Another look at what is possible and impossible in optics

I. I. Sobel'man

P. N. Lebedev Physical Institute, USSR Academy of Sciences

Usp. Fiz. Nauk. **113**, 701–705 (August 1974)

According to the Lagrange–Helmholtz law, no optical system can ever increase the brightness (and therefore the effective temperature) of a light beam. This limitation is frequently interpreted as a consequence of the second law of thermodynamics. Actually, thermodynamics imposes no limitations of practical importance on the possibilities of increasing the brightness of light beams: the decrease of the entropy of the light beam associated with the increase of its brightness can be compensated by a loss of a negligible fraction of the energy of the beam. The general discussions are illustrated by a number of specific examples.

The alluring prospect of producing an optical system capable of gathering divergent light fluxes into a narrow directed beam or of reducing the transverse dimensions of a light beam without increasing its divergence, i.e., capable, in the last analysis, of increasing the brightness (W/cm^2sr) of light beams, has been attracting the attention of inventors for centuries. The question whether such a system is possible in principle has naturally arisen. The theory of optical instruments answers this question in the negative: no optical system can increase the brightness of a light beam.

A proof of the above assertion based on the Lagrange-Helmholtz law can be found in any handbook of optics.[1)] An especially complete and detailed discussion will be found in [1], where we read: "The Lagrange-Helmholtz law is valid for any optical system, however it may be constructed of whatever number of both reflecting and refracting elements. No combination of optical systems can ever violate this law, which asserts that all hopes for achieving an ideal concentration of radiant energy are vain. Further, the law also holds for more general optical systems, consisting, for example, of separate zones (like the Fresnel lenses used in lighthouses), making use of refraction in layers of air in which the refractive index varies, or incorporating light pipes and the bundles of vitreous fibers ("fiber optics") that have recently begun to come into use."

The question would seem to have been thoroughly exhausted. Nevertheless, there are weighty reasons for taking another look at it. It should be emphasized at once that no question of any revision of the theory of optical instruments is being raised here; the question is rather, whether the limitations on the possible transformations of light beams of the type formulated in the Lagrange-Helmholtz law can be extended to all of optics, i.e., whether they are to be regarded as universal. One may frequently encounter assertions to the effect that the Lagrange-Helmholtz law is one form of the law conservation of energy, that an increase in brightness would violate the second law of thermodynamics, and so on (see [1], for example). Is this actually beyond the range of applicability of geometric optics?

Doubts arise as soon as we consider systems that include lasers. For example, in devices such as ruby or neodymium lasers, the radiation from a xenon lamp is transformed into a narrow sharply directed beam with an enormous increase in brightness. How does the ruby rod with its mirrors differ from "ordinary" optical instruments—telescopes, projectors, etc.? What limitations, if indeed any, does thermodynamics impose on the performance of optical systems in the broad sense of the word, i.e., optical systems that may include lasers? These are the questions that will be discussed below.

Let us consider a linearly polarized light beam with energy density $\Delta E (J/cm^3)$ concentrated in the spectral interval ω, $\omega + \Delta\omega$ and in the solid angle ΔO. When the values of ΔE, $\Delta\omega$, and ΔO are fixed, the beam also has a definite entropy. The entropy density ΔS (cm^{-3}) is given by the expression (see [2], for example)

$$\Delta S = \Delta g \cdot \Delta O \cdot [(n+1)\ln(n+1) - n\ln n], \tag{1}$$

$$\Delta S = \frac{\omega^2 \Delta\omega}{(2\pi c)^3}, \quad \Delta E = \Delta g \cdot \Delta O \cdot \hbar\omega n, \tag{2}$$

where Δg is the number of field oscillators in the frequency interval ω, $\omega + \Delta\omega$ per unit volume and unit solid angle, and n is the mean number of photons per field oscillator.

Let us consider how Eq. (1) can be used to find limitations on the possible transformations of light beams. According to the law of increasing entropy, the entropy of a closed system may increase or remain constant, but it cannot decrease. Let us apply this law to a light beam. When n increases, the function

$$f(n) = (n+1)\ln(n+1) - n\ln n \tag{3}$$

increases less rapidly; hence if $n = n_1 + n_2$, we have $f(n_1) + f(n_2) > f(n)$. A number of limitations can be derived from this inequality alone (see [3]). For example, if we use a nonabsorbing plane parallel plate to separate the beam into two parts 1 and 2 so that $\Delta E = \Delta E_1 + \Delta E_2$ and $n = n_1 + n_2$, the entropy will increase: $\Delta S_1 + \Delta S_2 > \Delta S$. Hence no optical system that would recover the initial beam can exist, since to recover the initial beam it would be necessary to decrease the entropy in violation of the law of increasing entropy.

It must be pointed out and emphasized that the entire discussion presented above is based on the assumption that the path difference between l beams 1 and 2 is so large that the beams cannot interfere. It is only in this case that the entropies ΔS_1 and ΔS_2 of the beams are additive. If the path difference is small or zero, on the other hand, the total entropy of beams 1 and 2 will not be equal to the sum $\Delta S_1 + \Delta S_2$, and the use made above of the law of increasing entropy will not be justified.

Let us send beams 1 and 2 back to the beam-splitting plate with the aid of ideal mirrors, as is done in the Michelson interferometer. Then each of the beams will be separated into two parts: 1′ and 1″, and 2′ and 2″, respectively. We denote the beams that propagate opposite to the initial beam by 1′ and 2′. If the mirrors are so set that the optical paths of beams 1 and 2 are strictly equal (zero path difference), then, as can be easily seen from the well known Fresnel reflection formulas, beams

1″ and 2″ interfere destructively, cancelling each other out, while beams 1′ and 2′ interfere constructively, leading to full recovery (neglecting diffraction losses) of the initial beam, and indeed, regardless of the spectral composition of the radiation. If the path difference is small enough but not zero, beams 1″ and 2″ will cancel each other out only at a certain value of the wavelength λ, for which the path difference is an odd multiple of $\lambda/2$. Complete cancellation of the beams throughout a finite wavelength interval $\Delta\lambda$ cannot be achieved, however small the interval. For an unpolarized light beam the energy and entropy densities are additive:

$$\Delta E=\sum_{\sigma=1,2}\Delta E_\sigma,\quad \Delta S=\Delta g\cdot\Delta O\cdot\sum_{\sigma=1,2}[(n_\sigma+1)\ln(n_\sigma+1)-n_\sigma\ln n_\sigma]. \quad (4)$$

Hence spatial separation of the polarization components of the beam involves no entropy change; consequently optical systems capable of recovering the initial beam from its spatially separated polarization components can exist, regardless of the path difference between the component beams.

Finally, let us consider the limitations on changes in the brightness of the beam. We denote the areas of the entrance and exit pupils of the optical system by F and F′, respectively. From conservation of the total energy flux we have $F\Delta E = F'\Delta E'$, whence $F\Delta O\cdot n = F'\Delta O'\cdot n'$, where the primed quantities $\Delta O'$ and n' pertain to the beam leaving the optical system.[2)] The entropy $F'\Delta S'$ of the beam leaving the optical system cannot be smaller than the entropy $F\Delta S$ of the beam entering it. Hence it follows from Eqs. (1) and (3) that $F'\Delta O'\cdot f(n') \geq F\Delta O\cdot f(n)$, or

$$\frac{1}{n'}[(n'+1)\ln(n'+1)-n'\ln n']\geqslant\frac{1}{n}[(n+1)\ln(n+1)-n\ln n]. \quad (5)$$

This inequality is satisfied provided $n' \leq n$. The brightness B of the light beam (W/cm^2sr) is related to ΔE and n by the equations

$$B=\frac{\Delta E}{\Delta O}c=c\hbar\omega\Delta g\cdot n, \quad (6)$$

where c is the velocity of light. Hence condition (5) is equivalent to the condition $B' \leq B$; thus we conclude in full conformity with the Lagrange-Helmholtz law that no optical system can increase the brightness of a light beam.

In the examples considered above we have assumed that there is absolutely no exchange of energy between the light beams and the optical system. This assumption is of cardinal importance; relaxing it changes the situation completely. Let us assume that the energy δE is transferred from the beam to the optical system as the former passes through the latter. Then (writing $F = F' = 1$ for simplicity, as we may without loss of generality) the condition on the entropy change must be written in the form

$$\Delta S' + \delta S \geqslant \Delta S,\quad \Delta E' + \delta E = \Delta E, \quad (7)$$

where δS is the increase in the entropy of the optical system consequent on its absorption of the energy δE from the beam.

It is evident that if satisfaction of the inequality $\delta S \geq \Delta S$ can be assured, no limitations at all will be imposed on $\Delta S'$, so that it will be possible to make the beam leaving the optical system as bright as may be desired. A question naturally arises: How much of the light-beam energy must be sacrificed? May not the required energy loss turn out to be too great? In other words, what is the greatest value of the efficiency

$$\eta=\frac{\Delta E'}{\Delta E}=\frac{\Delta E-\delta E}{\Delta E}, \quad (8)$$

consistent with the inequality $\delta S \geq \Delta S$, i.e., with the condition that the law of increasing entropy impose no limitations on the increase of the brightness of the beam? We shall find it convenient in what follows to express the entropy ΔS in terms of the temperature T of the light beam, defining the latter, as usual (see[2]), by the equation

$$kT=\frac{\hbar\omega}{\ln(1+n^{-1})},\ \text{where}\ n=\frac{1}{e^{\hbar\omega/kT}-1}. \quad (9)$$

From (1) and (9) we have

$$\frac{\hbar\omega}{kT}\gg 1\quad(n\ll 1),\quad \Delta S\approx\frac{\Delta E}{\hbar\omega}\ln\frac{\Delta g\cdot\Delta O\cdot\hbar\omega}{\Delta E}=\frac{\Delta E}{kT}, \quad (10)$$

$$\frac{\hbar\omega}{kT}\ll 1\ (n\gg 1),\quad \Delta S\approx\Delta g\cdot\Delta O\left(1+\ln\frac{\Delta E}{\hbar\omega\cdot\Delta g\cdot\Delta O}\right)=\frac{\Delta E}{kT}\left(1+\ln\frac{kT}{\hbar\omega}\right). \quad (11)$$

Assuming that the energy δE is released in the system at temperature T_0 (at constant volume), we write $\delta S = \delta E/kT_0$; then we find

$$\eta\leqslant 1-\frac{T_0}{T},\quad \frac{\hbar\omega}{kT}\gg 1, \quad (12)$$

$$\eta\leqslant 1-\frac{T_0}{T}\left(1+\ln\frac{kT}{\hbar\omega}\right),\quad \frac{\hbar\omega}{kT}\ll 1. \quad (13)$$

Relations (12) and (13) are similar to the well known expression for the efficiency of a Carnot cycle, the temperature T of the light beam corresponding to the temperature of the working substance. If we decrease ΔO while holding ΔE constant, i.e., if we increase the brightness B and temperature T of the beam, the entropy ΔS of the beam will decrease. In the limiting case of very bright beams, the beam entropy is virtually zero, and the formation of such a beam is in a certain sense analogous to the performance of mechanical work.

Radiation in the optical region of the spectrum always has a comparatively high temperature—much higher than room temperature. Hence the ratio T_0/T is usually small. For a xenon lamp, for example, $T \approx 10\,000^\circ K$, and for $T_0 = 300^\circ$ we have $T_0/T \approx 3\times 10^{-2}$.

Thus, the energy losses that must in principle be accepted in order to be able to form a light beam of arbitrary brightness are virtually negligible. Of course it does not follow from all this that just any sort of light absorption would facilitate the solution of this problem. General thermodynamic considerations cannot indicate specific means for realizing the systems discussed. It is extremely important, however, that thermodynamics does not impose any limitations on the increase in the brightness of light beams provided only that energy can be exchanged between the light beams and the optical system.

Fifteen years ago there were no optical systems capable of increasing the brightness of a light beam; now there are. For example, the ruby laser mentioned above converts the radiation flux from a xenon lamp into a laser beam of enormous brightness, the decrease in the entropy of the light flux being compensated by an increase of the entropy of the "optical system." The fact that the pumping process produces a population inversion in the ruby crystal and that the radiation is emitted as a result of induced transitions in no way invalidates the approach to the treatment of the system outlined above. For example, we may consider a pumping pulse during which the crystal is excited, radiates, and then returns to its initial equilibrium state, giving up the energy "trapped" within it to a thermostat. The increase δS in the entropy of the system naturally depends on specific features of this cycle (constant volume, constant pressure, chemical reactions, etc.), but in all cases the decisive feature is the smallness of T_0 as compared with T. When $T_0/T \ll 1$, thermodynamics imposes no limitations of practical significance on the characteristics of the laser beam or the magnitude of the conversion efficiency η. Of course large values of η, close to the limiting values given in (12) and (13), cannot always be achieved.

We note in concluding that it is not at all necessary to use a medium in which there is a population inversion in order to increase the brightness of light beams. Systems for transforming laser beams using Raman and Mandel'shtam-Brillouin scattering have already been designed and tested, and brightness increases of tens and hundreds of times have been achieved with them (see [4,5], for example). The scattering liquid (or gas) is in a state of thermodynamic equilibrium. The exchange of energy between the light beam and the medium takes place in the scattering process itself, which is accompanied by a decrease in the frequency of the light quanta: $\omega \rightarrow \omega'$ ($\omega' < \omega$). It is noteworthy that the main thing here is the frequency decrease, i.e., the energy loss, in the scattering process. It may be very small (for example, $(\omega - \omega')/\omega \sim 10^{-5}$ for Mandel'shtam-Brillouin scattering), but it is absolutely essential; the converter cannot work if there is no frequency shift in the scattering process, i.e., if $\omega = \omega'$. In more complicated cases the frequency may increase in the scattering process (e.g., in generating anti-Stokes lines), but the process always takes place in such a manner that the total entropy increases.

The nonlinear optical phenomena whose observation and study became possible only when powerful laser light sources became available have opened entirely new possibilities for the processing of light pulses, including the possibility of increasing the brightness of the pulse at the expense of its duration.

1)The Lagrange-Helmholtz law is usually written in the form $\mu l \sin u = \mu' l' \sin u'$, where μ represents the refractive index in the object space, l, the transverse dimension of the object, u, the angle subtended by the entrance pupil at the center of the object, and the primed letters represent the corresponding quantities pertaining to the image space, the image, and the exit pupil.

2)We assume for simplicity that the areas F and F' are perpendicular to the axes of the corresponding beams.

1. G. G. Slyusarev, O vozmozhnom i nevozmozhnom v optike (The Possible and Impossible in Optics), Fizmatgiz, Moscow, 1960.
2. L. D. Landau and E. I. Lifshitz, Statisticheskaya fizika (Statistical Physics), Nauka, 1964 [Addison-Wesley, 1971].
3. H. A. Lorentz, Lectures on Theoretical Physics. I. The Theory of Radiation (English Transl. MacMillan, London, 1927, 1931) (Russ. Transl., ONTI, Moscow, 1935).
4. A. Z. Trasyuk, V. F. Efimkov, I. G. Zubarev, V. I. Mishin, and V. G. Smirnov, ZhETF Pis. Red. 8, 474 (1968) [JETP Lett. 8, 291 (1968)].
5. V. I. Kovalev, V. I. Popovichev, V. V. Ragul'skiĭ, and F. S. Faĭzullov, ZhETF Pis. Red. 14, 503 (1971) [JETP Lett. 14, 344 (1971)].

Translated by E. Brunner