on the volume goes into coherent and partially coherent scattering.

If the medium shows true absorption, then one introduces the cross section C_{tr} for true absorption of the total wave, along with the absorption cross section C for coherent radiation. In this case, the optical theorem for the entire volume of the medium is written in the form of the equality

$$C = H + C_{tr}, \qquad (4.5)$$

According to this, the energy extracted from the incident wave goes not only into coherent and partially coherent scattering, but also into true absorption.

b) The Method of Group Expansions for the Mass Operator and the Intensity Operator. The Single-Group Approximation

The pictorial quality of the Feynman diagram technique consists in the fact that it depicts each elementary process of coherent and partially coherent scattering of waves in a medium with the aid of diagrams (see, e.g., [50]). These diagrams are derived by expanding the scattering operator T of the ensemble of particles and its bilinear combination $T\times T^*$ in series in terms of the number of scattering events. Then one averages these over the ensemble by using the class distribution functions. A further essential step is to transform from the class distribution functions of the particles to their correlation functions[6)] by formulas like (3.7). This

[5)]Rosenbaum [61] has also tried to derive such a relationship, as has Rozenberg [32] for the special case of an infinite, planar thin layer.

[6)]Germogenova [62] has proposed a method of group integrals in problems of scattering of electromagnetic waves that is based on expanding the Gibbs distribution for the ensemble of interacting particles in terms of the correlation functions of Jursel (see [48], p. 125).

permits one to classify all the diagrams as being strongly or weakly connected. Among these, the strongly connected diagrams that enter into the kernels M and K describe the scattering of waves by an individual effective inhomogeneity. However, the weakly connected diagrams portray the successive scattering of waves by several inhomogeneities.

All of the diagrams constituting the kernels M and K are classified into single-group and multigroup diagrams. The single group diagrams are constructed from particles that belong to one correlation group, and which are linked by one correlation function. These diagrams depend linearly on the correlation functions of the particles, and they give contributions to the kernels M and K that decline with increasing distance between their arguments at the rate of the correlation functions of the particles. Diagrams of this type decrease rapidly.[34] The multigroup diagrams contain several correlation groups of particles, and these groups interact with one another only by means of their mutual wave illumination. Hence the contributions of multigroup diagrams to the kernels M and K decline with increasing distance between their arguments as some integral power of the Green's function of free space. Such diagrams are slowly declining. Since the kernels M and K give the optical properties of an effective inhomogeneity that is correlated with the volume elements of transport theory, then at first we can naturally keep in the kernels M and K only the single-group, rapidly declining diagrams, and reject the multigroup, slowly declining diagrams. Here one gets the single-group approximation that Finkel'berg[50] has treated.

In the single-group approximation, the kernels M and K have the form

$$M_1=\sum_{n=1}^{\infty}\frac{1}{n!}T^{gr}_{1\ldots n}g_n(1\ldots n),$$
$$K_1=\sum_{n=1}^{\infty}\frac{1}{n!}(T\times T^*)^{gr}_{1\ldots n}g_n(1\ldots n); \tag{4.6}$$

Here the $g_n(1 \ldots n)$ are the correlation functions of the particles having their centers at the points 1, ..., n, over whose coordinates the integration is being performed. The superscript gr indicates the group scattering operators, which are defined by

$$T^{gr}_1=t,\quad T^{gr}_{12}=T_{12}-t_1-t_2,\ \ldots,$$
$$(T\times T^*)^{gr}_1=t_1\times t^*_1,\quad (T\times T^*)^{gr}_{12}=T_{12}\times T^*_{12}-t_1\times t^*_1-t_2\times t^*_2,\ \ldots$$

$T_{1\ldots n}$ ($T_1 = t_1$), n = 1, 2, ..., denotes the scattering operator of the system of n particles.

In the single-group approximation of (4.6), the kernels M and K satisfy the conservation law in the form of an optical theorem like Eq. (4.2), but with the Green's function G_0 of free space on the right-hand side of this relationship instead of the mean Green's function $\langle G\rangle$. This means that one neglects in the single-group approximation (4.6) the effect whereby an inhomogeneity lies in an environment of other inhomogeneities when one is expressing the law of conservation of energy for the inhomogeneity.

The single-group approximation (4.6) is very general, and it combines many other known approaches to the theory of multiple scattering of waves. If the particles are not correlated, then the single-group approximation (4.6) leads to a model of independent scatterers, according to which

$$M_1=t_1g_1(1),\quad K_1=t_1\times t^*_1g_1(1). \tag{4.7}$$

Foldy[47] has treated this model in the case of a scalar field for point, isotropic scatterers, and Gnedin and Dolginov[59] have applied it for studying quantum-mechanical scattering of a flux of particles by independent centers of force. For an electromagnetic field, the model of independent scatterers of (4.7) is used in the molecular optics of a rarefied gas, where the scatterers are taken to be point dipoles.

The independent-scatterer model of (4.7) takes no account of correlation of particles. The model of correlated scatterers with weak mutual illumination is more exact. If the effect of mutual illumination of the particles in each correlation group is small, then one can expand the scattering operators of the groups in terms of the number of scattering events. Here the formulas of the single-group approximation (4.6) in the second order of multiplicity of scattering take on the form

$$M_1=t_1g_1(1)+t_1G_0t_2g_2(12),$$
$$K_1=t_1\times t^*_1g_1(1)+(t_1\times t^*_2+t_1\times t^*_1G^*_0t^*_2+t_1G_0t_2\times t^*_1)\,g_2(12), \tag{4.8}$$

Here additional terms of the third order of smallness have been included on the right-hand side of the equation for K_1. They cause the expressions (4.8) exactly to satisfy an optical theorem of the type of (4.2) with the Green's function of free space. The expressions (4.8) are the basis of study of the molecular scattering of x-rays in liquids and light in liquids near the critical point,[48,51,52] 7) and they are also used in taking account of the effect of electrostatic interaction on scattering of electromagnetic waves by atmospheric aerosols.[53]

In the single-group approximation (4.6), the kernels M and K decline with increasing distance between their arguments at the rate of the correlation functions of the particles, while in the independent-scatterer model of (4.7), they have a scale of non-locality of the order of the dimensions of the particles.

If the linear dimensions of the volume of the scattering medium are large in comparison with the inhomogeneity scale, then the kernel $M_1(\mathbf{r}, \mathbf{r}')$ of the mass operator can be represented in the form $\mathscr{M}_1(\mathbf{r}-\mathbf{r}')$ outside a narrow zone of the volume near the boundary that has a width of the order of the inhomogeneity scale. Analogously, the kernel $K_1(\mathbf{r}_1, \mathbf{r}_1'; \mathbf{r}_2, \mathbf{r}_2')$ of the intensity operator can be written as $\mathscr{K}_1(\mathbf{R}-\mathbf{R}', \mathbf{r}, \mathbf{r}')$, where $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$, and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ are the coordinate of the center of gravity and the difference coordinate for the points $\mathbf{r}_1$ and $\mathbf{r}_2$; $\mathbf{R}'$ and $\mathbf{r}'$ pertain to the points $\mathbf{r}_1'$ and $\mathbf{r}_2'$.

Let the linear dimensions of the studied volume of the scattering medium be small in comparison with the extinction length, and let a plane wave be incident on it. We can take such a volume as the unit volume of transport theory. If we neglect the change in the incident wave throughout the chosen volume, its extinction and scattering cross sections have the form[34] of Ω/d and $\Omega f(\mathbf{s}, \mathbf{s}_0)$, where Ω is the size of the volume, and $1/d$ and $f(\mathbf{s}, \mathbf{s}_0)$ denote

$$\frac{1}{d}=-\frac{\operatorname{Im}\tilde{\mathscr{M}}_1(k_0)}{k_0},\quad f(\mathbf{s},\mathbf{s}_0)=\frac{1}{(4\pi)^2}\tilde{\mathscr{K}}_1(\mathbf{s},\mathbf{s}_0). \tag{4.9}$$

In these equations, $\tilde{\mathscr{M}}_1(k_0)$ is the Fourier image of the

7)Molecular light scattering in a liquid has been treated in [63-65] from the standpoint of fluctuations of the dielectric constant of a continuous scattering medium.

kernel $\widetilde{\mathscr{M}}_1(\mathbf{r})$ as calculated on the k_0^2 surface; $\widetilde{\mathscr{K}}_1(\mathbf{s}, \mathbf{s}_0)$ is the Fourier transform of the kernel $\mathscr{K}_1(\mathbf{R}, \mathbf{r}, \mathbf{r}')$ in the difference coordinates $\mathbf{r}$ and $\mathbf{r}'$ as calculated on the k_0^2 surface and integrated over the coordinates $\mathbf{R}$ of the center of gravity. As must be the case, the extinction and scattering cross sections of the volume element are proportional to its size. The quantities $1/d$ and $f(\mathbf{s}, \mathbf{s}_0)$ are the extinction and scattering coefficients of the volume element. They satisfy the law of conservation of energy in the form of the relationship that is customary in transport theory.

As we see from the formulas of the single-group approximation (4.6), we can calculate the extinction and scattering coefficients of the volume element of (4.9) if we know the scattering amplitudes of a plane wave for a system of one, two, etc., particles, and the correlation functions of the particles. Here the contributions of the correlations of the particles to the extinction and scattering coefficients depend nonlinearly on the density of particles. This leads to breakdown of the law of additivity of transverse cross sections[1] for monochromatic light, and it constitutes one of the manifestations of the cooperative effects. For a medium that consists of spherical particles, in calculating the extinction and scattering coefficients of (4.9) for light for the volume element, one can use Mie's solution[37] of the problem of scattering of a plane electromagnetic wave by a spherical particle, and also the results of Trinks[66] and Germogenova[67] from studying the scattering of a plane electromagnetic wave by two spherical particles.

The single-group approximation (4.6) has one defect concerning the law of conservation of energy, as noted by Frisch[68] and Howe.[69] This involves the fact that the kernels M and K in the single-group approximation satisfy an optical theorem of the type of (4.2) with the Green's function of free space, rather than the mean Green's function. Hence the solutions of the D and BS equations with these values of the kernels M and K do not satisfy the optical theorem for the entire volume of the scattering medium exactly, but only approximately. The reason for this defect in the single-group approximation (4.6) for the kernels M and K was revealed in[60]. Namely, this approximation is derived in the first order of the expansion of the exact values of the kernels M and K in terms of the small parameter of the group expansion.[50] Here the optical theorem (4.2) for an inhomogeneity is also expanded in terms of the small parameter of the group expansion. This gives relationships that play the role of the optical theorem of the first, second, etc., orders. Among these, the first-order optical theorem has the form of (4.2), but with the Green's function of free space. Hence the solutions of the D and BS equations with the single-group kernels M and K satisfy the optical theorem for the entire volume of the medium to the accuracy of the two-group terms.

The fact that the optical theorem for the entire volume of the scattering medium is satisfied only approximately in the single-group approximation (4.6) can be understood provisionally as being a manifestation of some effective "true absorption." The size of its cross section $C_{\text{eff.tr}}$ can be obtained by using a relationship[70] that connects the optical theorems for an inhomogeneity and for the entire volume of the medium, and it equals

$$C_{\text{eff.tr}} = \frac{1}{k_0}\int \operatorname{Im}\left[\mathscr{G}(\mathbf{r}_1, \mathbf{r}_2) - G_0(\mathbf{r}_1 - \mathbf{r}_2)\right] d^3\mathbf{r}_1\, d^3\mathbf{r}_2 K_1(\mathbf{r}_1, \mathbf{r}_1'; \mathbf{r}_2, \mathbf{r}_2')\, d^3\mathbf{r}_1'\, d^3\mathbf{r}_2' \Phi(\mathbf{r}_1', \mathbf{r}_2'); \tag{4.10}$$

Here $\mathscr{G}(\mathbf{r}, \mathbf{r}')$ and $\Phi(\mathbf{r}_1, \mathbf{r}_2)$ denote the mean Green's function and the covariance of the field that satisfy the D and BS equations with the kernels M and K in the single-group approximation (4.6). One of the conditions for applicability of the single-group approximation must be that we can neglect the value of the effective "true absorption" cross section of (4.10).

The formulas (4.6) of the single-group kernels M and K pertain to a model of a discrete scattering medium. One can derive[50] analogous formulas of the single-group approximation also for the model of a continuous scattering medium. In contrast to (4.6), the right-hand sides of these formulas contain the cumulants of the potential of the medium and the products of values of the Green's function of free space. If the potential of the continuous medium fluctuates according to a Gaussian law with a zero mean value, then the single-group approximation for the kernels M and K goes over into the approximation of Bourret[54] and the ladder approximation.[56]

Whenever the particles are weak scatterers, there is a simple relationship between the single-group kernels M and K in the discrete and continuous models of the scattering medium. It is established by expanding the scattering operators of the particles in the formulas of (4.6) in a Born series of perturbation theory in powers of their potential, and by using the formulas[49] for expressing the cumulants of the potential of the discrete medium in terms of the correlation functions of the particles.

5. COHERENT AND PARTIALLY COHERENT SCATTERING OF WAVES IN THE FRAUNHOFER APPROXIMATION

Effective inhomogeneities having the kernels M and K in the single-group approximation (4.6) are spatially localized. There is a physically graphic approximate method[71] of treating the coherent and partially coherent scattering of waves in a medium containing such inhomogeneities. This method is based on the assumption that the main contribution to the scattering of the waves comes from the long-range configurations of the inhomogeneities, in which they lie in one another's Fraunhofer zone. For coherent scattering, the Fraunhofer approximation is equivalent to neglecting the spatial dispersion of the waves, and for partially coherent scattering, it is equivalent to going over to transport theory.[8)]

a) Neglect of Spatial Dispersion of Waves. The Van de Hulst Approximation

The D equation in the Fraunhofer approximation reduces to the Helmholtz equation with the effective complex wave number k_1, whose square equals

$$k_1^2 = k_0^2 - \widetilde{\mathscr{M}}_1(k_0). \tag{5.1}$$

If the scattering medium is infinite, and a point source lies at the origin, then the mean Green's function $\mathscr{G}(\mathbf{r})$ in the Fraunhofer approximation equals

$$\mathscr{G}(r) = -\frac{e^{ik_1 r}}{4\pi r}. \tag{5.2}$$

Under the condition

8)A derivation is proposed in [72,73] of the transport equation from the D and BS equations that doesn't use the Fraunhofer approximation.

$$\frac{|\widetilde{\mathscr{M}}_1(k_0)|}{k_0^2} \ll 1 \tag{5.3}$$

the effective complex refractive index of the medium differs but little from unity. This permits one in calculating k_1 to take the approximate square root in the right-hand side of Eq. (5.1). Then, two times the imaginary component of the effective complex wave number equals the extinction coefficient $1/d$ of the volume element, which is equal to (4.9). The quantity d is called the extinction length. The mean Green's function (5.2) in the Fraunhofer approximation leads to an exponentially declining intensity of the mean field with a distance that corresponds to Bouguer's law[2,25] in transport theory.

The problem of the effective refractive index of the scattering medium, the effective acting field, and the effective dielectric-constant tensor is one of the principal problems in the theory of multiple scattering of waves, and a large number of studies has been devoted to solving it. In molecular optics, this problem has been studied by Ewald[74] and Oseen[75] (for their studies, see the book by Born and Wolf[76]). They have established an "extinction theorem" that leads to the Lorentz-Lorenz formula. Sharapov[77] has applied the Ewald method for calculating the interference of light in thin plates. Debye[78] has introduced the effective dielectric constant of a solution of low concentration. Maxwell Garnett[79] (see also[80]) has extended the Lorentz-Lorenz formula from molecular optics to the optics of colloidal solutions. Rozenberg[80,81] has studied the optical properties of a two-dimensional colloidal coating that has a nature different from the underlying medium. The cited studies, beginning with Ewald and Oseen, have treated cases of a medium showing dipole (Rayleigh) scattering by the particles. Rozenberg[32] has shown that the effective complex refractive index of the scattering medium can also be introduced in the case in which each particle of the medium is characterized by some scattering matrix. Finkel'berg[43] has derived a generalized variant of the Lorentz-Lorenz formula (more exactly, that of Maxwell Garnett) for the effective static dielectric constant of an emulsion with account taken on the right-hand side of the formula of the terms that are quadratic in the density of drops.[9)]

The conditions for applicability of the Fraunhofer approximation for solving the D equation with the single-group kernel M impose limitations on the properties of the scattering medium (and on the distance traveled by the wave or the dimensions of the volume of the medium). In order to elucidate these conditions, let us examine separately the two cases in which the scattering medium is infinite, or it occupies a half-space or a plane layer.

In an infinite scattering medium, the exact solution[33,50] of the D equation for the mean Green's function leads to the dispersion equation

$$k^2 = k_0^2 - c\widetilde{\mathscr{M}}_1(k). \tag{5.4}$$

This equation generally has several roots. However, let us assume the derivative of the Fourier transform of the kernel M with respect to the square of the wave number to be small:

$$\left|\frac{dc\widetilde{\mathscr{M}}_1(k)}{dk^2}\right|_{k^2=k_0^2} \ll 1, \tag{5.5}$$

Then the main root is k_1, which is closest to the wave number k_0 of free space.[50] Under the condition (5.5), the square of this root k_1 is approximately equal to its value (5.1) in the Fraunhofer approximation. When we use only one root k_1 of the dispersion equation (5.4) as calculated in the Fraunhofer approximation (5.1), this implies neglect of the spatial dispersion of the waves.

According to the condition (5.5) for neglecting spatial dispersion of the waves, the Fourier image of the single-group kernel M must be a sufficiently smooth function of the wave vector. This requirement can be satisfied, since the kernel M of the single-group approximation declines rapidly as the distance between its arguments is increased. One can establish the concrete value of the conditions for neglecting spatial dispersion of the waves with the example of a continuous scattering medium having the kernel M in the Bourret[54] approximation with an exponential cumulant of the potential. Here the solution of the D equation found by Tatarskiĭ and Gertsenshteĭn[56,83] implies that the dispersion equation (5.4) has two roots k_1 and k_2. The condition (5.5) of smallness of the derivative of the Fourier image of the kernel M takes on the form

$$\frac{l}{d} \ll 1, \tag{5.5a}$$

Here l is the effective-inhomogeneity scale, for fine-scale inhomogeneities ($k_0 l \ll 1$) and coarse-scale inhomogeneities ($k_0 l \gg 1$). According to the inequality (5.5a), the inhomogeneity scale l is small in comparison with the extinction length d. Under this condition, the wave having the wave number k_2 is exponentially small in intensity at a distance r from the source that exceeds the inhomogeneity scale l, when $r \gg l$.

Whenever the scattering medium has a boundary, the transition from the D equation to the Helmholtz equation with an effective complex refractive index faces the problem of the boundary conditions for the mean field.[84] This problem admits a simple solution if the effective complex refractive index of the medium differs little from unity. Let us assume a continuous scattering medium occupying a half-space, with a plane wave incident on its boundary. Then, according to the exact solution of the D equation with the kernel M in the Bourret approximation that was obtained by using[85],[10)] the refracted mean field in the medium is equal to the sum of two waves that propagate from the separation boundary. Under the condition (5.5a) that the derivative of the Fourier image of the kernel M should be small, one of these refracted waves is exponentially small in intensity outside a narrow zone near the boundary zone near the boundary whose width is of the order of the inhomogeneity scale l. Under the additional condition (5.3) of smallness of the deviation of the effective complex refractive index from unity, the other refracted wave has the same form as the solution of the Helmholtz equation having the same effective refractive index in the geometrical-optics approximation with neglect of reflection and refraction of the waves at the separation boundary. Here the effect of the medium is reduced to an additional complex phase shift of the wave. Van de Hulst[46] has used this case of the geometrical-optics approximation together with the

[9)]See the review of Ryzhov and Tamoĭkin [82] on the effective dielectric constant of a continuous scattering medium.

[10)]The problem of the mean field in a medium having a separation boundary has been treated by many authors (see, e.g., [32,59,76]). Exact solutions of the D equation for a continuous medium in the form of a plane layer or sphere with the kernel M in the Bourret approximation have been found in [84,86,87]. The D equation for a plane layer has been brought in [85] into a form in which it is easily solved.

Huyghens principle for studying attenuation and anomalous diffraction of light by large spherical particles having a complex refractive index close to unity.

The Van de Hulst approximation permits a simple solution of the problem of coherent scattering of a plane wave by a bounded volume of a scattering medium. If the volume has the shape of a sphere whose radius L is large in comparison with the extinction length d, $L \gg d$, then its absorption cross-section for coherent radiation will approach the geometric cross-section πL^2. This means that such an optically deep sphere will behave like a black body.[46] In the black-body limit, the extinction cross-section and the total coherent-scattering cross-section are $2\pi L^2$ and πL^2.

If the scattering medium is weakly inhomogeneous on the average on the scale of the wavelength and the effective inhomogeneity, then one can apply Kravtsov's[88] geometrical-optics method for media having spatial dispersion for solving the D equation.

b) Distinguishing the Effects of the Distribution of Inhomogeneities in the Close- and Long-Range Regions in Partially Coherent Scattering

In a medium whose effective inhomogeneities are given by the kernels M and K of the single-group approximation (4.6), the partially coherent scattering of waves is described by the solution of the BS equation for the covariance of the field $\Phi(\mathbf{R}, \mathbf{r})$, where $\mathbf{R}$ and $\mathbf{r}$ are the coordinates of the center of gravity and the difference coordinates of the points of observation. If the scattering medium occupies a bounded volume, it suffices to solve this equation within the medium, whereupon the covariance of the field outside the medium is found by quadratures. Let us denote by $\Phi_0(\mathbf{R}, \mathbf{r})$ the inhomogeneous term of the BS equation. It constitutes the coherent component of the covariance of the field, and it is expressed in terms of the mean field, which is calculated in the van de Hulst approximation in the problem of a plane wave incident on the scattering volume. One calculates the average Green's function $\mathcal{G}(\mathbf{r}, \mathbf{r}')$ in the same approximation for the radiation propagating in and exiting from the medium.

The integral term of the BS equation for the covariance of the field within the medium contains the bilinear combination of the average Green's function. It describes the propagation of mutual coherence of the field between successive events of partially coherent scattering by the inhomogeneities of the medium. If the inhomogeneities lie in one another's long-range zone, then the bilinear combination of the average Green's function is represented by a Fraunhofer expansion in the form of Aiken[89]:

$$\mathcal{G}\left(\mathbf{R}+\tfrac{1}{2}\mathbf{r}\right)\mathcal{G}^*\left(\mathbf{R}-\tfrac{1}{2}\mathbf{r}\right) = \left[1+O\left(\frac{k_0r^3}{R^2}\right)+O\left(\frac{r^2}{Rd}\right)+O\left(\frac{r^2}{R^2}\right)\right]|\mathcal{G}(R)|^2 e^{ik_0\mathbf{s}\mathbf{r}}, \tag{5.6}$$

Here the unit vector $\mathbf{s}$ lies along the vector $\mathbf{R}$. The principal term of this expansion gives the Fraunhofer approximation. The remaining terms constitute the Fresnel corrections. Let us transform in the integral term of the BS equation to the Fraunhofer approximation for the bilinear combination of the average Green's function, and also neglect the change in the covariance of the field $\Phi(\mathbf{R}, \mathbf{r})$ as a function of the coordinates of the center of gravity $\mathbf{R}$ on the effective-inhomogeneity scale. Consequently, the covariance of the field within the medium will admit a representation in the form[71,72]

$$\Phi(\mathbf{R}, \mathbf{r}) = \Phi_0(\mathbf{R}, \mathbf{r}) + \int_{4\pi} e^{ik_0\mathbf{s}\mathbf{r}} I_s(\mathbf{R}, \mathbf{s})\, d^2\mathbf{s}; \tag{5.7}$$

Here $I_s(\mathbf{R}, \mathbf{s})$ denotes the radiance of the scattered radiation. It satisfies the transport equation in integral form with the known inhomogeneous term that corresponds to single partially coherent scattering of the covariance $\Phi_0(\mathbf{R}, \mathbf{r})$ of the mean field, and with the extinction coefficient $1/d$ and scattering coefficient $f(\mathbf{s}, \mathbf{s}_0)$ of the volume element equal to (4.9).

Equation (5.7) shows that the correlation function of the field within the medium is expressed in terms of the radiance of the scattered radiation by a relationship like that found by Dolin.[90] [11)]

The problem is not obvious of the conditions for applicability of the Fraunhofer approximation for multiple, partially coherent scattering of waves, in which the mutual coherence of the scattered field in each elementary scattering event by an effective inhomogeneity succeeds in acquiring its asymptotic form before the next scattering event. According to Gnedin and Dolginov[69] and Borovoĭ,[91] these conditions for an ensemble of uncorrelated particles reduce to the requirement that the amplitude of the scattering by an isolated particle should be small in comparison with the mean distance between particles. Ryazanov[92] assumes for the same type of ensemble that the amplitude of scattering by an isolated particle is small in comparison with the extinction length. According to Watson,[7] who treated an ensemble of correlated electrons in a plasma, the wavelength should be small in comparison with the extinction length.

Elucidation of the conditions for applicability of the Fraunhofer approximation for treating partially coherent scattering of waves is reduced to estimating the contribution of the Fresnel corrections of the Aiken expansion (5.6) to the solution of the BS equation. The size of this contribution determines the accuracy of the Fraunhofer approximation, which can be estimated, e.g., from the error of calculating in the Fraunhofer approximation the differential cross-section for incoherent scattering by the volume of the medium.

In every sequence of events of partially coherent scattering by inhomogeneities, one can distinguish the effects of the distribution of inhomogeneities in the close- and long-range regions with respect to one another by using some parameter R_0 of the scale of the close-range region. An estimate[93] shows that the size of the effect of the close-range region of the inhomogeneity distribution is proportional to the ratio R_0/d of the scale R_0 of this region to the extinction length d. One uses the Fraunhofer approximation for the bilinear combination of the average Green's function for calculating the size of this effect on the long-range region of the inhomogeneity distribution, Here one drops those Fresnel corrections in the Aiken expansion (5.6) whose relative contribution is of the order of the largest of the three quantities: k_0r^3/R_0^2, r^2/R_0d, or r^2/R_0^2.

Neglecting the effect of the close-range region of the

11) Rozenberg [26] has derived a relationship of the type found by Dolin for a partially coherent wave field by spatially averaging the mutual-coherence function of the field over the coordinates of the center of gravity of the observation points.

inhomogeneity distribution and the Fresnel corrections to the effect of the long-range region transforms the partially coherent into incoherent scattering, and it imposes upper and lower bounds on the scale R_0 of the close-range region. In turn, this leads to restrictions on the scattering medium.

The final solution of the problem of the accuracy of the Fraunhofer approximation for partially coherent scattering can be found by estimating the resolvent of the BS equation within the scattering medium. One can make such an estimate for an optically deep volume of a medium[12)] by using the solution of the homogeneous BS equation in an infinite medium[94] and the theorem of Goursat on the simple pole of the resolvent.[95] Let us take an optically deep volume of a discrete scattering medium in the form of a sphere of radius L, whose particles are uncorrelated and are weak scatterers. Moreover, let upper bounds exist for the ratios K_0r_0 (of the radius of the particles of the particles to the wavelength) and L/d (of the dimension L of the volume of the medium to the extinction length d). Then the relative error of applying the Fraunhofer approximation for calculating the differential cross-section for partially coherent scattering by the volume of the medium proves to approach zero at the rate of the ratios r_0/d[13)] and $r_0L_{\Phi_0}$ of the radius r_0 of the particles to the extinction length d and to the spatial-inhomogeneity scale L_{Φ_0} of the coherent component $\Phi_0(\mathbf{R}, \mathbf{r})$ of the covariance of the field as a function of the coordinates $\mathbf{R}$ of the center of gravity. At fixed ratios r_0/d and r_0/L_{Φ_0}, the accuracy of the Fraunhofer approximation declines with increasing k_0r_0 and L/d, according to the resolvent method[93] of estimating it.

The representation (5.7) of the covariance of the field within the scattering medium in terms of the radiance establishes the nexus between the theory of multiple scattering of scalar waves and the classical transport theory. The derivation of the transport equation for polarized radiation from the multiple-scattering theory of electromagnetic waves has been treated in part by Rozenberg[32] in the case of a dispersed medium, and in greater detail by Watson[7] for a plasma with account taken of electron correlation; by Dolginov, Gnedin, and Silant'ev[96] for a model of independent scatterers; and by Apresyan[97] in a continuous scattering medium with account taken of interconversion of longitudinal and transverse waves.

An unstudied problem of special interest is that of the applicability of transport theory for electromagnetic radiation in a discrete scattering medium having close-packed particles.[1] This is realized, e.g., in powders, minerals, biological objects, snow, and also under conditions of critical opalescence. A peculiarity of this case is that the particles lie in a highly inhomogeneous field (in the non-wave zone), and one must take account of the longitudinal component of the field of the scattered wave, while mutual-shielding effects of the particles are also very substantial.

c) Estimating the Effective "True Absorption"

The effective "true absorption" cross-section $C_{\mathrm{eff.tr}}$, which is equal to (4.10), was introduced in connection with the defect of the single-group approximation (4.6) concerning the law of conservation of energy (see Sec. 4, b). It is expressed in terms of the difference between the imaginary components of the average Green's function and the Green's function of free space. When divided by the wave number in free space, this difference is of the order of the deviation of the effective refractive index of the scattering medium from unity. Let us take an optically deep volume of a discrete medium in the form of a sphere of radius L whose particles are uncorrelated and are weak scatterers. Then a more detailed estimate shows that the ratio of the cross-section $C_{\mathrm{eff.tr}}$ for effective "true absorption" to the total cross-section πL^2 for incoherent scattering by the volume approaches zero at the rate of the deviation of the effective refractive index from unity, provided that upper bounds exist for the ratios k_0r_0 (of the radius of the particles to the wavelength) and L/d (of the dimension L of the volume to the extinction length d). This implies that the conditions for neglecting the effective "true absorption" are of the same nature as those for applying the Fraunhofer approximation for partially coherent scattering.

d) Scattering Medium Having Large-Scale Inhomogeneities

Whenever the scale of the inhomogeneities of a discrete scattering medium, as defined by the dimensions of its particles and the scale of their correlations, is large in comparison with the wavelength, one can start with the parabolic equation of Leontovich[98] in treating the partially coherent scattering of waves.

The parabolic equation is an approximate substitute for the Helmholtz equation (3.2), and it has the form of a non-steady-state Schrödinger equation (see the review[21]):

$$i\frac{\partial}{\partial t}u(\rho, t) = [-\Delta + V(\rho, t)]\,u(\rho, t). \tag{5.8}$$

Here $u(\rho, t)$ is the complex amplitude of the field; the "time" variable t is equal to $t = x/2k_0$, where x is the longitudinal coordinate with respect to the initial direction of propagation of the wave; ρ represents the transverse coordinates; Δ is the Laplacian in the transverse coordinates; and $V(\rho, t)$ is the potential of the scattering medium, which proves to be a random function of the transverse coordinates ρ and the "time" t. The problem is set up for the parabolic equation (5.8) with "initial" data for the complex amplitude of the field.

Chernov and Dolin (see the review[21]) first applied the parabolic equation for studying the propagation of short waves in a continuous scattering medium. By starting with this equation and applying asymptotic perturbation theory, they derived an equation for the mutual transverse coherence function of the complex amplitude of the field that coincides in its spectral representation with the transport equation in the small-angle approximation.[99,100] Transport theory in the small-angle approximation is simply related to the Green's function method and the Feynman diagram technique.

Let us denote by $\gamma(\rho_1, \rho_2, t) = u(\rho_1, t)u^*(\rho_2, t)$ the

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12)We note that the total incoherent scattering cross-section of an optically deep volume of a medium in the form of a sphere of radius L ($L \gg d$) as calculated in the Fraunhofer approximation approaches the geometric cross-section πL^2 of the sphere, according to the approximation of van de Hulst for coherent scattering and the optical theorem (4.4) for the scattering volume.

13)According to B. I. Stepanov (see [1]), the condition $r_0/d \ll 1$ renders the transport equation inapplicable, e.g., to strongly absorbing powders. Yet in Rozenberg's opinion, [1,27] the transport equation can remain true even in this case if the quantities entering into it are subjected to a special averaging.

bilinear combination of the complex amplitude of the field. It satisfies a Liouville equation (see, e.g., [101]). By starting with the Liouville equation, we can write a Dyson-type equation (a tensor D equation) with a mass operator (a tensor kernel M) in a single-group approximation like (4.6). In cases of models [50] of a discrete or continuous scattering medium, we can write an equation for the approximate value of the mutual transverse coherence function $\langle \gamma(\rho_1, \rho_2, t)\rangle$ of the complex amplitude of the field, equivalently, for the density matrix.[101] As always, the angle brackets denote averaging over the ensemble of fluctuations of the potential of the medium.

Let us assume that the medium is continuous, and its potential $V(\rho, t)$ fluctuates according to a Gaussian law, homogeneously in the space ρ and the "time" t, and is delta-correlated in the "time" t. Then the equation for the density matrix of the complex amplitude of the field derived by Chernov and Dolin acquires the form of a tensor D equation with the tensor kernel M in an approximation of the Bourret type.[54] Tatarskiĭ[22] was able to use the Furutsu-Novikov formula to show that this equation is exact under the given assumption that the potential of the medium is delta-correlated.

Tatarskiĭ's result has stimulated an entire series of studies on propagation of short waves in a randomly-inhomogeneous medium that start with the parabolic equation. On the one hand, the approximation that assumes delta-correlation of the potential of the medium has been widely developed by the studies of Klyatskin and Tatarskiĭ (see their reviews[102-104]). On the other hand, Papanicolacu[23] has taken as a basis his joint studies with Hersh[105,106] and has used the method of time-averaging (see also Papanicolaou and Keller[16] on the two-time method of averaging) in order to obviate the assumption that the potential of the medium is delta-correlated in "time". This has also been done in[24,107-109] by the majorant-process method. Closed approximate equations were derived for the density matrix of the complex amplitude of the field with a rigorous estimate of their limits of applicability.

According to[107], such an approximate equation for the density matrix in the case of an arbitrary fluctuation law of the potential of the medium in space and in "time" has the form of a tensor D equation with a tensor kernel M in a single-group approximation of the type of[50] for a continuous-medium model. With the aid of this equation, while using the relationship mentioned in Sec. 4, b between the single-group approximations for the mass operator in the discrete and continuous models of the medium, one can study the conditions for applicability of transport theory in the small-angle approximation from the standpoint of the theory of multiple scattering of waves in a discrete medium that consists of particles that are large in comparison with the wavelength.

According to[24,108,109], the applicability of transport theory in the small-angle approximation as based on the parabolic equation for a continuous medium having Gaussian large-scale fluctuations of the potential is limited by the condition that the ratio of the effective-inhomogeneity scale to the extinction length should be small enough, while an upper bound should exist for the ratio of the distance traversed by the wave to the extinction length.

From the quantum-mechanical standpoint, the Liouville equation that follows from the Schrödinger equation (5.8) describes the non-steady-state, partially coherent scattering of a de Broglie wave packet in a randomly-variable medium.[109,110] In this problem, the assumption of delta-correlation of the potential of the medium with respect to time implies[109] that the fluctuations of the potential are rapid, i.e., $\omega_l t_0' \ll 1$, where ω_l is the frequency of the de Broglie wave in free space with a wavelength of the order of the spatial scale l of the potential fluctuations, and t_0' is the time scale of the potential fluctuations.[14)] At the opposite limit of slow (or quasistatic in the sense of Chernov[111]) potential fluctuations where the condition[109] $\omega_l t_0' \gg 1$ is satisfied, one obtains a Boltzmann kinetic equation for the density matrix of the wave packet in the mixed coordinate-momentum representation of Wigner.[112] Peierls[8] has applied this equation in studying electron scattering by crystal-lattice impurities.

6. ON THE APPLICABILITY OF THE SINGLE-GROUP APPROXIMATION

The problem of the conditions for applicability of the single-group approximation (4.6) for the kernels M and K is the most difficult one in the statistical basis of transport theory. It is currently amenable to study only by carrying out an asymptotic expansion in a small parameter.

There are several approaches[33,50,56,59,70,113,114] to studying the conditions of applicability of the single-group approximation for the kernels M and K. They all ultimately boil down to comparing the magnitudes of the Feynman diagrams that are dropped in the single-group approximation with those that are taken into account. One of the approaches[70] consists in constructing transformed perturbation-theory series. These series, when written for the exact value of the mean field $\langle \psi(\mathbf{r})\rangle$ and the covariance of the field $\langle \psi(\mathbf{r}_1)\psi^*(\mathbf{r}_2)\rangle$ have the following appearance in symbolic form:

$$\langle\psi\rangle = \varphi + \sum_n (\delta\varphi)_n \mu^n, \tag{6.1}$$

$$\langle\psi \times \psi^*\rangle = \Phi + \sum_n (\delta\Phi)_n \mu^n; \tag{6.2}$$

Here the principal terms $\varphi(\mathbf{r})$ and $\Phi(\mathbf{r}_1, \mathbf{r}_2)$ are obtained by solving the D and BS equations with the kernels M and K in the single-group approximation, the rest of the terms are correction terms, and μ is the small parameter in powers of which the correction terms are ordered. In a discrete medium, it is convenient to take μ to be the small parameter of the expansion in terms of multiplicity of scattering by the particles of the medium.

The correction terms of the transformed series (6.1) and (6.2) are expressed in terms of the multigroup, slowly declining diagrams of the exact values of the kernels M and K that are not taken into account in the single-group approximation. We can estimate those terms by assigning some quadratic functional of the average field and a linear functional of the covariance of the field. In other words, the correction terms of the series (6.1) and (6.2) are estimated by using certain square-law detectors[26,27] of coherent and partially coherent radiation. If the scattering medium occupies a limited volume upon which a plane wave is incident, then one can arrange the detectors in the remote zone of the

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14) The variable $t = (h/2m)\,t'$, where t' is the time, h is Planck's constant, and m is the mass of the particle being scattered.

volume. Here it is convenient to select for coherent scattering a detector that measures the absorption cross-section C for coherent radiation.

There are two fundamentally different types of detectors for partially coherent scattering: 1) those that average over scattering directions, and 2) those that don't. The first type includes a detector that measures the total cross-section H of partially coherent scattering, and the second type includes one that measures the differential cross-section $\tilde{U}(\mathbf{s}, \mathbf{s}_0)$ of partially coherent scattering. The fundamental distinction between the two cited types of detectors of partially coherent radiation stems from the fact that the contribution of the correction terms of the series in (6.2) to the differential cross section for partially coherent scattering can vary rapidly as a function of the scattering direction. Here it is not ruled out that these contributions for certain scattering directions are so large that they must not be neglected.

a) Energy Equivalence of the Conditions of Applicability of the Single-Group Approximation for Coherent and Partially Coherent Scattering

According to the optical theorem (4.4) for the entire volume of a scattering medium showing no true absorption, the exact values of the absorption cross section C for coherent radiation and the total cross section H for partially coherent radiation are equal to one another. The analogous cross sections that are found by solving the D and BS equations with the single-group kernels M and K, which we shall designate by C_1 and H_1, are interrelated by an equation of the type of (4.5). In this equation, we must replace the cross section C_{tr} for true absorption on the right-hand side by the cross section $C_{eff.tr}$ for effective "true absorption". Subtracting these two equations gives

$$\delta C = \delta H, \tag{6.3}$$

Here we have denoted $\delta C = C - C_1 + C_{eff.tr}$, and $\delta H = H - H_1$. Eq. (6.3) implies coincidence of the absolute errors δC and δH in calculating the absorption cross-section for coherent radiation and the total cross section of partially coherent scattering by using the D and BS equations with the single-group kernels M and K. This expresses the energetic equivalence [70] of the conditions of applicability of these equations.[15]

In line with Sec. 4, b, we must impose the requirement for the applicability of the BS equation with the single-group kernels M and K that the effective "true absorption" cross section $C_{eff.tr}$ should be small in comparison with the absorption cross section C_1 for coherent radiation or with the total incoherent-scattering cross section H_1 as calculated in the single-group approximation. Here the requirements on the quantities C_1 and H_1 practically coincide. This leads to energetic equivalence of the conditions of applicability of the D and BS equations with the single-group kernels M and K also in terms of the relative errors of calculating the absorption cross-section for coherent radiation and the total cross section for partially coherent scattering.

When one uses a detector of partially coherent radiation that averages over a rather broad range of scattering directions, the stated energetic equivalence favors the idea that the conditions for applicability of the sin single-group approximation for partially coherent scattering may prove to be the same as for coherent scattering. These ideas have been confirmed by Gnedin and Dolginov, [59] who say that the conditions for applicability of the D and BS equations with the kernels M and K found in the model (4.7) of independent scatterers are the same.

Let us estimate the relative error $\delta C/C_1$ of calculating the absorption cross section for coherent radiation in the single-group approximation for a volume of a discrete scattering medium in the shape of a sphere of radius L that consists of independent Rayleigh scatterers. At the same time, we shall estimate the relative values $C_{eff.tr}/C_1$ of effective "true absorption." Here we expand the quantities δC and $C_{eff.tr}$ in series in multiplicity of scattering. The volume of the scattering medium is considered to be deep, so that $C_1 \approx \pi L_2$. The absolute error δC is of the fourth order of smallness in terms of multiplicity of scattering, and the relative error $\delta C/C_1$ is equal in order of magnitude to

$$\frac{\delta C}{C_1} \sim \frac{1}{k_0 d}\frac{L}{d}, \tag{6.4}$$

Here d is the extinction length in the model of independent Rayleigh scatterers. The expansion of the cross section $C_{eff.tr}$ for effective "true absorption" in terms of multiplicity of scattering begins with terms of the third order of smallness. When the terms of the fourth order of smallness are also taken into account, its relative value $C_{eff.tr}/C_1$ is equal in order of magnitude to

$$\frac{C_{eff.tr}}{C_1} \sim \frac{|{}_0\tilde{M}_1(k_0)|}{k_0^2}\max\left(1, \frac{L}{d}\right)\frac{L}{d}. \tag{6.5}$$

The relative error in (6.4) of calculating the absorption cross section for coherent radiation in the single-group approximation approaches zero at the rate of the ratio $1/k_0 d$ of the wavelength to the extinction length d, if there is an upper bound to the ratio L/d of the dimensions L of the volume of the medium to the extinction length d. Gnedin and Dolginov [59] have given[16] the requirement that the wavelength should be small in comparison with the extinction length as the condition for applicability of the D and BS equations with the kernels M and K taken in the independent-scatterer model of (4.7). The relative value (6.5) of the effective "true absorption" approaches zero at the rate of the deviation of the effective refractive index of the scattering medium from unity when an upper bound exists for the ratio of the dimensions of the volume of the medium to the extinction length. This condition is more rigid that the one under which the relative error (6.4) is small for calculating the absorption cross section for coherent radiation.

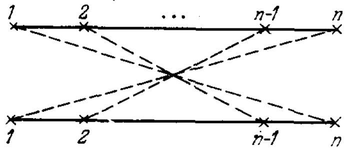
b) The Effect of Cyclic Diagrams for Partially Coherent Scattering in the Backward Direction

One gets a completely different result in estimating the accuracy of the BS equation with the single-group kernels M and K in terms of the error of calculating the differential cross section for partially coherent scattering by the volume of the medium than when one estimates it from the error of calculating the total cross section. This involves the fact that the expression for the kernel

[15]Here we must make the qualification that the error of applying the D equation in the single-group approximation is actually equal to the difference $C - C_1$, rather than δC.

[16]They give in their study a typological classification of a broad class of diagrams for partially coherent scattering, and estimate them in order of magnitude.

[17]In connection with this study by de Wolf, see the articles of Vinogradov, Kravtsov, and Tatarskiĭ. [118,119]



$$\begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ n & n-1 & \dots & 2 & 1 \end{pmatrix}$$

FIG. 1

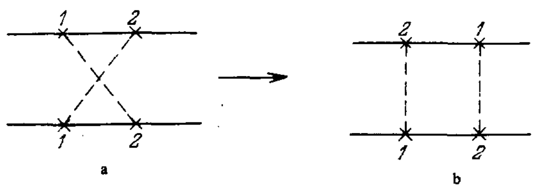


FIG. 2

K in the single-group approximation takes no account of a broad class of cyclic[115] multigroup, slowly-declining diagrams that contribute substantially to the partially coherent scattering in the backward direction. Gnedin and Dolginov[59] have noted such a role of the cyclic diagrams in phenomena of quantum-mechanical scattering of a flux of particles by independent force centers. Roffine and de Wolf[116] have noted it in scattering of electromagnetic waves by a turbulent plasma; Watson[7] has done so in scattering of electromagnetic waves by pairwise-correlated electrons of a plasma; and de Wolf[117]17) has done so in the scattering of short electromagnetic waves by a turbulent medium (Fig. 1, 2).

Figure 1 illustrates the cyclic diagrams. In this diagram, the two crosses joined by a dotted line depict the intensity operator K_1 of the single-group approximation, and the horizontal solid lines of the upper and lower rows represent the average Green's function $\mathscr{G}$ that satisfies the D equation with the mass operator M_1 of the single-group approximation, and its complex-conjugate $\mathscr{G}^*$. The linking of the crosses of the upper and lower rows is carried out by the cyclic substitution written to the right of the diagram. Upon assigning to the number n the values n = 2, 3, ..., we get the whole set of cyclic diagrams.

The cyclic diagrams, just like all the others that figure in the intensity operator, are strongly connected. Yet they have the property of being equivalent to weakly connected diagrams in a certain sense. This equivalence is established by adding to the cyclic diagram the outer horizontal lines $\mathscr{G}$ and $\mathscr{G}^*$. This gives diagram (a) in Fig. 2, where we have set n = 2 for simplicity. By using the properties of reciprocity for the average Green's function $\mathscr{G}$ and the kernel K_1, we can perform an inversion of the upper or lower rows. In such a transformation, e.g., of the upper row, the lower row remains fixed, while the upper row is rotated by 180° in a plane perpendicular to the plane of the drawing without breaking the dotted lines. Thereupon the diagram of Fig. 2a is transformed into Fig. 2b.

Let us denote by $(4\pi)^{-2}\tilde{U}_{cycl}(\mathbf{s}, \mathbf{s}_0)$ the contribution of the cyclic diagrams to the differential cross-section for partially coherent scattering by the volume of the medium. In the perturbation-theory approximation, this contribution for the backward scattering direction, $\mathbf{s} = -\mathbf{s}_0$, has the single form[7,115]

$$\tilde{U}_{cycl}(-\mathbf{s}_0, \mathbf{s}_0) = \tilde{U}_1'(-\mathbf{s}_0, \mathbf{s}_0), \qquad (6.6)$$

Here $\tilde{U}_1'(\mathbf{s}, \mathbf{s}_0)$ is the differential cross-section for partially coherent scattering as calculated by using the BS equation with the single-group kernels M and K after the single partially coherent scattering has been subtracted.

Equation (6.6) implies that the BS equation with the single-group kernels M and K gives too low a value for the partially coherent scattering cross section in the backward scattering direction. This defect of the single-group BS equation is fully revealed in a one-dimensional model of a scattering medium, where the wave is scattered only forward or backward in each elementary scattering event. As Gazaryan[15] and the authors of[16-19] have shown, the mean of the square of the modulus of the reflection coefficient of a layer of a one-dimensional scattering medium approaches unity exponentially, according to the solution of the Helmholtz equation, as the thickness of the layer is increased. However, it increases as a power function according to the solution of the BS equation with the single-group kernels M and K and the solution[14] of the transport equation.

In the three-dimensional scattering problem, the total contribution of all the cyclic diagrams to the differential partially-coherent scattering cross section varies rapidly as a function of the scattering direction.[115] Let the effective inhomogeneities of the scattering medium for the kernels M and K in the single-group approximation be small in scale. Then, with a deviation by the angle θ from the backward scattering direction, the relative contribution of the cyclic diagrams to the differential partially-coherent scattering cross section of the volume of a medium whose dimension L is small in comparison with the extinction length d, $L \ll d$, will decline as $1/(\theta k_0 L)$. That is, it is small outside a cone of backward scattering directions having a width of the order of the ratio of the wavelength to the dimensions of the scattering volume. The relative contribution of the cyclic diagrams to the total partially-coherent scattering cross-section of a volume of small optical depth is estimated to be of the order of the ratio $1/k_0L$ of the wavelength to the dimensions of the volume. If the scattering medium occupies a half-space, and a plane wave is normally incident on its boundary, then the relative contribution of the cyclic diagrams to the flux density of scattered radiation energy will decline as $1/(\theta k_0 d)^2$ upon deviation from the backward scattering direction by the angle θ. That is, it is small outside a cone of backward scattering directions of width of the order of the ratio of the wavelength to the extinction length. The relative contribution of the cyclic diagrams to the total flux of energy of the radiation scattered by the half-space is of the order of $(k_0 d)^{-2}\ln(k_0 d)$, and it approaches zero at the rate of the ratio of the wavelength to the extinction length.

These results of estimating the relative contribution of the cyclic diagrams to the differential and the total cross section of partially coherent scattering by a volume of a medium favor the idea that, when one uses a detector of partially coherent radiation that averages over a rather broad range of scattering directions, the effect in the three-dimensional multiple-scattering problem of the multigroup diagrams of the kernels M and K to the partially coherent scattering will be just as substantial as in the one-dimensional scattering problem.

7. CONCLUSION

The method presented in this review of group expansions of the optical properties of an effective inhomogeneity of a random medium in multiple scattering of waves, which leads to the photometric theory of radiation transport, is very general in nature, and it permits one to treat a large number of phenomena in a unified way. Thus, e.g., Bourret[54] has treated by this method the propagation of waves and molecular diffusion in a turbulent medium. Kubo,[120] who has essentially used Bourret's method,[54] has proposed an approach to studying Brownian movement (in particular, of a particle that interacts with a systems of free scatterers of a thermostat) by starting with a stochastic Liouville equation. In[121] (see also[16]), the group-expansion method has been applied for studying parametric resonance in an oscillatory system with random parameters without using the assumption[103] that these parameters are delta-correlated in time. One can also treat with the group-expansion method the problem of reciprocity relationships in the transport theory of light radiation in a randomly-variable medium, which Rozenberg[122] has studied by another method.

It is of considerable interest to study[109,110] non-steady-state partially-coherent multiple scattering of wave packets in a randomly-variable medium in the limit of quasistatic[111] fluctuations of its potential. The results of these studies seem to reveal the fundamental potentiality of rigorous justification of the method of group expansions as applied to steady-state partially-coherent multiple scattering of waves in a randomly-inhomogeneous medium.

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