Quantum-mechanical limitations in macroscopic experiments and modern experimental technique

V. B. Braginskiĭ and Yu. I. Vorontsov

Moscow State University

Usp. Fiz. Nauk **114**, 41–53 (September 1974)

Perfection of the technique of macroscopic physical experiments has recently been proceeding so intensively that we can now inquire naturally under what conditions in macroscopic experiments will an increase in sensitivity be limited by the quantum-mechanical properties of the test objects. In this article we determine the limiting values of the detectable accelerations (or forces) when free particles or oscillators are used as the test objects. The conditions for attaining the limiting sensitivity are discussed. It is shown possible to increase the sensitivity of a converter of mechanical into electrical oscillations by increasing the relaxation time of the electric resonator. The possibility is discussed of nondestructive recording of the *n*-quantum state of an oscillator. It is shown with the example of a concrete experimental design that one can determine the value of *n* (including *n* = 0) in such a way that the probability of transition to the adjacent levels after the measurement will be small.

CONTENTS

1. INTRODUCTION

Perfection of the technique of macroscopic physical measurements has recently been proceeding very intensively. A methodology has already been developed that permits one to detect an amplitude of mechanical oscillations at the level of 10^{-14} cm (an amplitude of relative elongation[1] of 10^{-16}), and a methodology is being developed that is designed for an amplitude of 10^{-18} cm.[2,3] Aluminum cylinders of mass 2.6 T are cooled to 3×10^{-3} K.[4] The short-term stability of the frequency standards was as much as 10^{-14};[5] the natural width of the line of optical generators has been measured in detail[6] (it amounts to hundredths of a Hertz). A Q-factor of electric resonators of 10^{11} has been attained in the centimeter range.[7] Accelerations of 10^{-9} cm/sec^2 are being recorded[8] in drift-free satellites, and systems are being developed[9] that are designed for 10^{-11} cm/sec^2. The distance between the space apparatus and the Earth, which is of the order of 100 million km, is fixed with an accuracy up to 1 m.[10] With such a vigorous development of measurement technique, it has become natural to pose the question: under what conditions in macroscopic experiments will the increase in sensitivity be limited by the quantum-mechanical properties of the test objects?

We shall restrict the treatment in this article to the problem of the minimum detectable variable (not quasistatic) accelerations (or forces). As we see it, this question is pertinent with regard to increasing the sensitivity of gravitational antennas, and to the problem of inventing a sensitive accelerometer.

2. THE EFFECT OF THE QUANTUM-MECHANICAL PROPERTIES OF TEST OBJECTS ON THE ACCURACY OF MEASUREMENT OF FORCES

The minimum force whose action can be detected by the response of a test object is determined by the quantum-mechanical features of the test object and the time of action of the force.

Let us first examine the case in which the test object is a free particle. The action of a force on a free particle can be detected by observing the variation of its coordinate or momentum. We must measure the coordinate x_0 and the momentum p_0 of the particle at some instant t_0 of time, and its coordinate at the instant $t_0 + \hat{\tau}$. We can calculate the value of the final coordinate $x(\hat{\tau})$ that it would have had in the absence of a force by using the known x_0, p_0, and $\hat{\tau}$. However, we cannot simultaneously measure x_0 and p_0 exactly. The root-mean-square errors Δx_0 and Δp_0 are related by the Heisenberg uncertainty relationship[11]

$$\overline{(\Delta x_0)^2}\,\overline{(\Delta p_0)^2} \geqslant \frac{\hbar^2}{4}. \tag{1}$$

Hence, owing alone to the uncertainty of the initial values of the momentum and the coordinate, the uncertainty of the final coordinate will be

$$\Delta x \geqslant \sqrt{\overline{(\Delta x_0)^2} + \left(\frac{\hat{\tau}}{m}\right)^2 \frac{\hbar^2}{4\,\overline{(\Delta x_0)^2}}} \tag{2}$$

(m is the mass of the particle).

Since the change in the final coordinate caused by the constant force F is $F\tau_1^2/2m$ (τ_1 is the time of action of the force), the root-mean-square error of measurement of the force will be

$$\Delta F \geqslant \frac{2m}{\hat{\tau}^2}\sqrt{\overline{(\Delta x_0)^2}+\left(\frac{\hat{\tau}}{m}\right)^2\frac{\hbar_2}{4\,\overline{(\Delta x_0)^2}}}. \qquad (3)$$

(We shall consider the time of action of the force to be equal to the time of observation $\hat{\tau}$.) The right-hand side of (3) has a minimum at

$$\overline{(\Delta x_0)^2}=\frac{\hbar\hat{\tau}}{2m}. \qquad (4)$$

Finally we get

$$\Delta F_{\min} \geqslant \frac{1}{\hat{\tau}}\sqrt{\frac{4m\hbar}{\hat{\tau}}}. \qquad (5)$$

Equation (5) implies that the minimum force $F_{\min}$ that can be detected by the response of the free particle is

$$F_{\min}=\zeta\cdot\frac{1}{\hat{\tau}}\sqrt{\frac{4m\hbar}{\hat{\tau}}}, \qquad (6)$$

Here ζ is a certain coefficient greater than unity that depends on the reliability of detection.

Let us examine another way of detecting a force using a free particle: by measuring its energy. Now we must measure the energy of the particle before the action of the force and after. The first measurement must give the value that the initial energy has after the process of measuring it. The accuracy of ΔE from such a measurement depends on the duration τ of the measurement:[11-14]

$$\Delta E=\frac{\hbar}{\tau}, \qquad (7)$$

and in principle, it can be as small as we wish with a long enough time τ of measurement. However, the total time of observation of a free particle will be limited by some quantity $\hat{\tau}$. Part of this time will be spent in measuring the energy, while the change in the energy caused by the force will take up the rest of the time. The work done by the force F during the time τ_1 with $F\tau_1 \ll p_0$ will be approximately equal to (we shall assume that $\mathbf{F} \parallel \mathbf{p}_0$):

$$\Delta E_0=\frac{p_0 F\tau_1}{m}. \qquad (8)$$

This change in the energy can be detected if $\Delta E_0 > \Delta E$. Upon accounting for the fact that $\tau + \tau_1 = \hat{\tau}$, we find from (7) and (8) the optimum relationship between τ and τ_1: $\tau = \tau_1 = \hat{\tau}/2$. The corresponding minimum detectable force is

$$F_{\min}=\zeta\cdot\frac{1}{\hat{\tau}}\sqrt{\frac{8m\hbar^2}{E_0\hat{\tau}^2}}, \qquad (9)$$

where $E_0 = p_0^2/2m$ is the mean initial energy of the particle.

In contrast to (6), we note that the right-hand side of Eq. (9) depends on E_0, and $F_{\min}$ decreases with increasing E_0.

Now let us examine the measurement of a force using another test object that fundamentally differs from a free particle: a harmonic oscillator. The action of a force on a harmonic oscillator can be detected by the change in the energy of the oscillator. In this case the major error of measurement will involve the discreteness of the energy levels. The work done by the force on the oscillator can be measured only to an accuracy of $\hbar\omega$, since the energy of the oscillator cannot be a fraction of $\hbar\omega$ (ω is the characteristic frequency of the oscillator). The error of determining the initial and final energies of the oscillator with a time of measurement $\tau \gg 2\pi/\omega$ can be in principle much smaller than $\hbar\omega$.

Let us consider a change in the state of the harmonic oscillator caused by a force of the type $F(t) = F_0\cos\omega t$. If the initial state of the oscillator was one having the assigned energy E_0, then its mean value over a time $\hat{\tau}$ will be[15]

$$E(\hat{\tau}) = E_0 + w, \qquad (10)$$

while the standard deviation is

$$\Delta E = \sqrt{2E_0 w}, \qquad (11)$$

where

$$w=\frac{F_0^2\hat{\tau}^2}{8m}; \qquad (12)$$

and m is the mass of the oscillator.

The oscillator will change its initial state with a reliability close to unity if

$$w \geqslant \hbar\omega, \qquad (13a)$$

or

$$\Delta E \geqslant \hbar\omega. \qquad (13b)$$

When $E_0 > w$, then the second condition will be satisfied earlier. We get from (11) and (13b) the following:

$$\Delta F_0 \geqslant \frac{1}{\hat{\tau}}\sqrt{\frac{4m\hbar\omega}{n_0}}, \qquad (14)$$

where $n_0 \approx E_0/\hbar\omega$. This inequality determines the root-mean-square error of measurement of the amplitude of the force from the response of the harmonic oscillator (under the above-stated restrictions). The minimum detectable force corresponding to Eq. (14) is

$$F_{0,\min}=\zeta\cdot\frac{1}{\hat{\tau}}\sqrt{\frac{4m\hbar\omega}{n_0}}. \qquad (15)$$

In the analysis presented above, we have not accounted for such processes in actual test objects and measuring devices as the process of relaxation and the process of fluctuational forces exerted by the measuring device. Let us find the conditions under which the stated processes will not be decisive.

3. THE EFFECT OF RELAXATION

One can detect the action of a force on a background of fluctuations if the change in the state of the oscillator that it causes is larger than the change caused by relaxation. The mean of the intrinsic value of the energy in the process of relaxation varies according to the law[16]

$$n_R(\hat{\tau}) = n_T - (n_T - n_0)\,e^{-\hat{\tau}/\tau^*}, \qquad (16)$$

while the variance is

$$\overline{(\Delta n_R)^2} = (2n_T+1)(n_R-n_T)(1-e^{-\hat{\tau}/\tau^*})\,e^{-\hat{\tau}/\tau^*} + n_T(n_T+1)(1-e^{-2\hat{\tau}/\tau^*}), \qquad (17)$$

Here $n_T = (e^{\hbar\omega/kT} - 1)^{-1}$, n_0 is the eigenvalue corresponding to the energy E_0, and τ^* is the relaxation time, T is the equilibrium temperature, and k is the Boltzmann constant. If $n_T \gg 1$ (i.e., $\hbar\omega \ll kT$) and $n_0 \gg 1$, then when $\hat{\tau}/\tau^* \ll 1$, we get from (17):

$$\Delta n_R \approx \sqrt{\frac{\hat{\tau}}{\tau^*}\,2n_T n_0}. \qquad (18)$$

We can find the condition for detecting the force on the background of fluctuations by comparing (18) with (11):

$$F_0=\zeta\sqrt{\frac{8mkT}{\hat{\tau}\tau^*}}. \qquad (19)$$

We note that (19) agrees with Nyquist's formula, which was derived by a purely classical method (for more details, see[17]).

A force equal to $F_{0,\min}$ can be detected on the background of fluctuations if $\Delta n_R \leq 1$, i.e., in our case, when

$$\frac{\hat{\tau}}{\tau^*} 2n_T n_0 \leqslant 1 \tag{20}$$

(this inequality was first derived in[18] by a somewhat different method).

Let us consider the fact that Eq. (15) can be derived classically, just as Eq. (19) can, provided only that we consider the minimum work that the force can do on the classical oscillator to be $\hbar\omega$. Moreover, the initial relationships (10), (11) and (16), and (18) themselves have their classical analogs. In fact, the change in the energy of a classical oscillator acted on by a harmonic force is (when $\omega\hat{\tau} \gg 1$):

$$W_{\text{cl}} = \frac{F_0^2\hat{\tau}^2}{8m} + \frac{a_0\omega F_0\hat{\tau}\sin\varphi}{2}, \tag{21}$$

Here φ is the initial phase difference between the force and the displacement, and a_0 is the initial amplitude of the oscillations.

If the phase φ is random, we get from (21):

$$\overline{W}_{\text{cl}} = \frac{F_0^2\hat{\tau}^2}{8m} = w, \tag{21a}$$

$$\overline{(\Delta W_{\text{cl}})^2} = 2E_0 w. \tag{21b}$$

In particular, the classical analog of Eq. (18) is (1)–(4) in the book[17].

We can conclude from the above said that we can use the classical method for calculating the quantities of interest to us when $\hbar\omega \ll kT$, while the quantum limitations are introduced by restricting the minimum change in the energy of the oscillator to be the quantity $\hbar\omega$.

4. CONDITIONS FOR EXPERIMENTAL DETECTION OF FORCES OF THE ORDER OF $F_{0,\min}$

For practical realization of the limit (15) for a given observation time, one must:

a) decrease the intrinsic fluctuations of the oscillator by: 1) increasing the relaxation time τ^*, and 2) decreasing the equilibrium temperature T;

b) find a way of measuring that will permit one to detect the corresponding change in the energy of the oscillator.

First let us discuss the possibility of carrying out the program mentioned under point a). The final goal of this program is to satisfy the inequality (20). Let us assume that $\hbar\omega \ll kT$, and write (20) in the following form:

$$\hat{\tau} < \frac{A\hbar^2}{(kT)^2}\alpha, \tag{22}$$

Here $A = Q\omega$ (Q is the Q-factor of the oscillator), and $\alpha = n_T/n_0$. If the only cause of losses in the oscillator is the heat conductivity of the material, then the quantity A is the characteristic constant of the material.[19] For example, it is about $10^{20}\ \text{sec}^{-1}$ for sapphire at T = 0.4 K.

If we assume that T = 0.4 K and $\alpha = 1$, then we get from (12) that the minimum detectable force with $\hat{\tau} < 0.3$ sec will be determined by Eq. (15). In the absence of an external force, the oscillator will remain in the initial level over a period of 0.3 sec with a probability close to unity.

We shall not discuss here the ways of artificially transferring the oscillator to the level n_0. However, such an energy change is possible in principle (a diminution of the initial oscillation energy has already been carried out, e.g., in[20]).

5. ON SOME POSSIBILITIES OF INCREASING THE SENSITIVITY OF A CONVERTER OF MECHANICAL INTO ELECTRICAL OSCILLATIONS

The fundamental methods of measuring small mechanical displacements are described in the book[17]. Estimates of the maximum sensitivity of these methods are also derived there under certain conditions of measurement. According to[17], when one uses an electronic converter in a quasi-steady-state mode ($\tau_e^* < 1/\omega$, where τ_e^* is the relaxation time of the oscillator and ω is the frequency of capacity modulation; see Figs. 1 and 2a), the intrinsic thermal fluctuations of the resonator do not permit one to measure a force smaller than

$$F_{0,\min} = \zeta \cdot \frac{1}{\hat{\tau}} \sqrt{4m\hbar\omega \frac{kT_e}{\hbar\Omega}} \tag{23}$$

(T_e is the temperature of the resonator, and Ω is its natural frequency). The relationship (23) holds when $\hbar\Omega \ll kT_e$. Comparison of (23) with (15) shows that, when one uses an electronic transducer in a quasi-steady-state mode, one cannot attain the limit of (15). In order to increase the sensitivity of the detector, one must increase the relaxation time τ_e^* of the resonator. However, when $\tau_e^* > 1/\omega$, the quasi-steady-state method of conversion is not applicable. In this case one can perform the conversion by the "upward frequency conversion" method. V. I. Panov is responsible for the idea of applying this method for detecting mechanical oscillations. In this method, the following relationship holds between the frequencies of the resonator Ω, that of the pumping generator p, and that of the mechanical oscillator ω (Fig. 2b):

$$\omega + p = \Omega. \tag{24}$$

In the absence of capacity modulation of the resonator, the oscillations at the intrinsic frequency of the resonator will be determined only by its Nyquist noise and by the noise of the generator that falls within the band of the resonator. Capacity modulation at the frequency ω gives

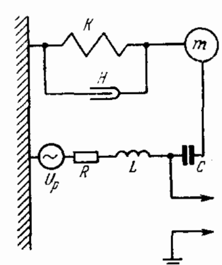

FIG. 1. Schematic diagram of converter of mechanical oscillations into electrical ones. m, K, H are the mass, elasticity coefficient, and friction coefficient of the oscillator; U_p is the pump-generator emf; C is the modulated capacitance; L and R are the inductance and loss resistance of the resonator.

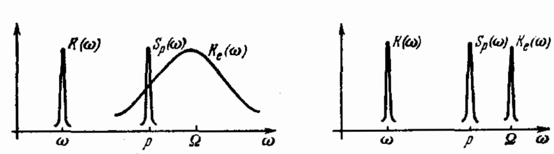

FIG. 2. Relative position, on the frequency axis, of the stationary characteristics of the mechanical oscillator $K(\omega)$ and of the electric oscillator $K_e(\omega)$ and of the pump generator radiation density $S_p(\omega)$ at $\tau_e^* < 1/\omega$ (a) and $\tau_e^* > 1/\omega$ (b).

rise to a voltage at the combination frequency $\omega + p$. The amplitude of this voltage will be

$$U_{p+\omega}=\frac{1}{4}U_p\frac{\Omega}{\Omega-p}\frac{a}{d_0};\qquad(25)$$

Here U_p is the amplitude of the voltage of the pumping generator, d_0 is the mean distance between the plates of the condenser, and a is the amplitude of the mechanical oscillations. (We have assumed in deriving (25) that $\tau_e^* \gg 1/\omega$, and $U_{p+\omega} \ll U_p$.) The change in the amplitude of the mechanical oscillator δa during the time that the force acts on it (with $\hat{\tau} \ll \tau^*$, and $\delta a \ll a_0$) is determined by the relationship

$$\delta a\approx\frac{1}{2a_0}\left(\frac{F_0\hat{\tau}}{2m\omega}\right)^2+\frac{F_0\hat{\tau}}{2m\omega}\sin\varphi.\qquad(26)$$

With random φ, the root-mean-square variation of the amplitude will be

$$\Delta a\approx\frac{F_0\hat{\tau}}{2\sqrt{2}m\omega}.\qquad(27)$$

Hence the root-mean-square variation in the amplitude of the voltage at the combination frequency over the time $\hat{\tau}$ is

$$\delta U_{p+\omega}=\frac{1}{8\sqrt{2}}U_p\frac{\Omega}{\omega d_0}\frac{F_0\hat{\tau}}{m\omega}.\qquad(28)$$

The action of this voltage on the resonator can be detected on the background of Nyquist noise if its time average obeys

$$\frac{1}{2}\delta U_{p+\omega}>\sqrt{\frac{2kT_eR}{\hat{\tau}}};\qquad(29)$$

Here R is the equivalent resistance of the resonator losses.[17] The detector exerts a force effect on the mechanical oscillator.[17] In the studied system, the spectral component of this force at the frequency ω arises as a combination component of the pumping voltage and the intrinsic oscillations of the resonator. The amplitude of this component is

$$F_i=U_pU_\Omega\frac{\Omega}{\Omega-p}\frac{C_0}{4d_0},\qquad(30)$$

where C_0 is the mean capacitance. $U_{p+\omega}$ varies with varying amplitude of the mechanical oscillations, and hence, so do U_Ω and F_i.

Equation (28) continues to hold until $\delta F_i < F_0$. Let us determine the value of U_p for which the quantity δF_i attains the value F_0 in the time $\hat{\tau}$. Since when $\hat{\tau}$ is small enough,

$$\delta U_\Omega=\frac{1}{2}\delta U_{p+\omega}\frac{\Omega\hat{\tau}}{2},\qquad(31)$$

then we obtain from the condition

$$\delta F_i=F_0\qquad(32)$$

and the relationships (30) and (31) the following:

$$U_{p,\,\mathrm{opt}}=\frac{8\omega d_0}{\hat{\tau}\Omega}\sqrt{2m\omega\Omega L}.\qquad(33)$$

If we assume that (28) holds to a certain approximation during the entire time $\hat{\tau}$, and substitute U_p from (33) into (28), we get the minimum detectable amplitude of the force

$$F_{0,\,\min}=\zeta\,\frac{4}{\hat{\tau}}\sqrt{\frac{mkT_e\omega}{\Omega}\frac{\hat{\tau}}{\tau_e^*}}.\qquad(34)$$

The given value of $F_{0,\min}$ is in order of magnitude smaller by a factor of $\sqrt{\hat{\tau}/\tau_e^*}$ than (23).

Thus, using a converter having a large time constant permits one to obtain a substantial gain in sensitivity. When

$$\frac{Q}{T}=\frac{Q_e}{T_e}\qquad(35)$$

the value from (34) agrees with (19). That is, the noise of the electrical part of the system then plays the same role as the noise of the mechanical part.

Although we have for the sake of simplicity used rather crude approximations in deriving (34), an exact calculation confirms the derived result. However, thorough analysis shows that the process of establishment of oscillations in the system is accompanied by beating, and in order to detect forces of the order of $F_{0,\min}$, one must know either the prior history of the system, or the period of the beats.

Equation (35) has been derived purely classically. However, it also holds for a quantum oscillator when $\hbar\Omega \ll kT_e$. We shall introduce the quantum limitations here in the same way as we did in deriving (15). Namely, we shall assume that the minimum change in the energy of the resonator over the time $\hat{\tau}$ must be no smaller than $\hbar\Omega$. The change in the energy at the frequency Ω when $\delta U_\Omega \ll U_\Omega$ is

$$\delta E=C_0U_\Omega\delta U_\Omega.\qquad(36)$$

If the energy of the oscillations in the resonator is kT_e, then

$$U_\Omega=\sqrt{\frac{2kT_e}{C_0}}.\qquad(37)$$

We get the following equation from the condition $\delta E = \hbar\Omega$ and Eqs. (31) and (34)–(37):

$$n_T^e\sqrt{\frac{2\hat{\tau}}{\tau_e^*}}=1,\qquad(38)$$

where

$$n_T^e=\frac{kT_e}{\hbar\Omega}.\qquad(39)$$

By using (38) and (35), we can find the limiting measurable force amplitude

$$F_{0,\,\lim}=\zeta\cdot\frac{1}{\hat{\tau}}\sqrt{\frac{8m\hbar\omega}{n_T^e}}.\qquad(40)$$

Thus, in the described detector system, the Nyquist noise of the resonator allows one to measure a force at the level given by (15) under the condition (35) and when

$$n_T^e>2n_0.\qquad(41)$$

Let us discuss Eqs. (38)–(40) and (41). They imply that one must increase the temperature T_e of the resonator in order to increase the sensitivity. If we consider the fact that we must increase τ_e^* upon increasing T_e, in line with (38), then this result is easily explained on the basis of (36).

Actually, the larger the initial amplitude U_Ω of the oscillations is, the smaller the value of δU_Ω at which the condition $\delta E \geq \hbar\Omega$ will be satisfied. These arguments imply that it is not necessary to increase the equilibrium temperature of the resonator in order to increase the sensitivity. It suffices to produce in some way intense oscillations at the intrinsic frequency of the resonator.

Let us assume that the energy of the initial oscillations of the resonator is E_0^e. Then, according to (18), the dispersion of the energy that arises from relaxation will be

$$\Delta E=\sqrt{\frac{\hat{\tau}}{\tau_e^*}2kT_eE_0^e},\qquad(42)$$

while the corresponding change in the amplitude when $\Delta E \ll E_0^e$ is independent of E_0^e, and is equal to (32). Since the change in amplitude does not depend on E_0^e, the inequalities (29) and (33) that serve to define $U_{p,opt}$ and $F_{0,min}$ are not altered. However, $U_\Omega = \sqrt{2E_0^e/C_0}$ enters into (36), and instead of (39) we get

$$\sqrt{\frac{2\hat{\tau}}{\tau_e^*} kT_e E_0^e} \geqslant \hbar\Omega. \quad (43)$$

Upon substituting (43) into (35), we find

$$F_{0,\,\lim} = \zeta \frac{1}{\hat{\tau}} \sqrt{\frac{8m\hbar\omega}{n_0^e}}, \quad (44)$$

where $n_0^e = E_0^e/\hbar\Omega$. Now, instead of (41), we get

$$n_0^e \geqslant 2n_0. \quad (45)$$

This calculation shows that it is possible in principle to measure a force at the level of Eq. (15) by observing a transition process in the system.

We note that, since the amplitude of the total force acting on the mechanical oscillator is approximately equal to $F_i + F_0$, and $F_i \gg F_0$, the energy of the mechanical oscillator will vary over the time $\hat{\tau}$ by an amount greater than $n_0\hbar\omega$. The change in the energy of the oscillator caused by the work done by the force F will depend on n_0^e and U_p. Its minimum value (i.e., when $n_0^e = 2n_0$) is approximately equal to $4\hbar\omega$. One can easily prove this by substituting the sum $F_i + F_0$ into (21) instead of F_0, and using Eqs. (30), (34), (38), (43), and (45).

6. REQUIREMENTS ON THE PUMPING GENERATOR

In the calculations performed above, we have neglected the noise of the pumping generator. Let us examine whether this assumption is possible in such a case. As we know, the spectral emission line of an autogenerator has a narrow peak and a broad but relatively low pedestal.[21] The noise of the generator can affect the oscillations at the intrinsic frequency of the resonator in two ways:

a) directly (owing to the "tails" of the spectral line);

b) owing to combination interaction with the mechanical oscillations.

We can make the direct effect negligibly small by: 1) increasing the difference $\Omega - p$; and 2) decreasing the spectral density in the tails of the line by using an auxiliary narrow-band resonator.

The total variation of the combination voltage $U_{p+\omega}$, as we see from (25), is

$$\delta U_{p+\omega} = \frac{U_p}{4d_0}\frac{\Omega}{\Omega - a}\left(\frac{\delta a}{a} + \frac{\delta U_p}{U_p} + \frac{\delta p}{\Omega - p}\right). \quad (46)$$

Hence, we can neglect the noise of the generator in (46) under the condition

$$\frac{\delta U_p}{U_p} + \frac{\delta p}{\Omega - p} < \frac{\delta a}{a}, \quad (47)$$

Here δa, δU_p, and δp are the variations of the corresponding quantities over the time $\hat{\tau}$. In measurements at the level of $F_{0,min}$, $\delta a/a$ will be $\sim 1/2n_0$. When $n_0 \sim 10^3$, the variation of the generator frequency over the time $\hat{\tau}$ must be no greater than $\delta p = 10^{-4}\,\omega$, while the change in the amplitude must be no greater than $\delta U_p = 10^{-4}\,U_p$. If, e.g., $\omega \sim 10^4$ while $\Omega \sim 10^{11}$, then the frequency stability of the pumping generator must be of the order of 10^{11}. Hence, one can use here an ordinary ammonia maser. When $n_0 \sim 10^5$, one now needs an atomic hydrogen beam maser.

7. ON NONDESTRUCTIVE RECORDING OF THE n-QUANTUM STATE

We started above with the idea that the error of measurement of the initial and final values of the energy of the oscillator can in principle be much smaller than $\hbar\omega$. In other words, we assumed that we could measure the energy of the oscillator while practically not perturbing it. However, up to now no way, even putative, has been proposed for making such a measurement.

We shall show with the example of a concrete experimental system that one can determine the value of n (including the case n = 0) for a chosen resonator mode in such a way that the probability of transition to the adjacent levels after the measurement is small.

Figure 3 shows the fundamental diagram of the experiment under discussion. An electron beam having the horizontal velocity v_x passes through the condenser of a klystron-type UHF resonator in such a way that the electric field deflects the passing electrons in the direction of the axis Oy. Then the electrons pass into a system of lenses A_1 and A_2, which carries out a mirror reflection of the trajectories of the electrons with respect to the symmetry plane of the condenser. Thereupon the electrons again enter a condenser of the studied resonator. The resonator in this case must have two spatially separated condensers. If the resonator is designed with only one condenser, then we must supplement the diagram shown in Fig. 3 with a system of mirrors.

The second passage through the condenser of the electrons that hadn't struck the screens B_1 and B_2 is used to compensate the effect that they exert as they pass through the resonator the first time. In order to permit such a compensation, the time between the first and second passages must correspond to a phase change of the oscillations by $2\pi m$, where m is an integer. This condition presupposes a rather precise knowledge of the frequency ω and good monokineticity of the electrons in the beam. The receiving electrodes, which are made in the form of two screens (B_1 and B_2), are arranged perpendicular to the axis Oy near the focus of the lenses, symmetrically with respect to their optic axis. When an a.c. field is applied in the left-hand condenser, the focal spot will oscillate in the plane of the screens. Consequently, if the edges of the screens lie closer to the optic axis than the first diffraction minimum, the mean number of electrons striking the screens will increase with increasing amplitude of the oscillations in the resonator. The oscillations of the focal spot will be detectable when the increase $\delta\overline{N}_B$ of the mean number of electrons striking the screens exceeds the fluctuation ΔN_B of the number of electrons striking the screens in the absence of a field in the condenser. Since $\delta\overline{N}_B \sim N$, while $\Delta N_B \sim \sqrt{N}$ (where N is the total number of electrons passing through the condenser), the detectable oscillation of the focal spot will become smaller as N increases. However, an increased number of electrons

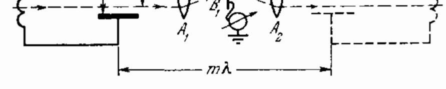

FIG. 3. Schematic diagram of "non-perturbing" measurement of the resonator energy.

passing through the resonator increases the probability of a change in the state of the resonator.

Calculation shows that, if the oscillator was in the ground state (n = 0) before the measurement, then after one electron has passed through only the left-hand condenser, the probability of finding it in this state will be

$$P_0 = e^{-w},$$

where in our case

$$w = \frac{e^2(\omega\tau_f)^2}{2C\hbar\omega}\left(\frac{\bar{y}^2}{Y^2} + \frac{\bar{v}_y^2\tau_f^2}{4Y^2}\right); \tag{48}$$

Here $\overline{y^2}$ is the mean-square coordinate of the electron with respect to the symmetry plane of the condenser, Y is the distance between the plates of the condenser, e is the charge of the electron, τ_f is the time of flight of the electron through the condenser, and v_y is the uncertainty of the velocity of the electrons along the Oy axis.

Equation (48) holds when $\omega\tau_f \leq 1$. The second term in (48), which corresponds to the divergence of the beam during the time of passage, is much smaller than y^2/Y^2 in the cases of practical interest to us. If the probability density of the coordinate y is independent of y over almost the entire cross-section H of the beam, then $\overline{y^2} = H^2/12$. Then, e.g., if $\omega\tau_f = 1$, $C \approx 0.3$ pF, $\omega = 2 \times 10^{10}$ sec^{-1}, and $H/Y \approx 1$, we get $w = 1.5 \times 10^{-3}$. That is, one electron perturbs the ground state of the oscillator but little. However, if αN electrons out of N electrons transmitted through the system strike the screens, then w increases by a factor of αN.

The necessary number N of electrons depends on the amplitude of the oscillations y_0 of the focal spot. In the general case it is

$$y_0 = \frac{eU_0}{\mu Y}\,\frac{\tau_f L}{v_x},$$

where U_0 is the amplitude of the voltage oscillations on the condenser, μ is the mass of an electron, and L is the focal length of the lens. If n = 0, then $U_0 = \sqrt{\hbar\omega_0/C_0} = 2.5 \times 10^{-6}$ V. Then if $v_x = 10^{10}$ cm/sec, Y = 0.2 cm, L = 10^2 cm, and $\tau_f = 5 \times 10^{-11}$ sec, $y_0 = 10^{-8}$ cm. As a simple analysis shows, one can detect such an oscillation of the spot if $N \approx 10^3$. Then 10^2 electrons strike the screens on the average, i.e., $\alpha \approx 0.1$. Hence, the described experimental system permits one to determine the ground state of the oscillator, and the probability of finding the oscillator in this state after the measurement will be P = 0.85. These estimates refer to the case in which all the electrons passing outside the main diffraction maximum are stopped by the screens. However: a) the screens can be made in the form of narrow strips having an angular width smaller than the width of the diffraction maximum; and b) in order to detect the electrons, it suffices to draw from them an energy of only several electron volts. Since the energy of the electrons is of the order of 3×10^4 eV, the screens can be made almost transparent to the electrons.

Hence, the decompensation coefficient α can be diminished at least by another factor of ten. Then the probability of finding the oscillator after the measurements in the ground level will be $P_0 = 0.98-0.99$.

The minimum value of α is determined not only by the width and transparency of the screens, but also by the diffraction perturbation of the electrons passing alongside the screens. Estimates show that this effect plays a smaller role than the absorption of electrons by the screens does.

We note that the mechanical degrees of freedom of the resonator play a substantial role in the described system. Measurement of the momentum of the electrons is accompanied by a change in the mechanical momentum of the resonator. However, since the mass of the latter is large, its position remains well defined.

The significance of the discussed experimental system consists not only in the fact that it can be realized, even with the current state of experimental technique, but the main thing is that it answers the fundamental question of whether one can record nondestructively the n-quantum state of an oscillator.

8. CONCLUSIONS

The quantum-mechanical properties of test objects limit the minimum detectable force to the quantity

$$F_{\min} = \zeta B(\hbar,\ \hat{\tau}).$$

The concrete form of $B(\hbar, \hat{\tau})$ depends on the properties of the test object and the method of observation.

When the test object is a harmonic oscillator, the minimum amplitude of the force is

$$F_{0,\min} = \zeta\cdot\frac{1}{\hat{\tau}}\sqrt{\frac{4m\hbar\omega}{n_0}}.$$

The natural fluctuations of the oscillator will not mask the action of this force if the following inequality holds:

$$\frac{\hat{\tau}}{\tau^*}\cdot 2n_T n_0 \leqslant 1.$$

The change in the state of a mechanical macroscopic oscillator can be converted without loss of information into an electromagnetic high-frequency signal by using an electronic transducer if its relaxation time is much greater than the time of observation. Using transducers having long relaxation times ($\tau_e^* \gg \hat{\tau}$) makes it possible to increase the sensitivity by a factor of $\sqrt{\hat{\tau}/\tau_e^*}$ as compared with a transducer that has $\tau_e^* < 1/\omega$. In the classical approximation, such a transducer will permit one to measure a force of the order of

$$F_{0,\min} = \zeta\cdot\frac{4}{\hat{\tau}}\sqrt{\frac{mkT_e\omega}{\Omega}\,\frac{\hat{\tau}}{\tau_e^*}}.$$

An account for the discrete nature of the energy levels of the transducer transforms the latter relationship into (40).

Nondestructive recording of the n-quantum state of an oscillator is possible in principle.

The authors thank V. I. Panov for a useful discussion on the problems taken up.

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Translated by M. V. King