METHODOLOGICAL NOTES

Some "moving boundaries paradoxes" in electrodynamics

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Several nontrivial questions are discussed which arise in obtaining and utilizing boundary conditions along moving surfaces separating two media and which are associated with characteristics features of electrodynamic material equations. It is noted that the boundary conditions themselves can depend on the relationship between the thickness of the boundary and the proper times of the motion of the particles of the medium (and even simply on the velocity of the boundary). In view of the inertial properties of the medium all the electrodynamic quantities remain continuous (the law of continuity) at an ideally sharp discontinuity of its parameters in time. An exception is presented by the case of the motion of the boundary with velocity c, and also by discontinuities "frozen into" the medium. For a more smoothly varying (although sharply varying compared to the external scale of variation of the field) boundary layer one can neglect the inertial properties (dispersion) of the medium; in this case the field and the polarization undergo a "discontinuity." However, in this case difficulties of another kind arise when the velocity of the boundary is "above light velocity" on one side and "below light velocity" on the other side (in particular, the interaction of small perturbations with shock waves belongs to such a case); these cases require a separate investigation. The effect of the inertial properties of the medium on the boundary conditions is illustrated on the example of electromagnetic waves in a dielectric with elastic oscillators.

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CONTENTS

1. INTRODUCTION

Phenomena arising in the course of interaction of electromagnetic waves with a moving boundary separating two media occupy an important place in physics. We remind the reader that the problem of the reflection of a wave packet from a moving mirror was investigated by Einstein in his famous paper [1] as an illustration of the effects of the special theory of relativity. During the last twenty years these questions have been repeatedly discussed, most frequently from a more utilitarian point of view: it was proposed to utilize the Doppler frequency shift arising at the boundary in order to transform the spectrum of electromagnetic signals. For example, the sharp boundary of a plasma beam can serve as a moving mirror [2]. Since in order to obtain a large frequency shift it is necessary that the velocity of the boundary would be comparable with the phase velocity of the wave, then it is useful to have the beam propagating in a dielectric (or a retarding system); this relieves one of the requirement of having to achieve a relativistic motion of the medium [3]. An even more radical solution consists of arranging that there is no macroscopic motion of the medium at all, and the boundary is created by the sharp front of an intense running wave (pulse) which acts on a nonlinear medium (ferrite, semiconductor, plasma), producing a relativistically moving sharp change in its parameters [4,5]. We note that if the boundary moves faster than the phase velocity of any one of the waves, then new quanta of the field are produced [6]. With the aid of a distributed interaction one can even produce a "parametric wave" propagated faster than the velocity of light in vacuo (the problem of emission of such objects moving with velocity faster than light has recently been discussed specially [7]). A review of specific results relating to the transformation of waves at moving boundaries is given in [8].

Here we would like to direct the attention of the reader to certain special features of this phenomenon, instructive in our opinion, which usually remain unnoticed. The interaction of a field with moving objects is generally one of the most complicated and interesting problems of electrodynamics to which already quite a few books and reviews have been devoted. A moving boundary is a rather unique object of this kind. For example, the incidence of a plane wave on a plane boundary transforms the latter into a moving oscillator, a kind of a plane particle which moves with relativistic velocity and emits waves—a transmitted and a reflected wave—into different media. The properties of such a "particle" are determined by the well-known boundary conditions relating the values of the electromagnetic field and of the polarization of the media on both sides of the moving boundary. These conditions have the form [9]

$$\{\mathbf{E} + [\boldsymbol{\beta}, \mathbf{B}]\} = \{\mathbf{H} - [\boldsymbol{\beta}, \mathbf{D}]\} = 0, \qquad (1)$$

here $\boldsymbol{\beta} = \mathbf{V}/c$, c is the velocity of light in vacuo, $\mathbf{V}$ is the velocity of motion of the boundary; brackets indicate the difference in the values on the two sides of the boundary. For the sake of simplicity we do not take into account surface charges and currents and consider only the components of the field tangent to the boundary.

Conditions (1) are universally applicable and are valid even for a nonlinear medium. But they, just as the Maxwell equations themselves, acquire a nonformal meaning only taken together with the material equations connecting the quantities **D** and **B** with the intensities **E** and **H** (for the sake of argument we shall have in mind the connection between **D** and **E**). From this follow, generally speaking, additional "material" boundary conditions the number and the form of which depend not only on the parameters of the medium "outside" the boundary, but also, possibly, on finer features of the latter—the structure of the boundary region or the velocity of its motion. It is just at this stage that different nontrivial situations arise which sometimes appear to be nonobvious and even paradoxical. Of course, we nowhere go outside the framework of the usual equations of macroscopic electrodynamics, and one can speak of "paradoxes" only with respect to a definite level of understanding of a phenomenon (but it is just in this sense that paradoxes were interpreted by Mandel'shtam who ascribed to them, in particular, considerable instructional importance[10]). Probably the problems discussed below could be given a more rigorous mathematical formulation (and in some cases this presents no difficulty), but our aim is only to show by simplest possible means that one should sometimes approach with great care the study of phenomena which at first sight are quite simple and which have been discussed for a long time in order not to arrive at contradictory or simply incorrect conclusions.

2. ARE FIELD DISCONTINUITIES AT THE BOUNDARY POSSIBLE?

First of all we make the following assertion (we shall refer to it as the law of continuity)! at a moving sharp separation boundary both the field and the polarization of the medium are continuous, i.e., all four vectors **E**, **H**, **D**, and **B** are continuous. This assertion might appear paradoxical already because it contradicts the well-known results for a stationary medium where the tangential components of **E** and **H** are continuous, but $\mathbf{D} = \epsilon\mathbf{E}$ is necessarily discontinuous since the values of the parameter ϵ are different on the two sides of the boundary. Moreover, as can be seen from (1), at a moving boundary ($\beta \neq 0$) a discontinuity in one of the quantities immediately leads to discontinuities in the remaining quantities. Nevertheless, the underscored assertion turns out to be just the rule (which, it is true, has important exceptions), which is based on the obvious fact of the inertial nature of any medium. Indeed, no real medium has time to react instantaneously to a stepwise (i.e., infinitely rapid) variation of the field in time. The appearance of a discontinuity in a field comoving with the boundary signifies the appearance of infinitely high frequencies in the spectrum of the waves. But, as is well known[9], in the high frequency limit the dielectric susceptibility of any medium is described by the asymptotic expression $\epsilon = 1 - (A/\omega^2)$, where A are constants possibly different on opposite sides of the boundary. We see, that at very high frequencies these dielectric constants are close to unity and differ little from each other. In other words, a sharp discontinuity in the field "does not notice" either the medium or the separation boundary, and must be propagated as in vacuo with the velocity c (this circumstance has already been noted at the beginning of this century by A. Sommerfeld). But if $V \neq c$, then there can be no sharp change in the field or in the induction comoving with the boundary, and the field must remain continuous independently of the value of V, as has been asserted above.

It is necessary immediately to qualify the above statement: the continuity of the instantaneous values of all the quantities does not at all mean that the wave incident on the boundary will not be reflected or refracted. Since for waves of finite frequency the parameters of the medium on the two sides of the boundary are different the conditions of continuity can be satisfied only by adding to the field of the incident wave also the fields of the secondary (transmitted and reflected) waves[1)]. In this case the law of continuity can also be of definite practical use. For example, in solving problems on the reflection of an electromagnetic wave from a moving half-infinite plasma[2,3], where use was made of a relativistic transformation of the fields into the comoving system of coordinates and back, it would have been sufficient and would have produced the same result to assume the total values of E and H to be continuous at the boundary. Another problem of the same kind is discussed below.

Of course, the case V = c is a special one. Such a boundary can carry a field discontinuity which "does not notice" the medium and is propagated just as in vacuo. Therefore the discontinuities in all the functions E, H, D, and B must be the same (in the CGSE system of units), but the magnitude of discontinuity is arbitrary and depends on the initial conditions of the problem (the solution of a specific problem of such kind—concerning a sharp ionization front moving in a medium with a velocity c can be found in [11]).

The foregoing refers to the general case when the boundary moves with respect to the medium. In order to sort out the apparent contradiction with the case of a stationary boundary we consider the following situation. Let the medium with a sharp boundary move as a whole, i.e., there exists such a system of coordinates in which both the medium and the boundary are stationary. It is obvious that in this case a fixed element of the medium does not undergo any sharp variations in time, although the parameters of neighboring elements separated by the boundary differ from each other. In such a case a discontinuity in the electromagnetic quantities is possible which is "frozen into" the medium, and which therefore does not affect its inertial properties. This also applies to the case when the media are moving parallel to the boundary (tangential discontinuity)[2)]. This, of course, also leads to understanding the special case of a stationary boundary separating two stationary media. But the law of continuity again becomes valid if one takes into account spatial dispersion, i.e., nonlocality of the material equations. Therefore, in speaking of "frozen-in" discontinuities, we are dealing rather with a pseudo-exception—an ideal field discontinuity always moves with velocity c.

3. WHAT IS MEANT BY A SHARP SEPARATION BOUNDARY?

In the foregoing we have been speaking all the time of a sharp boundary, having in mind essentially a discontinuity in the properties of the medium and correspondingly a discontinuity in the field. But in real cases the situation is more complicated. In order to be convinced of this we consider more closely those idealizations which are associated with the concept of a boundary (the hypothesis of a discontinuity). It is clear that in actual fact there always exists a certain transition region of a finite, although possibly small, thickness d. In deriving the basic boundary conditions (1) it is assumed only that d is small compared to the characteristic dimensions of the field itself (the wavelength in space and the period in time), so that the field can be regarded as constant over intervals considerably larger than d.

However, this requirement is insufficient to conclude that the field is continuous. It is necessary that the variation in the parameters of the medium over the boundary would be rapid also compared to all the times of the characteristic oscillations or relaxation of the particles of the medium—only then do the latter not have time to "move off the spot," and the polarization is not altered. In the opposite case the law of continuity is, generally speaking, inoperative and, although the relationships (1) remain valid, the complete system of boundary conditions is altered. In particular, the boundary layer can be sufficiently thick for the inertial nature of the medium (dielectric) not to have any effect at all. If at the same time the wave incident on the boundary is such that for it one can neglect dispersion (inertial properties) of the medium then the latter turns out to be nondispersive everywhere including the transition layer itself. In order to obtain the desired transformation of the field in this case it is sufficient to substitute into (1) the relationship $\mathbf{D}_{1,2} = \epsilon_{1,2}\mathbf{E}_{1,2}$, where the subscripts 1 and 2 refer to values on opposite sides of the boundary, while $\epsilon_{1,2}$ are given at the outset. Now, of course, **D** and **E** cannot be simultaneously continuous.

Thus, in the general case, the boundary conditions themselves are not completely determined by the parameters of the medium external with respect to the boundary, but also depend on the internal parameters of the boundary layer, in this case on the relationship between its duration and the characteristic times for the motion of the particles of the medium. From the foregoing it is clear that in considering the transformation of the field at an ideally sharp boundary it is in principle not possible to regard the medium as nondispersive. But if dispersion is not taken into account and the boundary is not rigidly "attached" to the medium, then it should be assumed that the thickness of the boundary layer is not too small. An example of such boundaries which is important in physics is a shock wave—an intense discontinuity in the field which moves with respect to the medium and alters its parameters. Shock waves are a classical object of study in the mechanics of continuous media; comparatively recently electromagnetic shock waves have become known created by powerful pulses in a nonlinear dielectric or magnetic medium (for example a ferrite)[13]. The boundary conditions (1) remain valid at such a shock discontinuity. At the same time one can neglect dispersion (it is true, that instead of $\mathbf{D} = \epsilon\mathbf{E}$ one has to adopt the nonlinear relation $\mathbf{D} = \mathbf{D}(\mathbf{E})$, since the wave itself alters the value of ϵ), while the thickness of the shock front turns out to be just such that the polarization of the medium would have time to "readjust" from one constant value to another. We shall return to this case again later.

The solution to the problem as to which boundary is sufficiently sharp and which is not in the sense indicated above depends not only on its thickness d, but also on the velocity of its motion. In a medium without spatial dispersion (the word "without" means, of course, that the scale of the dispersion is small compared to the thickness of the boundary) polarization suffers a discontinuity at a stationary boundary, since the latter has an infinite extension in time; in the case of small values of V a discontinuity in the field intensity is also possible. Only in the case of a sufficiently high velocity of the boundary, when the time d/V becomes smaller than the characteristic time scale of the medium, does the law of continuity begin to operate.

The question of "thick" and "thin" boundaries is essential already because in practice it is not always simple to produce a sharp moving boundary in a medium. Thus, no "proper" wave of a field propagating in a medium and altering its properties can have an infinitely short front—it is smeared out to a duration of the order of the relaxation time of the medium (and converts into the shock wave that has already been mentioned). In order to produce a boundary shorter than the characteristic time of the medium external sources are required—for example, ionizing radiation, which transforms a dielectric into a plasma, or a wave propagating in another auxiliary system (as is accomplished in radiophysics with the aid of transmission lines for electromagnetic waves). In such cases one can obtain a sufficiently thin boundary which satisfies the law of continuity.

Example. As an example we consider a simple, but still very useful Lorentz model of a dielectric medium as a collection of identical elastic oscillators (dipoles). The polarization of such a medium satisfies the oscillatory equation

$$\ddot{P} + \omega_0^2 P = \frac{\omega_p^2}{4\pi} E, \tag{2}$$

where ω_0 is the characteristic frequency of the oscillators, ω_p is the plasma frequency proportional to their density. We assume the magnetic susceptibility to be equal to unity ($B = H$).

Let some external factor produce in the dielectric a moving boundary along which some parameter of the oscillators is varied. For the sake of simplicity we assume that the medium remains macroscopically stationary, but that it is acted upon by a strong external field in the form of a wave with a sharp front. This field alters the elasticity (potential energy) of the oscillators, i.e., the magnitude of ω_0, from ω_{01} to ω_{02}.[3)]

We consider plane waves propagating along the normal x to the boundary ($E = E_y$, $H = H_z$). Then from (1) we have

$$\{E - \beta H\} = \{H - \beta D\} = 0. \tag{3}$$

It is now necessary to add the relationships between the discontinuities in the magnitudes of D and E. For this we integrate the material equation (2) twice with respect to t over the time τ of the passage by the boundary past a given point. Since E and P are finite then as $\tau \to 0$ the integrals over them vanish, and only the integrals involving $\ddot{P}$ remain, and therefore

$$\{\dot{P}\} = \{P\} = 0, \tag{4}$$

For $\{P\} = 0$, evidently, $\{D\} = \{E\}$ and the conditions (3) yield at once

$$\{E\} = \{H\} = \{D\} = 0, \tag{5}$$

i.e., the field and the induction are continuous at the boundary, in complete agreement with the foregoing.

The continuity conditions (4) and (5) are sufficient for the solution of specific problems. The usual formulation of the problem consists of the following (Fig. 1a). Let a plane monochromatic wave of frequency ω_i fall from region I on the boundary, and, for the sake of definiteness, in the direction towards it ($V < 0$). Then all the secondary waves (reflected and transmitted) will also be monochromatic, and the frequency and the wave number of each of them will be related by the dispersion equation

$$ck = \pm\omega\sqrt{1 + [\omega_p^2/(\omega_0^2 - \omega^2)]} = \pm\omega\sqrt{\varepsilon(\omega)}, \qquad (6)$$

with the values of ω_0 being different in regions I and II.

What waves, reflected and transmitted, can appear due to the interaction with the boundary? The dispersion equation (6) enables us to answer this question. From the continuity of the instantaneous values of the field it follows, in particular, that the phases of all the waves are equal at the boundary (with $x = Vt$), i.e., for all the waves the values of $\omega - kV$ are the same. Using (6) one can determine what frequencies satisfy this condition in each region; from them, as usual, one should select only those for which the condition for radiation is satisfied: the group velocities of the secondary waves must be such that the energy should be travelling away from the boundary[4]. This is convenient to carry out graphically: we obtain all the points of intersection of the dispersion curves $\omega(k)$ in both regions with the straight line $\omega - \omega_i = V(k - k_i)$. Such a construction is shown in Fig. 1b, where the point 1 corresponds to the incident wave[5]. From the diagram it can be seen that for a sufficiently small $|V|$ the conditions for radiation are satisfied by the values of ω denoted by the numbers 2, 3, 4 and 5 (the group velocity $d\omega/dk$, evidently, is determined by the slope of the dispersion curve). If $\omega_i \ll \omega_0$, then for the incident wave it follows from (6) that $ck = \omega\sqrt{\bar{\epsilon}_1}$, where $\bar{\epsilon} = \sqrt{1 + (\omega_p^2/\omega_0^2)}$ corresponds to zero frequency and dispersion can be neglected. But if in addition $|V| \ll c/\sqrt{\bar{\epsilon}_{1,2}}$, then the frequencies of all the waves can be found analytically without difficulty. In region I there exists one reflected wave of frequency

$$\omega_2 \approx \omega_i\,(1 + 2\beta\sqrt{\bar{\varepsilon}_1}), \qquad (7a)$$

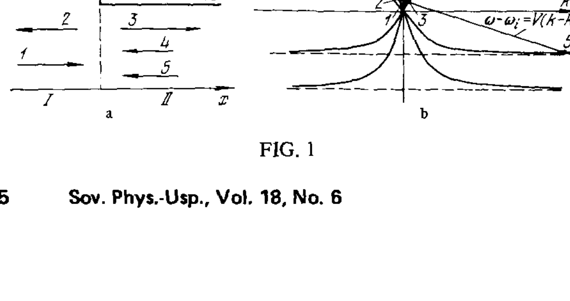
and in region II there exist three waves of frequencies

$$\omega_3 \approx \omega_i\left[1 + \beta\left(\sqrt{\bar{\varepsilon}_1} - \sqrt{\bar{\varepsilon}_2}\right)\right], \qquad (7b)$$

$$\omega_4 \approx \omega_5 \approx \omega_{02}\left(1 - \frac{1}{2}\beta^2\frac{\omega_p^2}{\omega_0^2}\right). \qquad (7c)$$

Thus, in addition to waves of frequency ω_2 and ω_3 for which dispersion can be neglected there appear two resonant essentially dispersive waves of frequencies close to ω_{02}. These waves are propagated in the same direction as the discontinuity, but lag behind it with respect to their group velocities and therefore satisfy the condition for radiation[6]. The field in region I is a superposition of the fields of the waves 1 and 2, and in region II it is a superposition of the waves 3, 4 and 5. The amplitudes of all four unknown waves 2, 3, 4 and 5 can now be easily determined in terms of the known incident wave 1 from the conditions of continuity for the four quantities E, H, P and $\dot{P}$, if one takes into account that for each wave $H = \pm E\sqrt{\epsilon}$, while $4\pi P = (\epsilon - 1)E$. Without reproducing here the corresponding solutions we note only that the amplitudes of the "resonance" waves 4 and 5 turn out to be small of order β^2.

Going to the limit $\beta = 0$ we formally obtain for E and H in waves 2 and 3 the usual Fresnel formulas at a stationary boundary, and in the resonance waves $E = H = 0$. However this transition, which appears natural at first sight, is not justified. This can already be seen from the fact that the polarization of the resonance wave remains finite, ($E_{4,5} \sim \beta^2$, but $\chi_{4,5} \sim \epsilon - 1 \sim \beta^{-2}$). It can be shown that the energy density of these waves is also finite. Consequently, no matter how slow is the motion of the boundary, it leaves a finite trace in the form of oscillations of particles of the dielectric with a characteristic frequency ω_{02}. This can be understood: for an arbitrarily small β the motion of an infinitely thin boundary alters discontinuously the properties of each oscillator, and this in the presence of the field of an incident wave leads to a "shock" excitation of the characteristic oscillations of the medium. At the same time, if from the outset one regards the boundary as stationary nothing of this kind occurs.



a b

FIG. 1

The reason for this seeming contradiction was indicated in the previous section. Since the boundary region always possesses a certain finite thickness d, then, evidently, time variations of all the quantities at the boundary have a finite duration $\tau \approx d/V$. But then P and $\dot{P}$ are continuous only if the velocity of the boundary is not too small. Indeed, the term $\ddot{P}$ in Eq. (2) is of order P/τ^2 and is larger than the term $\omega_0^2 P$ under the condition $V^2 \gg (\omega_0 d)^2$ (i.e., the path traversed by the boundary during a period of the characteristic oscillations of the medium $2\pi/\omega_0$ must be greater than the thickness of the boundary). If the reverse inequality holds (together with $\omega_i \ll \omega_0$) one can neglect the term $\ddot{P}$ entirely, then over the whole transition region we have $P = \bar{\epsilon}E$, and as $\bar{\epsilon}$ varies at the boundary the polarization has time to "keep track" of the field. It is clear that in this case the dielectric can be regarded as being nondispersive. Substituting $D_{1,2} = \epsilon_{1,2}E_{1,2}$ and $B_{1,2} = H_{1,2}$ into the initial relations (1) we easily obtain

$$E_2 - E_1 = E_1\frac{\beta^2(\bar{\varepsilon}_1 - \bar{\varepsilon}_2)}{1 + \beta^2\bar{\varepsilon}_2}, \qquad H_2 - H_1 = E_1\beta\frac{(\bar{\varepsilon}_2 - \bar{\varepsilon}_1)}{1 + \beta^2\bar{\varepsilon}_1}, \qquad (8)$$

where the subscripts 1 and 2 refer to values on opposite sides of the boundary. Thus, as has already been stated above, over a sufficiently thick and slowly moving boundary layer the field experiences a discontinuity. As $\beta \to 0$ only the discontinuity of the induction D remains, and as a result Fresnel formulas are obtained without any singularities of any kind.

The same holds if the boundary (in this case an arbitrarily sharp one) is created at the junction of similarly moving dielectrics with different parameters ω_0 or ω_p. In the comoving system of coordinates the material equation (2) is a local relationship in which $\ddot{P}$ remains finite and which for the same E determines different values of P in regions I and II. But in the initial system of coordinates, where the medium moves as a whole, E and H also vary discontinuously at the boundary. All this, of course, agrees with the considerations stated above.

4. BOUNDARIES MOVING WITH SPEEDS BELOW AND ABOVE THE SPEED OF LIGHT

In those cases when the boundary is sufficiently "thick" and one can neglect dispersion, the problem appears to be quite obvious—the relationships (8) must lead to an elementary generalization of the Fresnel formulas since no "unforeseen" resonance waves can appear. However, it turns out that also in this case difficulties can arise which either require the introduction of additional data for a unique solution of the problem, or even force one in general to give up the simple boundary conditions in the form of (8). In this case the problem is associated with rapidly moving boundaries. As has been noted already, in principle, and also in practice, the boundary can have any arbitrary velocity both smaller than and greater than the velocity of the waves interacting with it, and this, in turn, affects the solution of the problem in an essential manner.

We illustrate this by utilizing once again the example of a stationary dielectric in which a boundary is moving, thereby changing its parameters ϵ and μ. We shall take as given a wave incident normally and moving towards the discontinuity. Since ϵ and μ do not depend on the frequency, the field on both sides of the boundary (regions I and II) satisfies the one-dimensional wave equation and is given in the general case by a superposition of two waves propagated in opposite directions with velocities $v_{1,2} = c/\sqrt{\epsilon_{1,2}\mu_{1,2}}$ (Fig. 2). Each of these waves is characterized by one unknown function, for example, $E(x, t)$, since $H = \pm E/Z$, where $Z_{1,2} = \sqrt{\mu_{1,2}/\epsilon_{1,2}}$ is also a known quantity (the wave impedance of the medium). The field of one wave, $E_1^+ = E_i$ is given, and in order to determine E in the remaining three secondary waves there are two boundary conditions (8) (or their generalization to the case $\mu_1 \neq \mu_2$). Therefore the problem will be completely correct only in those cases when the existence of one of the secondary waves is impossible as a result of the condition for radiation. Just such a situation exists in the case of a stationary boundary of separation, when in region II there exists only one transmitted wave E_2^+, while the wave E_2^- is evidently excluded. The possibility for the existence of a given wave depends on the relation between its velocity and the velocity of the boundary (in view of the absence of dispersion the variation of the frequency is of no significance). In the general case four variants are possible:

$$\text{a)}\ |V| < v_1, v_2, \quad \text{b)}\ |V| > v_1, v_2, \quad \text{c)}\ v_2 < |V| < v_1, \quad \text{d)}\ v_1 < |V| < v_2. \tag{9}$$

The characteristic features of these cases can be demonstrated visually if in the plane of the variables x and t one constructs characteristics, i.e., trajectories of waves moving with velocities $\pm v_1$ and $\pm v_2$, and also the trajectory of the boundary itself $dx = Vdt$ (Fig. 3). It is evident that outside the boundary region these characteristics are straight lines (V is constant), but with different slopes on opposite sides of the boundary. Those waves arise in reality whose characteristics move away from the boundary (independently of their direction with respect to a stationary system of coordinates).

The simplest is the case a) (motion with velocity less than that of light), which also includes the case of a stationary boundary. Here the wave E_2^- is excluded, while the fields E_1^- and E_2^+ can be easily found from (8) in terms of the known E_i if we set $E_1 = E_1^- + E_i$, $E_2 = E_2^+$ [7]. In the case b) of motion with velocity greater than that of light the reflected wave E_1^- is absent, but in region II both

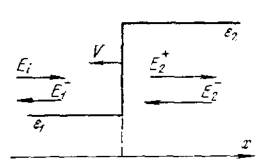


FIG. 2

transmitted waves E_2^+, E_2^- remain, and the problem again has an elementary solution. The appearance of the wave E_2^- which falls back with respect to the boundary means that quanta of the field are created at the discontinuity. We note that as $|V| \to \infty$ we obtain from this the case of a one-time stepwise change in the parameters of a homogeneous medium (which is, of course, possible only in the presence of a distributed interaction.)

In both cases indicated above there exist in the xt plane two families of characteristics crossing the boundary. However, the cases c) and d) are not so simple when the velocity of the discontinuity is smaller than the velocity of light on one side of the boundary and is greater than the velocity of light on the other side of the boundary. From Fig. 3 it can be seen that in these cases one of the families of characteristics remains separated by the boundary which, thus, is itself a characteristic, or, to be more precise, moves synchronously with the wave with the velocity of the wave having a value intermediate between v_1 and v_2. Consequently, the boundary conditions (8) are specified in the neighborhood of the characteristic, and such cases are always special in mathematical physics[8]. At the same time, these singularities turn out to be quite different for the cases c) and d).

In the case c) there is no basis for excluding one of the three waves, and then the two boundary conditions (8) are insufficient to determine them—some kind of additional information is needed. In the present case one can remove this difficulty in the following manner (Fig. 4). We consider instead of a single discontinuity two discontinuities (1 and 2) separated by a region with comparatively smooth variation of width d, in which the value of the velocity $v = c/\sqrt{\epsilon\mu}$ varies from $|V| + \delta V$ to $|V| - \delta V$, where δV is a sufficiently small positive quantity. Then the discontinuity 1 is completely characterized by a velocity smaller than that of light, while the discon-

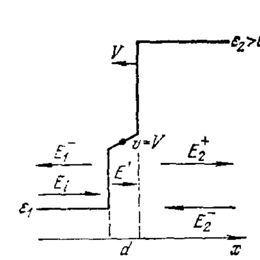


FIG. 4

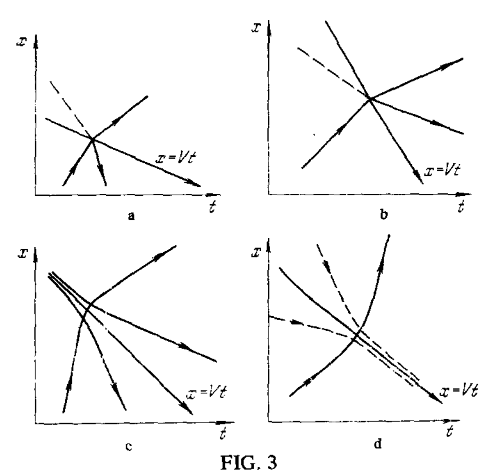


FIG. 3

tinuity 2 is characterized by a velocity greater than the velocity of light, and a fourth unknown quantity is added—the field E′ in the region d. Utilizing the boundary conditions (8) in sequence for both discontinuities we obtain all four unknown waves. Then by letting d and δV approach zero we obtain a unique solution for the initial discontinuity (for details cf.,[15]). This solution contains the additional internal parameter $Z' = \sqrt{\mu'/\epsilon'}$—the wave impedance at the point of synchronism where $v = |V|$. Of course if only one quantity varies, for example ϵ, while μ is constant, then Z' is known in advance, since ϵ' is evidently equal to $c^2/\mu V^2$. In this case the solution takes on an extremely simple form:

$$E_1^- = -E_i, \quad E_2^+ = -E_2^- = \frac{v_2}{v_1} E_i. \qquad (10)$$

It is curious that the velocity of the boundary does not enter this at all. But if both ϵ and μ vary, then Z' is an independent parameter.

Thus, we have to give up the completely "discontinuous" description of the problem (at least in favor of two discontinuities) and to make more specific the internal parameter in the boundary region Z' (we recall that in neglecting dispersion one cannot treat the boundary as being infinitely thin). One can see the reason for this from the picture of the trajectories in Fig. 3c. In this case the characteristics diverge from the boundary which serves them in the manner of a "watershed." As a result we partially lose connection between processes in regions 1 and 2, and it is just because of this that we require additional data for the problem.

Still worse is the situation in case d), when the incident wave E_i gives rise to only one transmitted wave E_2^+, while the waves E_1^- and E_2^- are impossible. Now the problem turns out to be overdetermined—two boundary conditions (8) for one unknown E_2^+. Here a division of the discontinuity into two offers no help. In order to obtain a unique result we must in some manner correct the boundary conditions (8) themselves. In order to understand what is happening here we turn to Fig. 3d corresponding to this case. One of the families of characteristics converges towards the boundary in such a manner that the waves moving in the same direction as the boundary are grouped at the boundary. Consequently, the perturbations produced by it are accumulated and must increase without limit. Therefore the problem with a uniform completely autonomous motion of the boundary, in essence, does not in general have in this case a bounded solution.

Thus, we again arrive at a contradiction, for the resolution of which we must return to the question of the manner in which the moving boundary is produced. As long as we are dealing with a "thick" boundary in a stationary medium then the natural possibility consists of the fact that in a nonlinear medium there is created a powerful pulse of the field with a sharp front propagated in it as a wave characteristic of it and altering its parameters. In order to describe the motion of such a front it is necessary, in essence, to solve the nonlinear problem, since the variation of the parameters of the medium itself affects the wave. But the initial boundary conditions (1) or, in the one-dimensional case, (3) are valid as before, and one cannot utilize only linear material equations of the type $D = \epsilon E$. But such a nonlinear variation of the field is just the electromagnetic shock wave which has been mentioned earlier. The linear problem which is of interest to us now reduces to the interaction of small perturbations of the field with the shock wave. Such problems are not new in physics—they have been studied in connection with the problem of the stability of shock waves. In such a case in order to obtain a unique solution one must take into account that the incident wave perturbs the motion of the shock front, so that $\beta = \beta_0 + \delta\beta$ (it is just here that the effect of the "accumulation" of the field at the boundary is felt—it leads to oscillations of the boundary). Then, taking into account the fact that (3) describes the sum of the fields of the weak waves and of the strong shock field and that the latter also satisfies (1), we obtain for the perturbations

$$\{E - \beta_0\mu H - B_0\delta\beta\} = \{H - \beta_0\epsilon E - D_0\delta\beta\} = 0, \qquad (11)$$

where D_0, and B_0 are the unperturbed values different in regions I and II, while ϵ and μ now denote the derivatives dD/dE and dB/dH also taken for the unperturbed values corresponding to the shock wave.

Now our problem in the case d) can be solved in an elementary manner. The unknown that is lacking is $\delta\beta$, while the field E_2^+ in the only transmitted wave is determined easily and in a finite manner together with $\delta\beta$ from the two equations (11) (the corresponding solution is given in[4]).

It is of interest to note that it is specifically in case d), and only in this case, that the shock wave turns out to be stable, since a finite E_i gives rise to finite perturbations. In the remaining cases $\delta\beta$ can be arbitrary (among other possibilities it can be arbitrarily large), and this indicates an instability of the shock wave (this question has been studied in detail in the mechanics of continuous media). Consequently, if the boundary is created by the "proper" wave in the given nonlinear medium, then we, in essence, always have the case d), while in the remaining cases something artificial is required for the creation of the boundary: an independent system giving rise to pumping, an external source of a different nature, etc. (this now refers also to "thick" boundaries).

Thus, in the case of an interaction of a weak wave with a strong (shock) wave the former would grow without limit if it were not for its reaction on the motion of the shock front. Here there exists a certain analogy with the phenomenon of parametric resonance in an oscillatory circuit with a periodically variable capacitance, when the amplitude of the proper oscillations increases without limit until the effect of these oscillations on the law governing the variation of capacitance becomes felt, i.e., a nonlinear effect appears.

For sharper discontinuities of a parameter which are produced externally such characteristic features do not arise, since the velocity of the secondary waves varies due to dispersion, and when this is taken into account the equations of electrodynamics all have the same characteristic velocity c which is the same for both regions I and II, while for $V \neq c$ the problem of the reflection and transmission of the wave must always be correct in this sense. But if $V = c$, then in this case also, as has been mentioned previously, additional conditions are required for a unique solution of the problem.

5. CONCLUSION

Idealization is a necessary step in the theoretical description of any phenomenon which enables one to set aside details of little significance. At the same time, when the same phenomenon is studied more deeply the neglected factors sooner or later "obtain revenge" leading to different kinds of contradictions and paradoxes the resolution of which requires at least a partial departure from the idealizations adopted. This situation, which is usual in physics, also arises for the problems discussed here. A sharp boundary of separation between two media is one of the most fundamental idealizations in the electrodynamics of continuous media. It is generally accepted that if the thickness of the boundary transition region is small compared to a wavelength, then the boundary is completely described by universal boundary conditions which contain only the parameters of the field and of the medium outside this region. From what has been stated above it follows that this, generally speaking, is not so, in particular in the case of moving boundaries. In order to obtain a unique solution one must "peek inside" the boundary introducing additional data concerning its thickness, velocity, structure, etc. In doing so we in essence reject the initial idealization and replace it by other more complicated idealizations. Thus, it turns out that even such a common idealization as the concept of a sharp separation boundary, encompasses within itself a quite complicated physical content.

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1)If the velocity of the boundary is close to the phase velocity of the reflected wave then, because of the Doppler effect, the frequency of this wave increases without limit. As a result, as was noted by B. M. Bolotovskiĭ, already the whole wave with a limited spectrum (wave packet) can penetrate the boundary without being reflected. Thus, if a stationary plasma completely reflects all waves with frequencies below a certain finite plasma frequency, then a wave incident on the boundary of a plasma moving with a velocity close to c practically always penetrates across the boundary. [3]

2)In this latter case there exists the possibility of amplification of waves unconnected with the Doppler effect, if one of the media is dissipative [12].

3)The limiting case when $\omega_{01} \to \infty$, $\omega_{02} \to 0$ corresponds, essentially, to an ionization front which converts a nondispersive dielectric into a plasma.

4)This requirement (for a stationary boundary between two dispersive media) was first formulated, probably, by Mandel'shtam [14].

5)Here the plasma region of transparency ($\omega > \sqrt{\omega_0^2 + \omega_p^2}$) has not been indicated since in the present case all the waves lie on the low frequency branch ($\omega < \omega_0$).

6)We note that the frequencies ω_4 and ω_5 differ by quantities of order $\beta^2 (\omega_i/\omega_{02})$. At the same time it also turns out that the phase velocity of the wave 4 is greater and the phase velocity of the wave 5 is smaller than |v|, i.e., the latter wave lies in the region of the anomalous Doppler-effect (it is just because of this that ω changes sign in it; cf., Fig. 1b). As has been stated already, this means that at the boundary quanta of the field are created.

7)We do not reproduce here the elementary solutions of this and subsequent problems; these solutions can be found in [4,15] and in the review [8].

8)However, the present formulation of the problem and the singularities associated with it are not quite usual for the theory of hyperbolic equations in mathematical physics.

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