Covariant quantization of the gravitational field

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Usp. Fiz. Nauk **111**, 427–450 (November 1973)

A review dedicated to the contemporary methods of quantization of the gravitational field. In view of possible applications to elementary particle theory, the authors consider only asymptotically flat gravitational fields. The basis of the exposed method of quantization is the method of quantization of gauge fields in the functional integration formalism. The main result is the formulation of covariant rules for a diagrammatic perturbation theory. Its elements are the lines representing gravitons and the vertices of graviton-graviton interaction, as well as the lines and interaction vertices of fictitious vector particles ("Faddeev–Popov ghosts") characteristic for the theory of gauge fields. The expressions for the propagators and vertex functions are given explicitly. It is shown that the presence of fictitious particles in the covariant diagram technique guarantees the unitarity of the theory and the agreement between the covariant quantization with the canonical quantization. The bibliography contains 44 entries (54 names).

CONTENTS

INTRODUCTION

The interest in a quantum theory of gravitation is maintained to a large extent by hopes that inclusion of gravitation into the scheme of quantum field theory will allow us to construct a self-consistent closed theory of elementary particles. In this connection there are two directions. One is related to cosmology and to the use of cosmological considerations in the theory of elementary particles (cf. the papers of Wheeler[1], Misner[2], Markov and collaborators[3]). The other direction, which does not make use of cosmological considerations, considers the field of the particles concentrated effectively in a finite volume and vanishing at spacelike infinity. In the latter approach the gravitational field is considered on the same footing as any other field (cf.[4]) and the theory is a variant of the theory of gauge fields. The role of gauge transformations is played by coordinate transformations which do not affect the spacelike infinity, and the role of the gauge group is played by the Poincaré group (in more detail this problem is discussed in the report by one of the authors[5]). The role of Lorentz-invariance in the general theory of relativity was underlined by Fock[6].

The interest in elementary particle theory is mainly related to the hope that it may be just the gravitational field which could play the role of a natural "physical regularizer" which removes the singularities and infinities from quantum field theory. The first results which supported this point of view were obtained by De Witt[7] and Khriplovich[8]. At the present time this direction is actively developed by Salam and collaborators[9], making use of methods developed by Efimov, Fradkin, Volkov and Filippov[10-13].

In this review article we restrict our attention only to the problem of constructing covariant rules for a diagrammatic perturbation theory. The most convenient tool for this purpose is Feynman's functional integral. In this formalism the physical degrees of freedom, which are quantized, and the gauge-field degrees of freedom, which remain c-numbers, are treated on the same footing. This makes it possible to use a large class of different transformations.

The first correct formalism for the quantization of the gravitational field was constructed by Dirac in 1958[14], within the framework of the Hamiltonian method. Other methods, essentially equivalent to that developed by Dirac have been developed by the Arnowitt-Deser-Misner group[15], by Schwinger[16] and also some others[17-19]. The absence of explicit covariance in the Hamiltonian formalism makes perturbation theory very cumbersome. It is appropriate to recall the analogous situation in quantum electrodynamics. The noncovariant perturbation methods of the thirties did not allow one to go beyond the first nonvanishing approximation of perturbation theory. The creation of a covariant perturbation theory toward the end of the forties and the development of the diagram technique associated to it have simplified the calculations considerably, so that at present the theory can be compared to experiment up to the seventh decimal place. In the theory of gravitation the technical difficulties are so great that a noncovariant quantization scheme becomes practically useless in concrete perturbation-theoretic computations.

The first attempts of constructing a covariant quantization scheme for the gravitational field were contained in the papers of Gupta[20]. He followed the scheme developed for quantum electrodynamics, and to overcome the difficulties related to the singularity of the free Lagrangian of the gravitational field, he made use of a trick analogous to the well-known Fermi method of quantization for the electromagnetic field (cf., e.g.,[21,22]).

It turned out, however, that an uncritical transfer of the Fermi quantization method (which is justified in quantum electrodynamics) to more complicated systems can lead to a violation of the unitarity condition of the theory. This was first discovered by Feynman in 1963[23] on the examples of the Yang-Mills field and gravitation theory. Feynman has mapped out the path to overcome the difficulties which he had pointed out. He has shown that the unitarity of a closed-loop diagram can be recovered if one subtracts from the appropriate matrix element the contribution of another diagram, also having the form of a loop and describing the propagation of a fictitious particle. It was not possible to extend the Feynman method to more complicated diagrams.

The solution of the problem for arbitrary diagrams was found in 1967 by De Witt[24] and by the authors of the present review[25], using basically different methods. Both methods are unified by the use of the method of functional integration, which provides a scheme of covariant perturbation theory for gauge fields. In the case of a non-Abelian gauge group the method involves lines corresponding to fictitious particles and vertices describing their interactions with the quanta of the gauge field. The fictitious particle lines enter into the diagram as closed loops, each identical with the diagram introduced by Feynman in order to recover the unitarity of the one-loop diagram for the real particles. The sign (−1) in front of the contribution of the closed loop shows that the fictitious particles (which are also known under the name of "Faddeev-Popov ghosts"—transl.) are subject to Fermi statistics. This makes a direct interpretation of the fictitious particles difficult, since they have integral spin (they are scalar for the Yang-Mills field and vectorial for the gravitational field). The results obtained in[24,25] have been repeatedly rederived by various authors, and other methods of solving the problem have been proposed. We note here the papers by Mandelstam[26], Veltman[27], Fradkin and Tyutin[28], Khriplovich[29], Boulware[30], 't Hooft[31], Altukhov and Khriplovich[32].

The construction of a correct covariant quantization scheme for gauge fields has generated a series of papers where the (massless) gauge theory is compared to the corresponding massive (non-gauge) theory and the problems of the limit $m \to 0$ are discussed (the papers of Faddeev and Slavnov[33], Vaĭnshteĭn and Khriplovich[34], Zakharov[35], Veltman[27], Fradkin and Tyutin[36]. The basic result of the majority of these papers reduces to the fact that the limit of a massive theory for $m \to 0$ does not, in general, coincide with the corresponding massless theory.

As we already noted, methods of quantization of the gravitational field have been developed within the framework of the canonical Hamiltonian formalism and in explicitly covariant form. The functional integration method allows one to relate both formulations. The canonical Hamiltonian yields a unitary normalized theory and the covariant formulation allows one to construct a perturbation theory which is convenient for concrete calculations.

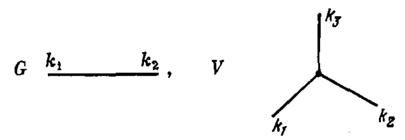
Here are some details about the plan of our review article. Chapter 1 deals with the scheme of derivation of rules of covariant perturbation theory in the functional integration formalism, on the example of a scalar self-interacting field theory. The Feynman rules for the gravitational field are formulated in Chap. 2. The derivation of these rules in Chap. 3 is based on the form of the functional integral for the gravitational field postulated there. A justification for this is the possibility of reducing the theory to an explicitly Hamiltonian form. For this reason we deal in Chap. 4 with the canonical Hamiltonian form of the theory of gravitation, and in the last, fifth, chapter we show that the functional integral following from this theory can be reduced to the relativistic form which was used in the derivation of the Feynman rules.

1. THE FEYMAN RULES FOR COVARIANT PERTURBATION THEORY AND THE FEYNMAN FUNCTIONAL INTEGRAL

We recall and discuss here the recipe for the derivation of the rules of covariant perturbation theory according to Feynman[37]. We start with the simplest example of a self-interacting scalar field φ with the Lagrangian

$$\mathcal{L}(x)=\frac{1}{2}(\partial_\mu\varphi(x)\,\partial_\mu\varphi(x)-m^2\varphi^2(x))-(\lambda/3!)\,\varphi^3(x). \tag{1.1}$$

The diagrams corresponding to the terms of perturbation expansions is constructed out of two elements: the line G (propagator) and the vertex V:



(1.2)

The scattering amplitude $M_n(k_1, \ldots, k_n)$ describing n (incoming or outgoing) particles with momenta $k_1, \ldots, k_n$ is represented by a sum of contributions corresponding to diagrams with external (non-closed) lines. The contribution of a given diagram is obtained by associating to its elements the following expressions: to internal lines one associates the propagators

$$G(k_1, k_2)=\frac{\delta(k_1-k_2)}{k_1^2-m^2}, \tag{1.3}$$

to the vertices one associates the vertex function (coupling constant)

$$V=\lambda\delta(k_1+k_2+k_3), \tag{1.4}$$

and after integrating over all the momenta k of the internal elements of the diagram, the final result must be multiplied by

$$\frac{1}{r}\left[\frac{i}{(2\pi)^4}\right]^{l-v+1}, \tag{1.5}$$

where l is the number of internal lines, v is the number of vertices and r is the order of the symmetry group of the diagram[1)]. To obtain the amplitude for a real process one has to go onto the energy shell $k^0 = \pm(\mathbf{k}^2 + m^2)^{1/2}$; the sign depends on whether the particle is incoming or outgoing.

The described elements of the diagram technique are determined by the Lagrangian (1.1) in the following manner. We consider the action

$$S[\varphi]=\int\mathcal{L}(x)\,d^4x \tag{1.6}$$

as a functional of the Fourier transform $\tilde{\varphi}(k)$ of the field

$\varphi(x)$. This functional consists of a quadratic form in the field and a form of third degree:

$$S[\varphi] = S_2[\varphi] + S_3[\varphi]. \tag{1.7}$$

The function $G(k_1, k_2)$ is the kernel of the integral operator inverse to the operator which defines the quadratic functional form S_2; the function V is the coefficient function determining the form of third degree.

This correspondence can also be seen in many other examples for which there exists a diagram technique, e.g., for the pseudoscalar theory of pions and nucleons. The only difference is the appearance of the factor $(-1)^s$, in the weight (1.5) of the diagram, where s is the number of closed loops formed of fermion lines.

The described recipe is not directly applicable to theories where the corresponding quadratic form is singular. The simplest example is provided by electrodynamics, where the action integral of the free photons

$$S_{\text{ph}} = -\frac{1}{4}\int (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 d^4x \tag{1.8}$$

is degenerate owing to gauge invariance, and does not depend on the longitudinal component $\partial_\mu A_\mu$ of the potential A_μ. However, in this case it is known (cf.[21,22]) that one can use as photon propagator a generalized inverse operator of the quadratic form (1.8); the momentum-space representation of this propagator is the following

$$G_{\mu\nu}(k, k') = \delta^{(4)}(k+k')\left(-\frac{\delta_{\mu\nu}}{k^2} + \frac{\beta k_\mu k_\nu}{k^4}\right), \tag{1.9}$$

where the constant is arbitrary. The scattering amplitudes for real processes do not depend on this constant.

However, as we have already pointed out in the introduction, a direct adaptation of this recipe to theories with nonabelian gauge groups leads to incorrect results. In order to make clear the reason for these difficulties it is useful to analyze the derivation of the rules of covariant perturbation theory. In our opinion the most convenient approach is the formalism of the Feynman path (functional) integral[2].

We recall the main features of this method on the example of the scalar field with the Lagrangian (1.1). The scattering amplitude is obtained as an integral over all possible fields with a given asymptotic behavior for $t \to \pm\infty$ of the functional exp(iS), where S is the action integral. We shall use the following notation for such integrals

$$\int \exp\{iS[\varphi]\}\prod_x d\varphi(x), \tag{1.10}$$

where the symbol $\Gamma_x d\varphi(x)$ signifies that the integration variables $\varphi(x)$, which are considered independent at each point x, take on all possible values from $-\infty$ to $+\infty$, and the integration measure with respect to these variables is Lebesgue measure. Moreover, we omitted an inessential (infinite) normalization factor.

A rigorous mathematical theory of such objects does not exist as yet. We shall use the functional integral only for the derivation of the rules of perturbation theory. In order to explain the formal rules of manipulation of such objects we shall appeal to finite-dimensional analogues.

Expansion of the functional exp(iS) in powers of the coupling constant λ leads to integrals of the form

$$\int \exp\{iS_2[\varphi]\}(S_3[\varphi])^n \prod_x d\varphi(x), \tag{1.11}$$

where the integrands have the form

exp[i(quadratic form)] × polynomial.

Such integrals can be explicitly calculated and can be expressed in terms of the operator which is the inverse of the operator of the quadratic form as well as the coefficients of the polynomial.

Indeed, let us consider an integral of this type for a finite number of variables:

$$J = \int_{-\infty}^{\infty} dx_1 \ldots \int_{-\infty}^{\infty} dx_n \exp\left(\frac{i}{2}\sum_{ij} A_{ij}x_ix_j\right) \sum_{i_1\ldots i_n} c_{i_1,\ldots,i_n} x_1^{i_1}\ldots x_n^{i_n}. \tag{1.12}$$

It can be calculated by differentiating the generating function

$$Z(\eta_1, \ldots, \eta_n) = \int_{-\infty}^{\infty} dx_1 \ldots \int_{-\infty}^{\infty} dx_n \exp\left(\frac{i}{2}\sum_{ij} A_{ij}x_ix_j + i\sum_i \eta_i x_i\right) \tag{1.13}$$

namely:

$$J = \sum_{i_1,\ldots,i_n} c_{i_1,\ldots,i_n}(-i)^{i_1+\ldots+i_n}\frac{\partial^{i_1+\ldots+i_n}}{\partial\eta_1^{i_1}\ldots\partial\eta_n^{i_n}} Z(\eta_1,\ldots,\eta_n)\Big|_{\eta=0}.$$

The generating function can be computed by a shift of the variable, yielding

$$(2\pi i)^{n/2}(\det A)^{-1/2}\exp\left(-\frac{i}{2}\sum_{i,j} B_{ij}\eta_i\eta_j\right), \tag{1.14}$$

where the matrix B is the inverse of the matrix A. The first factor here plays the role of the normalizing constant which we have agreed to neglect. The differentiation of the second factor leads to a result which in words can be formulated in the following manner: The integral J is represented by the sum of the coefficients $c_{i_1\ldots i_n}$, contracted with products of the elements B_{ij} of the matrix B, carrying the same set of indices.

In our infinite-dimensional case the analogues of the coefficients $c_{i_1\ldots i_n}$ are products of the coefficient-functions of the cubic form in the action. The role of the matrix A is played by the hyperbolic differential operator (the Klein-Gordon operator) $(\Box - m^2)$, or, in momentum space, the multiplication operator by $(k^2 - m^2)$. A natural generalization of our finite-dimensional result leads then to the Feynman rules formulated above.

Two remarks are in place here. The first refers to the definition of the inverse of the operator A, i.e., of the Green's function of the operator $(\Box - m^2)$. There are many such Green's functions: the retarded one, the advanced one, the causal one, etc. The Feynman rules require the use of the causal Green's function. One can justify this within the framework of the functional integral method, by using the asymptotic conditions on the integration variables. The same condition determines the form of the external lines of the diagram. Both these circumstances are characteristic for all field theories. We have no room to discuss them in detail in this review, which is dedicated mainly to the distinguishing features of the quantization of gauge fields, to which we go on, on the example of the gravitational field.

2. FEYNMAN RULES FOR THE GRAVITATIONAL FIELD

The peculiarities of the gravitational field are related mainly to its self-interaction. Therefore the majority of this chapter will deal with the "free" self-interacting gravitational field. The main result, namely the diagram technique, is listed at the end of the chapter. We also indicate there the changes made necessary by the presence of a matter field or the electromagnetic field.

Among the most frequently used parametrizations of the gravitational field the following two are the most important: the metric tensor and the moving-frame or tetrad (Vierbein) formalism. We summarize both of these.

In the metric tensor formalism the gravitational field is described by the potentials $g_{\mu\nu}(x)$ and the Christoffel symbols $\Gamma^{\rho}_{\mu\nu}(x)$. The latter can be considered as independent dynamical variables (the Palatini formalism) or as functions of the $g_{\mu\nu}$:

$$\Gamma^{\rho}_{\mu\nu}=\frac{1}{2}g^{\rho\sigma}(\partial_{\mu}g_{\nu\sigma}+\partial_{\nu}g_{\mu\sigma}-\partial_{\sigma}g_{\mu\nu}). \quad (2.1)$$

The contravariant $g^{\mu\nu}$-matrix is the inverse of $g_{\mu\nu}$, g denotes the determinant of the matrix $g_{\mu\nu}$.

In this review we shall only consider asymptotically flat gravitational fields. In this case the space-time manifold is topologically equivalent to four-dimensional Euclidean space and can be parametrized by global coordinates x^{μ} $(-\infty < x^{\mu} < +\infty,\ \mu = 1, 2, 3)$. These coordinates shall be compatible with the conditions at spacelike infinity, such that

$$g_{\mu\nu}=\eta_{\mu\nu}+O\left(\frac{1}{r}\right),\quad \Gamma^{\rho}_{\mu\nu}=O\left(\frac{1}{r^2}\right),\qquad r\to\infty, \quad (2.2)$$

where $r = ((x^1)^2 + (x^2)^2 + (x^3)^2)^{1/2}$, and $\eta_{\mu\nu}$ is the metric tensor of Minkowski space, with the signature $(+---)$.

The action functional has the form

$$S=\frac{1}{2\varkappa^2}\int[-\Gamma^{\rho}_{\mu\rho}\partial_{\nu}(\sqrt{-g}\,g^{\mu\nu})+\Gamma^{\rho}_{\mu\nu}\partial_{\rho}(\sqrt{-g}\,g^{\mu\nu}) + \sqrt{-g}\,g^{\mu\nu}(\Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\rho\nu}-\Gamma^{\rho}_{\mu\nu}\Gamma^{\sigma}_{\rho\sigma})]\,d^4x, \quad (2.3)$$

where κ is the Newton constant, is invariant under the group of coordinate transformations acting on the quantities $g^{\mu\nu}$, $\Gamma^{\rho}_{\mu\nu}$ according to the rules

$$\begin{aligned}\delta g^{\mu\nu}&=-\eta^{\lambda}\partial_{\lambda}g^{\mu\nu}+g^{\mu\lambda}\partial_{\lambda}\eta^{\nu}+g^{\nu\lambda}\partial_{\lambda}\eta^{\mu},\\ \delta\Gamma^{\rho}_{\mu\nu}&=-\eta^{\lambda}\partial_{\lambda}\Gamma^{\rho}_{\mu\nu}-\Gamma^{\rho}_{\mu\lambda}\partial_{\nu}\eta^{\lambda}-\Gamma^{\rho}_{\nu\lambda}\partial_{\mu}\eta^{\lambda}+\Gamma^{\lambda}_{\mu\nu}\partial_{\lambda}\eta^{\rho}-\partial_{\mu}\partial_{\nu}\eta^{\rho}.\end{aligned} \quad (2.4)$$

We have written here the equations for infinitesimal transformations; η^{μ} are the infinitesimal components of a vector field which generates the coordinate transformations

$$\delta x^{\mu}=\eta^{\mu}(x). \quad (2.5)$$

Variation of the action (2.3) with respect to the $\Gamma^{\rho}_{\mu\nu}$ leads to equations, the solutions of which are the functions (2.1). In this sense one may consider the $\Gamma^{\rho}_{\mu\nu}$ as independent variables, which is sometimes convenient to do.

Substituting into the expression (2.3) the explicit form (2.1) of the Christoffel symbols $\Gamma^{\rho}_{\mu\nu}$ in terms of the metric tensor, it becomes

$$S=\frac{1}{4\varkappa^2}\int\left(h^{\rho\sigma}\partial_{\rho}h^{\mu\nu}\partial_{\nu}h_{\mu\sigma}-\frac{1}{2}h^{\rho\sigma}\partial_{\rho}h^{\mu\nu}\partial_{\sigma}h_{\mu\nu}+\frac{1}{4}h^{\rho\sigma}\partial_{\rho}\ln h\,\partial_{\sigma}\ln h\right)d^4x, \quad (2.6)$$

where for convenience we have introduced the contravariant tensor density

$$h^{\mu\nu}=\sqrt{-g}\,g^{\mu\nu},\qquad h=\det h^{\mu\nu}. \quad (2.7)$$

In the tetrad formalism (also known as the "moving-frame formalism" or "Vierbein formalism") the gravitational field is described by the components of the tetrad (frame) $e^{\mu a}(x)$ and the torsion coefficients $\omega_{\mu,ab}(x) = -\omega_{\mu,ba}(x)$. The set of $e^{\mu a}(x)$ form a matrix with positive determinant e(x). The action functional

$$S=(1/2\varkappa^2)\int[\omega_{\nu ab}\partial_{\mu}(e^{-1}e^{\mu a}e^{\nu b})-\omega_{\mu ab}\partial_{\nu}(e^{-1}e^{\mu a}e^{\nu b}) + e^{-1}e^{\mu a}e^{\nu b}(\omega_{\mu ac}\omega^{c}_{\nu b}-\omega_{\nu ac}\omega^{c}_{\mu b})]\,d^4x \quad (2.8)$$

is invariant with respect to coordinate transformations

$$\begin{aligned}\delta e^{\mu a}&=-\eta^{\lambda}\partial_{\lambda}e^{\mu a}+e^{\lambda a}\partial_{\lambda}\eta^{\mu},\\ \delta\omega_{\mu ab}&=-\eta^{\lambda}\partial_{\lambda}\omega_{\mu ab}-\omega_{\lambda ab}\partial_{\mu}\eta^{\lambda}\end{aligned} \quad (2.9)$$

and with respect to local Lorentz transformations

$$\begin{aligned}\delta e^{\mu a}&=\eta^{a}_{b}e^{\mu b},\\ \delta\omega_{\mu ab}&=\eta^{c}_{a}\omega_{\mu cb}+\eta^{c}_{b}\omega_{\mu ac}+\partial_{\mu}\eta_{ab}.\end{aligned} \quad (2.10)$$

A variation of S with respect to ω leads to equations which allow us to express ω in terms of e. The solution is conveniently written in the form

$$\omega_{\mu ab}=e^{c}_{\mu}\omega_{cab}\equiv\frac{1}{2}e^{c}_{\mu}(\Omega_{abc}+\Omega_{bca}-\Omega_{cab}), \quad (2.11)$$

where

$$\Omega_{abc}\equiv e_{\mu a}\Omega^{\mu}_{bc}\equiv e_{\mu a}(e^{\nu}_{b}\partial_{\nu}e^{\mu}_{c}-e^{\nu}_{c}\partial_{\nu}e^{\mu}_{b}).$$

If necessary, one may assume that this is already done ahead of time, so that S may be considered as a functional only of the functions $e^{\mu a}$.

We shall talk about a formalism of the first order if the variables $g_{\mu\nu}$ and $\Gamma^{\rho}_{\mu\nu}$ (or $e^{\mu a}$ and $\omega_{\mu,ab}$) are considered as independent. If the Γ are expressed in terms of the g, and the e in terms of the ω we shall talk about a formalism of the second order.

The descriptions of the free gravitational field in terms of the $g_{\mu\nu}$ or the $e^{\mu a}$ are equivalent. The difference in the number of components—10 in the first case and 16 in the second—is compensated by the difference in the gauge groups, which in the first case is parametrized by four functions and in the second case by ten. The tetrad formalism is convenient for the description of interactions with spinor fields.

The equivalence between first- and second-order formalisms may disappear when the interaction with other fields is switched on. Geometrically Eq. (2.11) defines a connection without torsion. The minimal interaction of the gravitational field with the spinor field in a first-order formalism leads to the appearance of torsion (cf. [38]).

The remainder of the exposition of this chapter will be mainly on the example of a tensor formalism of the second order. We set

$$h^{\mu\nu}=\eta^{\mu\nu}+\varkappa u^{\mu\nu} \quad (2.12)$$

and consider $u^{\mu\nu}$ a tensor field describing the gravitational field. The action functional (2.6) takes on the form

$$S=S_2+\sum_{n=1}^{\infty}\varkappa^n S_{n+2}, \quad (2.13)$$

where S_2 is a quadratic form and S_n is a form of n-th degree in the variables $u^{\mu\nu}$ and their first derivatives.

The linearization (2.12) is in many respects not a natural one. It can violate the signature of the metric tensor if $u^{\mu\nu}$ is not sufficiently small. Recently exponential parametrizations have become popular, e.g., for the tetrad matrix $e^{\mu a} = \exp(\kappa\chi^{\mu a})$. In principle the expansion (2.13) can be computed in this parametrization. We note that the quadratic form S_2 does not depend on the parametrization.

A direct application of the recipe from Chapter 1 to the case of gravitation would lead to the following formulation of covariant perturbation theory rules: the quadratic form S_2 defines the propagator and the coefficient-functions of the forms S_n give the expressions for the vertices, which in this case are infinite in number. We note here that in a first-order formalism the linearization (2.12) and the substitution

$$\Gamma^{\rho}_{\mu\nu} = \varkappa \gamma^{\rho}_{\mu\nu} \tag{2.14}$$

transforms the action into a sum of forms of second and third degrees, so that in this formalism the number of vertices is finite.

As a consequence of the invariance of the action with respect to transformations (2.4) the quadratic form

$$S_2 = \frac{1}{4} \int \left(-\eta_{\nu\sigma}\delta^{\alpha}_{\rho}\delta^{\beta}_{\mu} + \frac{1}{2}\eta^{\alpha\beta}\eta_{\mu\rho}\eta_{\nu\sigma} - \frac{1}{4}\eta_{\mu\nu}\eta_{\rho\sigma}\eta^{\alpha\beta} \right) \partial_\alpha u^{\mu\nu} \partial_\beta u^{\rho\sigma} d^4x \tag{2.15}$$

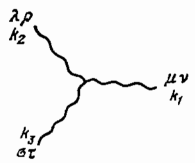
is degenerate. It does not contain the longitudinal components $\partial_\nu u^{\mu\nu}$. The example of electrodynamics suggests the idea to use as propagator for the gravitons a generalized inverse operator, e.g., the one-parameter family

$$\begin{aligned} G^{\mu\nu,\,\rho\sigma}(k) = {} & \frac{2}{k^2}\left(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} + (\alpha^{-1}-2)\eta^{\mu\nu}\eta^{\rho\sigma}\right) \\ & + \frac{2(1-\alpha^{-1})}{k^4}\left(2k^\mu k^\nu\eta^{\rho\sigma} + 2k^\rho k^\sigma\eta^{\mu\nu} - k^\mu k^\rho\eta^{\nu\sigma}\right. \\ & \left. - k^\nu k^\rho\eta^{\mu\sigma} - k^\mu k^\sigma\eta^{\nu\rho} - k^\nu k^\sigma\eta^{\mu\rho}\right), \end{aligned} \tag{2.16}$$

with the elements of the S-matrix independent of the choice of the constant α.

However, Feynman[23] has shown that the matrix elements calculated according to the naive rules depend in an essential way on this constant, and the unitarity condition is violated. Feynman has also outlined the way out of this difficulty. As a result of the efforts of a large number of authors, as described in the introduction, correct rules for the perturbation theory have been obtained, rules which we describe here. Their derivation will be given in the following chapters.

The diagram technique contains, to be sure, all the elements of the "naive approach": graviton lines, for which the expressions have the form (2.16) and vertices generated by the forms S_{n+2} of expansions of the type (2.13). Here is the explicit expression for the third-order vertex, corresponding to the linearization (2.12):



$$\begin{aligned} = \frac{\varkappa}{2^5} \Big\{ \Big[\frac{k_1^2}{2} & (\eta_{\mu\lambda}\eta_{\rho\sigma}\eta_{\tau\nu} + \eta_{\mu\tau}\eta_{\nu\lambda}\eta_{\rho\sigma} + \eta_{\mu\sigma}\eta_{\nu\lambda}\eta_{\rho\tau} + \eta_{\nu\sigma}\eta_{\mu\lambda}\eta_{\rho\tau} \\ & + \eta_{\mu\tau}\eta_{\nu\rho}\eta_{\mu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}\eta_{\lambda\tau} + \eta_{\nu\tau}\eta_{\mu\rho}\eta_{\lambda\sigma} + \eta_{\nu\sigma}\eta_{\mu\rho}\eta_{\lambda\sigma}) \\ & + k_1^2\eta_{\mu\nu}(\eta_{\rho\sigma}\eta_{\lambda\tau} + \eta_{\rho\tau}\eta_{\lambda\sigma}) + (k_{2\mu}k_{3\nu} + k_{2\nu}k_{3\mu})\eta_{\lambda\rho}\eta_{\sigma\tau} \\ & + (k_{2\mu}k_{3\nu} + k_{2\nu}k_{3\mu})(\eta_{\lambda\sigma}\eta_{\rho\tau} + \eta_{\lambda\rho}\eta_{\sigma\tau}) \\ & - k_{1\nu}k_{1\tau}(\eta_{\mu\lambda}\eta_{\rho\sigma} + \eta_{\mu\rho}\eta_{\lambda\sigma}) - k_{1\nu}k_{1\sigma}(\eta_{\mu\lambda}\eta_{\rho\tau} + \eta_{\mu\rho}\eta_{\lambda\tau}) \\ & - k_{1\mu}k_{1\tau}(\eta_{\nu\lambda}\eta_{\rho\sigma} + \eta_{\nu\rho}\eta_{\lambda\sigma}) - k_{1\mu}k_{1\sigma}(\eta_{\nu\lambda}\eta_{\rho\tau} + \eta_{\nu\rho}\eta_{\lambda\tau}) \\ & - k_{2\nu}k_{3\rho}\eta_{\lambda\sigma}\eta_{\mu\tau} - k_{2\nu}k_{3\lambda}\eta_{\rho\sigma}\eta_{\mu\tau} - k_{2\mu}k_{3\rho}\eta_{\lambda\tau}\eta_{\nu\tau} \\ & - k_{2\mu}k_{3\lambda}\eta_{\rho\sigma}\eta_{\nu\tau} - k_{2\nu}k_{3\rho}\eta_{\lambda\tau}\eta_{\mu\sigma} - k_{2\nu}k_{3\lambda}\eta_{\rho\tau}\eta_{\mu\sigma} \\ & - k_{2\mu}k_{3\rho}\eta_{\lambda\tau}\eta_{\nu\sigma} - k_{2\mu}k_{3\lambda}\eta_{\rho\tau}\eta_{\nu\sigma} \Big] \end{aligned} \tag{2.17}$$

+ the sum over permutations of the pairs $((\mu, \nu), (\lambda, \rho), (\sigma, \tau)) \Big\}$.

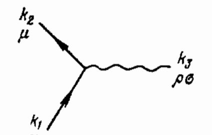
In addition to these elements, in the internal parts of the diagrams one must make use of additional elements which can be interpreted in terms of vector particles interacting with gravitons. Such an interpretation is introduced only for convenience. The mentioned vector particles have no independent physical meaning and they are usually called "fictitious" (or, by other authors: "Faddeev-Popov ghosts"—Transl.). The propagator of such a fictitious particle has the form

$$G^{\mu\nu} = -\eta^{\mu\nu}/k^2, \tag{2.18}$$

and its interaction vertex with gravitons is generated by the trilinear form

$$\varkappa \int \bar{\theta}_\mu (u^{\nu\lambda}\partial_\nu\partial_\lambda\theta^\mu + \partial_\nu u^{\nu\lambda}\partial_\lambda\theta^\mu - \partial_\nu u^{\mu\nu}\partial_\lambda\theta^\lambda - \partial_\lambda\partial_\nu u^{\mu\nu}\theta^\lambda)\, d^4x \tag{2.19}$$

and has the expression



$$= -\frac{\varkappa}{2}\left[\delta^\mu_\nu (k_{1\rho}k_{2\sigma} + k_{1\sigma}k_{2\rho}) - k_{1\nu}(\delta^\mu_\sigma k_{3\rho} + \delta^\mu_\rho k_{3\sigma})\right]$$

$$(k_1 + k_2 + k_3) = 0). \tag{2.20}$$

Fictitious elements participate only in closed loops and the external lines are always graviton lines.

The contribution from a given diagram is obtained if the product of expressions of the form (2.16)—(2.18), (2.20) associated to its elements is integrated over the internal momenta, and the result is multiplied by

$$[i/(2\pi)^4]^{l-v+1} r^{-1} (-1)^s, \tag{2.21}$$

where s is the number of closed loops formed by fictitious particles. A comparison of this equation with (2.5) shows that the fictitious particles act like fermions, i.e., they violate the spin-statistics theorem. This shows that their role reduces to a subtraction of the contributions from unphysical degrees of freedom.

In addition to the described diagrams perturbation theory involves infinite contributions of the renormalization type, contributions which are proportional to powers of the delta-function $\delta^{(4)}(0)$. The structure of these terms will be described below in Chap. 3. In the first-order formalism the elements associated to fictitious particles do not change. In addition to the tensor propagator $\langle uu \rangle$ the perturbation theory also involves the propagators $\langle u\gamma \rangle$ and $\langle \gamma\gamma \rangle$. In the momentum-space representation these propagators have the form:

$$\begin{aligned} G^{\rho,\,\mu\nu}_{\rho\tau}(k) &= \frac{i}{2}(\eta_{\sigma\alpha}\delta^\rho_\beta k_\tau + \eta_{\tau\alpha}\delta^\rho_\beta k_\sigma - \eta_{\mu\alpha}\eta_{\nu\beta}k^\rho)\, G^{\alpha\beta,\,\mu\nu}(k) = \Omega^\rho_{\sigma\tau,\,\alpha\beta}(k)\, G^{\alpha\beta,\,\mu\nu}(k), \\ G^{\rho}_{\sigma\tau,}{}^{\lambda}_{\mu\nu}(k) &= \frac{1}{4}(\delta^\rho_\mu\delta^\lambda_\sigma\eta_{\nu\tau} + \delta^\rho_\nu\delta^\lambda_\sigma\eta_{\mu\tau} + \delta^\rho_\mu\delta^\lambda_\tau\eta_{\nu\sigma} + \delta^\rho_\nu\delta^\lambda_\tau\eta_{\mu\sigma}) \\ & \quad - \frac{1}{6}(\delta^\lambda_\nu\delta^\rho_\tau\eta_{\mu\sigma} + \delta^\lambda_\mu\delta^\rho_\tau\eta_{\nu\sigma} + \delta^\lambda_\nu\delta^\rho_\sigma\eta_{\mu\tau} + \delta^\lambda_\mu\delta^\rho_\sigma\eta_{\nu\tau}) \\ & \quad + \Omega^\lambda_{\mu\nu,\,\alpha\beta}(k)\, \Omega^\rho_{\sigma\tau,\,\gamma\delta}(-k)\, G^{\alpha\beta,\,\gamma\delta}(k), \end{aligned} \tag{2.22}$$

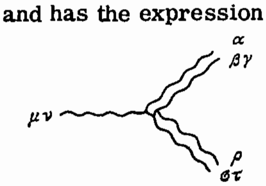
where

$$\Omega^\rho_{\sigma\tau,\,\alpha\beta}(k) = \frac{i}{2}(\eta_{\sigma\alpha}\delta^\rho_\beta\delta^\gamma_\tau + \eta_{\tau\alpha}\delta^\rho_\beta\delta^\gamma_\sigma - \eta_{\mu\alpha}\eta_{\nu\beta}\eta^{\rho\gamma})\, k_\gamma. \tag{2.23}$$

The only graviton vertex is generated by the trilinear form

$$\frac{\varkappa}{2}\int u^{\mu\nu}(\gamma^\rho_{\mu\sigma}\gamma^\sigma_{\rho\nu} - \gamma^\rho_{\mu\nu}\gamma^\sigma_{\rho\sigma})\, d^4x \tag{2.24}$$

and has the expression



$$= \frac{\varkappa}{8}\left[(\delta^\sigma_\mu\delta^\tau_\nu + \delta^\sigma_\nu\delta^\tau_\mu)(\delta^\beta_\alpha\delta^\gamma_\rho + \delta^\beta_\rho\delta^\gamma_\alpha)\right.$$

$$+(\delta^{\sigma}_{\alpha}\delta^{\tau}_{\rho}+\delta^{\sigma}_{\rho}\delta_{\alpha})(\delta^{\beta}_{\mu}\delta^{\gamma}_{\nu}+\delta^{\beta}_{\nu}\delta^{\gamma}_{\mu})-\delta^{\sigma}_{\mu}\delta^{\tau}_{\alpha}\delta^{\gamma}_{\nu}\delta^{\beta}_{\rho}-\delta^{\sigma}_{\mu}\delta^{\beta}_{\nu}\delta^{\gamma}_{\rho}\delta^{\tau}_{\alpha}-\delta^{\sigma}_{\nu}\delta^{\tau}_{\alpha}\delta^{\beta}_{\mu}\delta^{\gamma}_{\rho}$$
$$-\delta^{\sigma}_{\nu}\delta^{\tau}_{\alpha}\delta^{\gamma}_{\mu}\delta^{\beta}_{\rho}-\delta^{\tau}_{\mu}\delta^{\sigma}_{\alpha}\delta^{\beta}_{\nu}\delta^{\gamma}_{\rho}-\delta^{\tau}_{\mu}\delta^{\sigma}_{\alpha}\delta^{\gamma}_{\nu}\delta^{\beta}_{\rho}-\delta^{\tau}_{\nu}\delta^{\sigma}_{\alpha}\delta^{\beta}_{\mu}\delta^{\rho}_{\gamma}-\delta^{\tau}_{\nu}\delta^{\sigma}_{\alpha}\delta^{\gamma}_{\mu}\delta^{\beta}_{\rho}]. \quad (2.25)$$

We do not write out the elements of the diagram technique in the tetrad formalism. The reader can obtain them by himself using the methods described below. The inclusion of matter fields does not lead to the appearance of new fictitious particles and the corresponding elements of the diagram technique, as long as the quadratic forms in the action functionals of the matter fields are nondegenerate.

As an example we list the interaction Lagrangians of the gravitational field with a scalar field and with a spinor field. In the first case it is convenient to use the metric formalism and in the second case one must use the tetrad formalism:

$$\begin{aligned} S[\omega] &= \frac{1}{2}\int d^4x\sqrt{-g}\,(g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - m^2\varphi^2),\\ S[\psi,\bar{\psi}] &= \int d^4xe\,[i\bar{\psi}e^{\mu a}\gamma_a(\partial_\mu\psi+\omega_{\mu,\,cd}\sigma^{cd}\psi)-m\bar{\psi}\psi]; \end{aligned} \quad (2.26)$$

here we have utilized the standard notations for the components of the scalar and spinor fields; γ^a and $\sigma^{cd} = \frac{1}{4}(\gamma^c\gamma^d - \gamma^d\gamma^c)$ are the usual Dirac matrices. We note that if there is no mass term it is easy to choose the parametrization of the gravitational field in such a manner that these functionals generate only a finite number of vertices.

3. A DERIVATION OF THE MODIFIED PERTURBATION THEORY RULES

Functional integration is a convenient heuristic means for the explanation and heuristic derivation of the rules of perturbation theory for the gravitational field, rules which have been enumerated in the preceding chapter. Within the framework of this approach the additional terms mentioned there are interpreted as a consequence of the nontriviality of the measure with respect to which the functional exp(iS) is being integrated.

Let us explain this in more detail, making use of natural geometric terms. The functional exp(iS) is invariant with respect to the infinite group of coordinate transformations in the metric formalism or with respect to the semidirect product of this transformation group and the group of local Lorentz transformations in the moving frame (tetrad) formalism. Thus, this functional is a function on (equivalence) classes of fields, where one class unites all the gravitational fields which can be transformed into one another by transformations from the indicated group.

We shall assume that it is a class rather than an individual field which describes a concrete physical situation. This is the content of the principle of general covariance. Keeping such a formulation of this principle in mind we can consider that in quantum theory the Feynman functional exp(iS) should be integrated with respect to such classes of fields, rather than individual fields. The nontriviality of the measure which we have mentioned is related just to this circumstance.

Let us discuss how one can describe measures on the set of classes of fields. Following our method, we first consider the finite-dimensional case. Mathematically we are dealing with the following situation: we are given a manifold M (in the sequel this will be the set of all fields) and a group G (in the sequel this will be the group of gauge transformations) acting on M. Let ξ be a point in M and a a group element, ξ^a denotes the action of the group element a on the point ξ. Consider the quotient manifold $M/G = M^*$, formed by the class of all points of the form ξ^a, where ξ (the representative of the class) is fixed and a runs over the whole group (ξ^a is also called the orbit of the point ξ and M^* is the orbit-space of G—Transl.).

Any measure μ^* on the orbit space M^* can be extended, in view of its being constant on the classes, to a measure μ on M which is invariant under the group action. Conversely, given an invariant measure μ on M it is not hard to construct a measure μ^* on M^*, which extends to μ in the sense indicated. One can make the selection of representatives from the classes concrete by defining in M a hypersurface which intersects each orbit (class) once. This means that if the hypersurface is defined by the equation

$$f(\xi) = 0, \quad (3.1)$$

then for given ξ the system of equations

$$f(\xi^a) = 0 \quad (3.2)$$

must have a unique solution a (depending in general on ξ). If one uses such a parametrization the measure μ^* looks as follows:

$$d\mu^*(\xi) = \delta(f(\xi))\,\Delta_f(\xi)\,d\mu, \quad (3.3)$$

where the function $\Delta_f(\xi)$ is defined by the relation

$$\Delta_f(\xi)\int\delta(f(\xi^a))\,da = 1, \quad (3.4)$$

and da is the invariant measure on the group G. The function $\Delta_f(\xi)$ is invariant, i.e., $\Delta_f(\xi^a) = \Delta_f(\xi)$. Equation (3.3) can be explained in the following way. We go over to the new variables

$$\xi \to (\xi_T, a), \quad (3.5)$$

where a is the group element defined by Eq. (3.2) and $\xi_T = \xi^a$. Let the invariant measure on M have the following expression in terms of the coordinates ξ:

$$d\mu = M(\xi)\,d\xi. \quad (3.6)$$

In the new variables it will have the form

$$M(\xi)\,D_f\,d\xi_T\,da, \quad (3.7)$$

where

$$d\xi_T = \delta(f(\xi))\,d\xi \quad (3.8)$$

is the measure generated by the measure $d\xi$ on the hypersurface, D_f is the Jacobian of the transformation to the variables (ξ_T, a). The measure on the orbit space M^* is obtained from (3.7) by omitting the invariant measure da in the group coordinates.

We show that the Jacobian D_f coincides with the invariant function $\Delta_f(\xi)$ defined by the relation (3.4). We consider the integral

$$\int\Phi(\xi)\,M(\xi)\,d\xi \quad (3.9)$$

of the arbitrary invariant function $\Phi(\xi)$, integrated with respect to the measure $M(\xi)d\xi$. The invariance condition

$$\Phi(\xi^a) = \Phi(\xi)$$

allows one to assume that Φ depends only on ξ_T. Indeed, since $\xi_T = \xi^a$,

$$\Phi(\xi) = \Phi(\xi^a) = \Phi(\xi_T).$$

Making use of the expression (3.7) for the measure $M(\xi)d\xi$ in the integral (3.9), we transform the integral to the form

$$\int \Phi(\xi) M(\xi) D_f(\xi) d\xi_T da = \mu(G) \int \Phi(\xi_T) M(\xi_T) D_f(\xi_T) d\xi_T, \qquad (3.10)$$

where $\mu(G) = \int da$ is the "volume of the group." Another expression for the integral (3.9) can be obtained by introducing into the integrand the factor (3.4) which equals one, then carrying out the substitution $\xi^a \to \xi$, with respect to which the functions Φ, Δ_f and the measure $Md\xi$ are invariant:

$$\int \Phi(\xi) M(\xi) d\xi = \int \Phi(\xi) M(\xi) [\Delta_f(\xi) \int \delta(f(\xi^a)) da] d\xi \qquad (3.11)$$
$$= \int \Phi(\xi) \Delta_f(\xi) M(\xi) \delta(f(\xi)) d\xi da = \mu(G) \int \Phi(\xi_T) M(\xi_T) \Delta_f(\xi_T) d\xi_T.$$

In view of the arbitrariness of the function $\Phi(\xi)$ the measures in the integrals (3.10) and (3.11) must coincide:

$$\mu(G) M(\xi_T) D_f(\xi_T) d\xi_T = \mu(G) M(\xi_T) \Delta_f(\xi_T) d\xi_T,$$

which yields the equality $D_f = \Delta_f$.

Let us return to the gravitational field. From what we just said it is clear that in order to define a measure in a class of fields it suffices to define the measure on a manifold of fields which is invariant with respect to coordinate transformations (and local Lorentz transformations) and to specify the equations which parametrize the classes. These equations should be Lorentz-invariant if we wish to obtain a covariant perturbation theory. In the metric formalism we choose as such equations the harmonicity conditions of de Donder-Fock:

$$\partial_\nu (\sqrt{-g}\, g^{\mu\nu}) = l^\mu(x), \qquad (3.12)$$

where $l^\mu(x)$ is a prescribed vector field. The arbitrariness in the choice of $l^\mu(x)$ will be useful in the sequel for formal transformations. The condition (3.12) is not generally covariant and therefore can serve for the parametrization of classes. The analog of Eq. (3.2) is a complicated nonlinear equation for the parameters of the coordinate transformation which takes a given metric into a harmonic one. Within the framework of perturbation theory this equation has a unique solution.

In the following chapters it will be shown that one must select as the invariant measure the expression[3)]

$$\prod_x \Big[g^{5/2}(x) \prod_{\mu \leq \nu} dg^{\mu\nu}(x) \Big] = \prod_x \Big[h^{-5/2}(x) \prod_{\mu \leq \nu} dh^{\mu\nu}(x) \Big], \qquad (3.13)$$

where

$$g = \det g_{\mu\nu}, \quad h^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu}, \quad h = \det h^{\mu\nu} = g. \qquad (3.14)$$

This will be done on the basis of an investigation of the Hamiltonian formulation, which we shall interpret as an alternative, non-Lorentz-invariant method of parametrization of field classes.

The advantage of the Hamiltonian formalism is the fact that in it the unitarity condition leads to the standard expression for the integration measure.

Having the parametrization (3.12) of the classes and the measure (3.13) we obtain the following expression for the functional integral:

$$\int \exp(iS) \Delta_h[g] \prod_x \Big[\prod_\mu \delta(\partial_\nu h^{\mu\nu} - l^\mu) \Big] \Big(g^{5/2} \prod_{\mu \leq \nu} dg^{\mu\nu} \Big), \qquad (3.15)$$

where according to (3.4) the functional $\Delta_h[g]$ is defined by the equation

$$\Delta_h[g] \int \prod_x \Big[\prod_\mu \delta(\partial_\nu (h^{\mu\nu})^a - l^\mu) \Big] da = 1 \qquad (3.16)$$

and thus is expressed in terms of the integral of the δ-functional $\prod_{x,\mu} \delta(\delta_\nu (h^{\mu\nu})^a - l^\mu)$ over the gauge group.

Let us discuss the calculation of this integral. The expression $\Delta_h[g]$ enters into the integral (3.15) only on the hypersurface determined by the equations (3.16). For such $g^{\mu\nu}$ the total contribution to the integral from (3.16) comes from an infinitesimal neighborhood of the unit element of the group. In this neighborhood the action of group transformations on the $h^{\mu\nu}$ and the measure da can be parametrized by means of the infinitesimal functions $\eta^\mu(x)$ introduced above in (2.5). With this parametrization

$$\partial_\nu (h^{\mu\nu})^{a(x)} - l^\mu(x)$$
$$= h^{\nu\lambda}(x) \partial_\nu \partial_\lambda \eta^\mu(x) + \partial_\nu h^{\nu\lambda}(x) \partial_\lambda \eta^\mu(x) - \partial_\nu \eta^{\mu\nu}(x) \partial_\lambda \eta^\lambda(x) \qquad (3.17)$$
$$- \partial_\lambda \partial_\nu \eta^{\mu\nu}(x) \eta^\lambda(x).$$

At the unit element of the group, the measure da has the simple form

$$da = \prod_x \prod_\mu d\eta^\mu(x). \qquad (3.18)$$

Consequently, the integral in which we are interested has the form

$$\int \prod_{x,\mu} \delta((A\eta)^\mu(x)) d\eta^\mu(x). \qquad (3.19)$$

Formally this integral equals $(\det A)^{-1}$ where A is the operator acting on the quartet of functions η^μ:

$$(A\eta)^\mu = \partial_\nu (h^{\nu\lambda} \partial_\lambda \eta^\mu) - \partial_\lambda (\eta^\lambda \partial_\nu h^{\mu\nu}). \qquad (3.20)$$

Thus we have found that

$$\Delta_h[g] = \det A. \qquad (3.21)$$

For the formulation of perturbation theory it is convenient to represent det A as an integral over auxiliary fields of a functional of exponential type. These fields must be anticommuting fields, since we need an integral which yields the first power of the determinant. These requirements are satisfied by the expression

$$\det A = \int \exp\Big\{ i \int \bar\theta^\mu(x) A_{\mu\nu}[g(x)] \theta^\nu(x) d^4x \Big\} \prod_{x,\mu} d\theta^\mu(x) d\bar\theta^\mu(x), \qquad (3.22)$$

where θ and $\bar\theta$ are classical anticommuting fields satisfying the relations

$$\theta^\mu(x) \theta^\nu(y) + \theta^\nu(y) \theta^\mu(x) = 0 \qquad (3.23)$$

and similar relations for the pairs $(\theta, \bar\theta)$, $(\bar\theta, \bar\theta)$. A definition and rules of operation with integrals over anticommuting variables can be found, e.g., in the monograph of Berezin[39].

Returning to the integral (3.15) we can not write it in the form

$$\int \exp\Big\{ iS[g] + i \int \bar\theta^\mu A_{\mu\nu}[g] \theta^\nu d^4x \Big\} \qquad (3.24)$$
$$\times \prod_x \Big[\prod_\mu \delta(\partial_\mu h^{\mu\nu} - l^\mu) \Big] \Big(g^{5/2} \prod_{\mu \leq \nu} dg^{\mu\nu} \Big),$$

which can be used directly for the formulation of perturbation theory. However, we still transform it, making use of the arbitrariness in the selection of μ. The integral (3.24) does not depend on the choice of l^μ, by definition. We may therefore average it over l^μ with an arbitrary weight. Let us use as a weight the exponential of the quadratic form in the fields

$$\exp[(i\alpha/4) \int l^\mu \eta_{\mu\nu} l^\nu d^4x], \qquad (3.25)$$

where $\eta_{\mu\nu}$ is the Minkowski metric tensor. The averaging can be done explicitly and yields the expression

$$\int \exp\left\{iS[g]+\frac{i\alpha}{4}\int \partial_\rho h^{\mu\rho}\eta_{\mu\nu}\partial_\sigma h^{\nu\sigma}\,d^4x + i\int \bar{\theta}^\mu A_{\mu\nu}\theta^\nu d^4x\right\}\prod_x g^{5/2}\Big(\prod_{\mu\leqslant\nu} dg^{\mu\nu}\Big)\Big(\prod_\mu d\theta^\mu d\bar{\theta}^\mu\Big), \quad (3.26)$$

which contains a quadratic form in the longitudinal parts of the fields $h^{\mu\nu}$ with the arbitrary coefficient α. It follows from our reasoning that the integral does not depend on α. A method for seeing this directly has been proposed by DeWitt[24]. Here we have followed the simpler method of 't Hooft[31].

The diagram technique discussed in the preceding chapter follows from the expression (3.26) in the same manner as explained in Chap. 1 for the example of the scalar field. By introducing the "fictitious" fields θ^μ and $\bar{\theta}^\mu$ we have managed to make the quadratic form in the exponent nondegenerate. Thus the inverse operators corresponding to the operators of the quadratic forms in $h^{\mu\nu}$, θ^μ and $\bar{\theta}^\nu$ become well defined, i.e., we obtain the propagators of the particles corresponding to the lines of the diagrams. The graviton propagator $\langle h^{\mu\nu}h^{\rho\sigma}\rangle$ contains the arbitrary constant α. The propagator $\langle\theta^\mu\bar{\theta}^\nu\rangle$ of the fictitious vector particles in the k-representation is given by Eq. (2.18). The anticommutation of the fields θ^μ, $\bar{\theta}^\mu$ leads to the factor $(-1)^s$ for a diagram containing s closed fermion loops. The higher-degree forms in the expansion of the action (3.26) in powers of the fields $h^{\mu\nu}$, θ^μ, $\bar{\theta}^\nu$ give rise to the vertex functions of the diagram technique, as described in Chap. 2. Their concrete form depends on the choice of linearization used.

We note also the role of the local factor $\prod_x h^{-5/2}(x)$ in the measure. For the linearization (2.12) we have

$$\prod_x dh^{\mu\nu}=\prod_x du^{\mu\nu}, \quad (3.27)$$

and thus, this factor should be taken into account in the construction of perturbation theory. Formally its role reduces to the appearance of a contribution of the form

$$\Delta S = (5/2)\, i\delta^{(4)}(0)\int \ln h(x)\, d^4x, \quad (3.28)$$

in the action giving rise to vertices which are proportional to $\delta^{(4)}(0)$. The appearance of such renormalization terms is noted in many papers treating nonlinear theories (cf., e.g.,[40,41]). We note that they are absent in the exponential parametrization. In this parametrization the measure (3.13) has, up to a constant factor, the simple form

$$\prod_x \prod_{\mu\leqslant\nu} d\Phi^{\mu\nu} \quad (3.29)$$

without any local additions.

We have considered in detail the case of the gravitational field in vacuo. Introducing interactions with other fields does not change substantially the scheme of construction of perturbation theory. For matter fields with nondegenerate Lagrangians interacting with the gravitational field no new fictitious particles appear. Such particles and their corresponding diagrams appear only when a field with larger gauge group than the gravitational field is included, e.g., the electromagnetic field or fields of the Yang-Mills type. We shall not consider this case in detail here. We just list, as an example the expression of the functional integral corresponding to the electromagnetic and gravitational fields:

$$\int \exp\left\{iS[g^{\mu\nu}, A_\mu]\,\Delta[g]\prod_x \delta(\partial_\mu h^{\mu\nu}A_\nu)\prod_\mu \delta(\partial_\nu h^{\mu\nu})\,|g|^{5/2}\prod_{\mu\leqslant\nu} dg^{\mu\nu}\prod_\mu dA_\mu\right\},$$

$$S[g^{\mu\nu}, A_\mu]=S_g-\frac{1}{4}\int(\partial_\mu A_\nu-\partial_\nu A_\mu)(\partial_\lambda A_\rho-\partial_\rho A_\lambda)\,g^{\mu\lambda}g^{\nu\rho}\sqrt{-g}\,d^4x,$$

where S_g is the action of the free gravitational field and $\Delta[g]$ equals the product of the determinants

$$\det A\cdot\det(\partial_\mu h^{\mu\nu}\partial_\nu),$$

where A is the operator (3.20). The presence of a nontrivial second factor in this product shows that the inessential scalar fictitious particle which could be introduced for the description of the electromagnetic field also interacts with the gravitational field. Thus, in covariant perturbation theory for the electromagnetic and gravitational fields a fictitious neutral scalar particle participates in addition to the elements which have been described above.

The reader who has understood the basic principles of construction of the diagram technique for gauge fields may, if he wishes, perform the appropriate calculations for the more complicated case.

4. THE HAMILTONIAN FORMULATION OF GRAVITATION THEORY

A justification of the correctness of the expression (3.13) for the invariant measure is based on the Hamiltonian formulation of gravitation theory. This formulation has been developed by Dirac[14]. A series of variants for this formulation were obtained by diverse authors[15-19]. The construction of an explicitly Hamiltonian form of the Einstein equations runs into the difficult problem of finding solutions to the constraint equations. For us it will be sufficient to consider a generalized Hamiltonian formulation of gravitation theory, where it is not necessary to solve the constraint equations, and one can restrict one's attention to a verification of the commutation relations.

We explain the generalized Hamiltonian formulation of the example of a system with a finite number of degrees of freedom. In this formulation the action functional of the system under consideration has the form

$$S=\int\Big(\sum_{i=1}^n p_i\dot{q}^i-\mathscr{H}(p, q)-\sum_a \lambda_a\varphi^a(p, q)\Big)\,dt. \quad (4.1)$$

Here p, q denote canonically conjugate coordinates and momenta which form a phase space of dimension 2n, $\mathscr{H}$ is the Hamiltonian, φ^a are the "constraints," λ_a are Lagrange multipliers ($a = 1, \ldots, m$, $m < n$). The constraints φ^a and the Hamiltonian $\mathscr{H}$ are in involution, i.e., satisfy the conditions

$$\{\mathscr{H}, \varphi^a\}=\sum_a c^a_b\varphi^b,$$
$$\{\varphi^a, \varphi^b\}=\sum_d c^{ab}_d\varphi^d. \quad (4.2)$$

In these equations the notation $\{f, g\}$ is adopted for the usual Poisson brackets

$$\{f, g\}=\sum_{i=1}^n\Big(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i}-\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}\Big). \quad (4.3)$$

The conditions (4.2) lead to a reduction of the dimension of phase space to $2(n-m)$. This space can be realized as a submanifold Γ^* in the space Γ, determined by the m constraint equations

$$\varphi^a(p, q)=0, \qquad a=1, \ldots, m \quad (4.4)$$

and the m supplementary conditions

$$\chi_a(p, q)=0, \qquad a=1, \ldots, m. \quad (4.5)$$

The functions χ_a are subject to the conditions

$$\det\|\{\chi_a, \varphi^b\}\|\neq 0. \quad (4.6)$$

In addition it is convenient to assume that the functions χ_a commute with one another:

$$\{\chi_a, \chi_b\} = 0. \qquad (4.7)$$

In this case it is simple to introduce canonical variables on the submanifold Γ^*. Indeed, in view of (4.7), a canonical transformation in Γ allows one to go over to new variables, where the χ_a take the form

$$\chi_a = p_a, \qquad a = 1, \ldots, m, \qquad (4.8)$$

where p_a are part of the canonical momenta of the new system of variables. Let q^a denote the coordinates conjugate to them and p^*, q^* the other canonical variables. In the new variables the condition (4.6) has the form

$$\det \left\| \frac{\partial \varphi^a}{\partial q^b} \right\| \neq 0$$

and can be interpreted as the condition for solvability of the constraints $\varphi^a = 0$ with respect to the coordinates q^a. As a result of this the surface Γ^* is defined in Γ by the equations

$$p_a = 0, \quad q^a = q^a(p^*, q^*),$$

so that the p^* and q^* play the roles of independent variables on Γ^*. By construction these variables are canonical. A more detailed discussion of the generalized Hamiltonian formulation for the finite-dimensional case with applications to the theory of the Yang-Mills field is contained in a paper by one of the authors[42].

Let us return to the gravitational field. We shall show that its action can be reduced to a form which is a field-theoretic analog of (4.1), with the appropriate constraints and the Hamiltonian satisfying conditions of the type (4.2). We shall follow the general method proposed by one of the authors in a form especially adapted for the gravitational field[19].

For our purposes it is convenient to make use of a formalism of the first order. We consider the expression of the action for the gravitational field in the form (2.3) and collect in the corresponding Lagrange function all the terms which involve derivatives with respect to time:

$$\frac{1}{2\varkappa^2}(\Gamma^0_{\mu\nu}\,\partial_0 h^{\mu\nu} - \Gamma^\rho_{\mu\rho}\,\partial_0 h^{\mu 0}) = \frac{1}{2\varkappa^2}[\Gamma^0_{ik}\,\partial_0 h^{ik} + (\Gamma^0_{i0} - \Gamma^k_{ik})\,\partial_0 h^{i0} - \Gamma^i_{0i}\partial_0 h^{00}]. \qquad (4.9)$$

This expression does not contain the variables Γ^μ_{00} which occur in $\mathscr{L}(h, \Gamma)$ linearly and play the roles of Lagrange multipliers. The factors (denoted by A^{00}_μ) in front of Γ^μ_{00} are the constraints. The constraint equations

$$\begin{aligned} A^{00}_0 &= h^{ik}\Gamma^0_{ik} + h^{00}\Gamma^i_{0i} + \partial_i h^{i0} = 0,\\ A^{00}_i &= 2h^{k0}\Gamma^0_{ik} + h^{00}(\Gamma^0_{i0} - \Gamma^k_{ik}) + \partial_i h^{00} = 0 \end{aligned} \qquad (4.10)$$

allow us to express the variables Γ^i_{0i}, $(\Gamma^0_{i0} - \Gamma^k_{ik})$ in terms of the Γ^0_{ik} and $h^{\mu\nu}$. Then the terms containing time-derivatives take the form

$$\frac{1}{2\varkappa^2}\frac{\Gamma^0_{ik}}{h^{00}}\,\partial_0(h^{00}h^{ik} - h^{i0}h^{k0}), \qquad (4.11)$$

if one omits the terms

$$\frac{1}{2\varkappa^2 h^{00}}(\partial_0 h^{00}\,\partial_i h^{i0} - \partial_i h^{00}\,\partial_0 h^{i0}) = \frac{1}{2\varkappa^2}(\partial_0 \ln h^{00}\,\partial_i h^{i0} - \partial_i \ln h^{00}\,\partial_0 h^{i0}), \qquad (4.12)$$

which vanish upon integration by parts. Equation (4.11) suggests that the natural dynamical variables are the quantities

$$q^{ik} = h^{i0}h^{k0} - h^{00}h^{ik}, \quad \pi_{ik} = -\frac{1}{h^{00}}\Gamma^0_{ik}. \qquad (4.13)$$

The variables $\Gamma^\rho_{\mu\nu}$ which differ from the Γ^0_{ik} are not dynamical variables. They can be excluded with the help of the constraint equations

$$\frac{\partial \mathscr{L}(h, \Gamma)}{\partial \Gamma^\rho_{\mu\nu}} = 0 \qquad (\Gamma^\rho_{\mu\nu} \neq \Gamma^0_{ik}). \qquad (4.14)$$

The system (4.14) contains the equations (4.10) together with the equations

$$\begin{aligned} \partial_k h^{i0} + h^{is}\Gamma^0_{sk} + h^{00}\Gamma^i_{k0} + h^{s0}\Gamma^i_{ks} - h^{i0}\Gamma^s_{ks} = 0,\\ \partial_k h^{ij} + h^{i\sigma}\Gamma^j_{\sigma k} + h^{j\sigma}\Gamma^i_{\sigma k} - h^{ij}\Gamma^\sigma_{k\sigma} = 0. \end{aligned} \qquad (4.15)$$

The solution of the system (4.10), (4.15) expressing the "nondynamical" quantities Γ^0_{i0}, Γ^k_{i0}, Γ^k_{ij} in terms of the Γ^0_{ik} and $h^{\mu\nu}$ is of the form

$$\left.\begin{aligned} \Gamma^0_{i0} &= \Gamma^s_{is} - \frac{\partial_i h^{00}}{h^{00}} - \frac{h^{s0}}{h^{00}}\Gamma^0_{is},\\ \Gamma^k_{i0} &= -\frac{1}{h^{00}}(\partial_i h^{k0} + h^{s0}\Gamma^k_{is} - h^{k0}\Gamma^s_{is} + h^{ks}\Gamma^0_{is}),\\ \Gamma^k_{ij} &= \overset{*}{\Gamma}{}^k_{ij} + \frac{h^{k0}}{h^{00}}\Gamma^0_{ij}; \end{aligned}\right\} \qquad (4.16)$$

where $\overset{*}{\Gamma}{}^k_{ij}$ are the three-dimensional connection coefficients, defined by the three-dimensional metric g_{ik} (i, k = 1, 2, 3).

Let us substitute the expression (4.16) for the Γ^0_{i0}, Γ^k_{i0}, Γ^k_{ij} into the Lagrange function $\mathscr{L}(h, \Gamma)$. After omitting several terms of the type of a divergence, which vanish when integrated over three-space when the asymptotic conditions (2.2) are taken into account, the result of the substitution reduces to the form

$$\frac{1}{2\varkappa^2}\left[\pi_{ik}\partial_0 q^{ik} - \mathscr{H}(x) - \left(\frac{1}{h^{00}} - 1\right)T_0(x) - \frac{h^{i0}}{h^{00}}T_i(x)\right], \qquad (4.17)$$

where

$$\left.\begin{aligned} T_0(x) &= q^{ij}q^{kl}(\pi_{ik}\pi_{jl} - \pi_{ij}\pi_{kl}) + g_3 R_3,\\ T_i(x) &= 2[\nabla_i(q^{kl}\pi_{kl}) - \nabla_k(q^k\pi_{il})],\\ \mathscr{H}(x) &= T_0(x) - \partial_i\partial_k q^{ik}; \end{aligned}\right\} \qquad (4.18)$$

here $g_3 = \det g_{ik}$, R_3 is the three-dimensional curvature scalar associated to the three-dimensional metric g_{ik} (i, k, = 1, 2, 3). The symbol ∇_k in the expressions for the constraints T_i denotes the covariant derivative with respect to the metric g_{ik}.

As pointed out by Arnowitt, Deser and Misner[15], the canonical variables and the expressions for the constraints have an intuitive geometric meaning. The functions q^{ik} and π_{ik} serve as the coefficients of the first and second quadratic forms associated to the surface x^0 = const, submerged in the four-dimensional space-time with the metric $g_{\mu\nu}$ and connection $\Gamma^\rho_{\mu\nu}$. More precisely, q^{ik} are a covariant metric density of weight +2, and the π_{ik} form a covariant density of weight −1. The constraints are then the well-known Codazzi-Gauss relations in the theory of surfaces (cf. e.g., [43]).

Equation (4.17) solves the problem of reducing the action of the gravitational field to the generalized Hamiltonian form, analogous to (4.1) for finite-dimensional systems with constraints The constraints T_μ, as is easily checked, commute with one another. In order to write explicit expressions it is convenient to introduce the quantities

$$T(\eta) = \int T_k(x)\,\eta^k(x)\,d^3x, \quad T(\varphi) = \int T_0(x)\,\varphi(x)\,d^3x; \qquad (4.19)$$

where η is a vector field, φ is a scalar field (a scalar density of weight −1). The following relations hold

$$\left.\begin{aligned} \{T(\eta_1), T(\eta_2)\} &= T([\eta_1, \eta_2]),\\ \{T(\eta), T(\varphi)\} &= T(\eta\varphi),\\ \{T(\varphi), T(\psi)\} &= T(\varphi\eta_\psi - \psi\eta_\varphi); \end{aligned}\right\} \qquad (4.20)$$

here $[\eta_1, \eta_2]$ is the Lie bracket of the vector fields, i.e., the vector field with the components $\eta_1^l \partial_l \eta_2^k - \eta_2^l \partial_l \eta_1^k$, $\eta\varphi = \eta^l \partial_l \varphi - \partial_l \eta^l \varphi$, $\eta\varphi$ is the vector field with components $q^{ik}\partial_{i\varphi}$. The relations (4.20) are the field-theoretic analogs of the equality (4.2). The first row in (4.20) shows that the constraints $T_k(x)$ $(k = 1, 2, 3)$ play the role of generators of coordinate transformations. The other relations do not have a simple group-theoretic interpretation.

It is easy to see that the action written in the generalized Hamiltonian form (4.17) gives rise to the correct canonical equations of motion for the dynamical variables q^{ik} and π_{ik}, with the constraints $T_\mu = 0$ $(\mu = 0, 1, 2, 3)$.

We note the role of the divergence $\partial_i \partial_k q^{ik}$ in the Hamiltonian density $\mathscr{H}(x)$. If the constraint equations $T_\mu = 0$ hold, the Hamiltonian $\mathscr{H}$ reduces to the three-dimensional integral of the divergence, i.e., to an integral over an infinitely remote surface. The latter is determined by the asymptotic behavior of the functions q^{ik} for $r = |\mathbf{x}| \to \infty$. For an asymptotically flat gravitational field we have[6]

$$q^{ik} = \delta^{ik}\left(1 + \frac{\varkappa^2 M}{2\pi r}\right) + O\left(\frac{1}{r^2}\right),$$

where M is the total mass which can be obtained by integrating $\mathscr{H}(x)$:

$$\mathscr{H} = \int \mathscr{H}(x)\, d^3x = \frac{-1}{2\varkappa^2}\int \partial_i \partial_k q^{ik}\, d^3x = \frac{-1}{2\varkappa^2}\lim_{R\to\infty} \oint (\partial_k q^{ik}\, dS_i) = M.$$

Thus, one may consider that $\mathscr{H} = \int \mathscr{H}(x) d^3x$ plays indeed the role of energy. The integrand

$$\mathscr{H}(x) = T_0(x) - \partial_i \partial_k q^{ik}(x),$$

which plays the role of energy density, has the form of a sum of two quadratic forms: one in the derivatives of q^{ik} and another in the "momenta" π_{ik}, as required for the energy density of a wave field. In our case this is the energy of the gravitational field having two polarization states in agreement with the usual counting:

$$2 = 6(\text{coordinates}) - 4(\text{constraints}).$$

We remark moreover, that in the weak field approximation the Hamiltonian is represented by a quadratic form in the densities of the transverse components of the linearized field.

We now discuss the selection of supplementary conditions. It is widely accepted to use the conditions which were first proposed by Dirac[14]:

$$\partial_k q^{-1/3} q^{ik} = 0 \quad (i = 1, 2, 3), \quad \pi \equiv q^{ik}\pi_{ik} = 0, \tag{4.21}$$

where $q = \det q^{ik} = (\det g_{ik})^2$. These conditions have a simple geometric sense: the surface $x^0 = \text{const}$ is minimal and the coordinates x^1, x^2, x^3 on it are "harmonic" coordinates.

For us the following supplementary conditions will be more convenient:

$$\ln q = \Phi(x), \quad q^{ik} = 0 \quad (i \neq k), \tag{4.22}$$

where Φ is a function with the asymptotic behavior C/r at infinity. The commutation relations (4.7) are satisfied for these conditions. The Poisson bracket matrix of the conditions (4.21) with the constraints is determined by the equations

$$\left.\begin{aligned}
(C\eta)^0 &= \{T_\eta, \ln q - \Phi(x)\} = -\eta^s \partial_s \ln q - 4\partial_s \eta^s + 4\pi\eta^0,\\
(C\eta)^1 &= \{T_\eta, q^{23}\} = -\eta^s \partial_s q^{23} + q^{2s}\partial_s \eta^3 + q^{3s}\partial_s \eta^2 -\\
&\quad - 2q^{23}\partial_s \eta^s - 2(\pi^{23} - q^{23}\pi)\eta^0,\\
(C\eta)^2 &= \{T_\eta, q^{31}\} = -\eta^s \partial_s q^{31} + q^{3s}\partial_s \eta^1 + q^{1s}\partial_s \eta^3 -\\
&\quad - 2q^{31}\partial_s \eta^s - 2(\pi^{31} - q^{31}\pi)\eta^0,\\
(C\eta)^3 &= \{T_\eta, q^{12}\} = -\eta^s \partial_s q^{12} + q^{1s}\partial_s \eta^2 + q^{2s}\partial_s \eta^1 -\\
&\quad - 2q^{12}\partial_s \eta^s - 2(\pi^{12} - q^{12}\pi)\eta^0
\end{aligned}\right\} \tag{4.23}$$

and is nondegenerate if the curvature of the metric g_{ik} is nonzero.

5. THE HAMILTONIAN FORM OF THE FUNCTIONAL INTEGRAL

The functional integral for the quantization of a classical system defined in a generalized Hamiltonian formulation has the following form (cf.[42]):

$$\int \exp\left[i\int\left(\sum_{i=1}^{n} p_i \dot{q}^i - H(p, q)\right) dt\right] \prod_t d\mu(p(t), q(t)). \tag{5.1}$$

The integration measure is here defined by the equation

$$d\mu(t) = (2\pi)^{m-n} \det \|\{\chi_a, \varphi^b\}\| \prod_a \delta(\chi_a)\,\delta(\varphi^a) \prod_{i=1}^{n} dp_i\, dq^i. \tag{5.2}$$

To prove this we reduce the integral (5.1) with the measure (5.2) to an integral over paths in the physical phase space Γ^*. For this purpose we go over to the coordinates p_a, p^*, q^a, q^* described in Chap. 4. In these coordinates the measure has the following form:

$$d\mu(t) = (2\pi)^{m-n} \det \left\|\frac{\partial \varphi^a}{\partial q^b}\right\| \prod_a \delta(p_a)\,\delta(\varphi^a) \prod_{i=1}^{n} dp_i\, dq^i,$$

which can be rewritten:

$$\prod_a \delta(p_a)\,\delta(q^a - q^a(p^*, q^*))\, dp_a\, dq^a \prod_{j=1}^{n} \frac{dp_j^*\, dq^{*j}}{2\pi}.$$

The integration with respect to the p_a and q^a is reduced by the delta-functions. As a result the integral takes the form

$$\int \exp\left\{i\int\left[\sum_{i=1}^{n-m} p_i^* \dot{q}^{*i} - \mathscr{H}(p^*, q^*)\right] dt\right\} \prod_t \prod_{j=1}^{n-m} \frac{dp_j^*\, dq^{*j}}{2\pi}, \tag{5.3}$$

which is standard for the usual Hamiltonian formulation[44]. This proves the correctness of Eq. (5.1), which has the advantage over (5.3) of not requiring a solution of the constraint equations.

The integral (5.1) can be represented in the form

$$\int \exp(iS) \prod_t \det \|\{\chi_a, \varphi^b\}\| \prod_a \delta(\chi^a)\, d\lambda_a \prod_{i=1}^{n} dp_i\, dq^i, \tag{5.4}$$

where the functional exp(iS) (S is the generalized action of the system (4.1)) is integrated over all independent variables p_i, q^i, λ_a. Indeed, the Lagrange multipliers λ_a enter linearly into this action and the integral with respect to them yields a delta function in the constraints φ^a. Starting with this formula we shall not write out the factors of the type of π or the volumes of the integration lattices.

Let us return to the case of the gravitational field. We select the supplementary conditions in the form (4.22) and introduce the notations

$$\ln q - \Phi = \chi_0, \quad q^{23} = \chi_1, \quad q^{31} = \chi_2, \quad q^{12} = \chi_3. \tag{5.5}$$

The analog of the integral looks as follows

$$\int \exp\left[i\int\left(\pi_{ik}\partial_0 q^{ik} - \frac{h^{i0}}{h^{00}} T_i - \frac{1}{h^{00}} T_0 + \partial_i \partial_k q^{ik}\right) d^4x\right]$$

$$\times \det\{T_\mu, \chi_a\} \prod_{a=0}^{3} \delta(\chi_a) \Big(\prod_{i \leqslant k} d\pi_{ik}\, dq^{ik} \Big)\, d\frac{1}{h^{00}} \prod_{i=1}^{3} d\frac{h^{i0}}{h^{00}}. \qquad (5.6)$$

We intend to reduce this expression to a form where the integration is only over the field $g^{\mu\nu}$. This will allow us to identify the invariant measure we are looking for. For this purpose we must integrate with respect to the fields π_{ik}. This integration can be done explicitly, since the expression

$$\det\{T_\mu, \chi_a\} \equiv \det C \qquad (5.7)$$

depends linearly on π_{ik}, and thus the integral turns out to be gaussian; here C is the operator defined by the relations (4.22). Let us explain in more detail this feature of det C. The functions π_{ik} enter only into the coefficients c_{0k} of the operator C, coefficients which do not contain derivatives; moreover the dependence on the π_{ik} is linear. Thus, the operator C can be written in the form

$$C = C_1(h) + C_2(h, \pi),$$

where the operator C_2 does not contain derivatives and for each π is determined by a matrix of rank 1. It is known from linear algebra that the determinant of a matrix A + B, where B is one-dimensional, is linear in the matrix elements of B. The analog of this assertion in our case leads to the indicated linearity of det C with respect to the π_{ik}.

The Gaussian integration over π_{ik} reduces to the substitution

$$\pi_{ik} = \pi_{ik}(h),$$

where $\pi_{ik}(h)$ is an expression which follows from formulas of the type (2.1), expressing the Christoffel symbols in terms of the metric. After such a substitution the action corresponding to the Lagrangian (4.17) turns into the initial covariant action (2.3). The determinant det C turns into the product

$$\det B \prod_x h^{00}(x), \qquad (5.8)$$

where B is the operator defined by the equations

$$\begin{aligned}
(B\eta)^0 &= -\eta^\lambda \partial_\lambda \ln q - 4\partial_s \eta^s - \Big(\frac{h^{s0}}{h^{00}} \partial_s \ln q + 4\partial_s \Big(\frac{h^{s0}}{h^{00}}\Big)\Big)\eta^0,\\
(B\eta)^1 &= -\eta^\lambda \partial_\lambda q^{23} + q^{2s}\partial_s \eta^3 + \eta^{3s}\partial_s \eta^2 - 2q^{23}\partial_s \eta^s +\\
&\quad + \Big(-\frac{h^{s0}}{h^{00}} \partial_s q^{23} + q^{2s}\partial_s \Big(\frac{h^{30}}{h^{00}}\Big) + q^{3s}\partial_s\Big(\frac{h^{20}}{h^{00}}\Big) - 2q^{23}\partial_s\Big(\frac{h^{s0}}{h^{00}}\Big)\Big)\eta^0,\\
(B\eta)^2 &= -\eta^\lambda \partial_\lambda q^{31} + q^{3s}\partial_s \eta^1 + q^{1s}\partial_s \eta^3 - 2q^{31}\partial_s \eta^s + \qquad (5.9)\\
&\quad + \Big(-\frac{h^{s0}}{h^{00}} \partial_s q^{31} + q^{3s}\partial_s \Big(\frac{h^{10}}{h^{00}}\Big) + q^{1s}\partial_s\Big(\frac{h^{30}}{h^{00}}\Big) - 2q^{31}\partial_s\Big(\frac{h^{s0}}{h^{00}}\Big)\Big)\eta^0,\\
(B\eta)^3 &= -\eta^\lambda \partial_\lambda q^{12} + q^{1s}\partial_s \eta^2 + q^{2s}\partial_s \eta^1 - 2q^{12}\partial_s \eta^s +\\
&\quad + \Big(-\frac{h^{s0}}{h^{00}} \partial_s q^{12} + q^{1s}\partial_s \Big(\frac{h^{20}}{h^{00}}\Big) + q^{2s}\partial_s\Big(\frac{h^{10}}{h^{00}}\Big) - 2q^{12}\partial_s\Big(\frac{h^{s0}}{h^{00}}\Big)\Big)\eta^0.
\end{aligned}$$

Finally, the local factors in the products of differentials together with the local factor which appeared in the integration over π_{ik} and the differentials themselves collect into the expression

$$\prod_x [(h^{00})^4 q^{-2} \prod_{\mu \leqslant \nu} dh^{\mu\nu}]. \qquad (5.10)$$

Here the factor in front of the differentials can be reduced to the form

$$(h^{00})^{-1} h^{-5/2} q^{1/2},$$

and the last factor $q^{1/2}$ can be omitted owing to the constraint $q = \exp \Phi$. As a result our functional integral takes the form

$$\int \exp(iS[h]) \det B \prod_x \Big[\prod_a \delta(\chi_a) \Big] h^{-5/2} \prod_{\mu \leqslant \nu} dh^{\mu\nu}. \qquad (5.11)$$

We now show that this integral is an integral over classes of gravitational fields in the sense of Ch. 3, the classes being parametrized by the condition (4.22), and the invariant measure having the form

$$\prod_x (h^{-5/2} \prod_{\mu \leqslant \nu} dh^{\mu\nu}). \qquad (5.12)$$

For this it is sufficient to verify that det B coincides with the factor $\Delta_\chi[h]$ obtained according to the rules of Chap. 3

$$\Delta_\chi[h] \int \prod_x \Big[\prod_b \delta(\chi_b^{a(x)}) \Big] da(x) = 1.$$

The integral in this expression can be calculated in in the same way as the integral (3.16) in Chap. 3. This yields

$$\Delta_\chi[h] = \det B', \qquad (5.13)$$

where the operator B′ is defined as follows:

$$\left.\begin{aligned}
(B'\zeta)^0 &= -\zeta^\lambda \partial_\lambda \ln q - 4\partial_s \zeta^s + 4\frac{h^{s0}}{h^{00}}\partial_s \zeta^0,\\
(B'\zeta)^1 &= -\zeta^\lambda \partial_\lambda q^{23} + q^{2s}\partial_s \zeta^3 + q^{3s}\partial_s \zeta^2 - 2q^{23}\partial_s \zeta^s -\\
&\quad - \Big(\frac{h^{20}}{h^{00}} q^{3s} + \frac{h^{30}}{h^{00}} q^{2s} - 2\frac{h^{s0}}{h^{00}} q^{23}\Big)\partial_s \zeta^0,\\
(B'\zeta)^2 &= -\zeta^\lambda \partial_\lambda q^{31} + q^{3s}\partial_s \zeta^1 + q^{1s}\partial_s \zeta^3 - 2q^{31}\partial_s \zeta^s -\\
&\quad - \Big(\frac{h^{30}}{h^{00}} q^{1s} + \frac{h^{10}}{h^{00}} q^{3s} - 2\frac{h^{s0}}{h^{00}} q^{31}\Big)\partial_s \zeta^0,\\
(B'\zeta)^3 &= -\zeta^\lambda \partial_\lambda q^{12} + q^{1s}\partial_s \zeta^2 + q^{2s}\partial_s \zeta^1 - 2q^{12}\partial_s \zeta^s -\\
&\quad - \Big(\frac{h^{10}}{h^{00}} q^{2s} + \frac{h^{20}}{h^{00}} q^{1s} - 2\frac{h^{s0}}{h^{00}} q^{12}\Big)\partial_s \zeta^0.
\end{aligned}\right\} \qquad (5.14)$$

it is easy to see that

$$\det B' = \det B.$$

Indeed, one can go over from one operator to the other by means of the triangular substitution

$$\zeta^0 = \eta^0, \quad \zeta^i = \eta^i + (h^{i0}/h^{00})\eta^0 \quad (i = 1, 2, 3). \qquad (5.15)$$

Let us summarize. Starting from the obviously unitary Hamiltonian formulation of the functional integral, after formal changes of the integration variables we have transcribed it in the form of an integral over equivalence classes of fields with a concrete parametrization of the classes. The corresponding invariant measure has the form (5.12). This justifies the Lorentz-invariant expression for the functional integral in Chap. 3, which represents another writing of the same integral with another parametrization of the classes. With these considerations we conclude the derivation of the covariant rules of perturbation theory for the quantization of the gravitational field.

The next problem which appears here is considerably more difficult. It consists in a consistent performance of the renormalization procedure based on an invariant regularization. The difficulties are caused by the unwieldiness of the theory as well as by the fact that from a formal point of view the theory is non-renormalizable. We hope that symmetry and general covariance considerations will help in solving this problem.

1)For instance, r = 2 for the diagram  .

2)The use of functional (path) integrals in various problems of quantum mechanics and field theory is discussed in the recent review article of Blokhintsev and Barbashov in Uspekhi [37b].

3)Any measure $\prod_x h^\alpha \prod_{\mu \leqslant \nu} dh^{\mu\nu}$ is invariant with respect to the group of coordinate transformations. Indeed, under coordinate transformations this expression acquires a factor of the type $\prod_x [\det(\partial x'^\mu/\partial x^\mu)]^\gamma$. This factor should be considered equal to one, since for infinitesimal transformation it is equal to $\exp[\gamma \operatorname{Tr} \ln(1 + \partial_\mu \eta_\mu)] = \exp(\gamma \delta^{(4)}(0) \int \partial_\mu \eta_\mu d^4x) = 1$.

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Translated by Meinhard E. Mayer