Hydrodynamical theory of multiple production processes

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Usp. Fiz. Nauk **116**, 271–302 (June 1975)

This paper contains an exposition of the classical hydrodynamical interpretation of multiple hadron production processes. Special attention is given to the clarification of the fundamental aspects of the hydrodynamical theory, the analysis of nucleon-nucleus collisions, and e^+e^- annihilation into hadrons. The conclusions of the theory are compared with experimental results. Some heuristic aspects of the theory are outlined. Use is made of the literature up to the beginning of 1975.

PACS numbers: 12.40.E, 13.65., 13.80.

CONTENTS

1. INTRODUCTION

The production of many particles in two-hadron collisions (multiple production processes) is a privilege of the strong interactions. These two characteristics of multiple production processes–a large number of particles in the final state and the strong interactions–determine the specific character of this phenomenon. Its description is complicated by lingering doubts about the applicability of modern field theory to the strong interactions. On the other hand, the fact that many particles take part in the process offers hope that one can successfully apply the quasi-classical approximation. It is therefore expedient to approach the description of multiple production processes from the standpoints of both quantum and classical ideas.

It is appropriate here to point out an analogy with nuclear models: the shell model, for instance, is a quantum model and, at the same time, a gas model, i.e., a classical approach. However, this analogy has a very important limitation: whereas the nucleus is a system of real particles, a major role is played by virtual particles in multiple production processes. This fact does not facilitate an understanding of the situation. Nevertheless, the main point of this analogy is that both quantum and classical approaches to the description of these two phenomena (the nucleus and multiple production processes) are still possible.

Historically, the theory of multiple production processes began with the prediction of the phenomenon even before it was observed experimentally. In an attempt to explain the showers which had been seen long before in cosmic rays, Heisenberg[1] made use of a now-forgotten variant of the β interaction. However, what is most important is that he had already pointed out in his early works[1, 2] the possibility of a statistical description of multiple production processes, owing to the large value of the coupling constant.

This idea was subsequently expressed in the work of Fermi[3], who adopted the basic hypothesis that a statistical equilibrium is established in a Lorentz-contracted volume.

However, Pomeranchuk[4] noted that there is an inconsistency in this theory, which is based on the assumption that there is a strong interaction of many particles concentrated in a volume whose dimensions are much smaller than the range of the forces. This inconsistency was aggravated by the fact that calculations of the characteristics of multiple production processes made use of the ideal-gas model. The Lorentz-contracted volume in the classical approach should therefore be only the initial state of the system.

On the basis of these observations, Landau[5] developed the hydrodynamical theory, which in fact constitutes the main content of the present review; the greatest emphasis here will be on those matters which have been discussed inadequately or not at all in previous reviews: the interaction of nucleons with complex nuclei, the annihilation of leptons into hadrons, and the fundamental principles of the hydrodynamical model.

The hydrodynamical theory has had a curious history. Although the quantum approach was developed together with the classical description[6,7], the hydrodynamical theory was undoubtedly dominant until the late 1950s. The situation changed after the formulation of the multiperipheral theory[8], when it seemed that a way had been found of solving all the problems of multiple production processes within the framework of current theory. These hopes were strengthened when the method of complex angular momenta[9] was incorporated in this theory.

Nevertheless, it was not possible to overcome the fundamental difficulty of formulating a self-contained theory of multiple production processes.

On the other hand, the latest precision experiments which have been carried out with large accelerators (in particular, the ISR[1)]) have demonstrated that many of the currently observed characteristics of multiple production processes were predicted previously in the framework of the hydrodynamical theory.

All these facts have led to a surge of interest in the hydrodynamical theory.

This rebirth has produced a curious paradox: questions and problems which confronted physicists about two decades ago are now being solved anew (and not always correctly) and rediscovered. This situation was the motive for the reinterpretation and exposition of the hydrodynamical theory.

2. FERMI'S STATISTICAL THEORY

Let us recall briefly the basic idea of the statistical theory. In the c.m.s., the colliding nucleons are contracted disks with transverse dimensions $1/\mu$ (where μ is the pion mass) and longitudinal dimensions $\sim(2/\mu)M/\sqrt{s}$ (where M is the nucleon mass and $\sqrt{s}$ is the total energy in the c.m.s.).

The basic hypothesis of the model is that during the collision within the characteristic volume

$$V\approx\frac{8}{3}\pi\frac{M}{\mu^3\sqrt{s}} \qquad (1)$$

the observed real particles appear with a distribution which, for sufficiently large particle numbers $N\gg 1$ (the thermodynamic approximation), correspond to black-body radiation with allowance for the isotopic factor 3/2, which reflects the fact that there exist three types of charged pions. The number of particles, $\overline{N}$, is then given by

$$\overline{N}=n_1 V, \qquad (2)$$

where the concentration is

$$n_1\approx 0.4\ T^3. \qquad (3)$$

We shall henceforth take $\hbar = c = k = 1$ (here k is the Boltzmann constant); T is the temperature of the system.

Since the energy density is given by the expression

$$\varepsilon=\frac{\sqrt{s}}{V} \qquad (4)$$

and

$$\varepsilon=0.7\ T^4, \qquad (5)$$

we have

$$\overline{N}\approx 1.5\sqrt{\frac{\sqrt{s}}{M}}, \qquad (6)$$

or[2)]

$$\overline{N}\approx 2.5\sqrt[4]{\frac{E_0}{M}}, \qquad (7)$$

where E_0 is the energy of the incident particle in the c.m.s.

3. POMERANCHUK'S STATISTICAL MODEL

Shortly after Fermi's work appeared, it was pointed out[4] that there is an inconsistency in the space-time description of N particles on the basis of the ideal-gas model ($N > 1$) when the interaction range is greater than the overall dimensions of the system[3)]. To overcome this inconsistency within the spirit of the whole statistical picture, it must be assumed that the description of multiple production processes is divided into three stages:

1) the formation of an initially contracted disk;

2) the expansion of the system;

3) the decay of the system into real particles.

It is important to emphasize here that no real particles are involved in the first two stages. Thus, the above-mentioned inconsistency is eliminated. However, a high price is paid for achieving this, since there arises the fundamental question as to the concrete meaning of the idea that the expansion takes place during the collision. We shall return to this problem later.

In order to assign a quantitative form to the statistical model with an expanding volume, we must specify the characteristic volume of the system. From physical arguments, there exists in our approach, apart from the initial Lorentz-contracted disk, only a single volume, equal to the "volume" V_0 of an elementary particle in its proper coordinate system. In order of magnitude,

$$V_0\approx\frac{4}{3}\pi\frac{1}{\mu^3}. \qquad (8)$$

The total volume of the system is then

$$V_T=NV_0. \qquad (9)$$

Since the total energy is $\sqrt{s}=\epsilon V_T$, we have

$$\overline{N}\approx\frac{\sqrt{s}}{\mu}. \qquad (10)$$

This result was obtained in the earliest works on the theory of multiple production processes[2].

During the expansion, the temperature T decreases. In the final phase, the temperature T_f is equal to the pion mass in order of magnitude (compare this with (10)):

$$T_f\sim\mu. \qquad (11)$$

Subsequent analysis of experimental data showed that the relation (10) is in conflict with the data and that Eq. (7) is much closer.

Owing to this fact, as well as to the appearance of a more consistent hydrodynamical theory, the statistical theory with an expanding volume was completely forgotten.

However, it was found quite recently[11] that a careful comparison of the conclusions of the statistical theory with an expanding volume with the experimental data up to $E_0 \lesssim 30$ GeV yields good agreement, provided that one introduces a new phenomenological parameter–an inelasticity coefficient K, which was found to have the value 0.4.

The law (10) is no longer in agreement with the data at energies $E_0 > 30$ GeV.

4. THE PHYSICAL BASIS OF THE HYDRODYNAMICAL THEORY

The statistical model with an expanding volume is also not sufficiently consistent. The point is that thermodynamics and hydrodynamics have the same domain of applicability. The transition from the initial state (the first stage) to the final state–the decay into real particles (i.e., the expansion of the system)–is therefore the factor that determines the third and final stage. The hydrodynamical *velocity* of an element of the fluid determines, in the final analysis, the rapidity distribution of the decaying elements:

$$y=\frac{1}{2}\ln\frac{E+p_\parallel}{E-p_\parallel}, \tag{12}$$

or

$$y=\ln\left[\frac{p_\parallel}{m_\perp}+\sqrt{1+\left(\frac{p_\parallel}{m_\perp}\right)^2}\right]; \tag{13}$$

here $m_\perp = \sqrt{m^2+p_\perp^2}$, $p_\perp$ and $p_\parallel$ are the perpendicular and parallel components of the momentum of the particle, and m is the mass of the secondary particle.

Thus, we must resort to hydrodynamics for a consistent description of the second stage. In the framework of the ideas that are being developed, we must also postulate that there exists a local equilibrium in all three stages of the development of the system. In other words, the thermal motion in the proper system associated with an element of the fluid is superimposed on the hydrodynamical motion. The distribution of bosons is described by the Bose formula, while that of fermions is given by the Fermi distribution. The overall rapidity y_Σ is a sum of the rapidities corresponding to the collective hydrodynamical motion and the thermodynamical thermal motion in the coordinate system associated with a given element of the fluid. Thus, the consistent description of multiple production processes amounts to the following postulates:

a) in the first stage, a Lorentz-contracted disk is formed with an initial temperature T_0 and energy density ϵ_0 determined by Eq. (5);

b) this disk constitutes the initial stage of the hydrodynamical expansion; the quantities ϵ and T decrease during the expansion process;

c) when the temperature T reaches a certain final value $T = T_f$, which is independent of the initial energy E_0 but depends on the properties of hadronic matter, the decay of the system into real particles sets in; it is natural to assume that $T_f \approx \mu$ also holds in this case.

Although this set of postulates is logically complete, it does not, unfortunately, provide a unique solution of the problem. The reason for this is that the hydrodynamical description of the motion is determined by two types of terms[12] (see Eqs. (14) and (15) below): inertial terms, which are in essence a consequence of the law of energy-momentum conservation, and dissipative terms, which reflect the properties of the medium (in our case, hadronic matter). Relativistic hydrodynamics is contained in the system of equations.

$$\frac{\partial T_{ik}}{\partial x^k}=0, \tag{14}$$

$$T_{ik}=p_p g_{ik}+(\varepsilon+p_p)u_i u_k+\tau_{ik}. \tag{15}$$

where p_p is the pressure, g_{ik} is the metric tensor, u_i is the i-th component of the 4-velocity, and τ_{ik} is a dissipative term comprising a sum of terms of the form $\eta_V \partial u_i/\partial x^k$ and $\eta_V(\partial u_i/\partial x^l)u_i u_k$; η_V is the coefficient of viscosity.

In addition to the ambiguity associated with the relative contribution of the dissipative terms, it is obvious that even if we put

$$\tau_{ik}=0, \tag{16}$$

the system (14) does not have a unique solution, since the five equations (the system (14) and the equation $\sum_{i=1}^4 u_i^2 = 1$) contain six unknowns (ϵ, p_p, u_i). Consequently, to obtain a unique solution, we must specify the equation of state

$$p_p=\varphi(\varepsilon). \tag{17}$$

The dependences (6) and (7) are obtained if we assume (16) and

$$p_p=\frac{\varepsilon}{3}\,^{4)}. \tag{18}$$

5. DISSIPATIVE TERMS AND THE EQUATION OF STATE

The question arises as to the degree of generality of the conditions (16) and (18). These conditions seemed almost obvious when the hydrodynamical theory was formulated. The situation is now much less clear.

To examine the problems which arise here, it is convenient to turn to certain simple models for which practically everything can be calculated quantitatively.

Let us first consider the equation of state (18). This relation is a relatively general consequence of the isotropy of space and relativity. In fact, the pressure p_p for an ideal gas is the average flux of the root-mean-square momentum projection per unit area on the normal to the surface:

$$p_p=\frac{mn_1\sqrt{\overline{v_n^2}}}{\sqrt{1-v^2}}, \tag{19}$$

where

$$\overline{v_n^2}=\frac{\overline{v_x^2+v_y^2+v_z^2}}{3}=\frac{v^2}{3}. \tag{20}$$

Putting v = 1, we obtain Eq. (18). The assumption that space is isotropic is important for these considerations. There is no problem here for macroscopic bodies. However, this assumption becomes less obvious when we go down to distances which are comparable with the "dimensions" of the particles, because of the possible influence of spin effects (there may occur a distinguished direction because of spin correlations).

If we use field theory and consider only an interaction with no derivatives, Eq. (18) is obtained again in first-order perturbation theory and the ultra-relativistic approximation[13-15].

Nevertheless, for a concentration $n_1 \to \infty$ and mass $m \neq 0$, for example, it is found that for vector particles[16]

$$p_p = \varepsilon. \quad (21)$$

We turn now to the problem of determining the role of dissipative particles. Putting $\partial u/\partial x \sim u/L$, where L denotes the characteristic dimensions of the system, we find that the dissipative terms may be neglected; with the Reynolds number

$$\mathrm{Re} \sim \frac{(\varepsilon + p_p) L}{\eta_V u} \gg 1 \quad (22)$$

or, since $\epsilon \sim p_p$, we have

$$\mathrm{Re} \sim \frac{p_p L}{\eta_V u}. \quad (23)$$

For a nonrelativistic gas consisting of real particles, $\eta_V \sim n_1 mvl$ (where l is the free path length), so that

$$\mathrm{Re} \sim \frac{uL}{vl}. \quad (24)$$

Hydrodynamics is applicable under the condition $L \gg l$; it was therefore concluded in the fundamental works[5, 17] that viscosity may be neglected.

However, this reasoning involves an inconsistency: the coefficient η_1 is calculated for a nonrelativistic gas, whereas all the calculations are carried out in a relativistic approximation.

Another argument is even more important: for a medium consisting of strongly interacting particles, the very concept of a "free path length" has a conditional character. The relevant criterion is the relation between the interaction range and the average distance between the particles. The free path length is obviously meaningful when their ratio is less than 1.

Let us consider the situation in greater detail for system of photons and electrons, when (in contrast with the case of hadrons) numerical estimates can be made. For a system of relativistic electrons and photons in a thermodynamic equilibrium, the role of the dimensions is evidently played by the classical electron radius $r_e = e^2/m$. In fact, the cross sections for the processes and the distributions in such a system are usually obtained in the single-photon approximation. Multiphoton processes begin to play a role when the following inequality is satisfied:

$$n_1 \lambda \sigma \gtrsim \frac{1}{\alpha} \approx \frac{1}{137} \quad (25)$$

(see, e.g.,[18]); λ is the average wavelength of the radiation, given by $\lambda \approx 1/T$ for an ultra-relativistic gas, and σ is the characteristic cross section for the process evaluated in the single-photon approximation. In our case, this is the process of pair production: $\sigma \sim \alpha r_e^2$. Using (3), we obtain the condition

$$T \gtrsim \frac{1}{r_e} \sim 10^{11}\ \mathrm{deg}, \quad (26)$$

corresponding to an average distance between the particles equal to the classical electron radius.

We note that, even in the case when the dimensions of the system are taken to be $L \sim r_e$, the use of a criterion such as (24) leads to values **Re** ~ 1 for the critical condition $n_1 \lambda \sigma \sim 1$. Digressing from the fact that this estimate is unreliable, we would like to emphasize that there exists a relation between the coefficient η_V and the equation of state (17). For hadron interactions, both functions are determined by the properties of hadronic matter.

The foregoing considerations make quantitative estimates of the coefficient of viscosity, and at the same time the role of dissipative processes, highly problematical.

For example, in the case of a scalar interaction in the lowest order of perturbation theory[5)], it has been found[19] that

$$\eta_V \sim T^3. \quad (27)$$

There is a better basis for an estimate of the coefficient of viscosity according to dimensional arguments[20]. Since we have the dimensionality η_V = [mass]/[length][time], we obtain, in our system of units, the relation (27) in the ultra-relativistic case ($T \gg M$). The absence of a parameter with the dimensions of mass is important for this argument; its role is therefore played by the temperature. The largest mass is usually taken to be equal to the nucleon mass M. However, the validity of this choice is not obvious; moreover, the initial temperature satisfies $T_0 \gtrsim M$ only for energies $E_0 \gtrsim 10^{14}$ eV (see (5)). In other words, dimensional arguments are insufficient for an estimate of the coefficient η_V in the very interesting energy range $E_0 < 10^{14}$ eV.

Taking into account these uncertainties, it is therefore expedient to postulate definite dependences $\eta_V(T)$ and $p_p(\epsilon)$. From considerations of simplicity and the analogy with electrodynamics, we shall first of all adopt the relations (16) and (18). Estimates using different dependences for $\eta_V(T)$ and $p_p(\epsilon)$ will be made later.

6. THE HYDRODYNAMICAL INTERPRETATION OF NUCLEON-NUCLEUS COLLISIONS

Consider a collision of a relativistic nucleon with a nucleus of atomic number A. Owing to the highly anisotropic emission of the secondary particles, the nucleon and the particles produced by it should be expected to move approximately along the same trajectory within the nucleus. Thus, the nucleon, as it were, cuts out a "tube" of nuclear matter in the nucleus. The description of this process has two limiting variants. In the first of them, the nucleon and the secondary particles produced by it undergo successive collisions with each of the nucleons of the "tube." In the second variant, the nucleon collides with the "tube," whose matter is devoid of structure (the "tube" model[21]). In the "tube" model, the nuclear matter constitutes a single "elementary" particle with a density of matter equal to the density of the nucleon. The geometric shape of the "tube" is a cylinder whose base has radius $\sim 1/\mu$.

Estimates have shown that, if the nucleon loses a relatively small fraction of its energy in an elementary collision process, the effective longitudinal interaction length may exceed the dimensions of the nucleus[22]. The time of a single event may then exceed the time between interactions; consequently, there is a basis for the use of the "tube" model.

There are as yet no unambiguous arguments in favor of the "tube" model; its basic postulate, the lack of structure of the nuclear matter during the collision, lies outside the system of postulates of the hydrodynamical theory. Nevertheless, the "tube" model is quite consistent with it. Let us consider the hydrodynamics of the initial stage of a collision of a nucleon with a structureless "tube."

In the case in which the length of the "tube" is n = 1 (a single nucleon), this problem is merely of methodological interest, since the initial state is determined by a symmetry condition; this state is a stationary fluid contained within a Lorentz-contracted volume.

The picture becomes much more complicated when $n > 1$. At the first moment of contact of the nucleon with the "tube," shock waves begin to propagate in both directions through the nuclear matter. If we adopt the equation of state (17), the velocity D of the shock waves is equal to 1/3, while the velocity of sound in the medium is $c_0 = 1/\sqrt{3}$[23]. It is convenient to carry out the analysis in a coordinate system in which the nucleon and the "tube" have identical velocities in absolute value.

If the densities of nuclear matter of the nucleon and the "tube" are the same, the matter between the shock waves will be at rest (we shall suppose that the nucleon is incident from the left (Fig. 1a)). Then the shock wave traveling to the left reaches the edge of the nucleon before the wave moving to the right reaches the edge of the "tube." At this moment, a simple (traveling) rarefaction wave moves from left to right through the nuclear matter (Fig. 1b). This wave propagates with the velocity of sound c_0, i.e., faster than the shock wave. Estimates[23] have shown that, if $n < 3.7$, (and, in particular, if $n = 1$), the simple wave does not manage to overtake the shock wave before the latter reaches the edge of the "tube." If $n > 3.7$, the simple wave catches up with the shock wave and, in being reflected by it, produces the so-called first reflected wave. When the shock wave moving to the right reaches the edge of the "tube," emission of matter into the vacuum begins.

Let us now estimate the effect of the hydrodynamical emission on the average multiplicity $\overline{N}$. Owing to the ideal nature of the fluid (see the condition (15)), the entropy does not increase, so that the dependence (6) and (7) remains valid for the NN interactions. However, the multiplicity rises somewhat with increasing atomic number A. It is easier to see the physical origin of this effect by using the statistical theory[21].

Let the mass of the tube be $M_t = nM$. Then the volume V_t of the "tube" in its proper coordinate system and its mass M_t are proportional to $A^{1/3}$. Using Eqs. (3)–(5), it is easy to show that

$$\overline{N} \sim s^{1/2}A^{1/6}. \qquad (28)$$

Since $E_0 \sim s/M_t$, we obtain the final result

$$\overline{N} \sim E_0^{1/4}A^{1/4}. \qquad (29)$$

A solution of the hydrodynamical equations[23] leads to the slower dependence

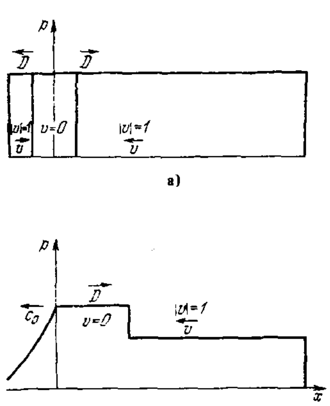


FIG. 1. Scheme of the hydrodynamical emission of particles. a) Scheme of the motion of the matter and shock waves in the initial stage. The matter between the shock waves is at rest, v = 0; "parts" of the nucleon (from the left) and the "tube" (from the right) continue to move with velocity v = 1 towards each other. b) The shock wave has reached the edge of the nucleon. Emission of a simple wave into the vacuum with velocity c_0 has begun. The dashed line indicates the plane of contact of the nucleon and "tube" at the initial moment.

$$\overline{N} \sim A^{0.19}. \qquad (30)$$

A steeper growth is obtained in the case of a collision of two identical ultra-relativistic nuclei of atomic number A. An argument similar to that leading to (28) and (29) gives the dependence

$$\overline{N} \sim A^{0.75}. \qquad (31)$$

It should be emphasized that this last estimate is a rough one, since the system contains many nucleons in a head-on collision (i.e., the initial entropy is very different from zero), a fact which is not taken into account in the derivation of (31).

Next, let us consider the effect of the hydrodynamical emission on the angular and momentum distributions or on the rapidity distribution.

Owing to the strong contraction at the initial moment, there exists a distinguished direction, which coincides with the direction of motion of the incident particles. The hydrodynamical emission is therefore predominantly one-dimensional in character. It has been shown[24] that, for energies up to $E_0 \lesssim 10^{13}$–10^{14} eV, one can employ one-dimensional motion to estimate the hydrodynamical motion in a first approximation. The longitudinal momentum components are then determined by the hydrodynamical velocity, while the perpendicular components are determined by the thermal motion[25]. In this approximation, the inclusive distribution function

$$f(p_\perp, p_\parallel, s) = \frac{E\, d^3\sigma}{dp^3} \qquad (32)$$

decomposes into two factors:

$$f(p_\perp, p_\parallel, s) = f_1(p_\perp, s)\, f_2(p_\parallel, s). \qquad (33)$$

This is an important point. Since the final temperature is T_f = const (s), the function $f_1(p_\perp, s)$ is independent of the initial energy and is determined entirely by the thermal motion. In that case, this distribution is the projection of the Bose distribution (for bosons) on the direction perpendicular to the motion[25]:

$$\int f\, dp_\parallel = \frac{dN}{dp_\perp} = \frac{gm^3}{2\pi^2} \sum_{r=1}^{\infty} (\mp 1)^{r-1} K_1\left[r\sqrt{1+\left(\frac{p_\perp}{m}\right)^2}\right] p_\perp; \qquad (34)$$

here g is the number of internal degrees of freedom of the particle, and K_1 is a Bessel function of an imaginary argument. Putting $T_f = m = \mu$, we obtain

$$\frac{dN}{dp_\perp} = B \sum_{r=1}^{\infty} (\mp 1)^{r-1} K_1\left[r\sqrt{1+\left(\frac{p_\perp}{m}\right)^2}\right] p_\perp. \qquad (35)$$

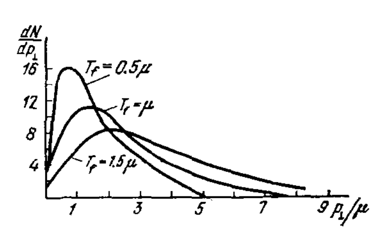
For $p_\perp \gg \mu$, we find

$$\frac{dN}{dp_\perp} = B p_\perp e^{-p_\perp/\mu}. \qquad (36)$$

Figure 2 shows the distribution $dN/dp_\perp$, calculated using various values of the parameter T_f.

In the next approximation, allowance must be made for the effect of the hydrodynamical expansion on the transverse motion of the matter and of the thermal

FIG. 2. The distribution $dN/dp_\perp$ for various values of the final temperature T_f [25].



fluctuations on the longitudinal motion. This complicated calculation was carried out in two papers[26, 27], whose results were in mutual agreement. To avoid a digression here involving cumbersome calculations (see the Appendix), we shall consider only the final conclusions[26]. In the c.m.s.,

$$\frac{dN}{dy^*}=N\frac{\exp[-(y^*)^2/2L]}{\sqrt{2\pi L}}, \tag{37}$$

where

$$L=0.56\ln\frac{E_0}{M}+1.6\ln\left(\frac{2}{n+1}\right)+1.6. \tag{38}$$

The accuracy of the solution of Eqs. (14) and (15) with the condition (16) is ~15–20%. The distribution (37) is correct for $n < 3.7$. In this case, the c.m.s. differs little from the equal-velocity system, and this difference can be neglected. The distribution dN/dy^* remains symmetrical with respect to $y^* = 0$. Collisions with nuclei in this case give only a slight narrowing of the distribution dN/dy^* and, accordingly, a broadening of the distribution $dN/d\vartheta^*$ (where ϑ^* is the angle of emission in the c.m.s.).

However, a curious effect occurs for $n > 3.7$[28]. In this case, the difference between the c.m.s. and the system in which the emission of the secondary particles is symmetrical becomes quite appreciable. This means that the symmetry with respect to $y^* = 0$ in the c.m.s. is broken. More particles are emitted in the fragmentation region of the target (the so-called backward cone) than in the fragmentation region of the incident particle.

The relative velocity V_c of the center-of-mass system with respect to the system in which the emission is symmetrical is determined by the expression

$$V_c=\operatorname{th}\left[\frac{\sqrt{3}}{2}+\frac{2n-4-2\sqrt{3}}{7+4\sqrt{3}}-\operatorname{Arth}\left(\frac{n-1}{n+1}\right)\right]. \tag{39}$$

$$\frac{dN}{dy^*}=\frac{N\exp[-(y^*-y_c^*)^2/2L]}{\sqrt{2\pi L}}, \tag{40}$$

where $y_c = \operatorname{Arctanh} V_c$.

Another important conclusion should be stressed. Allowance for lateral hydrodynamical motion leads to a weak s-dependence of the function $dN/dp_\perp$. Calculations[29] made on the basis of the results of[26] gave the following dependence $\bar{p}_\perp(s)$ in the energy range $10^{12} < E_0 < 10^{14}$ Ev:

$$\bar{p}_\perp(\mathrm{MeV})=250+40\ln\frac{\sqrt{s}}{M}. \tag{41}$$

If we try to approximate this dependence by a power function, we find in this interval

$$\bar{p}_\perp \sim s^{1/15}. \tag{42}$$

It is of interest to note that the dependence $\bar{p}_\perp(s)$ has been calculated quite recently[30] (see Eq. (105) in the Appendix) on the basis of the distributions obtained in the fundamental paper[5]. However, owing to the very crude approximation in estimating the final stage of the hydrodynamical emission, i.e., for the distribution $dN/dp_\perp$, the results of[5] are very inaccurate. The dependence $\bar{p}_\perp \sim s^{1/12}$ which was found[30] and the conclusion that the hydrodynamical theory is inconsistent with the experimental data are therefore incorrect.

7. THE DISTRIBUTION IN THE LOGARITHMIC COORDINATE η

For a long time, analyses of multiple production processes have been making use of the very convenient variable

$$\eta=-\ln\operatorname{tg}\frac{\vartheta^*}{2}, \tag{43}$$

which was introduced in high-energy physics in[5].

Since

$$\eta=\frac{1}{2}\ln\frac{p+p_\parallel}{p-p_\parallel}, \tag{44}$$

it is obvious that in the relativistic case ($p \gg m$)

$$\eta\approx y \tag{45}$$

(compare this with the definition of the rapidity, Eq. (12)). However, the situation is completely different for slow particles, $p \lesssim m$.

For example, if $p \ll m$, then $y \to 0$, while $0 < \eta < \infty$. This circumstance has a strong influence on the characteristics of the distributions dN/dy and $dN/d\eta$ as $y \to 0$ or $\eta \to 0$[6]. More accurate calculations (than those of[26]) of the distribution $dN/d\eta$ within the framework of the hydrodynamical theory have shown that the distribution dN/η exhibits a small minimum near the point $\eta = 0$[31], with a depth which becomes greater with decreasing $\bar{p}_\perp$, i.e., which depends on the value of the final temperature T_f.

Figure 3 shows the distributions $dN/d\eta$ for various temperatures T_f. The origin of the effect in question is easy to understand. As $\bar{p}_\perp$ decreases, the anisotropy becomes greater, leading to a decrease in the number of particles emitted at angles $\vartheta^* \sim \pi/2$, i.e., near $\eta = 0$. It was later pointed out[32] that the occurrence of a minimum in the distribution $dN/d\eta|_{\eta\to 0}$ is a very general phenomenon due to the boundedness of $\bar{p}_\perp$. In other words, if we accept that $\bar{p}_\perp(s)$ = const as an experimental fact, then for $s \to \infty$ there must necessarily be a minimum in the distribution $dN/d\eta$ at $\eta = 0$, for any distributions in the longitudinal momenta.

Subsequently, the occurrence of the minimum in the distribution was meticulously investigated in[33] within the framework of the hydrodynamical theory.

Qualitatively, the calculation of the distribution $dN/d\eta$ leads to an integration of the distribution (32) with respect to $p_\perp$ within the kinematic limits.

Using the dependence

$$y=\operatorname{Arcth}\left(\frac{p}{E}\operatorname{th}\eta\right), \tag{46}$$

we obtain

$$\frac{dN}{d\eta}=2\pi N\operatorname{sch}^2\eta\int_0^{p_m}\frac{p^2f[p\operatorname{sch}\eta,\,y(p,\eta),\,s]}{(p^2+m^2)^2}\,dp, \tag{47}$$

where p_m is the maximally allowed value of the momentum p.

8. THE EFFECT OF DISSIPATIVE TERMS AND THE EQUATION OF STATE ON THE VALUE OF $\bar{N}$

We pointed out earlier that, if the equation of state is approximated by a linear function

$$p_g=c_0^2\varepsilon, \tag{48}$$

then $0 \le c_0 < 1$.

Distributions for $c_0^2 \neq 1/3$ were calculated in[34] (see also the later estimates of[27]). It was found that

$$\frac{dN}{dy}\approx\frac{N\exp[-(y^*)^2/2L]}{\sqrt{2\pi L}}, \tag{49}$$

where

$$L=\frac{2c_0^2}{1-c_0^4}\ln\left(\frac{1+c_0^2}{2c_0^2}\frac{s}{\mu M}\right)^{c_0^2/(1+c_0^2)}. \tag{50}$$

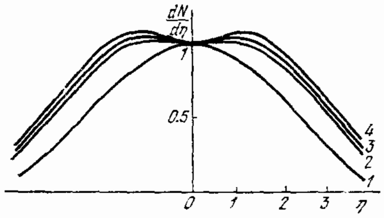


FIG. 3. The distribution $dN/d\eta$ for various values of the parameter T_f [31]. The curves are normalized so that $dN/d\eta = 1$ at $\eta = 0$. 1–$T_f \gg \mu$, 2–$T_f = \mu$, 3–$T_f = 0.75\mu$, 4–$T_f = 0.5\mu$.

In another form, the logarithmic term can be written

$$\ln\left(\frac{1+c_0^2}{2c_0^2}\frac{\sqrt{s}}{MV_0\mu^4}\right)$$

(see the Appendix), i.e., the Gaussian distribution has a universal character; however, the value of L, which characterizes the width of the plateau in the distribution dN/dy^* in the vicinity of $y^* = 0$, depends on the value of c_0.

The multiplicity $\bar{N}$ is determined largely by the quantity c:

$$\bar{N} \sim s^{(1-c_0^2)/2(1+c_0^2)}. \qquad (51)$$

This dependence is a consequence of the relations

$$s \sim \varepsilon^{1/(1+c_0^2)}, \quad \varepsilon \propto T^{(1+c_0^2)/c_0^2}, \qquad (52)$$

which follow from Eq. (48).

Thus, the allowed range of variation of the velocity of sound c_0 corresponds to a power-law dependence

$$\bar{N} \sim s^{\alpha}, \qquad (53)$$

with $0 \le \alpha \le 1/2$.

The maximum value of $\bar{N}$ corresponds to the parameter value $c_0 = 0$. The decrease in $\bar{N}$ with increasing c_0 is evidently explained by the fact that the relative value of the kinetic energy decreases with the pressure; the growth of the potential energy is due to the enhanced role of the dissipative processes, leading to a growth in the multiplicity (see below).

We turn now to the estimate of the dissipative processes[20, 28].

Assuming the relations $\eta_V \approx T^3$ and (52), we have

$$\mathrm{Re} \approx T^{(1-2c_0^2)/c_0^2} L; \qquad (54)$$

here L is the characteristic size of the system.

Putting $\mathrm{Re} \approx 1$, we obtain the new characteristic dependence

$$L_1 \approx \frac{T_1^{c_0^2/(1-2c_0^2)}}{c_0^4}. \qquad (55)$$

A detailed quantitative analysis of the effect of the viscosity on the characteristics of multiple production processes was given in[28, 34].

Qualitatively, this effect can be estimated by assuming that the inertial terms are negligible in comparison with the dissipative terms up to the limit defined by Eq. (55). The effect of these terms can be neglected for $L > L_1$[20].

Using (52) and (55), as well as Eq. (18), we obtain

$$T_1 \propto s^{1/6}, \quad \bar{N} \propto T_1^2 \propto s^{1/3}. \qquad (56)$$

Since the fluid is assumed to be ideal for $T < T_1$, the multiplicity remains unchanged in the subsequent emission stage.

The increase in the multiplicity (56) in comparison with the law (29) has a simple physical interpretation; the dissipative terms reflect an additional interaction of the elements of the fluid, which leads to an increase in the entropy.

The additional interaction also leads to a broadening of the angular distribution (or a corresponding shrinkage of the plateau in the distribution dN/dy). However, this change is relatively small and determines the pre-exponential factor in Eq. (49) or the logarithmic term in the definition (50).

The viscosity has a considerable effect on the entropy contained in the simple wave[34]. We note, however, that the entropy of the simple wave is relatively small[24, 35] and that estimates in this region are unreliable.

In conclusion, we mention that the expressions for the viscosity have been derived here, strictly speaking, only for the value $c_0^2 = 1/3$.

9. THE ROLE OF DISTINGUISHED PARTICLES IN MULTIPLE PRODUCTION PROCESSES

From studies of cosmic rays, it was concluded long ago that there exists one energetically distinguished particle among the secondary particles.

Subsequent careful investigation of this question has confirmed this conclusion; this distinguished particle is a baryon in baryon-baryon collisions, while it is a pion in πN collisions. The average inelasticity coefficient K for nucleon-nucleon collisions or for collisions of a nucleon with a light nucleus is approximately 0.5. The existence of distinguished particles and, even more so, their conservation of definite quantum numbers indicate that not all the energy of the colliding particles goes into the statistical system. The quantity K is a parameter of the theory which must be determined experimentally. However, we can give definite physical reasons why nucleons are distinguished in a complex statistical system[36]. If $E_0 \lesssim 10^{12}$–10^{13} eV, we have $T \lesssim M$ throughout the emission stage, so that the initial nucleons may be expected to preserve their quantum numbers in this energy range.

On the other hand, rough estimates show that at these energies the limiting Fermi energy is $\epsilon_F \gtrsim T_0$, so that the pressure due to the Pauli principle tends to force the nucleons to the periphery of the volume in which the hadronic fluid is concentrated[7].

As in the estimate of the effect of the viscosity, there occurs here a new characteristic energy $E_0 \sim 10^{13}$–10^{14} eV (corresponding to a value $T_0 \sim M$) at which we should expect a change in the dependence of the multiplicity $\bar{N}(E_0)$ towards higher values.

10. COMPOSITION OF THE SECONDARY PARTICLES

We shall refer to the relative yields of the various types of particles as the composition.

The statistical theory with a contracted volume predicted an incorrect composition of the secondary particles, since at $T = T_0$ the masses of the secondary particles no longer affect the composition; the relative statistical weights are therefore determined by the ratios of the internal quantum numbers, g_i/g_π, which are equal to 1 in order of magnitude. For example, the ratio of the number of antinucleons to the number of pions is 8/3, in gross conflict with the experimental data.

However, the situation is entirely different in the hydrodynamical theory, in which the composition of the secondary particles is determined by the final emission stage with $T = T_f$.

A calculation of the composition in the hydrodynamical approximation with $c_0 = 1/\sqrt{3}$ (the number of particles of type i is $n_i > 1$) leads to the expression

$$\frac{\bar{n}_i}{\bar{n}_\pi} = \frac{g_i}{g_\pi}\frac{F(m_i/T_f)}{F(\mu/T_f)}, \qquad (57)$$

where

$$F(z) = z^3 \int_0^\infty \frac{x^2\,dx}{\exp\left(z\sqrt{1+x^2}\right) \pm 1}. \qquad (58)$$

The signs "+" and "−" refer to fermions and bosons, respectively. For $m_i > T_f = \mu$,

$$\frac{\bar{n}_i}{\bar{n}_\pi} \sim \frac{g_i}{g_\pi}\frac{1}{F(1)}\left(\frac{\pi}{2}\right)^{1/2}\left(\frac{m_i}{\mu}\right)^{3/2}\exp\left(-\frac{m_i}{\mu}\right). \qquad (59)$$

This expression contains a characteristic exponential factor, which suppresses the occurrence of heavy particles (e.g., $\bar{n}_K/\bar{n}_\pi \sim 0.17$).

In the thermodynamic approximation, the composition is independent of the initial energy E_0. However, Eq. (57) was obtained with the very strong assumption that $n_i > 1$, which begins to hold for kaons and antinucleons only at energies $E_0 \gtrsim 10^{12}$ eV.

In calculating the composition at lower energies, it is therefore necessary to make use of the exact expressions for the statistical weight[11, 38-40]. This is an important point. Owing to the conservation of baryon number or strangeness, heavy particles appear in pairs. In the statistical approach, this means that, if such a pair appears with probability $W_i \ll 1$, the increase of phase space will involve a growth $W_i \sim r(r-1)$, where r is the number of elements of phase space. Since $\bar{n}_\pi > 1$, we have $\bar{n}_\pi \sim W_\pi \sim r$. Therefore, if $\bar{n}_i \ll 1$,

$$\frac{\bar{n}_i}{\bar{n}_\pi} \backsim \bar{n}_\pi \qquad (60)$$

and it is only for $\bar{n}_i > 1$ that we obtain the asymptotic relation

$$\frac{\bar{n}_i}{\bar{n}_\pi} = \text{const}.$$

11. CORRELATIONS AND FLUCTUATIONS IN MULTIPLE PRODUCTION PROCESSES

Owing to the relatively small number of secondary particles (this number is ~10 at the available accelerator energies), all characteristics of multiple production processes are subject to strong fluctuations. In addition to the difficulty due to this fact, there is another problem which also complicates attempts to understand the fluctuations in multiple production processes. It is usually necessary, in theoretical estimates of the fluctuations, to formulate assumptions which go beyond the postulates required for the calculation of average values.

We shall illustrate this point for the case of estimates of fluctuations in the framework of the hydrodynamical theory.

To be specific, we shall discuss here the fluctuations of two quantities: the multiplicity N at fixed energy E_0, and the relative spacing between the particles along the scale of rapidities y (assuming for definiteness that the rapidity of the i-th particle satisfies $y_i < y_{i+1}$). The following basic sources of fluctuations can be distinguished:

a) Deviations of the hydrodynamical velocity from the average value determined by the equations (14).

b) Fluctuations in the value of the inelasticity coefficient K, affecting the fraction of the energy imparted to the statistical system and hence the value of N.

c) The influence of resonance, whose decay affects both the multiplicity N and the relative rapidity y.[8)]

d) Thermodynamic fluctuations.

To estimate the fluctuations associated with the hydrodynamical velocity, allowance must be made for the kinetics of the emission, which, at least for the description of micro-physical processes, lies outside the scope of the phenomenological approach.

Crude estimates of the influence of the coefficient K show that variation of this quantity may lead to an appreciable change in the spread of the multiplicity distribution.

The influence of resonances could be estimated if we had at our disposal data on the partial cross sections for the yields of resonances. However, such data are already lacking at energies $E_0 >$ GeV. Finally, let us consider the role of thermodynamic fluctuations in greater detail. Since the first three factors are not taken into account here, we are estimating in this way a lower bound for the fluctuations.

Thermal fluctuations were studied in[42]. These fluctuations are determined largely by the dissipative terms. If, however, we assume in the framework of our system of postulates that emission of an ideal fluid takes place, then there is no thermal exchange between different elements of the fluid (adiabatic motion).

In the case of bosons with $c_0 = 1/3$, the ratio of the average number of particles $\bar{N}$ to the dispersion D is then found to be constant:

$$\frac{\bar{N}}{D(N)} \approx 2.$$

The effect of thermal fluctuations can also be extended to estimates of the rapidity shifts of the secondary particles with respect to the average values of the rapidity[43].

Let the sequence

$$\bar{y}_1 < \bar{y}_2 < \ldots < \bar{y}_N \qquad (61)$$

correspond to the average values of the rapidities of particles 1, 2, ..., N. We shall even assume that the distribution of Δy_i is determined by the deviation from $\overline{\Delta y_i}$ due to thermal fluctuations. Then the momentum distribution in the proper system associated with an element of the fluid is a Bose distribution (for pions), while in the c.m.s. it is a distribution transformed by a Lorentz factor equal to the component of 4-velocity u_0. Calculations[42, 44] have shown that, if the variable $\zeta = E/u_0 m$ is chosen, then for $u_0 \gg 1$ one obtains in the c.m.s. a universal distribution

$$\frac{dN}{d\zeta} \propto \left[\frac{m}{2T_f}\left(\zeta + \frac{1}{\zeta}\right) + 1\right] e^{-(m/2T_f)[\zeta + (1/\zeta)]}. \qquad (62)$$

Transforming from the quantity ζ to the rapidity y, it can be shown that

$$D(y) = \sqrt{\overline{y_i^2} - \overline{y}_i^2} \approx 1. \tag{63}$$

It is of interest to observe that this estimate is independent of the index i and is weakly dependent on the mass of the secondary particle. Since the average interval between two particles is[61a]

$$\overline{\Delta y} \approx \frac{\ln(E_0/M)}{N}. \tag{64}$$

we find that for energies $E_0 > 10^{12}$ eV.

$$D(y) \sim \overline{\Delta y}. \tag{65}$$

This relation implies that the thermal fluctuations are comparable in magnitude with the average interval $\overline{\Delta y}$. This leads to an important conclusion. As the motion is quasi-one-dimensional, the points in phase space lie within a narrow cylinder whose base is of radius $1/\mu$; therefore the state of a particle is practically determined by a single parameter–the rapidity y. Owing to the large fluctuations, there is a high probability that the value of Δy deviates strongly from its average value; consequently, thermal fluctuations may give rise to the production of clusters, which are usually defined as sets of particles with neighboring points in phase space.

12. SCALE INVARIANCE IN THE HYDRODYNAMICAL THEORY

In[45] a new invariance principle was formulated–scaling or scale invariance: in particular, if $s \to \infty$, the inclusive distribution function (32) becomes

$$f(p_\perp, y, s) = f(x, p_\perp), \tag{66}$$

where

$$x = \frac{2p_\parallel}{\sqrt{s}}. \tag{67}$$

Without going into the detailed history of how this principle was formulated (see[46]), we merely note that the principle of scale invariance is a consequence of ultra-relativity in the sense that it is satisfied when $\mathfrak{M}/\sqrt{s} \to 0$, where $\mathfrak{M}$ is the largest of the characteristic masses for a process.

However, this condition is not satisfied in the hydrodynamical theory. The largest mass here is the mass of the entire system, so that $\mathfrak{M}/\sqrt{s} \to \text{const} \neq 0$ if $s \to \infty$. It is therefore of fundamental interest to study scaling in the framework of the hydrodynamical theory[47, 48].

To do this, we transform to the scaling variable x in the distribution (37):

$$d\sigma \approx \frac{\overline{N}}{\sqrt{2\pi L}} \frac{dx}{x} \exp\left\{-\frac{[\ln(x\sqrt{s}/2m_\perp)]^2}{2L}\right\}. \tag{68}$$

Two asymptotic expressions are obtained in the limit as $s \to \infty$. In one region,

$$x \sim \frac{M}{\sqrt{s}} \qquad d\sigma \sim \frac{dx}{x}. \tag{69}$$

If $x \gg M/\sqrt{s}$, then

$$d\sigma \sim \frac{dx}{x^2}. \tag{70}$$

We note that, if we introduce in the distribution (37) the variable

$$u_1 = \frac{y}{\sqrt{L}}, \tag{71}$$

the dependence (37), expressed in the variable u_1, has a universal form which no longer contains the variable s[33]:

$$\frac{dN}{du_1} = N\frac{e^{-u_1^2/2}}{\sqrt{2\pi}}. \tag{72}$$

Thus, as was to be expected, scale invariance does not, strictly speaking, hold in the hydrodynamical theory. However, what is especially significant is that the practical deviation from the principle (66) is very small: it is reflected in a logarithmic factor L. In other words, the hydrodynamical theory is scale-invariant with logarithmic accuracy.

In practice, scale invariance holds in the hydrodynamical theory for $x \gtrsim 0.05$.

13. THE ACCURACY WITH WHICH THE CHARACTERISTICS OF MULTIPLE PRODUCTION PROCESSES ARE CALCULATED

In discussing the accuracy with which the parameters for a process are calculated on the basis of a strict physical theory, one usually has in mind the mathematical accuracy with which the equations are solved. This approach is completely inappropriate in our case. Equations (14) and (15) can be solved by numerical methods with practically any accuracy. However, such a procedure has little physical significance, owing to the uncertainty in the parameters of the theory. The first priority is therefore to estimate, at least approximately, the basic parameters and to improve the accuracy of the physical approximations. From this point of view, it is important to be aware of the physical accuracy of the calculations of various characteristics of multiple production processes. Estimates of these characteristics in the framework of the hydrodynamical theory may be intended to have quite different accuracies. The most reliable characteristics should be those which are practically independent of the poorly determined initial conditions. Primary consideration should be given here to the composition of the produced particles and the distribution $dN/dp_\perp$. These distributions are determined entirely by the equation of state and the final temperature T_f.[9)]

The dependence $\overline{N}(E_0)$ is a very important characteristic. However, apart from the value of the initial volume (whose order of magnitude is determined by physical considerations), calculations of this dependence usually neglect the dissipative terms, allowance for which would modify this dependence. Some uncertainty in the calculation of $\overline{N}(E_0)$ comes from the possible effect of quantum fluctuations in the initial stage of hydrodynamic expansion (see below).

The angular and energy distributions are determined by the initial conditions of the problem, the equation of state and the effect of the dissipative terms. Even with the assumptions which we have made (Eqs. (16) and (18)), the distribution dN/dy is, strictly speaking, governed by the distribution of hadronic matter within the initial volume. The postulate that this volume is spherical in shape in the proper coordinate system is obviously an idealization. However, attempts to specify this shape more precisely (for example, by allowing for edge effects in the form of an exponential fall-off of the density of matter in the initial state) are completely illusory.

Consequently, the question arises, for example, as to whether the thermal motion must be taken into account in calculating the distribution dN/dy. The problem is further aggravated by the fact that the calculation of dN/dy is carried out in the ultra-relativistic approximation, while $T \sim T_f \sim \mu$ in the final stage and $u_i \sim 1$ for slow particles. However, it is significant that, although all the factors which we have enumerated lead to an uncertainty in the final results, but this uncertainty has a logarithmic character (see Eqs. (37) and (38)) and therefore has little effect on the final results.

The problem of the fragmentation region is more complicated. We mentioned earlier that the hydrodynamical theory is not intended to describe the distinguished leading particles[10]. However, there arises here another fundamental problem concerning the limits of applicability of the hydrodynamical description. Does this approach provide generally adequate description of most of the fragmentation region? Evidently, two alternative answers are possible here at the present time, and only future experiments will be able to elucidate this problem.

The first possibility is to assume that the hydrodynamical theory can describe all multiple production processes throughout the allowed range of variation of y, apart from the region near the kinematic limits.

From the second point of view, the hydrodynamical theory may be expected to describe only collisions involving large values of the coefficient K (so-called central collisions), for which there are practically no distinguished particles.

However, it seems plausible a priori that these two alternatives are extreme points of view which to some extent reflect an inadequacy of our language.

Nevertheless, as the values of N and K increase, the possibility of a successful hydrodynamical interpretation undoubtedly becomes more likely[11].

14. COMPARISON OF THE HYDRODYNAMICAL THEORY WITH THE EXPERIMENTAL DATA

Very many comparisons of the distribution (35) for $T_f \approx \mu$ with the experimental data (see, e.g.,[11, 50, 51]) have shown good agreement. It is significant that, in accordance with the theoretical predictions, the distributions have been found to be independent of the type of incident particle and target and dependent on only the masses of the secondary particles[12].

While the distribution $dN/dp_\perp$ for pions with $p_\perp > \mu$ has been approximated by an exponential, this distribution for heavy particles has been represented[11] by a Gaussian function, in accordance with the experimental data. Figure 4 shows the calculated dependence of $\bar{p}(E_0)$ and the experimental values of this quantity[29]. Although we cannot as yet be completely confident about the observed weak growth of $\bar{p}_\perp$ with energy, such a tendency does show up clearly (see the latest data[29]).

Figure 5 shows the dependence $\bar{N}_S(E_0)$.[13] The dashed curve is the approximation $\bar{N}_S = 1.97E_0^{1/4}$ [48] (see also[52]).

A comparison of the experimental and theoretical distributions $dN/d\eta$ in a wide range of variation of the variable η was made in[33], where the experimental data of[53] were used (Fig. 6).

In Fig. 7 we show the results of a careful comparison of the calculated[54] distributions $dN/d\eta$ for various values of the coefficient c_0 in the equation of state (48). It can be inferred from this figure that the distribution $dN/d\eta$ depends strongly on the parameter c_0. From the fact that better agreement is obtained in the approximation $c_0 \to 1/\sqrt{3}$ with increasing s, the authors of[54] conclude that the asymptotic relation (18) has not yet been attained at energies $E_0 \sim 10^{12}$ eV; the calculations of[54] are evidently more accurate.

In Fig. 8 we compare the distribution $\ln[N/(dN/du_i)]$ for various values of E_0[33].

Very careful comparisons of the calculated and experimental distributions in y* and x at fixed $p_\perp$ have been made in[55] (Figs. 9 and 10). We see that very precise comparisons of many of the latest experimental characteristics of NN collisions with the hydrodynamical theory indicate good agreement.

It is a rather paradoxical situation that practically all the conclusions of the theory were obtained many years ago but that a detailed comparison became feasible only when large accelerators came into operation.

Comparisons of the "tube" model with the data are much more meager, although we can say that there is at least semi-quantitative agreement between experiment and theory.

As is well known, one of the most interesting features of the interactions of hadrons with complex nuclei is the relatively weak dependence of the characteristics of

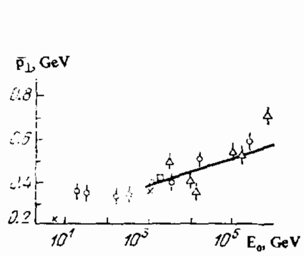


FIG. 4

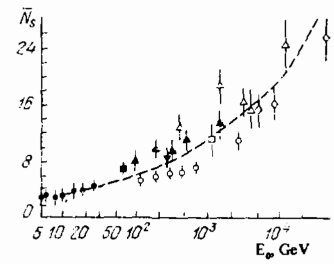


FIG. 5

FIG. 4. Plot of $\bar{p}_\perp(E_0)$ [29]. The solid curve shows the calculation according to the hydrodynamical theory.
FIG. 5. Plot of $\bar{N}_S(E_0)$ [48].

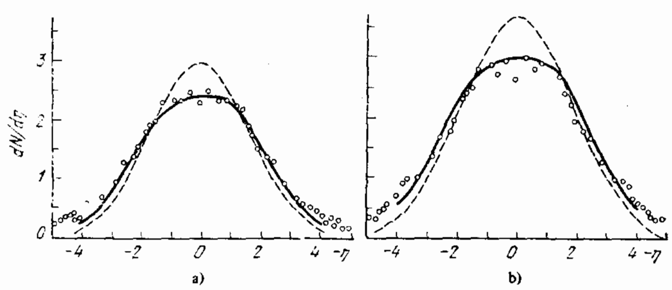


FIG. 6. Comparison of the theoretical [33] and experimental distributions [53] $dN/d\eta$ within a large range of variation of η. The solid curve is the theoretical distribution $dN/d\eta$. The dashed curve is the distribution dN/dy. a) $\sqrt{s}$ = 30.8 GeV, b) $\sqrt{s}$ = 53.4 GeV.

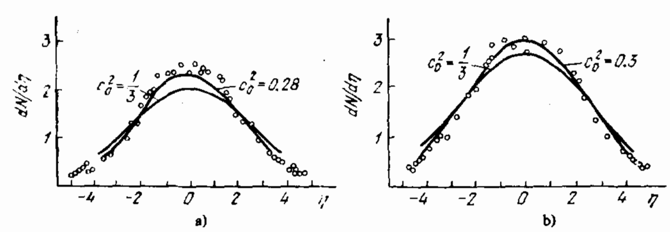


FIG. 7. Comparison of the theoretical and experimental distributions $dN/d\eta$ for various equations of state [54]. a) $\sqrt{s}$ = 30.8 GeV, b) $\sqrt{s}$ = 53.4 GeV.

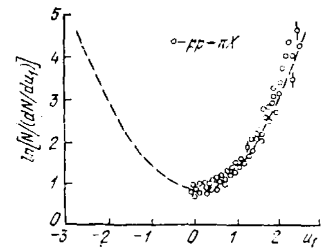


FIG. 8. The distribution dN/du [33]. The quantity $\ln[N/(dN/du_1)]$ is plotted on the vertical axis. The dashed curve is the calculated result. The experimental points correspond to the values E_0 = 12, 19, 21, 24 and 29 GeV.

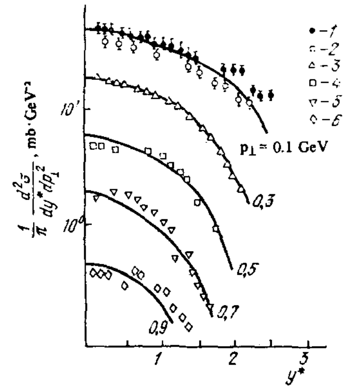


FIG. 9. Comparison of the theoretical and experimental rapidity distributions [55]. 1—$p_\perp$ = 0, 2 —0 < $p_\perp$ < 0.2 GeV, 3—0.2 < $p_\perp$ < 0.4 GeV, 4—0.4 < $p_\perp$ < 0.6 GeV, 5—0.6 < $p_\perp$ < 0.8 GeV, 6—0.8 < $p_\perp$ < 1.0 GeV.

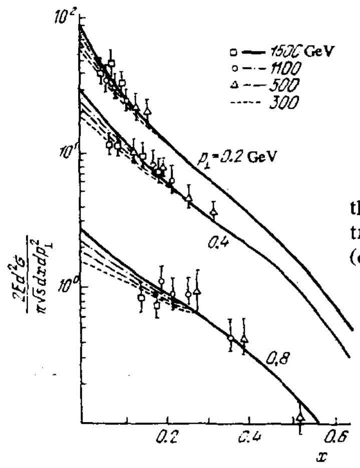


FIG. 10. Comparison of the theoretical and experimental distributions in the variable x [55] (on the vertical axis, in mb·GeV^{-2}).

multiple production processes on the atomic number A (see, e.g.,[29]). The "tube" model gives a good description of this property.

Thus, if the experimental dependence is approximated by a power function

$$\bar{N} \backsim A^{\alpha}, \tag{73}$$

then, according to the review[56], α = 0.15 ± 0.06. Photoemulsion studies[57a] have shown that, when the multiplicity for collisions of nucleons of energy E_0 = 200 GeV with heavy nuclei (Ag, Br) are compared with the value of $\bar{N}$ for nucleon-nucleon collisions, one finds α = 0.15 ± 0.01. Studies of the dependence (73) using cosmic rays[57a] have led to the value α = 0.30 ± 0.09. These data are consistent with the dependence (30).

Figure 11 shows the ratio $R = \bar{N}_{em}/N_H$ of the average multiplicities of charged relativistic particles in photoemulsion and in hydrogen (see[58]); the solid line gives the value of this ratio calculated from Eq. (30), assuming that the average atomic number in the emulsion is $\bar{A}$ = 70.

We note that the prediction of the "tube" model regarding the appearance of a pronounced asymmetry in

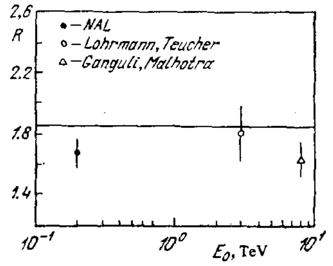


FIG. 11

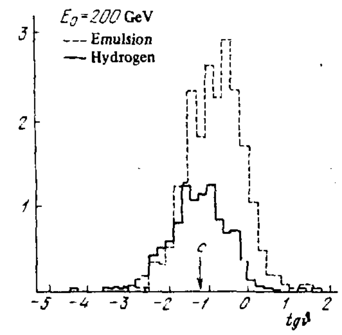


FIG. 12

FIG. 11. The ratio R of the average multiplicities in emulsion and in hydrogen [58].

FIG. 12. The distribution dN/d log tan ϑ in emulsion (dashed curve) and in hydrogen (solid curve). The point C corresponds to the c.m.s. for pp collisions at E_0 = 200 GeV (see [58]).

the c.m.s. for collisions of nucleons with heavy nuclei is also apparently confirmed experimentally[57-60]. This asymmetry manifests itself in the fact that the number of secondary particles rises sharply in the fragmentation region of the nucleus. To illustrate this, we show in Fig. 12 the experimental distributions in the logarithmic coordinates for emulsion (dashed histogram) and for hydrogen (solid histogram) (see[58]).

It is notable that a collective effect has been observed[61a]—the production (in the fragmentation region of the target nucleus) of particles whose velocity exceeds that of the nucleus before the collision in the c.m.s. This effect can be explained qualitatively by a hydrodynamical collective acceleration associated with the nucleon-nucleus collisions. However, a quantitative comparison of the characteristics of the collective effect with the predictions of the hydrodynamical theory is inappropriate for two reasons: 1) the incident nucleons in the experiments of[61a] have a relatively low energy ($E_0 \lesssim$ 10 GeV), and 2) the quantitative predictions of the hydrodynamical theory for the fastest particles (in the c.m.s.) are unreliable.

The detailed comparison of the "tube" model with the experimental data on nucleon-nucleus collisions is undoubtedly an interesting problem. Continuing along these lines, it would also be of interest to compare the characteristics of collisions of two relativistic nuclei (see[61b] for a discussion of this possibility) with the predictions of the hydrodynamical theory and, in particular, with Eq. (31), which gives a very strong dependence of $\bar{N}(A)$.

15. ANNIHILATION PROCESSES AND THE HYDRODYNAMICAL THEORY

The annihilation of relativistic baryon-antibaryon pairs evidently provides the richest data for the application of the hydrodynamical theory. The main point is that we know that annihilation processes do not have a peripheral character, so that the complex question of whether the hydrodynamical theory is applicable to the fragmentation region and, in particular, to the energetically distinguished particles is not as acute here as before.

Unfortunately, there are as yet no beams of sufficiently fast antinucleons. In the intermediate energy region, a comparison of the experimental data with the statistical theory of Pomeranchuk gives good agreement[11]. The statistical-hydrodynamical interpretation of the process of e^+e^- annihilation into hadrons has recently attracted much attention[62-66].

Experiments within the range $\sqrt{s}$ = 3–5 GeV have revealed a number of very interesting features (see[64]):

a) The energy distribution of secondary hadrons has a roughly exponential character

$$\frac{dN}{dE} \sim \exp\left(-\frac{E}{T}\right), \tag{74}$$

where T ~ 160 MeV. This distribution is independent of the value of E_0.

b) The yields of pions, kaons and nucleons are in the ratio

$$\bar{n}_\pi : \bar{n}_K : \bar{n}_{\bar{p}} \sim 1 : 0.1 : 0.01. \tag{75}$$

c) The average multiplicity of secondary particles rises sharply with energy, although the exact law of growth of the multiplicity is not clear.

d) The angular distribution is approximately isotropic in the c.m.s.

e) Any variation of the total cross section with the energy E_0 is weak, and this cross section is approximately equal to 20 nb.

It is difficult to account for all these facts from the point of view of field theories (see[64]). However, the hydrodynamical theory provides a good statistical explanation of all the above-mentioned experimental results (except, of course, the point about the cross section, which is outside the scope of the model). Nevertheless, the hydrodynamical interpretation of the process $e^+ + e^- \to$ hadrons entails a new and very important problem about the choice of the initial volume. Since the collision involves point particles (leptons), the initial state is naturally spherically symmetric, corresponding to the absence of anisotropy.

However, the size of the initial volume and its energy dependence are important here[14]. In the simplest variant[62, 63], it is assumed that the initial volume V is independent of the energy E_0 and has dimensions $\approx 1/\mu$. Using (4) and (5), one readily finds that in this case

$$\bar{N} \sim s^{3/8} V^{1/4}. \tag{76}$$

For a more general equation of state (see (52)), we find

$$\bar{N} \sim s^{1/2(1+c_0^2)} V^{c_0^2/(1+c_0^2)}. \tag{77}$$

The energy distribution of the secondary particles and their composition are explained in a natural way by the statistical character of the decay of the system.

However, certain objections can be raised against the simplest variant. First of all, the quantity $1/\mu$ is not the characteristic distance for the initial state in this process. For hadronic collisions, this value determines the magnitude of the total cross section ($\sim 1/\mu^2$), so that the quantity $1/\mu$ appears in the theory in a natural way; for e^+e^- annihilation, this is not at all the case.

Furthermore, dimensional arguments have suggested[64] that the initial volume should be energy-dependent, namely

$$V \sim s^{-3/2}. \tag{78}$$

In that case, one finds from Eq. (76) that

$$\bar{N} = \text{const}, \tag{79}$$

in poor agreement with the experimental data. This difficulty can apparently be resolved in two ways. The first solution[56] is to assume that the characteristic dimensions R_0 of the initial system are much smaller than $1/\mu$. For example, if we try to relate the initial volume to the experimental value of the cross section ($\sim 10^{-32}$ cm^2), we must assume that $R_0 \sim 10^{-16}$ cm; the multiplicity is then no longer determined by the initial volume, but by the dissipative terms and the condition $R_e \sim 1$. However, the characteristic equation in this case differs from the dependence (55). In particular, for the spherically symmetric case, the characteristic relation takes the form[15]

$$E^{(1-2c_0^2)/(1+c_0^2)} r_1^{(7c_0^2-2)/(1+c_0^2)} = 1. \tag{80}$$

Here r_1 is the distance at which the effect of the dissipative terms can be neglected. Using (77), we obtain

$$\bar{N} \sim s^{(3c_0^4+2c_0^2-1)/(7c_0^2-2)(1+c_0^2)}. \tag{81}$$

The relations (80) and (81) hold under the condition $c_0^2 > 2/7$.

A second way out of the difficulty is to abandon the attempt to relate the initial volume to the magnitude of the cross section for e^+e^- annihilation. Then the only remaining characteristic quantity in the initial state in our case is $V \to 0$. It follows from Eq. (77) that in general $\bar{N} \to 0$ in this case, which is obviously absurd.

However, there exists one case in which $\bar{N} \neq 0$, namely when $c_0 = 0$. In this case, the multiplicity has its maximum possible value (10). At the initial moment $T_0 \to \infty$ and $\epsilon \to \infty$, but the hydrodynamical emission comes to an end very rapidly (the picture here is very similar to the Pomeranchuk model).

An energy distribution of the Bose type (for pions) is established in the c.m.s., with a non-scaling character[16]. The composition is determined by the thermodynamic relations.

Future experiments on e^+e^- annihilation into hadrons (particularly measurements of the multiplicity) should determine whether a hydrodynamical description of this process is possible. It seems certain that studies of other forms of the electromagnetic nuclear interaction at high energies, namely $\gamma + N \to$ hadrons, $e + N \to e + N$ + hadrons and $\mu + N \to \mu + N$ + hadrons, will play a major role in substantiating the applicability of the statistical-hydrodynamical approach. In particular, it seems important to study the degree of universality of the spectra and the composition of the produced hadrons.

16. QUANTUM FIELD-THEORETIC INTERPRETATION OF THE HYDRODYNAMICAL THEORY

The hydrodynamical theory was constructed on the basis of quasiclassical ideas.

Two questions arise in this connection: does there exist an analogy with quantum field theory, and what are the limits of applicability of the classical approach?

The first question, which had already been raised in the early works on the theory of multiple production processes (see, e.g.,[2]), has a qualitative answer—a fluid, like a field, is a system with an infinite number of degrees of freedom, which extends throughout space-time.

However, it is not possible to establish a well-defined relation between the two approaches because the classical description is based on one set of concepts (entropy, temperature, etc.), while field theory uses entirely different ones (the field amplitude).

Nevertheless, the fact that the energy density or Lagrangian can be expressed in terms of classical and quantum concepts offers hope that it will be possible to find an analogy between the two approaches.

This problem was solved in[68], where scalar fields were studied. The analysis was based on the division of the Lagrangian into two terms:

$$L = L_0 + L_{\rm int}, \qquad (82)$$

where

$$L_0 = -\frac{1}{2}(\varphi_k^2 + m^2\varphi^2), \qquad (83)$$

$$\varphi_k = \frac{\partial\varphi}{\partial x_k}, \qquad (84)$$

$$L_{\rm int} = \lambda\varphi_k^{2t}; \qquad (85)$$

here λ and t are constants. The basic idea of the analogy is that the quantity L_0 can be neglected if the energy density ϵ is sufficiently large. If the energy density is small, then $L_{\rm int} < L_0$ and the particles can be regarded as free. The moment $L_{\rm int} = L_0$ corresponds to the time at which $T = T_f$ in the hydrodynamical picture. It was found that, within the framework of this analogy, one can introduce a conserved quantity Σ, which is analogous to the total entropy S:

$$\Sigma = 2\int \frac{\partial L_{\rm int}}{\partial\varphi_k^2}\varphi_l^2\, d\Omega^l. \qquad (86)$$

The integration is carried out over some space-like surface. Putting

$$\bar{N} \sim S \sim \Sigma, \qquad (87)$$

we can obtain the dependence

$$\bar{N} \sim s^{(t-1)/4t}. \qquad (88)$$

Comparing this expression with the dependence $\bar{N}(s)$ obtained using the generalized equation of state, we find

$$c_0^2 = \frac{1}{2t-1}. \qquad (89)$$

Thus, the hydrodynamical theory corresponds to a nonlinear interaction Lagrangian; in particular, the standard equation of state (18) corresponds to t = 2.

17. THE APPLICABILITY OF THE CLASSICAL APPROACH TO THE INITIAL STATE

Another fundamental problem is the restriction on the quasi-classical description of a micro-system with a Lorentz-contracted volume[69] 17).

This problem can be approached in the following way. Let us divide the contracted disk into n separate layers perpendicular to the direction of motion of the incident particle. Then the uncertainty principle for one of these layers leads to the condition

$$\varepsilon \gg \left(n\mu^2\frac{\sqrt{s}}{M}\right)^2. \qquad (90)$$

The conditions (4) and (90) imply that

$$n^2 \ll \frac{M}{\mu} \sim 7. \qquad (91)$$

Thus, the restrictions imposed by the uncertainty principle prevent us from dividing the contracted disk into an arbitrarily large number of separate layers. We must therefore give special consideration to the possibility of employing a quasi-classical description of the initial state of the hydrodynamical system.

There are several possible approaches to the resolution of the paradox[69], a fact which in itself already suggests that the solution of this problem is incomplete.

One approach is to accept that the discussion of[69] is strictly correct. However, the restrictions (90) and (91) obviously apply only to the initial state of the system. The difficulty associated with these restrictions disappears once the system expands to several times the contracted dimensions of the disk. This therefore raises the question of the role of the initial state in the hydrodynamical system.

This point has been considered previously (see (48), (50), (59) and Sec. 13), where we pointed out that, of all the basic characteristics of multiple production processes, the shape of the initial volume affects only the dependence $\bar{N}(E_0)$. The simplest interpretation of the restrictions (90) and (91) leads to the conclusion that there is no justification for the calculation of this dependence on the basis of the hydrodynamical approach18).

Let us now consider the reasoning of[69] in more detail. The postulate concerning the possibility of dividing the initial disk into separate layers is most fundamental here. This point was considered in detail in[13, 19], where it was pointed out that, owing to the strong interaction, each small cell is in strong interaction with its surrounding meson cloud, so that the division into separate layers in the initial stage can be made only under special conditions.

The situation is obviously completely different if a limited volume contains particles which have no strong interaction (such as electrons and photons).

The analysis of[13, 19] showed that a strongly interacting fluid involves a new characteristic length, which at high temperatures ($T \gg M$) is estimated to be $\tau \approx 1/T$ according to dimensional arguments[20]. These arguments are naturally confirmed by model calculations. However, if it is assumed that $\tau \approx 1/T$, then one cannot employ the concepts of ideal hydrodynamics in the vicinity of the initial state in this case.

A phenomenological method of allowing for the strong interaction in this case is to introduce a viscosity[20], which gives an increase in the multiplicity (this problem was discussed earlier; see (56)). However, it seems to us that dimensional arguments alone are insufficient for any firm conclusions about the quantitative values of the parameters τ or η. This is especially true of the region $E_0 < 10^{14}$ eV, where $T_0 \sim M$.

Consequently, the purely thermodynamic approach allows a consistent description of the initial state up to energies $E_0 \sim 10^{14}$ eV. However, this possibility is an independent postulate, which it does not seem possible to substantiate.

The complex problem of quantum fluctuations in the initial state also arises in the case of e^+e^- annihilation into hadrons. If we put $\tau \approx 1/T$, then the condition $\tau < R_0$ is satisfied only if $\sqrt{s} > 1/R_0$. It is easy to satisfy this condition if $R_0 \approx 1/\mu$; however, if $R_0 \lesssim 10^{-16}$ cm, then the thermodynamic concepts can be reconciled with the quantum fluctuations in the initial stage only by assuming that there is an appreciable effect of the dissipative terms, which determine the initial phase of the hydrodynamical emission19).

Because of certain difficulties in the interpretation of the initial state, it is expedient to make experimental tests of the various individual phases of the hydrodynamical description: the initial state, the hydrodynamical expansion and the final state. The initial phase largely determines the multiplicity of particles. If the equation of state has the form (48), then the dependence $\bar{N}(E_0)$ has a power form. On the other hand, a power dependence of $\bar{N}(E_0)$ is characteristic of models of the statistical-hydrodynamical type. It is therefore of paramount importance to make a precise test of this dependence over a sufficiently wide energy range (e.g., $10^{11} < E_0 < 10^{13}$ eV). The hydrodynamical emission determines the angular and energy distributions of the secondary particles. The study of these characteristics may therefore serve as a test for this phase.

The final stage determines the composition and distribution of $p_\perp$. The degree of universality of these characteristics is very important for the analysis of the possibility of a thermodynamic description.

Last (but not least), there is the question as to what it is that moves during the hydrodynamical emission.

As we have already noted, these are not real particles. It is apparently also not possible to speak unconditionally about virtual particles in the sense in which they are usually understood in quantum field theory.

The assumption that partons or quarks (see[71]) are "moving" is closer to current but quite tentative ideas. In this case, only sufficiently "energetic" partons with wavelengths $\lambda \approx 1/\sqrt{s} \ll M/m\sqrt{s}$ are quasi-free in the initial stage. Partons with large wavelengths $\lambda \gtrsim M/m\sqrt{s}$ have a strong mutual interaction and form a single system.

Such descriptive ideas undoubtedly have too great a linguistic element for them to be studied seriously.

18. CONCLUDING REMARKS

The hydrodynamical theory provides a good description of the experimental data on multiple production processes that have been obtained with modern large accelerators; most of the conclusions were obtained many years ago with practically the same parameters. It is paradoxical that such a crude model has such heuristic capacity. It is a curious fact that another approach to the theory of multiple production processes–multiperipheralism–is also in good agreement with the experimental data (see, e.g.,[72]). Undoubtedly, it also seems superficially paradoxical that such different theories give a sufficiently accurate description of the experimental results.

Of course, the results of this comparison can be attributed to the large number of free parameters in each of the two approaches.

However, it seems to us that there is a deeper explanation here. Both theoretical approaches are based on the principle of quasi-one-dimensional emission of the secondary particles[73].

Multiperipheralism is based on the assumption that $\bar{p}_\perp$ is bounded; in the hydrodynamical theory, this is a consequence of the relativistic contraction of the colliding hadrons. It is this fact which determines the main conclusions of the theory. Consequently, if the particles are ordered according to increasing rapidity values (the sequence (61)), the state of the i-th particle in the multiperipheral approach is

$$\varphi_i = \varphi(y_{i-1},\ y_{i+1}). \qquad (92)$$

Although the hydrodynamical theory leads to a different dependence

$$\varphi_i = \varphi(y_1,\ \ldots,\ y_{i-1},\ y_{i+1},\ \ldots,\ y_N,\ s), \qquad (93)$$

the state of the i-th particle is actually still determined almost entirely by the adjacent members of the sequence (61).

In this case, the larger the value of $i-k$ (where k is the index of the k-th particle), the smaller the correlations between the two particles.

The difference between the functional dependences (92) and (93) is also associated with a difference in the way in which the two approaches treat the physical state of a system of N particles. An equilibrium is also established in the multiperipheral approach, but this equilibrium occurs in rapidity space.

The hydrodynamical theory is based on the assumption of equilibrium in ordinary configuration space. In general, equilibrium is not established in rapidity space. The sole exception is hydrodynamics with the value $c_0 = 1$. This case, corresponding to $\bar{N}$ = const, comes closest to the ideology of the multiperipheral approach.

The hydrodynamical model is now undoubtedly an integral part of the theory of multiple production processes. The hydrodynamical theory accounts satisfactorily for the main experimental data and raises interesting problems for theorists. Thus, a recent detailed report[74] listed up to 20 problems whose solution would be useful for the development of the hydrodynamical theory (collisions of particles with different masses, estimates of correlations and fluctuations, etc.).

APPENDIX. THE FORMALISM OF HYDRODYNAMICAL EMISSION [26,27,68]

The solution of the equations (14) for collisions of a nucleon with a "tube" is based on the following principles:

1) Hydrodynamical emission has a mainly one-dimensional character.

2) A rigorous solution of the one-dimensional problem with the initial condition in the form of a contracted disk was obtained in[75] 20). The forward front here is a simple wave, which, on the one hand, borders on the vacuum and, on the other hand, borders on the region of the non-trivial solution.

3) The distribution in the transverse direction is determined mainly by the thermal motion. The solution based on these assumptions is called the quasi-one-dimensional approximation. It gives a good accuracy up to energies $E_0 \lesssim 10^{13}$–10^{14} eV[24].

4) Corrections due to the transverse hydrodynamical motion are obtained by averaging the hydrodynamical velocity over the cross section perpendicular to the axis of emission.

In the one-dimensional case, the system (14) is equivalent to the equation

$$c_0^2 \frac{\partial^2 \chi}{\partial a^2} - \frac{\partial^2 \chi}{\partial y^2} + (1 - c_0^2)\frac{\partial \chi}{\partial a} = 0, \qquad (94)$$

where a $\ln(T_0/T)$ (see[68, 75]).

Accordingly, the simple waves running to the right and to the left are described by the equations

$$\frac{x}{t} = \frac{v - c_0}{1 - vc_0}, \qquad a = -c_0 y \tag{95}$$

and

$$\frac{x - l}{t} = \frac{v + c_0}{1 + vc_0}, \qquad a = c_0 y, \quad l = \frac{1}{4}\frac{\sqrt{s}}{M}; \tag{96}$$

here x and t are the coordinate and time of an element of the fluid, which can be expressed in terms of the variable χ as follows:

$$x = e^{-a}\left(\frac{\partial\chi}{\partial a}\,\mathrm{sh}\,y - \frac{\partial\chi}{\partial y}\,\mathrm{ch}\,y\right), \qquad t = a^{-a}\left(\frac{\partial\chi}{\partial a}\,\mathrm{ch}\,y - \frac{\partial\chi}{\partial y}\,\mathrm{sh}\,y\right). \tag{97}$$

In the region of the non-trivial solution, the relation is rather complicated:

$$\chi = \frac{l}{2c_0} e^a \int_{-c_0 y}^{a} e^{-(1+c_0^2)/2c_0^2} I_0\left[\frac{1-c_0^2}{2c_0^2}\sqrt{x^2 - (c_0 y)^2}\right] dx; \tag{98}$$

here I_0 is a Bessel function.

Using the definition of the entropy,

$$dS = -s\,e^{-a}\left(\frac{\partial\psi}{\partial y}\,da + c_0^2\,\frac{\partial\psi}{\partial a}\,dy\right), \qquad \psi = \frac{\partial\chi}{\partial a} - \chi \tag{99}$$

(where s_e is the specific entropy), we finally obtain

$$\frac{dN}{dy} = N\,\frac{1-c_0^2}{4c_0}\left[I_0(r) + \frac{a I_1(r)}{\sqrt{a^2 - (c_0 y)^2}}\right], \qquad r = \frac{1-c_0^2}{2c_0^2}\sqrt{a^2 - (c_0 y)^2}. \tag{100}$$

By using the expansion of the Bessel functions, we obtain the solution (37) and (49). The dependence (100) is obtained in the quasi-one-dimensional approximation. Qualitatively, three-dimensional motion can be represented as follows[26]. At the initial moment, when the transverse coordinate is $z \approx 1/\mu$, the thermodynamic parameters experience a break, i.e., there is a strong discontinuity. This then breaks up into weak discontinuities, which move with the velocity of sound c_0. The region of three-dimensional motion is bounded by

$$\frac{1}{\mu} - c_0 t < z < \frac{1}{\mu} + c_0 t. \tag{101}$$

The transverse hydrodynamical motion becomes important when the longitudinal dimensions of the system are comparable with the transverse dimensions. To arrive at a solution of the complete system of equations (14), we introduce, instead of the components of 4-velocity, the new variables $u_0 = \cosh y \cosh \xi$, $u^1 = \sinh y \cosh \xi$, $u^2 = \sinh \xi \cos \varphi$ and $u^3 \sinh \xi \sin \varphi$ is a polar angle in the plane perpendicular to the axis of motion.

The system of equations (14) in the new variables can then be simplified, but it is still a difficult matter to solve these equations in an analytic form[21)].

A numerical solution was obtained for the value $c_0^2 = 1/3$ with an accuracy $\sim 15\%$. The angular and momentum distribution in the c.m.s. can be expressed in terms of the total 4-velocity $\sinh \xi$ in the transverse direction:

$$\mathrm{tg}\,\vartheta^* = \frac{\mathrm{tg}\,\xi}{\mathrm{sh}\,y}, \tag{102}$$

$$E^* = \mu\,\mathrm{ch}\,y \cdot \mathrm{ch}\,\xi, \tag{103}$$

$$p_\perp = \mu\,\mathrm{sh}\,\xi. \tag{104}$$

The average hydrodynamical velocity in the transverse direction is

$$\mathrm{sh}\,\xi = 0.5\,\frac{n+1}{2}\left(\frac{E_0}{M}\right)^{0{,}07}. \tag{105}$$

Consequently, the average transverse momentum $\bar{p}_\perp$ has a very slow growth with energy ($\sim E_0^{1/15}$).

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1)Intersecting storage rings.

2)We stress that the dependence (7) is obtained in the thermodynamic approximation, which is justified for $E_0 > 10^{11}$ eV. For energies $E_0 \ll 10^{11}$ eV, an exact calculation of the statistical weights [10] with a power-law approximation of the dependence $\bar{N}(E_0)$ leads to a function $N \sim E_0^{1/3}$.

3)It was soon found that the Fermi statistical theory is in conflict with the experimental data on the composition and angular distribution of secondary particles.

4)The physical conditions for the validity of this relation are discussed later.

5)A very crude estimate of multi-particle interactions was made.

6)Until recently, no distinction was generally made between the two distributions within the entire allowed range of y.

7)The quoted estimates were made for nucleon-nucleon collisions; they apparently remain valid in the "tube" model, where the problem of distinguished particles is still present.

8)To eliminate the effect of resonances, it is expedient to consider correlations of particles having charges of the same sign [41].

9)We are not concerned here with the distribution $dN/dp_\perp$ for $p_\perp \gtrsim 2$ GeV, which evidently belongs to the domain of deep inelastic processes.

10)This circumstance has practically no effect on the results of the calculations of the above-mentioned characteristics.

11)For example, it is difficult to explain the appearance of clusters in the fragmentation region for $N \ll \bar{N}$ in the hydrodynamical theory [49].

12)The comparison was made in [50,51] for the thermodynamic model. However, the two models yield the same conclusions for the distribution $dN/dp_\perp$.

13)$\bar{N}_s$ is the average multiplicity of charged particles.

14)We stress that, owing to the spherical symmetry of the annihilation process, the expansion stage plays a smaller role than in the case of hadron collisions. In the case of annihilation, the final stage, in which the system decays into real particles, sets in very quickly.

15)As before, we are assuming here that $\eta_v \sim T^3$.

16)An analogy of multiple production processes with a point-like explosion was proposed earlier to provide a basis for the principle of scale invariance [67]. However, owing to the decisive influence of thermal motion in this case, a point-like explosion should apparently not result in scale-invariant distributions.

17)This is a particular aspect of the most important problem of the micro-physics of the space-time limit of the extrapolation of our current concepts [70].

18)We note that the possible role of quantum fluctuations is much smaller for collisions of a nucleon with a "tube" and, even more so, for collisions of two complex nuclei.

19)The uncertainty relation also implies the enequality $\sqrt{s} > 1/R_0$.

20)The simple wave and shock wave in the one-dimensional problem were also studied in [76].

21)An analytic solution was obtained in [27] in a somewhat simplified form.

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Translated by N. M. Queen