Interaction of free electrons with electromagnetic radiation

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A study is made of the interaction of a random field of electromagnetic radiation with the free electrons of a plasma, with applications to astrophysical problems, in particular the theory of how thermodynamic equilibrium of the radiation in a hot universe is established. A kinetic equation describes the variation of the spectrum; special attention is devoted to induced scattering and the classical interpretation of induced energy and momentum transfer. In the spectra of radio sources with a high luminosity temperature, induced scattering can lead to a Bose condensation of photons, a shock wave and solitons. The scattering of strong low-frequency waves is considered in connection with their effect on pulsars and laboratory coherent generators.

CONTENTS

I. INTRODUCTION

Physicists have been studying the interaction of free electrons with electromagnetic radiation from the moment that J. J. Thomson formulated the theory of the electron. These investigations have had different goals at various stages, but they have always been of great significance.

In classical physics, Thomson himself determined the scattering cross section, as well as its angular dependence, polarization and phase. We speak of "Thomson scattering"; the "Thomson cross section" has the numerical value

$$\sigma=\frac{8\pi}{3}r_0^2=\frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2=0.665\cdot 10^{-24}\ \text{cm}^2.$$

In this approximation, scattering occurs with no change in frequency of the radiation in the rest system of the electron.

As was discovered later, Thomson's results constitute the asymptotic form of the quantum theory in the case when the photon energy $\hbar\omega$ is small in comparison with the unique characteristic quantity, the rest energy mc^2 of the electron.

Thomson's theory therefore remains meaningful for scattering of radio waves in the ionosphere and for scattering of light by a hot plasma. In astrophysics, the velocity with which the energy of radiation is transmitted depends on Thomson scattering. Scattering involves a transfer of momentum. A certain force acts on an electron in a flux of radiation. From the condition that this force is equal to the gravitational force, one obtains the limiting flux of radiation of a star—the so-called Eddington limit, which exceeds the solar luminosity by a factor 30,000. A stronger flux would strip off the stellar atmosphere.

In 1905 Einstein formulated the theory of light quanta. This theory was based on Planck's theory, but at the same time it represented a significant step forward—Planck originally assumed that only the emission and absorption of light are "quantized," without encroaching on its propagation, which is so well described by Maxwell's theory.

The study of scattering acquired the significance of a decisive experiment! The quantum theory, making use of nothing but the laws of conservation of energy and momentum, leads to a definite change in the energy and frequency of a quantum (photon) as a function of its scattering angle.

The experimental investigations of Arthur Compton confirmed the theoretical dependence

$$\lambda\!\!\!{}^{-}=\lambda\!\!\!{}^{-}_0+\frac{\hbar}{mc}(1-\cos\theta),$$

where $\lambda\!\!\!{}^{-} = \lambda/2\pi$, $\lambda\!\!\!{}^{-}$ is the wavelength after scattering, $\lambda\!\!\!{}^{-}_0$ is the initial wavelength, and θ is the scattering angle. This result was of profound significance for the entire subsequent development of physics. This is even reflected in our terminology: we speak of the Compton wavelength of the electron, $\hbar/mc = 3.86\times 10^{-11}$ cm, and of other particles.

It is customary to speak of Compton scattering (as opposed to Thomson scattering) when allowance is made for the change in frequency of the photon in the rest system of the electron.

We note that the picture of Compton scattering is essentially statistical. The same photon can be scattered by the same electron through different angles according to the laws of chance.

This statistical character was reflected fully only in quantum mechanics[1)].

It was not immediately recognized that the scattering angle is "statistical" but that the conservation laws are not. In 1924 Bohr, Kramers and Slater proposed a theory in which the law of conservation of energy and the law of conservation of momentum are violated in individual elementary scattering events and hold in macroscopic physics only as a result of the averaging over a large number of elementary events. Before long, they abandoned their theory. Later, however, in 1933, there appeared an experimental work in which it was claimed that the Compton formula is incorrect. Confusion developed. New careful experimental investigations were required (the first and best of which was the work of the brothers A. I. and A. I. Alikhanov and L. A. Artsimovich) to reestablish the truth and rehabilitate the Compton formula and the applicability of the conservation laws in elementary events. This episode is quoted here neither to confirm the aphorism of Artsimovich that "there is nothing worse than a 'dirty' experiment to confirm a 'dirty' theory" nor to compromise Bohr and his famous coauthors. The story of the supposed nonconservation of energy and momentum in elementary events is instructive in giving some idea of the predicament of theoretical physics during the period from 1905 (or from 1899) to 1925: quantum concepts had already been conceived, but there was no rigorous theory; "quantization" was more of an art than a strict science, its success depended on intuition, and it was necessary to create something new, but with due regard for the old and for the principle of correspondence with classical mechanics and electrodynamics. The modern reader sometimes sees the operator formulation of quantum mechanics as a practical complication, perhaps an unnecessary one. For the physicists of the first quarter of the twentieth century, quantum mechanics made its appearance as a long-awaited deliverance from the agonizing uncertainty in all the discussions and calculations.

In nonrelativistic quantum mechanics, the hamiltonian of a neutral particle contains the kinetic energy $p^2/2m$. The prescription for going over to a charged particle interacting with an electromagnetic field is to make the substitution $p \to p - (e/c)A$, where A is the vector potential of the electromagnetic field. In the resulting expression

$$\frac{p^2}{2m} - \frac{e}{mc} pA + \frac{e^2}{2mc^2} A^2$$

the last term proportional to A^2 contributes directly to Compton scattering: in the language of Feynman diagrams, this term corresponds to a four-point function—a vertex at which two electron lines and two photon lines meet (Fig. 1). The second term $(e/mc)pA$ corresponds to a three-point function involving one photon line (Fig. 2).

The study of Compton scattering played a major role in the relativistic theory of the electron. The Dirac equation is of the first order, corresponding to the first power of the momentum in the Hamiltonian. The equation for a charged particle involves A to the first degree, and only

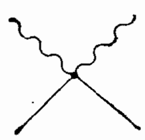

FIG. 1

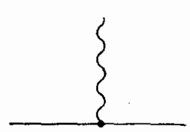

FIG. 2

FIG. 1. Diagram for the direct scattering of a photon by a spin-zero particle.

FIG. 2. Diagram for the emission or absorption of a photon by a charged particle.

the three-point vertex occurs (see Fig. 2). This means that the Compton effect in the Dirac theory corresponds to second-order diagrams involving the two sequences "absorption followed by emission" (Fig. 3) and "emission followed by absorption" (Fig. 4).

In the interval between the formulation of Dirac's equation (1928) and the discovery of the positron (1932), there was a period of 3 or 4 years during which there was no common attitude towards the prediction of negative-energy levels contained in the equation. The proton could obviously not be regarded as the antiparticle of the electron. Should one perhaps somehow forbid the negative-energy levels and give Dirac poor marks for not having done so himself?

Calculations of the Compton effect hinted at the correct answer long before the discovery of the positron. The point is that the diagrams (see Figs. 3 and 4) involve a dominant contribution which depends on the ways in which the intermediate-state electron finds itself in a negative-energy state[2]. These states cannot be forbidden without spoiling the agreement of the theory with the trivial limiting case of scattering of low-frequency electromagnetic waves. It was unambiguously concluded that all the predictions of Dirac's theory are real. Another few years passed, and this was also acknowledged by the Nobel Prize Committee.

The study of the relativistic problem provided not only a confirmation of the principles. At the same time, one obtained the scattering cross section as a function of the photon energy: the cross section drops to half its value at $\hbar\omega = 0.7\ mc^2$ and by a factor 10 at $\hbar\omega = 13\ mc^2$ (the Klein-Nishina-Tamm formula). The diagrams describing the scattering when viewed from another direction (the reader should lie on his side if he does not wish to turn the journal) correspond to two-photon annihilation of the electron and positron and the inverse process of e^+e^- pair production in a photon-photon collision. The foregoing historical reminiscences bear relatively little relation to the subject of the present review. We shall make direct use of only one fact here: all the variants of quantum theory (nonrelativistic, relativistic, the Dirac equation and other equations) give a result in agreement with the classical theory at low frequencies.

But there is still another aspect of the matter: the history of the study of the Compton effect shows how many facets are revealed by an intense, carefully considered and persistent investigation of a single phenomenon. This conclusion still holds today; it is corroborated by the subsequent investigations which are properly described below.

The motives for these subsequent investigations were as follows.

1) The development of lasers and masers, i.e., the emergence of powerful sources of highly coherent electromagnetic waves. One was led to the problem of scattering of an intense electromagnetic wave. After the discovery of pulsars, it became clear that a pulsar—a rotating magnetic dipole—is also a source of very long radio waves with a period between 0.03 sec and 5 sec.

This is the motivation for the investigations summarized in Chap. II (the "strong wave").

The basic criterion for a wave to be "strong" is the relativistic velocity of the oscillations of the electrons under the influence of the wave.

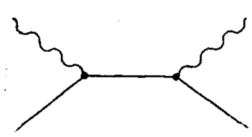

FIG. 3

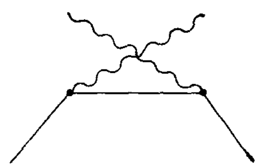

FIG. 4

FIG. 3. Diagram for the scattering of a photon by an electron. Variant 1: absorption of the incident photon and subsequent emission of the scattered photon.

FIG. 4. Diagram for the scattering of a photon by an electron. Variant 2: emission of the scattered photon and subsequent absorption of the incident photon.

Relativistic electrons radiate higher harmonics. The total scattering cross section becomes larger for a strong wave. At the same time, the contribution of the electrons to the index of refraction of a plasma becomes smaller, and a strong wave of a given frequency passes through the plasma more easily than a weak one.

2) In astrophysics, bodies and situations have been observed for which there is characteristically a strong predominance of Compton scattering over the emission and absorption of photons. This relationship holds when the temperature is high and/or the density of electrons and nuclei is low.

It is characteristic of these problems that photons are produced in numbers many times smaller than in equilibrium and with a spectrum which is unlike the equilibrium Planck spectrum. After this, each photon undergoes Compton scattering many times. In addition to the recoil effect, allowance is made for the Doppler effect, which depends on the motion of the electrons. At a nonrelativistic electron temperature, each scattering event changes the frequency of an individual photon only slightly and has a small effect on the overall spectrum. However, when repeated multiple scattering occurs, there results a spectrum having a quite specific form, which is essentially different from both the initial bremsstrahlung spectrum and the equilibrium Planck spectrum.

A. S. Kompaneets takes the credit for a lucid formulation of the problem as to the nature of the process by which thermodynamic equilibrium is established in a rarefied plasma [1].

A precise formulation of the problem leads to an integral equation for the photon spectrum. However, as the transfer of energy in an individual scattering event is small, the change in the spectral density at a given frequency depends only on the spectral density at neighboring frequencies, so that the integral equation can be effectively replaced by a differential equation. The Kompaneets differential equation has also served as a reliable basis for other problems which arose later.

We note some applications of the theory:

1) The laboratory analysis of a plasma with the aid of Compton scattering is in principle trivial. The plasma is exposed to a beam of highly monochromatic laser radiation. By determining the spectrum of radiation scattered at various angles, one can find the electron density and momentum distribution, i.e., in the simplest case, the electron temperature.

2) The clouds of hot plasma within our galaxy or beyond it cannot be "illuminated" at will. But these clouds are exposed to the relic equilibrium radio waves of temperature 2.7°K. Compton scattering by hot electrons displaces the entire spectrum on the average into the region of higher frequencies. Theory predicts [2] a reduction of the intensity in the low-frequency part of the spectrum ($\hbar\omega < kT_r = k \cdot 2.7°K$), since some of the photons from this region enter the region $\hbar\omega > kT$. The effect is proportional to the product of the path length, the density of electrons and their temperature. It appears that an effect of order $\Delta I/I = -2 \times 10^{-4}$ (where I is the intensity at a wavelength of about 3 cm) has been observed for the plasma in the great Coma cluster of galaxies [3]. This cluster is also an x-ray source, but x-rays depend on another combination of parameters, so that x-ray and radio measurements complement each other.

3) The universe—a hot universe filled with photons—is a particular case[3)] of an astrophysical system to which the Kompaneets equation can be applied [4-7].

It is well known that the density of protons and electrons in the universe amounts to only 10^{-8}—10^{-9} of the density of photons. Owing to the low density of p and e, the number of electron-proton collisions (which determines bremsstrahlung, i.e., free-free radiation) is extremely small and is far less than the number of electron-photon collisions, which lead to Compton scattering. At a plasma temperature above 3×10^9 °K, the equilibrium contains many positrons and electrons in addition to those electrons which at the present time compensate for the charges of nuclei (chiefly of hydrogen and helium). We are therefore sure of complete thermodynamic equilibrium and, in particular, the Planck radiation spectrum at a high temperature.

It is a remarkable fact that the expansion of the universe which occurs in accordance with the theory of A. A. Friedmann does not upset the equilibrium. In a uniform (everywhere identical) and isotropic (with the same expansion in all directions) universe, radiation undergoes a red shift and the energy density decreases with the passage of time. But the red shift is independent of the direction of propagation of the photons: the relative change in frequency is also the same for all photons. The Planck spectrum is transformed in the course of the expansion into a Planck spectrum with a lower temperature.

Current observations confirm the Planck spectrum (although with a limited accuracy of order 5—10%). But this does not mean that some processes are now sustaining the equilibrium. There are no such processes. The universe in its present state is transparent.

It is sufficient that there are no processes which upset the equilibrium. At the 1973 symposium of the International Astronomical Union in Cracow, the relic radiation was compared with an unattractive centenarian virgin: she remained a virgin, not because she was very virtuous, but because nobody encroached upon this virtue.

The real astrophysical interest in this problem is due to the fact that the universe does not correspond exactly to the Friedmann cosmological model. The existence of galaxies is evidence for a certain nonuniformity in the density of the plasma and for certain motions of this plasma (more or less random and irregular motions in addition to the overall cosmological expansion) during the early evolutionary stages of the universe.

A detailed investigation shows that the large-scale perturbations due to the nonuniform density become stronger in the course of time as a result of gravitational instability. However, small-scale perturbations are damped out because of the viscosity of the plasma. The kinetic energy of the random motions is transformed into heat. This is in fact an example of an "encroachment" on the Planck radiation spectrum.

Can we distinguish the "heat" energy produced in this way from the original energy of the plasma? Is it possible to ascertain the amplitude of perturbations that were damped out long before the present era? It turns out that a detailed study of the Compton effect and the application of the Kompaneets equation makes it possible to draw important conclusions. A comparison of the theory with observations leads to strong constraints on the energy of the perturbations. We quote only the main conclusion: the universe is practically uniform and isotropic, not only on a scale of thousands of megaparsecs (with deviations less than 10^{-3}), but it has always been just as uniform and isotropic to an accuracy of order 10^{-1} on a scale starting with 100 parsecs when converted to the present scale of distances [6,8].

The point is that the production of heat in the plasma after the loss of the positrons increases the energy density without changing the density of photons. After this, the Compton effect causes a displacement of the photons over the scale of frequencies and gives rise to an evolution of the spectrum towards equilibrium—but an incomplete equilibrium, with a given number of photons. Such a restricted equilibrium corresponds to a spectrum of the so-called Bose-Einstein type, which differs from the Planck spectrum. The theory provides clear recommendations as to the frequencies at which we should seek the deviations. The redistribution of the photons enhances the sensitivity of the observations: a 1% production of heat alters the spectral density in the long-wave region by more than 20% [6].

4) The requirements of astrophysics have led to the formulation of the problem of random radiation, but with a spectrum which differs appreciably from the Planck spectrum. The low-frequency part of the spectrum corresponds to a very high temperature T, which is also frequency-dependent; the high-frequency part of the spectrum is practically absent; and the total density of radiation is much smaller than the black-body radiation aT^4 at high temperatures.

Radio emission of pulsars is the most striking example of this type: at a wavelength of the order of meters, the effective temperature of the source is as high as 10^{30} °K for the first pulsar. The detailed picture of such sources of radiation is unclear. Most likely, there is some kind of transformation of the energy of macroscopic motion from the large scale to the small scale. The turbulent sea has a spectrum of oscillations corresponding to an effective temperature of order 10^{26} °K at a frequency of the order of an inverse second. The high-frequency part of the spectrum, at 10^{10} Hz and above, corresponds to a temperature of the water of 298°K. In itself, a temperature which varies over the spectrum is not extraordinary, particularly for the oscillations of a plasma. The chief theoretical difficulty is the mechanism by which the oscillations are transformed into electromagnetic waves. But the problem of the structure of the radiating region of a pulsar is beyond the scope of the present paper. Spectra having a high effective radio temperature with weak optical and x-ray emission are also characteristic for radio galaxies and quasars. A likely explanation in this case is synchrotron radiation of relativistic electrons. The maximum temperature is reached at a frequency at which a role is played by the inverse process of absorption of radiation by electrons in a magnetic field. At the maximum, kT is of the order of the electron energy; in the high-frequency region, the source of radiation is transparent and accordingly radiates little. In a sense, the question as to the origin of the high-energy electrons has not been answered, but has merely been brushed aside. Again, the source is ordered motion.

The subject of this review is the behavior of such radiation in a relatively cold plasma outside the source.

But apart from the astrophysical proposals there is also the internal logic of the investigations. Photons are bosons. The probability of a process which gives rise to a photon having given properties (frequency, direction and polarization) is proportional to (n + 1), where n is the number of photons which are already in the state in question. The celebrated factor (n + 1), which describes stimulated (we sometimes say "induced") emission, is the basis of the theory of masers and lasers. But this factor, or, more properly, the induction effect, must also occur in scattering. In this sense, it is highly instructive to consider the early works on the theory of thermodynamic equilibrium of a system consisting of free electrons and electromagnetic radiation.

One of these works [61] considered a gas consisting of two types of atoms—electrons and photons, which collide elastically with one another like hard spheres. Equilibrium is then attained when the electrons have a Maxwell spectrum and the photons have the so-called Wien spectrum, $F_\nu = \text{const}\ \nu^3 e^{-h\nu/kT}$, corresponding to the same temperature. The exact (Planck) radiation formula is not obtained.

Slightly earlier, the young Pauli ([62]; see also [63]) had already shown that, for thermodynamic equilibrium of the electrons and radiation obeying the Planck formula, allowance must be made for induced scattering.

The probability of a photon-electron collision also depends on the intensity of radiation having the direction and frequency acquired after the scattering. The dual—corpuscular and wave-like—nature of the photon and the way in which photons differ from hard spheres become apparent here.

The Kompaneets equation also includes the contribution from induced scattering in the simplest case of a uniform and isotropic problem. The limiting case of high temperature of the low-frequency part of the spectrum corresponds to large occupation numbers n, since in equilibrium $n = (e^{\hbar\omega/kT} - 1)^{-1}$ and hence $n = kT/\hbar\omega$ in the limit $kT \gg \hbar\omega$.

Thus, there arises a problem regarding the Kompaneets equation (i.e., the equation for the evolution of the spectrum) when $n \gg 1$ and the induced scattering proportional to n^2 is dominant over the spontaneous scattering proportional to n.

It turns out [9] that the equation subject to certain initial conditions leads to shock waves in phase space, i.e., discontinuities in the dependence $n(\omega)$. Further analysis [10] of the structure of a shock wave predicted the occurrence of sharp maxima, i.e., in essence broad lines which do not correspond to any proper frequencies of the plasma. Such lines have so far not been seen; it would be of particular interest to find them in nature or in the laboratory.

5) In addition to the problem concerning the evolution of the radiation spectrum, there arises the question as to how the electrons behave in a given radiation field[11–13]. Moreover, a consistent treatment of both problems is required in the majority of cases. The conservation of energy relates the change in the radiation energy (which follows from the evolution of the spectrum) to the contribution associated with the energy balance of the electrons. It turns out that there is no problem regarding the energy distribution for the electrons (analogous to the problem regarding the spectrum): the electrons gain and lose momentum in small random amounts, so that their momenta have a Gaussian distribution. But the Gaussian law for nonrelativistic electrons is the same as the equilibrium Maxwell-Boltzmann distribution. There need not be collisions between the electrons to establish this distribution, so that the distribution will hold even in a rarefied system. One quantity remains to be determined—the dispersion of the Gaussian distribution, i.e., the temperature of the electrons. A single energy-balance equation is sufficient for this purpose[4)].

6) The theory of induced scattering is also of interest because, as is well known, the limiting case of bosons with large occupation numbers represents a classical wave field. We obtain a typical example of how "classical theory helps us to understand quantum theory" (we remind the reader of Sov. Phys.-Uspekhi of a note by P. Paradoksov[14] bearing a similar title). It is necessary to rectify this indirect path (in German, "Herumführung" instead of "Einführung"—"round tour" instead of "introduction") and to demonstrate the significance of the results directly in the language of the classical theory of Maxwell and the equations of motion of Lorentz.

It turns out that second-order effects, such as the Lorentz forces of the magnetic field of one wave acting on an electron oscillating under the action of a second wave, play a decisive role. The case of anisotropic fluxes of radiation is also of great interest and of practical importance. On the whole, induced scattering does not tend to smooth out the anisotropy: if there is no radiation in some solid angle, then no radiation will get into that solid angle in the approximation $n \gg 1$. Anisotropic radiation produces an anisotropic electron temperature; whether this latter anisotropy is smoothed out depends on the interactions of the electrons with each other and, in particular, on the collective plasma interactions and instabilities.

Finally, anisotropic induced scattering produces a force acting on an electron which is proportional to the square of the intensity, i.e., to the fourth power of the field and accordingly to the fourth power of the electron charge[15]. It is a curious fact that this force vanishes both in the case of isotropy (which is obvious, by symmetry) and in the case of maximum anisotropy, when all the waves are strictly coincident in direction. This may be the reason why the force of induced scattering was not noticed until 1971, although this force is in fact a classical one and could have been calculated even in the nineteenth century.

7) A very important domain of applicability of the theory is the case of processes in the gas surrounding neutron stars and collapsed stars—black holes[16]. It can be regarded as established that x-rays are emitted by a gas which is heated as a result of its fall in the gravitational field of a super-dense star. The bremsstrahlung of the hot gas can then be significantly modified by scattering by the electrons. The radiation pressure largely determines the motion of the gas and its density.

In a real situation, particularly in the case of neutron stars, the theory is complicated to a great extent by the magnetic field.

In our brief introduction, we have enumerated only the main problems which are considered in the following chapters of the review. The introduction is no substitute for the review, just as the review is no substitute for the original papers. In addition, the arrangement of the material in the chapters which follow does not fully correspond to that of the introduction. We completely omit the theory of scattering by ultrarelativistic electrons accompanied by the conversion of radio and optical radiation into x-rays and gamma radiation.

A thorough study of this problem has been made by Ginzburg and others[17–19]. In the following chapters we also give references to the original literature. The astrophysical problems mentioned above are an exception: a detailed discussion of these problems within the framework of the present review would require excessive space and would be a digression from the physics of the phenomenon. We therefore confine ourselves to what has been said above and the references to the original papers. The astrophysical problems deserve a separate detailed exposition!

Radiation and scattering by electrons in a constant magnetic field are of great importance for astrophysics. The radiation (so-called synchrotron radiation) has been studied thoroughly and discussed in a number of monographs and reviews[20–23]. The scattering by magnetized electrons has been studied especially actively in recent years in connection with the conjectured fields of up to 10^{12}–10^{14} G at the surface of pulsars. It is very tempting to include this subject in our review, but, given the volume of this paper, this would adversely affect the completeness and intelligibility of the main part of the review. Let us hope that the authors of the original papers[57–59] on the properties of magnetized electrons will complete this series in their own way.

Finally, electromagnetic waves interacting with electrons may be regarded as a particular case of interacting plasma oscillations. Compton scattering in this context is discussed, in particular, in the reviews and monographs of Tsytovich and Kaplan[24,25]. Some of the results and methods are general ones. However, the peculiar properties of electromagnetic waves, particularly their velocity, which is equal to the velocity of light (pardon the tautology, for light ≡ electromagnetic waves), and their special significance in astrophysics justify the separate discussion given here.

I take this opportunity to thank my colleagues of the Institute of Applied Mathematics, the results of whose work became known to me long before their publication. I am particularly grateful to R. A. Syunyaev and A. F. Illarionov, in collaboration with whom many results on the subject of this review were obtained, for discussions and assistance in writing the review.

The review is organized in such a way that the reader who is interested in only the general aims of the series of works under consideration can dwell upon this.

The following chapters contain concrete calculations and methodological remarks of interest to theoretical physicists. These chapters can be considered as a supplement to the short popular article which is in essence contained in this introduction. It is for this reason, bearing in mind those who will read no further, that the acknowledgements have been inserted here, at the end of the introduction, and not at the end of the paper.

It is also appropriate here to touch upon one deeply personal matter. The original version of this paper was submitted for publication during the lifetime of A. S. Kompaneets. All that is said in the paper about his contribution remains strictly unchanged. But it is only now, after his sudden death, that this review is devoted to his memory. The author cannot help re-reading what has been written earlier with different feelings; but let it remain unchanged, let the sorrow of the loss not affect our appreciation of the work of A. S. Kompaneets on which this review is based.

II. THE STRONG WAVE

1. The Thomson theory (reminder). The classical Thomson theory begins with the solution of the equation of motion of the electron. A plane-polarized electromagnetic wave propagates along the z-axis, with

$$E_x = E_y = E\cos(kz-\omega t), \qquad k=\frac{\omega}{c};$$

the remaining field components are equal to zero. For the motion of the electron, we write

$$\ddot{x} = -\frac{eE}{m}\cos\omega t, \qquad \dot{x} = -\frac{eE}{m\omega}\sin\omega t, \qquad x=\frac{eE}{m\omega^2}\cos\omega t. \tag{1.1}$$

The other coordinates of the electron are taken to be y = z = 0. This equation is written without relativistic corrections, and no allowance is made for the Lorentz force in the magnetic field of the wave or for the radiation reaction.

In a wave which is weak with respect to the motion specified by (1.1), we find that the radiation of the electron has an instantaneous intensity W and a time-averaged intensity $\overline{W}$ given by

$$W=\frac{2e^2}{3c^3}\ddot{x}^2, \qquad \overline{W}=\frac{e^4E^2}{3c^3m^2}.$$

The flux of energy in a plane electromagnetic wave is

$$Q=\frac{c}{8\pi}(\mathrm{E}^2+\mathrm{H}^2), \qquad \overline{Q}=\frac{c}{8\pi}E^2.$$

The cross section is

$$\sigma=\frac{\overline{W}}{\overline{Q}}=\frac{8\pi}{3}\frac{e^4}{m^2c^4}=\frac{8\pi}{3}r_0^2, \qquad r_0=\frac{e^2}{mc^2}.$$

Numerically, $\sigma = 6.65\times10^{-25}$ cm^2 and $r_0 = 2.82\times10^{-13}$ cm. The scattered radiation is plane-polarized. Its intensity is proportional to the square of the projection of the vector **x** onto the plane perpendicular to the direction n of the scattered ray,

$$I\sim\sin^2\alpha, \qquad \alpha=\widehat{(\mathbf{nx})}.$$

Accordingly, the differential cross section is

$$\frac{d\sigma}{d\Omega}=\frac{3\sigma\sin^2\alpha}{8\pi}. \tag{1.2}$$

In the Euler coordinates ($\theta = (\mathbf{nz})$, φ)

$$\frac{d\sigma}{d\Omega}=\frac{3\sigma}{8\pi}(\cos^2\theta+\sin^2\theta\sin^2\varphi). \tag{1.3}$$

For an unpolarized ray, we must take an average with respect to the angle φ. We obtain

$$\frac{d\sigma}{d\Omega}=\frac{3\sigma}{8\pi}\left(\cos^2\theta+\frac{1}{2}\sin^2\theta\right)=\frac{3\sigma}{16\pi}(1+\cos^2\theta). \tag{1.4}$$

Here θ is the scattering angle. The scattered radiation is partially polarized (completely polarized for $\theta = \pi/2$).

The induced dipole moment per electron is given by

$$d_x=-ex=-\frac{e^2}{m\omega^2}E_x. \tag{1.5}$$

Accordingly, the dielectric constant ϵ and the square of the index of refrcction α for a plasma with a density n_e electrons/cm^3 have the values

$$\epsilon=\alpha^2=1-4\pi n_e\frac{e^2}{m\omega^2}, \tag{1.6}$$

from which we also obtain an expression for the critical (Langmuir) frequency ω_c, at which $\epsilon = 0$:

$$\omega_c=\sqrt{\frac{4\pi n_e e^2}{m}}=5.65\cdot10^4\sqrt{n_e}\ \mathrm{sec}^{-1}. \tag{1.7}$$

As the motion of the electron was determined without allowance for the radiation reaction, the corresponding dielectric constant was found to be real.

2. Criterion for the strength of a wave. Let us determine the amplitude E_c of a wave for which the velocity of the electron, when calculated naively in a nonrelativistic manner, becomes equal to the velocity of light. We denote by b the dimensionless ratio of the actual amplitude to E_c:

$$E_c=\frac{mc\omega}{e}, \qquad b=\frac{E}{E_c}. \tag{2.1}$$

A weak wave (for which the formulas of the preceding section are valid) obviously corresponds to $b \ll 1$, while a wave with $b \gtrsim 1$ must be called strong. According to[26], the required intensity of the ray per unit area has the value (at b = 1)

$$Q=\frac{10^{10}}{\lambda^2}\ \mathrm{W/cm^2}=10^{18}\ \mathrm{W/cm^2}\ \text{for}\ \lambda=10^4\ \text{Å}.$$

Suppose that the ray has a circular cross section and a radius $a\lambda$, where a is a dimensionless number that cannot be less than 1; λ is the wavelength divided by 2π (i.e., the inverse wave vector k^{-1}). The intensity of the ray is

$$W_r=SQ=\pi a^2\lambda^2\frac{c}{8\pi}b^2E_c^2=\frac{a^2b^2m^2c^5}{8e^2}=\frac{a^2b^2mc^2c}{8r_0}$$
$$=a^2b^2\cdot10^{16}\ \mathrm{erg\cdot sec^{-1}}=a^2b^2\cdot10^9\ \mathrm{W} \tag{2.2}$$

The critical intensity required to accelerate the electron to a relativistic velocity turns out to be independent of the frequency. Its absolute value is not large, corresponding to 1 J/nsec in the pulsed regime.

In a plasma exposed to a ray with $b > 1$, the electrons, in colliding with nuclei, can produce electron-positron pairs or can emit gamma rays (by bremsstrahlung). However, these processes are proportional to the square of the plasma density. For large $b > 137$, pair production due to electron-positron collisions exceeds annihilation and, for sufficiently long exposure and retention of the plasma, there occurs an avalanche-like process involving the growth of the number of pairs (cf.[27], where an analogous process is considered for electrons having a Boltzmann distribution). We recall that with two colliding light beams it is possible to have direct electron-positron pair production in a vacuum, without an initial plasma and, in addition, not as a result of the frequency of the colliding beams, but as a result of the quasi-static field. The corresponding critical field E_{pc} is determined by the condition

$$\frac{eE_{pc}\hbar}{mc} = mc^2, \quad E_{pc} = \frac{m^2c^3}{e\hbar} = \frac{E_c mc^2}{\hbar\omega}.$$

Consequently, for a quasi-static field, i.e., when $\hbar\omega \ll mc^2$, pair production in a vacuum is more difficult (requires a larger amplitude of the field) than the acceleration of electrons to a relativistic velocity and the production of pairs in a plasma.

3. Motion of an electron and scattering of a strong wave. Let us return to the subject of the review. Suppose that we have a rarefied plasma. We shall neglect the collisions and study the motion and radiation of the electrons in a strong electromagnetic wave in greater detail.

This problem has been solved in a number of papers with exact allowance for the relativistic mechanics of the electron and the Lorentz forces, but with neglect of the radiation reaction. The problem has been solved in quantum theory (see the very detailed papers of Nikishov and Ritus [28]) and in classical theory for a point charge (the most recent such work is [26]).

Qualitatively, the result of the classical calculation is that the motion of the electron actually becomes relativistic for $b > 1$:

$$\bar{\gamma} = \frac{\bar{\mathscr{E}}}{mc^2} = (1-\beta^2)^{-1/2} \approx b, \quad 1-\beta \sim b^{-2}, \quad \beta = \frac{v}{c}. \tag{3.1}$$

The Lorentz forces become comparable to the electric forces in order of magnitude. The electron describes a closed figure-of-eight trajectory in the xz-plane (Fig. 5).

This result refers to a plane-polarized wave with $E_x = H_y \neq 0$. The amplitude of oscillations along both the x- and z-axis reaches the value $\lambda = c/\omega$ (in order of magnitude, but remaining smaller).

It is significant that the trajectory is closed. More precisely, there exists a coordinate system in which the electron merely oscillates, and in this system[5)] its motion is periodic and the trajectory is closed; this closure property is obviously not preserved if we transform to another system in uniform motion with respect to the first one.

The law of motion which is found can be used to determine the radiation of the electron. The radiation of a relativistic electron is characterized by the fact that this radiation is confined at each moment to a narrow cone about the instantaneous direction of motion of the electron (the central angle of the cone is of order γ^{-1}). With the turning of the trajectory, the radiation at a given point of observation starts and stops abruptly. The radiation spectrum therefore contains higher harmonics of the fundamental frequency with which the electron rotates[6)].

The total radiation for $b > 1$ is found to be larger than that given by the Thomson formulas by approximately a factor b^2. Consequently, the scattering cross section is also enhanced by the corresponding factor. The law describing the cross section $\sigma(b)$ as a function of b for scattering of a strong wave is given below; in this formula, σ with no indices or function sign is the Thomson cross section. Omitting numerical factors, we write

$$\sigma(b) = \sigma b^2 = \frac{e^6E^2}{m^4c^6\omega^2} = \frac{e^6E^2}{m^4c^8}\lambda^2. \tag{3.2}$$

We note in particular that in the ultra-relativistic limit the dependence of the radiation and scattering on the rest mass of the scattering particle does not vanish, but becomes stronger.

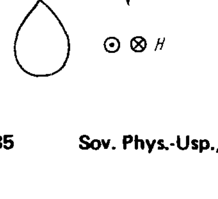

FIG. 5. Motion of an electron in a plane passing through the direction of a wave and the electric vector. The arrows at the side indicate the polarization of the wave and its direction.

The full problem of the interaction of an electromagnetic wave and an electron is solved by successive approximations: we are certain that the forward radiation of the electron in the direction of propagation of the strong wave, in interfering with the strong wave, weakens this wave precisely in accordance with the intensity scattered in other directions.

The forward radiation contains no higher harmonics! This fact [26,28] is non-trivial; it implies that a strong wave retains its sinusoidal form in passing through a rarefied plasma. The wave is weakened and altered in phase (in accordance with the real and imaginary parts of the dielectric constant), but there is no saw-tooth effect or tendency to form a "shock wave."

4. Radiation reaction and longitudinal acceleration. What change takes place in the next approximation, when allowance is made for the effect of the radiation reaction on the motion?

The main effect is a systematic acceleration of the electron in the direction of propagation of the wave. The origin of this effect is obvious: the electron, in absorbing energy from the strong wave, also absorbs a corresponding momentum, which is equal to the energy divided by c (more precisely, we should speak of the intensity W and the force F = W/c). The energy is not accumulated by the electron but is re-emitted. However, the re-emitted energy is not directly exactly forward. The momentum lost by the electron is proportional to $\overline{\cos\theta}$ and is less than the momentum gained by the electron. The average force acting on the electron is given by $\bar{F} = (W/c)(1 - \overline{\cos\theta})$. If allowance is made for this force, the trajectory of the electron is not closed in any coordinate system.

If $b \gg 1$ but is not too large ($b < \lambda/r_0$), i.e., $b < 10^{13}$ for centimeter radio waves or $b < 10^8$ for optical radiation, then an appreciable change occurs in the forward velocity after a large number of oscillations, and we can speak of the instantaneous coordinate system in which the electron only oscillates.

It is a remarkable fact that the parameter b characterizing the strength of a wave is Lorentz-invariant. In the system of an electron in motion (on the average) in the direction of the wave, the fields E_x and H_y are smaller than those in the initial rest system, but the frequency is also smaller in the same ratio! Denoting the average forward velocity (in units of c) by $\bar{\beta}$, we obtain $E_x \sim H_y \sim \omega \sim \sqrt{(1-\bar{\beta})/(1+\bar{\beta})}$, so that b = const ($\bar{\beta}$).

The electron always remains equally relativistic, its characteristic velocity of oscillations being constant, while the amplitude of these oscillations increases with time like ω^{-1}. The fact that b is constant means that there exists an automodelic solution for the motion of the electron in the field of a wave with allowance for the radiation reaction, this solution being similar to the solution[29] for a weak wave.

In a realistic situation, when we are dealing with not a single electron but a plasma, the overall average motion of the electrons is hindered by the longitudinal field arising from the separation of the electrons and nuclei, i.e., from the violation of electric neutrality.

5. Solution for a wave with circular polarization. However, there exists a formulation of the problem for which these collective effects strikingly facilitate an exact solution. Let us consider a strong wave with circular polarization and a longitudinal field, adjusted so as to eliminate the longitudinal drift of the electron[7)] [30].

It is then obvious from the symmetry of the problem that the electron undergoes circular motion with a period equal to the period of the wave. Thus, the functional form of the trajectory is determined, and it is not necessary to solve the differential equations! It remains to determine several numerical parameters: the radius of the orbit $R = v/\omega$ and the velocity of the electron v, as well as the shift in the phase of the rotation of the electron with respect to the phase of the wave, this shift being characterized, for example, by the angle φ between **E** and **R**, the radius vector of the electron (**E** is the electric field of the wave; the longitudinal component E_z is time-independent).

One obtains for these quantities a system of equations which follows from the energy and momentum balance. In essence, use is made of the well-known classroom method of determining the frequency of a circular pendulum by considering the centrifugal force, the tension in the string and gravity. One obtains finite equations, whereas a plane-polarized pendulum leads to a differential equation.

Let us return (from school) to the electron: we employ the well-known formulas for synchrotron radiation. The intensity of radiation, expressed in terms of the velocity and frequency, is

$$W=\frac{2}{3}\frac{e^2\omega^2}{c}\left(\frac{v}{c}\right)^2\left(1-\frac{v^2}{c^2}\right)^{-2}. \tag{5.1}$$

Moreover, it is obvious that the radiation is symmetric about the plane z = const.

The equations have the following form:

$$eEv\sin\varphi=W(v,\ \omega), \tag{5.2}$$

$$\frac{mv}{\sqrt{1-(v^2/c^2)}}\,\omega=eE\cos\varphi, \tag{5.3}$$

$$eE\frac{v}{c}\sin\varphi=eE_z. \tag{5.4}$$

Here (5.2) is the equation for the energy, (5.3) is the equation for the centrifugal force, and (5.4) is the equation for the Lorentz force (we have made use of the fact that $|\mathbf{H}| = |\mathbf{E}|$ and $\mathbf{H}\perp\mathbf{E}$), balanced by the longitudinal field. It is convenient to write the result in parametric form by expressing the dimensionless quantities

$$b,\ \beta=\frac{v}{c},\qquad \gamma=(1-\beta^2)^{-1/2},\qquad s=\frac{\sigma(b)}{\sigma},\qquad p=\frac{P(b)}{P}$$

as functions of an angle φ, where $0<\varphi<\pi/2$. The following explicit expressions are obtained[8)]:

$$b\beta\sin\varphi=\frac{2}{3}\frac{\beta^2\gamma^4 r_0}{\lambda\!\!\!^-},\qquad \gamma\beta=b\cos\varphi,\qquad s=\frac{W}{\sigma c}\frac{E^2}{4\pi}=\frac{\beta^2\gamma^4}{b^2},\qquad p=\frac{\beta}{b}\cos\varphi. \tag{5.5}$$

If we wanted to express all the quantities explicitly in terms of the natural parameter b characterizing the strength of the wave, we would have to solve a transcendental equation. If the radiation reaction is taken into account, there arises a fundamentally new circumstance: the solution as a whole (expressed in terms of dimensionless quantities) depends on the two initial dimensionless parameters b, characterizing the strength of the wave, and λ/r_0, characterizing the frequency. However, the limiting cases are described by the simple formulas

$$\varphi\ll 1,\quad b<\sqrt[3]{\frac{\lambda\!\!\!^-}{r_0}},\quad s=1+b^2,\quad p=\frac{1}{\sqrt{1+b^2}}, \tag{5.7}$$

$$\frac{\pi}{2}-\varphi\ll 1,\quad b>\sqrt[3]{\frac{\lambda\!\!\!^-}{r_0}},\quad s=\frac{3\lambda\!\!\!^-}{2r_0}\frac{1}{b},\quad p=\left(\frac{3}{2}\frac{\lambda\!\!\!^-}{r_0}\right)^{1/4}b^{-7/4}. \tag{5.8}$$

The first region yields results similar to those for a plane-polarized wave, but by a much easier method. The equations of motion of the electron in a strong wave are non-linear. Consequently, if the problem is solved for a plane-polarized wave, it is not possible by a direct superposition of solutions to arrive at the solution for a wave with circular polarization. Nor is it possible to go the other way. These two solutions together demonstrate that the basic properties of the solutions are stable—the increase of the scattering cross section in a strong wave and the absence of higher harmonics in the wave scattered in the forward direction apply to both cases. In the first region, λ/r_0 plays no role.

But the solution for circular polarization also takes into account the radiation reaction; this effect becomes dominant in the second region (φ is close to $\pi/2$). The maximum intensity which the electron can extract from a given field is eEc. Since the flux of energy of a wave is $cE^2/4\pi$, the scattering cross section cannot exceed $4\pi e/E$. It is this limit, which is inversely proportional to the amplitude of the wave, which is reached in the second region.

A wave with circular polarization[9)] also admits an exact solution for an electron in a longitudinal magnetic field; in this case, one can readily study the resonance behavior at the gyroscopic frequency in the non-linear region[31]. A circularly polarized wave combines properties which it would not seem possible to combine. It has a definite wavelength and frequency. In addition, the motion of an electron in such a wave is independent of time—all scalar quantities, such as the velocity or the angle between the velocity and the field, are constant.

The energy density and other scalar quantities characterizing the wave itself are constant in space. As a result, it is possible to construct a strictly uniform solution throughout all of infinite space, just as in the case in which gravitational effects are taken into account. In other words, there exists a spatially uniform combined solution of the equations of the general theory of relativity and Maxwell's equations corresponding to a circularly polarized wave[32].

If we took plane polarization instead of circular polarization, the uniform solution would only be an approximate one: the energy density in a plane wave depends on the coordinates in the form $\cos^2 kz = (1+\cos 2kz)/2$.

6. Estimate of the strength of a wave for a pulsar. From the hymn in honor of waves with circular polarization, let us return to concrete astrophysics. Consider a pulsar having a period 0.03 sec, $\omega\approx 200\ \mathrm{sec}^{-1}$ and a magnetic moment 3×10^{30} (dimensions 10^6 cm, field 3×10^{12} G at the surface, and moment $\mu = HR^3$ in order of magnitude). If the moment is perpendicular to the axis of rotation, such a pulsar radiates $W = (2/3)\mu^2\omega^4/c^3 = 3 \times 10^{38}$ erg/sec.

Let us determine the field E of the wave and the characteristic number b at a distance r from the pulsar. Averaging over the angle, we find $E = 10^{14}/r$ in CGSE units and $b = 10^{19}/r$ (since $E_c = mc\omega/e = 10^{-5}$ at $\omega = 200$).

The distance beyond which the field can be regarded as wave-like is $c/\omega = 1.5 \times 10^8$ cm. Consequently, the maximum value of the parameter b is approximately 10^{11}. The strong-wave zone extends to a distance of order $10^{18} = 0.3$ parsec, where the value $b = 1$ is attained.

On the axis of rotation, the wave is circularly polarized; the results outlined above then hold literally, and not merely in spirit. Let us also determine the value b_c at which the transition occurs from the dominant role of the radiation reaction (for $b > b_c$) to the possibility of allowing for the radiation as a small correction to the equations of motion (for $b < b_c$). We find $b_c = \sqrt[3]{\lambda/r_0} = 7 \times 10^6$ for $\omega = 200$ and $\lambda = 1.5 \times 10^8$. Thus, the radiation reaction dominates in the wide range $10^7 < b < 10^{11}$, $10^8 < r < 10^{12}$. However, b_c is not Lorentz-invariant. For a plasma escaping with relativistic velocity, λ is effectively increased and $r_0 = \text{const}$, in contrast with the parameter b, which remains unchanged.

In this review, we cannot analyze the concrete situation in pulsars in greater detail. We confine ourselves to the statement that the electrons and protons are subjected to a strong electromagnetic wave, in which the Thomson formulas are inapplicable. The selection of energy from the wave becomes larger (than in the Thomson theory), the energy is re-emitted in higher harmonics, and in a large region the radiation reaction has a profound influence on the motion of the electrons. The forces of longitudinal acceleration are large.

The theory outlined above is essential for an understanding of the ejection of relativistic particles by a pulsar. The long-wave radio emission of pulsars, by means of which pulsars were discovered, can apparently be explained only by collective effects and not directly by the theory of the strong wave.

We quote several references from the literature[33–35,60] (for the reader who wishes to delve into the theory of pulsars).

III. THE KINETIC EQUATION OF A PHOTON GAS

7. Formulation of the problem. Occupation numbers. Let us consider the problem of the evolution of a free electromagnetic field, which is on the average uniform in space. By the evolution, we mean the variation with time. This formulation of the problem applies directly to the hot universe[7,36–39], in which the uniformity of the electromagnetic field follows directly from observations. To be more precise, the observations prove only the isotropy of the radiation, but it is known that a strong nonuniformity would also destroy the isotropy, so that we infer from the observed isotropy that there is at least large-scale uniformity. The evolution of the universe is associated with the general expansion, in the course of which there is also a variation of the temperature, and with the occurrence of perturbations leading to a macroscopic motion of the plasma. The evolution does not result in a trivial propagation of photons in a space with a given metric (in the language of the general theory of relativity) because the radiation field contains free electrons, i.e., we are dealing with a plasma.

In many problems, the volume occupied by the plasma is not too large and one observes radiation emitted from the plasma, i.e., both the nonuniformity and the anisotropy of the radiation are important. Nevertheless, a given plasma cloud can be characterized approximately by a certain averate time during which the photons remain in the cloud, and one can solve the problem of the evolution of uniform radiation during this time. This formulation of the problem greatly reduces the number of parameters and variables and makes it possible to find the general regularities. We assume that the electromagnetic field is random and has a continuous spectrum.

We shall characterize a random free electromagnetic field by the distribution of the photon occupation numbers $n_i(\mathbf{k})$. The wave vector $\mathbf{k}$ also characterizes the direction of propagation (the unit vector is $\mathbf{r} = \mathbf{k}/|\mathbf{k}|$) and frequency of the wave, $\omega = c|\mathbf{k}|$ and hence also the energy of a single photon, $\hbar\omega$. The index i, which takes two values, characterizes the two orthogonal polarization states, for example left-handed and right-handed circular polarization.

Macroscopic quantities such as the density of photons, N, and the energy density per unit volume, $\mathscr{E}$, or the spectral density of radiation energy can easily be expressed in terms of n:

$$N = \sum_{i=1,2} \frac{1}{(2\pi)^3} \int n_i\, d^3k, \quad \mathscr{E}(\text{erg/cm}^3) = \sum_{i=1,2} \frac{1}{(2\pi)^3} \int n_i \hbar\omega\, d^3k, \tag{7.1}$$

$$F_{\omega,\mathbf{r}} = \sum_{i=1,2} \frac{n_i \hbar\omega^3}{(2\pi c)^3}. \tag{7.2}$$

The assignment of $n_i(\mathbf{k})$ obviously leads to a partial loss of information about the phase. For example, the statement $n_1(\mathbf{k}) = n_2(\mathbf{k})$ does not permit us to distinguish an unpolarized wave from a plane-polarized wave. However, we shall not dwell upon these details here; we shall return to them in part later, in connection with the situation $n \gg 1$, when the quantum-mechanical uncertainty in the phase is small, and, in particular, in connection with the theory of induced scattering of (spectrally and spatially) narrow lines.

8. Thomson scattering. The equations for Compton scattering in the lowest approximation, with no allowance for the change in the photon energy, have the form

$$\frac{\partial n_1(\mathbf{r},\omega)}{\partial t} = \rho c\sigma \left\{ -n_1(\mathbf{r},\omega) + \frac{1}{4\pi} \int \left[\alpha(\theta)\, n_1(\mathbf{r}',\omega) + \beta(\theta)\, n_2(\mathbf{r}',\omega)\right] dr' \right\}. \tag{8.1}$$

Instead of the 3-vector $\mathbf{k}$, we have taken the argument here to be the (unit) 2-vector having direction $\mathbf{r}$ and the frequency $\omega = c|\mathbf{k}|$. The integration is carried out only with respect to $\mathbf{r}$ or, more precisely, with respect to $\mathbf{r}'$, and the angle between $\mathbf{r}$ and $\mathbf{r}'$ is denoted by θ; $\alpha(\theta)$ is the differential cross section for scattering with no change in the polarization (i.e., left-handed to left-handed or right-handed to right-handed polarization), and $\beta(\theta)$ is the same but with a change of polarization. Obviously[10)]

$$(4\pi)^{-1} \int (\alpha + \beta)\, dr' \equiv 1, \quad \alpha + \beta = \frac{3}{16\pi}(1 + \cos^2\theta). \tag{8.2}$$

In the overall factor in Eq. (8.1), ρ is the density of electrons, σ is the Thomson cross section, and c is the velocity of light. The product $\rho c\sigma t = \tau$ is the so-called optical thickness for Thomson scattering, i.e., the average number of scattering events for each photon during the time t. With the Thomson cross section, τ is independent of the frequency. The stationary solution of Eq. (8.1) and the analogous equation for n_2 is $n_1 = n_2 = n(\omega)$, where $n(\omega)$ is an arbitrary (!) function of the frequency. It is easy to see that during a time interval corresponding to the thickness $\tau = 1$ there is a decrease by approximately a factor of two or e in the difference $n_1 - n_2$, as well as in the angular dependence of n_1 and n_2 (both with some constant ω). This result is readily obtained by expanding n_1 and n_2 in spherical harmonics. All the harmonics are damped out at roughly the same rate, since α and β have a weak angular dependence. We note in particular that in this approximation the induced scattering drops out identically from the considerations. In fact, if there are two beams with $n_i(\mathbf{r}_1, \omega)$ and $n_k(\mathbf{r}_2, \omega)$, where ω is the same for both beams and i and k are the same, for example, then in the expression for the transition from the first beam to the second we have

$$\frac{dn_i}{dt} = -\rho c\sigma \{\alpha n_i(\mathbf{r}_1, \omega)[1+n_k(\mathbf{r}_2, \omega)] - \alpha n_k(\mathbf{r}_2\omega)[1+n_i(\mathbf{r}_1, \omega)]\}. \quad (8.3)$$

The products $n_i n_k$ obviously cancel, since the angle θ and the function $\alpha(\theta)$ are the same for the transition $n_i(\mathbf{r}_1) \to n_k(\mathbf{r}_2)$ and the inverse transition $n_k(\mathbf{r}_2) \to n_i(\mathbf{r}_1)$.

This result should be borne in mind in order to compare it with the role of induced scattering with allowance for the change in frequency.

9. Integral equation for the variation of the spectrum. The change in frequency in an individual scattering event is small, since we shall be considering electrons with a nonrelativistic temperature and photons with an energy which is small in comparison with mc^2. Therefore appreciable effects take place when the interaction time corresponds to $\tau \sim (\hbar\omega/mc^2)^{-1} \gg 1$ or $\tau \sim (kT/mc^2)^{-1} \gg 1$. According to the results of the preceding section, it follows from this that for practically the whole time (except the first few units of τ) the radiation is isotropic and unpolarized, even if it did not have these properties initially. Consequently, for the main period of time we are considering the case $n_i = n_k = n(\omega, \tau)$, i.e., we have a single function of two scalars: the frequency ω and the thickness τ (by introducing the thickness instead of the time, we can get rid of the factors $\rho\sigma c$ in all the equations). For the convenience of astrophysicists, we shall go over from the angular frequency ω to the frequency in hertz. The normalization factor in the expression for the density of unpolarized photons in an isotropic distribution will be denoted by a single symbol A:

$$N = \frac{8\pi}{c^3}\int n\nu^2 d\nu = A\int n\nu^2 d\nu. \quad (9.1)$$

In an exact formulation of the integral equation, a major role is played by the "kernel" $K(\nu_i, \nu_f)$—the probability of a transition of a photon with initial frequency ν_i into a single cell of phase space with frequency ν_f.

The probability is normalized to the scattering cross section, so that $A\int K(\nu_i, \nu_f)\nu_f^2 d\nu_f \equiv 1$. We shall henceforth not write the indices i (initial) and f (final), bearing in mind, for example, that $K(\nu, \nu')$ involves $\nu = \nu_i$ in the first position and $\nu' = \nu_f$ in the second. The function $K(\nu, \nu')$ depends on the electron temperature T as a parameter: $K(\nu, \nu', T)$; for brevity, we shall sometimes omit this dependence on T. We shall also write $n(\nu)$ instead of $n(\nu, \tau)$, dropping the quantity τ, which is the same in all terms of an equation.

If no allowance is made for induced scattering, i.e., for the Bose factors $(1 + n_f)$ in the scattering probability, the kinetic equation would have the form

$$\frac{\partial n(\nu)}{\partial\tau} = -n(\nu)A\int K(\nu, \nu')\nu'^2 d\nu' + A\int K(\nu', \nu)n(\nu')\nu'^2 d\nu' \equiv -n(\nu) + A\int K(\nu', \nu)n(\nu')\nu'^2 d\nu'. \quad (9.2)$$

When induced scattering is taken into account, both terms are modified:

$$\frac{\partial n(\nu)}{\partial\tau} = -n(\nu)A\int K(\nu, \nu')[1+n(\nu')]\nu'^2 d\nu' + [1+n(\nu)]A\int K(\nu', \nu)n(\nu')\nu'^2 d\nu' = -n(\nu) + A\int K(\nu', \nu)n(\nu')\nu'^2 d\nu' + n(\nu)A\int [K(\nu', \nu) - K(\nu, \nu')]n(\nu')\nu'^2 d\nu'. \quad (9.3)$$

The formulas contain the limiting case of scattering with no change in frequency:

$$K(\nu', \nu) = \frac{\delta(\nu-\nu')}{A\nu^2}. \quad (9.4)$$

In this case, $\partial n(\nu)/\partial\tau = 0$, as was to be expected. The induced term vanishes not only for K equal to a δ-function, but for any symmetric $K_S(\nu, \nu') = K_S(\nu', \nu)$.

The function $K(\nu, \nu', T)$ is rather complicated; it is not easy to write it in an approximate form which preserves all the fundamentally important general properties. The small value of the change in frequency on scattering implies that K differs little from a δ-function. But this small difference is essential for the evolution of the spectrum. At T = 0 only a decrease in frequency is possible, in fact by not more than the amount $\Delta\nu_m = 2h\nu^2/mc^2$. Thus, $K_0(\nu, \nu')$ is non-zero only in the range $0 \le \nu - \nu' \le \Delta\nu_m$ (the index 0 on K means T = 0). At a finite electron temperature, there is also the Doppler effect, depending on the motion of the electrons. The corresponding function $K_T(\nu, \nu')$ has an approximately Gaussian structure $\sim\exp[(\nu-\nu')^2/2\Delta^2]$ with width $\Delta \sim \sqrt{\overline{v^2}/c^2} \sim \sqrt{kT/mc^2}$. The complete function $K(\nu, \nu', T)$ is neither a sum nor a product of K_0 and K_T. At small T, the complete function is a convolution of K_0 and K_T. However, when we later turn from $K(\nu, \nu', T)$ to the calculation of physical effects, these effects will naturally prove to be sums of zero-temperature and finite-temperature effects.

10. The Kompaneets differential equation. As there is little change in frequency for each scattering, it is natural to expect that $\partial n(\nu)/\partial\tau$ will depend only on the values of $n(\nu')$ in neighboring regions of the spectrum, where $|\nu' - \nu| \ll \nu$. Consequently, it will be possible to express $\partial n(\nu)/\partial\tau$ in terms of $n(\nu)$, $n'(\nu) = \partial n/\partial\nu$, $n''(\nu) = \partial^2 n/\partial\nu^2$, ..., i.e., as a function of the occupation number and its derivatives at the same value of ν. The function $n(\nu)$ is required to be smooth: in order of magnitude, we must have $n \sim \nu\partial n/\partial\nu \sim \nu^2\partial^2 n/\partial\nu^2$. Mathematically, the transition from the exact integral equation to the differential equation for $\partial n/\partial\tau$ is accomplished by means of the formulas

$$A\int K(\nu, \nu')n(\nu')\nu'^2 d\nu' = a_0 n(\nu) + a_1\frac{\partial n}{\partial\nu} + a_2\frac{\partial^2 n}{\partial\nu^2} + \ldots,$$
$$a_0 = A\int K(\nu, \nu')\nu'^2 d\nu', \quad a_1 = A\int K(\nu, \nu')(\nu' - \nu)\nu'^2 d\nu', \quad (10.1)$$
$$a_2 = \frac{A}{2}\int K(\nu, \nu')(\nu' - \nu)^2\nu'^2 d\nu'.$$

However, the obvious physical requirements on the equation actually impose such strong constraints on the parameters that the equation can be derived without explicitly writing down the cumbersome function $K(\nu, \nu', T)$.

This method was elegantly and systematically exploited by A. S. Kompaneets in 1956[1], in a paper containing the first lucid formulation of the problem of how the equilibrium spectrum of radiation is established in a rarefied plasma.

Let us first consider the case of cold electrons and small occupation numbers, when the n^2 terms can be neglected. The coefficient a_1 (taken with the + sign) is the average decrease in frequency for a single scattering; for scattering at an angle θ, it is equal to $\Delta\nu = \Delta\nu_m(1-\cos\theta)/2$, i.e., it has the value[11)] $a_1 = \overline{\Delta\nu} = h\nu^2/mc^2$. The coefficient a_2 is equal to the mean squared change in frequency. In our case,

$$a_2 = \overline{(\Delta\nu)^2} \sim (\overline{\Delta\nu})^2 \sim \left(\frac{h\nu}{mc^2}\right)^2 \nu^2 \ll \nu\overline{\Delta\nu}. \quad (10.1a)$$

Therefore a_2 can be neglected.

Thus, we are left with the equation

$$\frac{\partial n}{\partial\tau} = (a_0-1)\,n + \frac{h\nu^2}{mc^2}\frac{\partial n}{\partial\nu}. \quad (10.2)$$

Let us make use of the fact that the density of photons is conserved in the scattering. The equation must satisfy the condition

$$\frac{dN}{d\tau} = A\frac{d}{d\tau}\int n\nu^2\,d\nu = A\int\frac{\partial n}{\partial\tau}\nu^2\,d\nu \equiv 0.$$

If this relation is to hold for any $n(\nu)$, the right-hand side of Eq. (10.2) must have the form

$$\frac{\partial n}{\partial\tau} = \frac{1}{\nu^2}\frac{\partial}{\partial\nu}\,q\left(\nu, n, \frac{\partial n}{\partial\nu}\right). \quad (10.3)$$

We say that the divergence structure of the equation ensures that a conservation law holds: we can think of q as the flux of photons through a spherical surface in phase space corresponding to a given frequency ν. Comparing this with what we already know from Eq. (10.1), we obtain

$$q = f(\nu)\,n, \quad \frac{1}{\nu^2}f(\nu)\frac{\partial n}{\partial\nu} = \frac{h\nu^2}{mc^2}\frac{\partial n}{\partial\nu}, \quad f = \frac{h\nu^2}{mc^2},$$
$$\frac{\partial n}{\partial\tau} = \frac{h}{mc^2}\frac{1}{\nu^2}\frac{\partial}{\partial\nu}\nu^4 n. \quad (10.4)$$

It is worth noting how the general principles have enabled us to avoid a difficult calculation of a_0 from the function $K_0(\nu, \nu')$: this difficulty is due to the fact that we must calculate the small difference $a_0 - 1 = 4h\nu/mc^2$, for which the approximation $K \sim \delta(\nu-\nu')$ is inadequate.

Equation (10.4) describes a very specific physical situation, from which we can draw conclusions regarding the propagation of x-rays in a cold plasma. Here we shall proceed with the derivation of the general equation. Let us turn to the quadratic term describing induced scattering, keeping the electrons cold as before. It follows from the integral expression that

$$\left(\frac{\partial n}{\partial\tau}\right)_{\text{ind}} = b_0 n^2 + b_1 n\frac{\partial n}{\partial\nu},$$

where b_0 and b_1 are functions of ν; moreover, $b_1 = 2h\nu^2/mc^2 = 2a_1$. Applying the divergence principle, we finally obtain

$$\frac{\partial n}{\partial\tau} = \frac{h}{mc^2}\frac{1}{\nu^2}\frac{\partial}{\partial\nu}\nu^4(n+n^2). \quad (10.5)$$

Many of the applications of the theory involve this equation and its limiting form for $n \gg 1$, when allowance is made for only induced scattering:

$$\frac{\partial n}{\partial\tau} = \frac{h}{mc^2}\frac{1}{\nu^2}\frac{\partial}{\partial\nu}\nu^4 n^2. \quad (10.6)$$

Let us turn to the case of hot electrons, assuming, however, that $kT \ll mc^2$ and retaining the higher-order terms in the expansion in the small parameter kT/mc^2. The shift in frequency in an individual scattering event is $\sim\nu v/c$, and its square is $\nu^2v^2/c^2 \sim \nu^2 kT/mc^2$. Thus, the Doppler broadening of a line gives a certain large value of the coefficient a_2. We recall that this corresponds to a term proportional to $\alpha(kT/mc^2)\nu^2\partial^2 n/\partial\nu^2$ in the equation for $\partial n/\partial\tau$. However, the frequency shift, i.e., $\overline{\Delta\nu}$, does not contain a term $\nu v/c$ have both signs, as the frequency increases and decreases, and the term of order $\nu v/c$ vanishes when the average is taken. The symmetry of $K_T(\nu, \nu')$ also implies that there is no temperature-dependent induced contribution to the scattering which is quadratic in n.

The equation which takes into account the temperature must satisfy a thermodynamic principle: the equilibrium Planck distribution of photons should remain unchanged when interacting with electrons having the same temperature. This means that $\partial n/\partial\nu \equiv 0$ for $n = (e^{h\nu/kT}-1)^{-1}$. Without any calculations, this thermodynamic principle together with the principle of conservation of photons (the divergence principle) fix the factor α accompanying $\partial^2 n/\partial\nu^2$ and the remaining terms proportional to the temperature. We finally obtain the Kompaneets equation:

$$\frac{\partial n}{\partial\tau} = \frac{h}{mc^2}\frac{1}{\nu^2}\frac{\partial}{\partial\nu}\nu^4\left(\frac{kT}{h}\frac{\partial n}{\partial\nu} + n + n^2\right). \quad (10.7)$$

For a constant temperature of the electrons, it is convenient, following the author of the equation, to replace the frequency by a dimensionless photon energy x and to change the variable which characterizes the interaction time:

$$x = \frac{h\nu}{kT}, \quad y = \tau\frac{kT}{mc^2}. \quad (10.8)$$

In the new variables, the equation contains no parameters:

$$\frac{\partial n}{\partial y} = \frac{1}{x^2}\frac{\partial}{\partial x}x^4\left(\frac{\partial n}{\partial x} + n + n^2\right). \quad (10.9)$$

In the general case, there are the effects of the production of new photons and the absorption of photons in addition to scattering effects. If the production and absorption are due to a plasma with a given temperature, their contribution can be written in the form

$$\left.\frac{\partial n}{\partial y}\right|_{e,a} = B(x)(1+n) - nB(x)e^x \quad (10.10)$$

This equation takes into account the fact that, in addition to the spontaneous emission B(x), there is also induced emission (the term nB(x)); it also takes into account the thermodynamic relation between the emission B(x) and absorption $nB(x)e^x$. The quantity B(x) is proportional to the square of the electron density. In most of the problems involving a rarefied plasma which we consider below, B(x) can be completely neglected, or neglected everywhere except in a small region $x \ll x_0$, as a result of the fact that B, being small, grows like x^{-3} as $x \to 0$.

A brief historical survey. The earliest work involving a partial (without induced processes) formulation of the problem of the evolution of the spectrum in the presence of Compton scattering was due to Dirac[40]. Kompaneets carried out his investigation in 1949; a major contribution to this work was due to the prematurely deceased young gifted physicist S. D'yakov. The work came to the notice of the author of this review. For reasons beyond the control of the participants in this work, its publication was delayed until 1956. The astrophysical significance of the work remained unclear until 1964. Weyman[4], who was unaware of the work of Kompaneets, arrived at the same equation much later, in 1965. The priority of Kompaneets is now universally recognized.

11. Properties of the Kompaneets equation. Two of the most important properties (namely, the fact that $dN/dt = 0$ and the fact that $P(x) = (e^x - 1)^{-1} = \text{const}\,(y)$ is a solution) were used in deriving the equation and hence are satisfied identically. Here $P(x)$ is the equilibrium Planck radiation function corresponding to the electron temperature; we also use the notation $P(\nu) = (e^{h\nu/kT} - 1)^{-1}$. We naturally have in mind the Kompaneets equation (10.7) without absorption and emission. However, $P(x) = \text{const}\,(y)$ is not the only solution: it is easy to see that the Bose-Einstein expression $BE(x) = (e^{x+\mu} - 1)^{-1}$, which differs from the Planck expression by the presence of the chemical potential μ, also reduces the quantity $(\partial n/\partial x + n + n^2)$ to zero. This result is not unexpected: $BE(x)$ solves the problem of thermodynamic equilibrium of a specified number of photons in a specified volume at a given temperature. Since N remains constant in a scattering process, we are justified in asking what frequency distribution the photons will have (i.e., what form the function $n(\nu)$ or $n(x)$ will have for a given electron temperature and a given number N). The Planck formula gives a definite value of N for a given T and hence does not answer this question. We require one more parameter, and μ is such a parameter. The chemical potential μ can be expressed in terms of T and N by means of the equation

$$\int_0^\infty (e^{x+\mu} - 1)^{-1} x^2\,dx = \frac{\text{const}\cdot N}{T^3}.$$

Only $P(x)$ and $BE(x)$ are equilibrium solutions which reduce the photon flux q in ν-space to zero.

There also exist the stationary but non-equilibrium solutions $q = \text{const} \neq 0$, $\partial q/\partial\nu = 0$, $\partial n/\partial\tau = 0$; however, these solutions obviously do not play such a fundamental role as the equilibrium solutions. The stationary solutions with $q \neq 0$ exist only because of an influx of photons at $\nu = \infty$ and an outflow of photons at $\nu = 0$ (or an influx at $\nu = 0$ and an outflow at $\nu = \infty$).

The Kompaneets equation enables us to determine the rate of change of the density of radiation energy for a given spectrum and a given radiation temperature. It is convenient to revert to the dimensioned equation[12]

$$\frac{d\mathscr{E}}{dt} = \frac{d}{dt} A \int h\nu n(\nu)\nu^2\,d\nu = Ah \int \frac{\partial n}{\partial t}\nu^2\,d\nu = \frac{\rho c\sigma}{mc^2}\left(4kT\mathscr{E} - Ah^2 \int_0^\infty (n+n^2)\nu^4\,d\nu\right). \tag{11.1}$$

The energy gained by the radiation is obviously lost by the electrons, so that we can write the following equation for the electron temperature:

$$\frac{3}{2}k\rho\frac{dT}{dt} = -\frac{d\mathscr{E}}{dt}. \tag{11.2}$$

In particular, the preceding equation enables us to calculate the stationary electron temperature (from the condition $dT/dt = 0$):

$$T_{st} = \frac{Ah^2 \int (n+n^2)\nu^4\,d\nu}{4k\mathscr{E}} = \frac{h^2}{4kh}\frac{\int (n+n^2)\nu^4\,d\nu}{\int n\nu^3\,d\nu}. \tag{11.3}$$

Equation (11.3) makes it possible to determine T_{st} in an isotropic and unpolarized radiation field, but with an arbitrary spectrum. In the special case of a Planck or Bose-Einstein spectrum having a definite radiation temperature T_r, this equation gives $T_{st} = T_r$ identically.

IV. INDUCED SCATTERING AND THE CLASSICAL THEORY OF THE INTERACTION OF WAVES WITH ELECTRONS

12. Evolution of the spectrum and the Bose condensation. Let us make use of the Kompaneets equation to study the evolution of the spectrum in the limiting case in which the quadratic term (n^2) describing induced scattering is dominant.

The equation reduces to

$$\frac{\partial n}{\partial\tau} = \frac{h}{mc^2}\frac{1}{\nu^2}\frac{\partial}{\partial\nu}\nu^4 n^2. \tag{12.1}$$

We introduce the quantity $f = hn\nu^2$. The equation for the new function $f(\nu, \tau)$ has the form

$$\frac{\partial f}{\partial\tau} = \frac{2f}{mc^2}\frac{\partial f}{\partial\nu}. \tag{12.2}$$

But such an equation can be solved in terms of characteristics; this means that it can be subjected to the further transformation

$$\frac{df}{d\tau} = 0 \quad \text{along} \quad \frac{d\nu}{d\tau} = -\frac{2f}{mc^2}. \tag{12.3}$$

The solution $\nu(f, \tau)$, given implicitly, has the form

$$\nu(f, \tau) = \nu_0(f) - \frac{2f}{mc^2}\tau. \tag{12.4}$$

The evolution of the spectrum corresponding to these equations is very easy to visualize. Let us construct the spectrum in the $f-\nu$ coordinates at the instant $\tau = 0$ (Fig. 6a). Each point of the curve moves to the left with a constant, time-independent velocity. However, this velocity is different for different points—it is proportional to the ordinate of a point.

Thus, for each point of the initial curve $f_0(\nu)$, it is easy to determine the instant at which it intersects the vertical axis. As a result of induced scattering, photons of all species (excuse me—all colors, all wavelengths) "harmoniously" reduce their frequency and hence their energy, losing it to the electrons. Each group (with an initial frequency in a given interval from ν_0 to $\nu_0 + d\nu_0$) moves independently (in this approximation!) and disappears after a certain interval of time, when $\nu \to 0$ (Fig. 6b). Now what are zero-frequency photons?! Some mechanisms of genuine absorption are bound to appear as $\nu \to 0$, so that the rhetorical question about a photon with $\nu = 0$ no longer arises.

But the situation is not so simple for any form of the initial spectrum.

Under certain conditions, we can expect a spectrum f which has a bend (Fig. 7). In that case, even before the Bose condensation, the formal application of the rule governing the evolution of the spectrum leads to the formation of a characteristic three-valued structure or "overspill"[9]. This phenomenon is completely analogous to the formation of a shock wave in gas dynamics or the

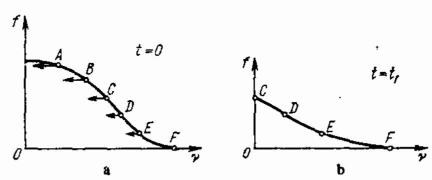

FIG. 6. a) The initial photon spectrum (on the vertical axis we plot the product $n\nu^2$, which is proportional to F_ν/ν, where F_ν is the spectral energy density); b) Bose condensation of photons whose frequency vanishes.

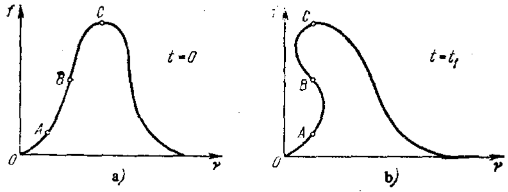

FIG. 7. a) An initial photon spectrum with a bend; b) the formation of a region of triple-valuedness during the evolution of the spectrum in a).

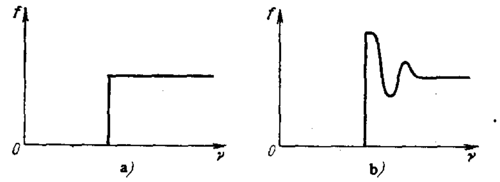

FIG. 10. a) An initial photon spectrum with a discontinuity; b) the formation of quasi-lines during the evolution of the spectrum in a).

overspill of waves in the sea onto a shoal. The formation of "shock waves" in the spectrum has been noted previously in connection with the study of plasma turbulence, i.e., the study of the spectrum of random plasma oscillations [41,42].

But what is the structure and subsequent fate of a shock wave?

To study this question, we must return to the integral equation. We recall that the Kompaneets differential equation was derived under the assumption that the spectrum is smooth, and this smoothness of $f(\nu)$ is destroyed when a shock wave is produced. We recall the expression for induced scattering:

$$\frac{\partial n(\nu)}{\partial \tau} = \mathrm{const}\cdot n(\nu)\int A(\nu, \nu')\, n(\nu')\, \nu'^2\, d\nu', \quad A(\nu, \nu') = K(\nu, \nu') - K(\nu', \nu). \tag{12.5}$$

The kernel A is antisymmetric (the dependence of A on the difference $\nu - \nu'$ is shown qualitatively in Fig. 8), and the characteristic width of A is of order

$$|\nu - \nu'| \sim \nu\frac{v}{c} \sim \nu\sqrt{\frac{kT}{mc^2}}. \tag{12.6}$$

This quantity represents a sort of free path length of the photons in the scale of frequencies. It would be natural to conclude that the structure of a shock wave in the photon spectrum in momentum space is similar to the structure (in ordinary coordinate space) of a shock wave in a gas with a given path length. However, it turns out [10] that these two structures have nothing in common. The integral equation—or, more properly, the induction process itself—is such that, instead of the smoothing of the S-type wave in a gas, one finds an accumulation of photons within a narrow frequency range, an oscillatory structure of $n(\nu)$ and quasi-lines in the photon spectrum (Fig. 9).

Let us begin with the simplest case: suppose that initially $f = 0$ for $\nu < \nu_0$ and that a curve having a discontinuity is specified at $t = 0$ (Fig. 10a).

The number of photons which get into a given cell is proportional to the number of photons which are already in that cell: we are neglecting the "spontaneous" unity in the Bose factor $(n + 1)$, and the "induced" n remains. In that region ($\nu < \nu_0$) in which $n = f = 0$ at the initial instant, the condition $n = f = 0$ will be preserved at all times. But to the right, in the region $\nu > \nu_0$, where $n \neq 0$ and $f \neq 0$, there is a flux of photons moving towards lower frequencies[13)].

These phonons must accumulate near the discontinuity, for $\nu_0 < \nu < \nu_0 + \Delta\nu$, and in the course of time must form a pronounced maximum there—a quasi-line (Fig. 10b).

We can consider a finite discontinuity in the initial conditions:

$$n = n_1,\ \nu = \nu_0 - 0,\ n = n_2,\ \nu = \nu_0 + 0,\ n_2 > n_1.$$

It follows from the integral equation that the discontinuity is not washed away: $n_2/n_1 = \mathrm{const}(\tau) > 1$. In fact, the integral $I(\nu, \tau) = \int A(\nu, \nu')n(\nu', \tau)\nu'^2 d\nu'$ is a smooth function of ν even if n in the integrand is discontinuous, and it follows from the equation that the rate of growth $\partial \ln n/\partial\tau$ depends only on I and hence is the same on both sides of the discontinuity.

We have made a detailed study (including numerical analyses) of the evolution of an initially smooth $n(\nu)$ or $f(\nu)$. We have considered a case in which a simple differential equation predicts the formation of a shock wave and is subsequently no longer applicable. The solution of the integral equation leads to the formation of several quasi-lines, whose number and amplitude grow with time. So far, the possible observation of quasi-lines remains an open question. If induced scattering is to dominate, a high effective radiation temperature is required. It is natural to suppose that the source of radiation is a turbulent and possibly magnetized plasma and that the scattering is geometrically separated from the radiation, i.e., that it takes place in different shells of the plasma, which are colder and more quiescent, practically in equilibrium and non-magnetized. But in that case there arise problems regarding the angular distribution of radiation: for induced scattering, there is no tendency towards isotropy!

13. The electron temperature for induced scattering. So far, we have been concerned almost exclusively with the evolution of the radiation spectrum in the presence of thermal electrons. There is another problem which is no less fundamental: the evolution of the energy spectrum of the electrons in a given radiation field. The answer is trivial if the radiation is the equilibrium Planck radiation corresponding to a definite radiation temperature T_r. The electrons will clearly acquire this temperature: $T \to T_r$. Since the average photon energy in the Planck distribution is $\sim 2.7kT_r$, we can also say that the average energy of an electron in equilibrium is of the same order of magnitude as the average energy of each photon.

However, the problem becomes non-trivial if the radiation spectrum is not an equilibrium spectrum. In astrophysical applications, we frequently encounter a

FIG. 8

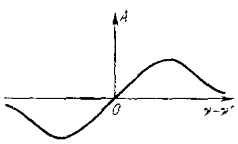

FIG. 9

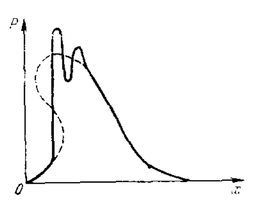

FIG. 8. The kernel for induced scattering.
FIG. 9. Structure of a shock wave formed instead of the triple-valued region shown by the dashed curve.

situation in which the effective temperature is very high, but only in the low-frequency region. There is an effective cut-off in the spectrum at a frequency ν_0 which is many times smaller than the frequency $\nu = kT_r/h$ at which the quantum character of light shows up.

A high effective temperature certainly indicates the presence of radiating "units" with a large energy. These "units" may be relativistic electrons or clusters with large effective charges (see the literature on pulsars[33-35]). At low frequencies, the absorption of radiation by these same "units" is important, and the radiation temperature does not exceed the average energy of each unit. At high frequencies, the system is transparent and the absorption is small, but the radiation of the "units" is also small. The radiation is exponentially small, but the exponent here is not of Maxwell-Boltzmann origin; the exponent comes from a convolution of a rapidly varying function (characterizing a high-frequency wave) and a smooth function characterizing the dimensions or the trajectory of the radiating "unit."

Thus, we come back to the problem of scattering for an unusual radiation field. In astrophysics, particularly in "plasma" astrophysics, a typical radiation field has $n = kT_{r1}/h\nu \gg 1$ for $\nu < \nu_0$ and $n \ll kT_{r1}/h\nu$ for $\nu > \nu_0$, with $h\nu_0 \ll kT_{r1}$.

Accordingly, the total density of radiation in such a field is much less than the equilibrium value. Omitting dimensionless factors, we write

$$\mathscr{E}\,(\text{erg/cm}^3) \approx kT_{r1}\frac{\nu_0^3}{c^3} \ll aT_{r1}^4 = \frac{(kT_{r1})^4}{h^3c^3}. \quad (13.1)$$

Now that the situation is completely determined, we try to choose the correct answer from the three possible solutions: the electron in the given radiation field will acquire a temperature:

a) such that kT is of the order of the average energy $h\nu_0$ of a single photon;

b) equal in order of magnitude to T_{r1}, i.e., to the radiation temperature in the region of the spectrum where the radiation is concentrated;

c) corresponding to the total density of radiation, i.e., T_{eff} such that $\mathscr{E} = aT_{eff}^4$, so that $T_{eff} = T_{r1}^{1/4}(h\nu_0/k)^{3/4}$.

The difficulty in choosing the correct answer is due to the fact that a free electron which gives rise to Thomson scattering experiences the radiation of all frequencies.

Einstein solved the problem of the Brownian motion of an oscillator in a radiation field. In this case, the answer is clear: the oscillator experiences the effective radiation temperature at the resonance frequency and nothing more. Knowing this answer in advance, Einstein drew far-reaching conclusions about the quantum structure of the radiation field itself.

Let us return to the electron. The equation written above yields

$$kT = \frac{h^2\int n^2\nu^4\,d\nu}{4h\int n\nu^3\,d\nu} = \frac{kT_{r1}}{4}. \quad (13.2)$$

Thus, the second answer proves to be correct (in order of magnitude); the electron "feels" the temperature of the low-frequency radiation.

Let us introduce the spectral density of the flux of radiation energy:

$$F_\nu\,(\text{erg/cm}^2\text{sec-Hz-sr}) = \frac{2h\nu^3}{c^2}n,\quad \mathscr{E} = \frac{4\pi}{c}\int F_\nu\,d\nu. \quad (13.3)$$

The expression for the electron temperature in an isotropic radiation field, written in terms of F_ν, takes the form

$$kT = (c^2/8)\int F_\nu^2\nu^{-2}\,d\nu \Big/ \int F_\nu\,d\nu. \quad (13.4)$$

We note that, for an admissible F_ν (such that $\int F_\nu d\nu$ converges), even if $F_\nu \to 0$ as $\nu \to 0$, it is possible to have the solution $T \to \infty$, and the integral in the numerator may diverge.

14. The classical interpretation of the theory of induced scattering. As is well known, large values of the "occupation number" $n \gg 1$ correspond to the possibility of making the transition to the classical (non-quantum) Maxwell theory of electromagnetic waves.

As regards the class of phenomena under consideration, a formal confirmation of the classical domain is provided by the disappearance of Planck's constant in the final equations after the variable n is replaced by the spectral density F_ν, which is a classical quantity. This disappearance of Planck's constant in the final equations applies to both the evolution of the radiation spectrum and the behavior of electrons in a given classical radiation field.

The foregoing derivation of the equations was carried out by considering the quantum problem, introducing the Bose factors (1 + n) and making the transition to the limit $n \gg 1$. It is obvious that there must also be a direct approach to the classical problem[14)].

On logical grounds, it is curious that this direct approach proves to be much more complex than the circuitous route (cf. the paper[14] mentioned in the introduction). From the point of view of the history of science, it may be noted that all the initial conditions required for the solution of the problem were already in existence towards the end of the last century among the works of Maxwell, Lorentz, Rayleigh and Jeans. If the problem was not formulated and solved 100 years ago, it is mainly because there was no taste (or fashion) for plasma, turbulence, or random fields and processes.

Let us return to the problem of the interaction of radiation and electrons. We shall begin the direct classical investigation of this problem with the simpler problem concerning the behavior of an electron in a random field; we shall only briefly touch upon the more complex problem of the evolution of a random field, i.e., the evolution of the radiation spectrum.

It is easy to solve the problem of the motion of an electron in the lowest approximation, which is linear in the field. This solution had already been obtained in the derivation of the Thomson scattering formula. The mean-square velocity of the electron is given by the formula

$$\overline{v^2} = \frac{\overline{E^2}e^2}{m^2\omega^2}. \quad (14.1)$$

For a random field, i.e., for uncorrelated waves, the mean-square velocity is composed of the mean squares corresponding to each individual wave. The square of the field can be expressed in terms of the energy flux. Thus, we obtain several equivalent formulations for the total energy of oscillatory motion:

$$\mathscr{E}_0 = \frac{m\overline{v^2}}{2} = \frac{2e^2}{mc}\int\frac{F_\nu}{\nu^2}\,d\nu = \frac{e^2}{2\pi m}\int\frac{d\mathscr{E}}{\nu^2}. \quad (14.2)$$

In particular, in the case of a Planck radiation spectrum, the electron energy has the order of magnitude

$$\mathscr{E}_0 \approx kT_r \frac{kT_r}{mc^2}\frac{e^2}{\hbar c} \ll kT_r. \tag{14.3}$$

Just as a long sentence in German may end with the particle of negation "nicht," we conclude the present calculation with a firm denial. The energy of oscillatory motion is not the energy of the electrons in which we are interested. The energy of the oscillations must be classed as a correction to the energy of the electromagnetic waves due to the fact that the index of refraction of the plasma is different from unity.

But the foregoing calculation made no allowance for the translational motion of the electron. In a theory which is linear in the field, this translational velocity may be arbitrary. It is only in the next approximation that the translational velocity varies with time; this process represents a gain and loss of energy by the electron.

What effects must be taken into account in the next approximation?

Let us first consider a single plane wave propagating along the z-axis, with fields E_x and H_y. In the linear approximation, the electron oscillates, with $v_x = \sin\omega t \cdot eE/m\omega$. In the next approximation, there occurs a Lorentz force directed along the z-axis and having the order of magnitude

$$F_L = \frac{eHv}{c} = \frac{e^2E^2}{mc\omega}. \tag{14.4}$$

However, this force is proportional to $\sin\omega t \cdot \cos\omega t \sim \sin 2\omega t$; its average with respect to time is equal to zero. It is only allowance for the radiation of the electron which shifts the phase of the velocity, $v \sim \sin(\omega t + \varphi)$, and as a result there is a non-zero average value $\overline{\sin(\omega t+\varphi)\cos\omega t} = (1/2)\sin\varphi$ and also $\overline{F} = (1/2)(e^2\overline{E^2}\sin\varphi/mc\,\omega) \sim \sigma\overline{E^2}$.

In order of magnitude,

$$\frac{\overline{F}}{|F_L|} \approx \sin\varphi \approx \frac{e^2\omega}{mc^3} \approx \frac{r_0}{\lambda}. \tag{14.5}$$

In essence, the result obtained earlier from the conservation laws is explained here in detail in the language of forces. The scattering of a flux of energy by an electron produces a force in the direction of the flux. This force is small, many times weaker than the Lorentz force. The average flux of energy is zero in an isotropic radiation field. But the concept of an isotropic field singles out a particular rest system. An observer moving together with the electron with respect to this system perceives the radiation as anisotropic: the radiation moving against him is shifted towards the blue and more energetic end of the spectrum. The radiation moving with the observer is "reddened" and weakened. The flux of energy relative to the electron is

$$q = -c\mathscr{E}\frac{4\beta}{3(1-\beta^2)} \approx -\mathscr{E}\frac{4v}{3}, \tag{14.6}$$

where $\mathscr{E}$ is the density of radiation energy in the system in which the radiation is isotropic. Consequently, the force acting on the electron and corresponding to the energy loss is

$$F = -\frac{4\sigma\mathscr{E}v}{3c}, \quad W_- = Fv = -\frac{4\sigma\mathscr{E}v^2}{3c} = -\frac{8}{3}\frac{\mathscr{E}\sigma c}{mc^2}\frac{mv^2}{2} = -4\mathscr{E}\sigma c\frac{kT}{mc^2}. \tag{14.7}$$

In the last expression, we have substituted the value of the kinetic energy of the electrons with temperature T; we find the expression for the energy lost by the electrons. The preceding lengthy arguments would have been useless if they had not suggested an effective mechanism of heating the electrons by the Lorentz forces: it is sufficient to get rid of the condition which relates E and H in a plane wave! In a random (on the average, isotropic) radiation field, there exists a Lorentz force of the magnetic field H_1 of one wave acting on an electron whose velocity v_2 depends on the electric field of a second wave E_2. When the wave vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ of the two waves do not coincide in direction, there is no reason why the force should be precisely zero over a long period of time. The force is proportional to the square of the amplitude, i.e., to the intensity of radiation. The change in energy of the electron is proportional to the square of the impulse of the force; thus, the classical heating mechanism considered here leads to an expression for W_+ proportional to $F_\nu F_{\nu'}$. A difficult statistical calculation gives—for a broad radiation spectrum—the expression const $\cdot \int (F_\nu^2/\nu^2)d\nu$, in agreement with that found previously.

The picture becomes clearer if we introduce the concept of a quasipotential (QP). The role of a QP is played by the kinetic energy of the oscillations of the electron, evaluated in the lowest approximation.

As has been shown by Gaponov and Miller [44], the time-averaged force, which is quadratic in the amplitude of the field[15)], is equal to the gradient of the kinetic energy of the oscillations, evaluated in the lowest approximation.

Following the ideas of [44], let us consider a standing wave in which $E_x = E\cos\omega t \cdot \cos kz$, with $k = \omega/c$. The energy of the oscillations of an electron in such a standing wave is

$$\mathscr{E}_{(1)} = \frac{e^2E^2}{4m\omega^2}\cos^2 kz \tag{14.8}$$

and, accordingly, the averaged force[16)] directed along the axis is

$$F_{z(2)} = \frac{e^2E^2}{2mc\omega}\cos kz \sin kz. \tag{14.9}$$

Cold electrons accumulate at the nodes, i.e., at the planes $z = n\pi/2k$ on which $E_x = 0$. A standing wave is a superposition of two oppositely travelling waves of the same frequency. If the frequencies of the latter are slightly different, the nodes undergo a slow displacement in space; strictly speaking, the resulting field is not a standing wave.

This is the basis of certain ideas of employing laser light beams to accelerate particles [47,48].

The subject of the present paper is a random field. In such a field, both the quasi-potential $\mathscr{E}_{(1)}$ and the corresponding[17)] force $\mathbf{F}_{(2)} = -\nabla\mathscr{E}_{(1)}$ are random functions of the coordinates and the time.

It is possible to expand $\mathscr{E}_{(1)}$ and $\mathbf{F}_{(2)}$ in elementary waves in the same way that the field **E**, **H** itself is expanded in individual waves.

What is the difference between these two expansions?

The quantity $\mathscr{E}_{(1)}$ is a scalar; the vector $\mathbf{F}_{(2)}$ is potential and can be expanded in longitudinal waves. For a given wave vector **k**, the expansion of $\mathbf{F}_{(2)}$ contains all frequencies ω from 0 to $c|k|$. This means that the field of the force $\mathbf{F}_{(2)}$, regarded as a function of the coordinates and the time, contains components having an arbitrary phase velocity, including small values. Thus, each slow electron finds in the field $\mathbf{F}_{(2)}$ a matching component (a wave synchronized with its translational velocity) which continually accelerates it. To first order, when the force is $F_{(1)} = eE$, there is no such synchronous wave—the field E is transverse, and its phase velocity is equal to c. It is for this reason that the effect of heating the electrons in the classical theory takes place only in second order and is quadratic in the intensity.

Thus, it is actually possible to construct a classical theory of the heating of electrons by a random field; in the course of the development of the theory, there emerge the important qualitative concepts of a quasipotential, Gaponov-Miller forces and synchronism. If we are only dealing with the derivation of the expression for W_+, then the quantum "tour" ("Herumführung") is shorter. However, it is not possible to attain a complete understanding without combining the exact quantum equations with the classical approach.

15. The classical theory of the evolution of the spectrum. Let us return to an elementary gedanken experiment: the scattering of a monochromatic beam with frequency ν_0 by a plasma situated in a radiation field having a broad spectrum directed at an angle to the monochromatic beam (Fig. 11).

Let us first consider electrons at rest in the quantum theory of induced scattering. The change in intensity is

$$\frac{dn_0}{dt} = \text{const}\,(n_0 n_1 \nu_1^2 - n_0 n_2 \nu_2^2), \tag{15.1}$$

where n_1 is the occupation number at the frequency ν_1 which gives the frequency ν_0 after scattering:

$$\nu_1 = \nu_0\left[1 + \frac{h\nu_0}{mc^2}(1 - \cos\theta)\right]. \tag{15.2}$$

Similarly, n_2 is the occupation number at the frequency ν_2 which results after scattering,

$$\nu_2 = \nu_0\left[1 - \frac{h\nu_0}{mc^2}(1 - \cos\theta)\right]. \tag{15.3}$$

Consequently, the change in intensity of the beam depends on the difference of n_1 and n_2 at nearby frequencies; for a broad spectrum, this difference can be replaced by the derivative:

$$\frac{d\ln n}{dt} = \frac{d\ln I_0}{dt} = \text{const}\cdot\frac{2h\nu^2}{mc^2}(1 - \cos\theta)\frac{\partial}{\partial\nu}(\nu^2 n). \tag{15.4}$$

Thus, depending on the spectrum of the "illumination" (in particular, depending on the sign of the derivative of $n\nu^2$), the plasma either effectively weakens or strengthens the beam.

If $\partial(n\nu^2)/\partial\nu > 0$, the plasma is like a providing medium, i.e., it strengthens the beam. It has been proposed to use this effect under laboratory conditions [49], but it turns out that the maximally possible amplitude of the enhanced waves is small [50].

How can the effect of enhancement of a monochromatic beam be explained in classical electrodynamics?

When a plasma is exposed to the action of two waves with frequencies ν_0 and ν and amplitudes E_0 and E, there occur Gaponov-Miller forces with the difference frequency $\nu_0 - \nu$ and the amplitude E_0E. These forces produce a perturbation of the electron density which is proportional to $E_0E(\nu_0 - \nu)^{-2}$ in amplitude and time-varying with a frequency $\nu_0 - \nu$. The index of refraction of the plasma and its dielectric constant depend on the electron density. The perturbation in the density implies a perturbation in the index of refraction.

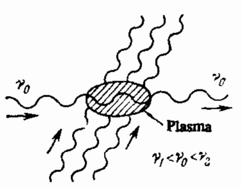

FIG. 11. Interaction of monochromatic radiation with a spectrally broad beam in a plasma.

The enhancement wave (ν, E) is scattered by the perturbations in the index of refraction[18]. The frequencies $(\nu_0 - \nu)$ and ν then add, and there results a wave with the frequency of the beam, ν_0.

Thus, we obtain

$$\frac{dE_0}{dt} \sim \delta\rho E \sim (E_0 E) E \sim E_0 E^2 \tag{15.5}$$

or

$$\frac{dI_0}{dt} \sim \frac{dE_0^2}{dt} \sim E_0^2 E^2 \sim I_0 I f(\nu, \nu_0). \tag{15.6}$$

The dependence of dI_0/dt on I_0 and the enhancement intensity is quadratic, i.e., it corresponds precisely to the induced scattering. The dependence on the frequency ν (which we have not followed) is such that we have the function $f(\nu, \nu_0) \approx d[\delta(\nu - \nu_0)]/d\nu$ for electrons at rest (T = 0) in the classical theory. For a finite electron temperature, $f(\nu, \nu_0)$ is an antisymmetric function of $\nu - \nu_0$ differing from zero in the region $|\nu - \nu_0| \lesssim \nu_0\sqrt{kT/mc^2}$. For a broad enhancement spectrum, $dI_0/dt \sim I_0 \int If\, d\nu$ reduces to the expression involving a derivative written at the beginning of this section.

16. Anisotropic fields, narrow beams and the conditions for the applicability of the theory. The Kompaneets equation and the conclusions drawn from it apply to an isotropic radiation field. The study of an isotropic field was justified by the fact that each scattering event halves the anisotropy, while multiple scattering is required for the evolution of the spectrum: the number of events needed to change the frequency by the order of magnitude of the frequency itself ($\Delta\nu \sim \nu$) is given by mc^2/kT, i.e., is much larger than unity.

However, while induced scattering plays a major role, it may happen that for large n the rate of induced evolution of the spectrum is even greater than the rate of the spontaneous tendency towards isotropy of the radiation field. The criterion is (for a broad spectrum, when $\partial n/\partial\nu \sim n/\nu$)

$$n > \frac{mc^2}{h\nu}, \tag{16.1}$$

i.e., the effective luminosity temperature of radiation is $kT > mc^2$. However, we recall that this temperature refers to the low-frequency part of the spectrum $\nu \ll mc^2/h$, so that neither relativistic effects nor e^+e^- pair production occur.

If there is a high luminosity temperature, we can consider induced processes for anisotropic beams. The total angular width of a beam does not increase, but the angular distributions of various wavelengths evolve differently. With decreasing scattering angles, all effects are suppressed in proportion to θ^2. The details of anisotropic effects can be found in [53,54].

The polarization of the radiation is also not smoothed out in the regime of induced scattering; the problem requires a strict investigation. A number of papers [54-56] have been devoted to the evolution of spectrally narrow lines in the presence of induced scattering. This problem is of interest in connection with laboratory experiments; lasers provide a very high effective temperature within a narrow spectral interval.

In this regime, not all the electrons of a Maxwell distribution take part in the scattering. Accordingly, in the case of heating by spectrally narrow radiation, there results a distribution of electrons which falls off more rapidly at large electron velocities than the Maxwell law [56].

The limitations of the theory are evident from the classical picture of induced scattering: the perturbations in the density result in an electrostatic repulsion of the excess electrons. If this effect is to be neglected, the frequency difference $|\nu_0 - \nu|$ must be greater than the Langmuir plasma frequency.

For a broad spectrum, the condition on the frequencies leads to the requirement that the wavelength of the radiation is less than the Debye radius.

The limitations due to plasma effects have been studied in [55, 56]. The authors find that the conditions quoted above are sufficient but not necessary, i.e., they are too stringent.

Finally, we note that induced scattering in an anisotropic radiation field results in an average force acting on the electrons which is proportional to the square of the intensity, i.e., to the fourth power of the amplitude of the field and the fourth power of the electron charge [15]. In order of magnitude, this force is $F_2 \sim \sigma q \mathscr{E}/m\nu^3$. However, the dependence of F_2 on the angular distribution and the spectrum is given by a complicated double integral with respect to the angle and an integral with respect to the frequency.

For an isotropic distribution, $\mathbf{q} = 0$ and $\mathbf{F}_2 = 0$; for an axially symmetric distribution, $\mathbf{q}$ and $\mathbf{F}_2$ are directed along the axis, but it is not excluded that their directions are opposite; both signs of the ratio F/q are possible. The force $\mathbf{F}_2$ vanishes for a narrow beam (with respect to the angle), in spite of the fact that $\mathbf{q}$ is then maximal (for a given $\mathscr{E}$). The force $\mathbf{F}_2$ is of the same order of magnitude as the known $\mathbf{F}_1 = \sigma \mathbf{q}/c$ if the luminosity temperature of the radiation is $T_r \sim mc^2/k$.

An important case is the small anisotropy which results from a slow ($v/c = \beta \ll 1$) motion of an electron with respect to an isotropic field. It can be shown that this average force is proportional to the velocity and is opposite in direction, i.e., that it decelerates the electron[19]. A comparison of the average decelerating force with the random force which produces the heating of the electrons shows that the deceleration effect is unimportant for a nonrelativistic electron temperature; the Kompaneets equation and all the foregoing calculations and arguments are inapplicable to relativistic electrons.

A classical determination of $\mathbf{F}_2$ (which may be called the super-Gaponov-Miller force) would be difficult. The calculation in terms of the quantum expressions presupposes the random phase approximation in the theory of a random field. This approximation must be verified when considering spectrally narrow (and narrow in the angle) fluxes of radiation: the initial radiation may not be completely random, and in fact a correlation may occur in the process of scattering. On the other hand, the results referring to isotropic or almost isotropic radiation with a broad spectrum are very reliable, and they provide a firm basis for the solution of many astrophysical problems.

17. Nonuniform distribution of electrons in space. So far we have been concerned exclusively with the spontaneous or induced scattering by a single electron.

It was understood that, under the realistic conditions of scattering by an electron gas, one must add the radiation fluxes, i.e., the radiation intensities of many individual electrons. It is obviously possible to have a situation in which the amplitudes add. Suppose that there are N electrons inside a sphere of radius much greater than the wavelength of the incident radiation. All N electrons oscillate in step, their radiation is coherent, and the amplitude of the scattered wave is N times as large as the scattering amplitude of a single electron, so that the intensity of the scattered radiation is increased by a factor N^2. Thus, a group of N electrons has a cross section $\sigma_g = \sigma N^2$, where σ is the Thomson cross section of a single electron.

It is easy to derive this result formally: we shall regard the group as a single particle with charge Ne and mass Nm. The cross section is $\sigma \sim e^4 m^{-2}$; substituting Ne and Nm instead of e and m, we obtain precisely the above-mentioned relationship between σ_g (for the group) and σ.

What happens in the case of a unifom (homogeneous) distribution of electrons in the plasma? We must stipulate that statistical homogeneity is implied here: the probability ΔP of finding one electron in a small volume ΔV is given by $\Delta P = n\Delta V$, where the coefficient n_e is independent of the coordinates x, y, z and the positions of the other electrons and ions; n_e is the density of electrons.

In this case, is it not necessary to mentally divide space into cells of dimensions $\lambda/2$ and volume $\lambda^3/8$ and to combine into groups the electrons found within the same cell in the quantity $N = n_e \lambda^3/8$? Such a procedure would lead to an erroneous result; if we mention it here, it is only in connection with the saying of Niels Bohr: "A good specialist is one who knows the most common errors in his field and is able to avoid them."

In calculating the radiation scattered at a definite angle from a wave with wave vector $\mathbf{k}_1$ into a wave with wave vector $\mathbf{k}_2$, there occurs an expression of the form

$$\sigma_g = \sigma \left| \sum_{j=0}^{j=N_1\to\infty} e^{i(\mathbf{k}_2-\mathbf{k}_1)\mathbf{r}_j} \right|^2.$$

The cross-terms in this expression give a contribution of the form

$$A = \sum_{j,l} \cos[(\mathbf{k}_2-\mathbf{k}_1)(\mathbf{r}_j-\mathbf{r}_l)].$$

When an average is taken over a statistically homogeneous ensemble of electrons in a large volume (of dimensions $L \gg \lambda$), we obtain $A = 0$, leaving $\sigma_g = N$, which corresponds to the addition of the intensities. This result clearly has a simple explanation: the electrons in a layer whose thickness is of order λ or, more precisely, $(\lambda/2)\cos(\mathbf{k}_1, \mathbf{k}_2)$, actually "interfere constructively" and the amplitudes add for them. However, the next layer gives an interference of the opposite sign—"destructive"—and so on. The terms depending on the interference of the waves vanish on the average.

This complicated way of looking at a simple result is useful because it gives us a better feeling for the domain of applicability of the equations. As a rule, we are dealing with the case $n\lambda^3 \gg 1$. Even in interstellar gas, we have $n_e \sim 0.1-1\ cm^{-3}$, and one studies the scattering of radio waves with $\lambda \leq 10^4$ cm, so that $n_e\lambda^3$ is as high as 10^{12}! The addition of the intensities is as a result of an exact cancellation of the large effects of constructive and destructive interference. A relatively small perturbation in the electron density, such as an acoustic wave, is sufficient for the scattering cross section to rise sharply in certain directions. This effect has recently been confirmed experimentally. The general formulas also contain a reflection of a wave from the boundaries of an abrupt variation in n_e. In an equilibrium plasma, there are corrections associated with the electrostatic interaction of the electrons; the characteristic dimension here is the Debye wavelength. However, the development of these topics would lead us deep into plasma theory. The effect of a nonhomogeneity on spontaneous scattering is discussed here to emphasize the nature of induced scattering.

A spatial nonhomogeneity in the distribution of electrons has no effect on induced scattering!

Let us begin with a limiting case: the kinetic equation for photons involves the electron density and the Thomson cross section divided by the mass of the electron, i.e., the combination

$$a = \frac{n_e \sigma}{mj} \approx \frac{n_e e^4}{m^3}.$$

Let us combine the electrons into compact groups of N. The density of the groups is $n_g = n_e/N$, and their charge and mass are Ne and Nm. Substituting these values, we conclude that a remains unchanged. A different argument involves the heating of electrons in a given external radiation field. If allowance is made for effects nonlinear in the field (but not for the self-radiation of the electron or its "radiation resistance"), we obtain an expression for the rate of heating which is independent of where the neighboring electrons are situated. But the process of heating the electrons is the reverse side of induced scattering. The energy balance of the electrons and the radiation interrelates the evolution of the spectrum in the presence of induced scattering and the heating of the electrons described just above. In a somewhat different way, we deduce once again that the spatial distribution of electrons does not affect induced scattering.

A particular conclusion which follows from this is that it is possible to make the limiting transition to a smeared electron fluid having a given mass density $\rho = n_e m$ and charge density $q = n_e e$—the transition opposite to that which considers groups of electrons.

The evolution of the spectrum can be studied by writing Maxwell's equations and the equations of motion of the continuous electron fluid. Without considering individual point electrons, we automatically exclude spontaneous Thomson and Compton scattering and, at the same time, quantum effects. It is well known that the plasma frequency and the index of refraction of a plasma may be determined in the electron-fluid approximation; they depend on ne^2/m and are unchanged if the electrons are grouped.

This review will not have been written in vain if there exists a reader who will carry out the program which we have sketchily outlined above.

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1)It is important for what follows that the classical and quantum theories differ less in this respect (change of frequency) than one might think at first sight. In the classical theory, an electron which undergoes scattering acquires a momentum in the direction of the incident wave and the Doppler effect causes a spread of frequency that grows with time in the system in which the electron was initially at rest. The quantum and classical average spreads of frequency are the same.

2)In fact, this conclusion was drawn from old-fashioned calculations without Feynman diagrams, which appeared 20 years later; during the period of controversy about the positron, Feynman was a high school student. On the other hand, high school students nowadays know about these diagrams, which justifies the violation of the chronology in the text.

3)The blasphemous phrase "The universe is a particular case" is nevertheless true from the narrow point of view of a physicist studying the interaction of electrons with radiation.

4)The situation is essentially different if the radiation is anisotropic or has a narrow spectrum (see Sec. 16 and [56]).

5)The choice of this system is not unique. In particular, if we stipulate that the electron is at rest before the arrival of the wave and consider a wave which rises gradually from zero to some large amplitude, it turns out that the electron (with no allowance for the radiation reaction!) acquires a relativistic longitudinal velocity. If the radiation reaction is taken into account, this longitudinal velocity varies with time, so that a strictly stationary solution of this type does not exist.

Later we shall consider a different formulation of the problem—the radiation in a coordinate system in which the oscillating electron is at rest on the average as a result of the longitudinal field; for details about this field, see Sec. 5 below.

6)These properties are well known from the theory of synchrotron radiation; see the reviews and monographs mentioned in the introduction.

7)We are considering only a strictly stationary state, and the interval of time during which the wave is inktiated is not taken into account.

8)P(b) and P denote the projection of the electron dipole moment on the electric field of the wave; in a strong wave $P(b) = -eR\cos\varphi = -e(v/\omega)\cos\varphi$, and $P = -e^2E/m\omega^2$ according to the Thomson formula. The real part of the dielectric constant in a strong wave is

$$\mathrm{Re}\,\epsilon = 1 - P(b)\frac{4\pi e^2 n_e}{m\omega^2}. \tag{5.6}$$

9)And with an appropriate longitudinal electric field.

10)Specifically, $\alpha = (3/32\pi)(1+\cos\theta)^2$ and $\beta = (3/32\pi)(1-\cos\theta)^2$. Here α and β refer to the average Thomson cross section. It is much more convenient to consider circulat polarization than plane polarization; the statement that a wave has a given circular polarization is invariant with respect to any proper coordinate transformation (rotation or Lorentz transformation).

11)It is readily verified that in our case (owing to the symmetry of the indicatrix of scattering $\sim(1+\cos^2\theta)$) the average change in ν is equal to the change for scattering at $\pi/2$ and has half the maximum value.

12)The derivation of the last equation involves an integration by parts, which requires that n decreases faster than ν^{-3} as $\nu\to\infty$ and grows more slowly than $\nu^{-2.5}$ as $\nu\to 0$. These conditions are not satisfied for the stationary but non-equilibrium solutions with $q \neq 0$.

13)We recall that, in accordance with the definition of the flux q (see Chapter III), $q = \mathrm{const}\cdot\nu^4 n^2$, the corresponding average photon velocity in the scale of frequencies is $d\nu/dt = -q/n\nu^2 = -f/mc^2$. For induced scattering, the velocity depends on the amplitude (n or f) but not on the gradient $(\partial n/\partial\nu)$; this is different from the diffusion of photons for spontaneous scattering by moving electrons. The velocity of motion is half the translational velocity of the point with f = const, i.e., the "mass" velocity is half the phase velocity (the velocity of a characteristic).

14)An interesting example is the work of Dirac and Kapitza [43] on the diffraction of electrons by a standing wave. Here the electron is regarded as a quantum object, and the standing wave is classical. This wave can be represented as a superposition of two travelling waves of the same frequency but opposite directions. The authors first cinsider the spontaneous scattering of a photon corresponding to one of the travelling waves into the state of the other travelling wave. Allowance is then made for the induced character of the scattering: this explains the scattering into the other travelling wave and not into any other state. However, at the same time, the transition is made to the classical theory of the electromagnetic field.

15)This refers to the Lorentz force described above and other similar terms which must be taken into account together with it.

For a simpler mechanical system with one-dimensional motion—a pendulum—this principle of introducing a quasi-potential was formulated even earlier by Kapitza [45] and has found its way into textbooks (see [46]).

16)The indices in parentheses show the order of the approximation: 1) the first estimate of the velocity, linear in the field, from which the energy of the oscillations is determined, 2) the approximation quadratic in the field; we note that $F_{Z(1)} \equiv 0$.

17)The kinetic energy from the first approximation is quadratic in the field.

18)This is the general method of studying forced scattering processes [51-52].

19)This force can be represented in the form $\mathbf{F}_2 = -(v\sigma/m)p \int_0^\infty \nu^a \times [\partial(F\nu^b)/\partial\nu]^2 d\nu$; according to a calculation of A. F. Illarionov and D. A. Kompaneets (Jr.), $p. = 7/60\pi$, $a = 1 \pm\sqrt{15/7}$ and $b = -a/2$.

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