Nonlinear waves and their interaction

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The review is devoted to a number of problems connected with the propagation of strongly nonlinear periodic waves in the presence of various types of perturbing factors. The problems considered can be divided into three groups. The first includes questions connected with the employed formalism. These include the characteristic properties of nonlinear waves, perturbation-theory methods, canonical variables, and the Hamiltonian formalism. The second group of questions is devoted to the propagation of nonlinear waves in the presence of external perturbations. A description is given of the resonant interaction between the wave and an external force, the stochastic instability of a nonlinear wave, the change of the adiabatic invariant of a linear wave in a weakly-inhomogeneous medium, and the propagation of a nonlinear wave in the presence of random perturbations, particularly in a medium with random inhomogeneities. Finally, the third group of considered questions include problems connected with weak interaction of strongly nonlinear waves. The conditions under which the interaction of the waves is weak are determined, and the interaction of two waves and resonant interaction of three waves is considered. This group includes an investigation of an ensemble of a large number of nonlinear waves and its description with the aid of the kinetic equation. The Appendix discusses problems connected with the energy-momentum tensor of the nonlinear wave equation.

CONTENTS

1. INTRODUCTION

Nonlinear wave processes are involved in phenomena that occur in a great variety of branches of physics. It suffices to mention elastic properties of media, surface oscillations of liquids, wave (and turbulent) motion of a plasma, nonlinear optics and electrodynamics, some problems of quantum theory, etc. Significant progress in the investigation of this region has led in recent years not only to the understanding of many nonlinear wave phenomena from a unified point of view, but also to the development of a number of general methods for their investigation. Many of these results are covered in the reviews[1-6].

An attempt to classify the problems that arise in strongly nonlinear wave processes would entail many difficulties and ambiguities. Nonetheless, one can point to two large regions in which the interests in this field "congregate." One of them is connected with the analysis of some particular type of nonlinear equation, to which various physical problems reduce. Examples are the Korteweg-de Vries equation, the self-focusing equation, and the equation of a nonlinear string. Closely linked with each of these equations are not only the characteristic properties of the wave processes, but also the research methods. Another "congregation" region is analysis of the evolution of definite types of nonlinear wave processes. Examples are shock-wave formation, the onset of wave modulation, focusing, decays, etc. The analysis of this type is based on the study of the behavior of a definite class of nonlinear motions. The methods employed show if a new class of wave processes is produced as a result of the evolution of the motion. The two approaches have a region where they overlap.

It appears that the simplest classification of methods of analyzing nonlinear wave processes is the following:

1) Construction of exact solutions. The simplest of them are nonlinear stationary waves of the Riemann type. Significant progress in this direction resulted from[7a] in which an exact solution of the Korteweg-de Vries equation was constructed for initial conditions of a definite type. Later on Lax[7b] proposed a regular method of reducing the Cauchy problem for a number of nonlinear wave equations to a linear eigenvalue problem.

2) The construction of approximate methods such as perturbation theory or the WKB method.

3) The determination of the limits of applicability of the employed methods. This question, which is natural in the second case, arises also in the case of exact solutions, since the equations that they satisfy are approximate. What makes this method timely are the distinguishing features of the nonlinear problems and the methods for their analysis. Unlike in linear problems, in the nonlinear case the existing methods of the theory usually employ not general solutions, but one or several particular solutions. In this connection, some physical processes can fall out of consideration, and, conversely, some physical properties of motion, which follow from the approximate analysis, may actually be missing. Although these questions are discussed in part in the literature, there is nevertheless a serious gap in their understanding. The discussion of problems that arise in this respect will be carried out in part in this review and in the conclusion (Chap. 8).

The group of problems to which the present paper is devoted is connected principally with the second aspect, i.e., with approximate methods of analyzing the evolution of nonlinear wave processes. In different physical problems, the evolution of waves arises as a result of the nonstationary character (and, in particular, the inhomogeneity) of the medium in which the waves propagate, their interactions with the external field and with one another, etc. For the results described below, the following two limitations are of importance: we are considering primarily the evolution of periodic nonlinear waves; the nonlinearity of the latter is not small.

A certain part of the review is only an adjunct to the main material, and is described briefly only in the form needed to understand the sequel. This includes the discussion of several properties of nonlinear stationary waves and the cursory mention of approximate methods that employ the Lagrangian formalism.

2. NONLINEAR TRAVELING WAVE

We consider here certain typical nonlinear wave equations and the properties of their solutions in the form of a traveling wave

$$y = y(x - ut), \tag{2.1}$$

where the wave velocity u is a parameter of the problem. We shall dwell here only on those wave properties which will be essentially employed later on.

a) **Phase plane**. If the solution sought in the form of (2.1) is substituted in the initial partial differential equation, then the latter becomes an ordinary differential equation. It is convenient to analyze the ensuing properties of the solution on the phase plane.

By way of example, we turn to the nonlinear Klein-Gordon equation

$$y_{tt} = c^2 y_{xx} + F(y), \quad F(y) \equiv dV(y)/dy. \tag{2.2}$$

In this equation is a convenient model for the investigation of different nonlinear phenomena[8,9] 1). Equations of similar type arise also for electromagnetic oscillations in nonlinear media[11] 2). Substitution of (2.1) in (2.2) yields

$$y'' + \frac{1}{c^2 - u^2} F(y) = 0, \tag{2.3}$$

where the prime denotes differentiation with respect to the argument $\xi = x - ut$. Equation (2.3) is equivalent to the equation of motion of a particle in a field with a potential

$$\frac{1}{c^2 - u^2} V(y),$$

and with the aid of the energy integral

$$\frac{1}{2} y'^2 + \frac{1}{c^2 - u^2} V(y) = \text{const} = C \tag{2.4}$$

its motion can be represented in the usual manner on the phase plane (y', y).

For a more detailed description of a solution of type (2.1) and its connection with the trajectories on the phase plane, let us consider the equation of the oscillations of a nonlinear string

$$y_{tt} = c^2 (1 + \varepsilon y_x^2) y_{xx} + \delta^2 y_{xxxx}. \tag{2.5}$$

Equation (2.5) arose in connection with the well known Fermi-Pasta-Ulam problem[12-14] and in certain problems of nonlinear acoustics[15,16]. The quantity y in this equation describes the displacement of the string, and the term y_{xxxx} takes into account the dispersion. Seeking the solution in the form $y = y(x - ut)$, we obtain

$$\delta^2 v'^2 + \frac{1}{2} \varepsilon v^4 - \left(\frac{u^2}{c^2} - 1\right) v^2 = C, \quad v = y'. \tag{2.6}$$

The trajectories on the phase plane are shown in Fig. 1. At $C < 0$ we have a periodic wave with $v > 0$ or with $v < 0$. The case $C = 0$ describes a solitary wave (soliton), and at $C > 0$ the periodic wave is of alternating sign. Equation (2.6) is integrated in elliptic functions, and, in particular, at $C < 0$ and $v > 0$ we obtain (Fig. 2)

$$\left.\begin{aligned} v &= \gamma_2 dn\left[\frac{\gamma_2}{\delta\sqrt{2}}(x - ut), \varkappa\right], \\ \gamma_{1,2} &= \left[\frac{1}{\varepsilon}\left(\frac{u^2}{c^2} - 1\right) \mp \frac{1}{\varepsilon}\sqrt{\left(\frac{u^2}{c^2} - 1\right)^2 - 2\varepsilon|C|}\,\right]^{1/2}, \\ \varkappa &= \frac{1}{\gamma_2}\sqrt{\gamma_2^2 - \gamma_1^2}. \end{aligned}\right\} \tag{2.7}$$

The spatial period of the solution (2.7) is determined by the relation

$$\lambda = 2\pi/k = (2\sqrt{2}/\gamma_2)\, \delta F(\pi/2, \varkappa). \tag{2.8}$$

Relation (2.8) plays the role of a dispersion equation in the nonlinear case, which is conveniently represented in the form

$$k = k(u, C). \tag{2.9}$$

It will be discussed in greater detail later on. We note here merely that the transition to the linear case is effected in the limit

$$u = \frac{\omega}{k}, \quad |C| - \frac{1}{2\varepsilon}\left(\frac{u^2}{c^2} - 1\right) \to 0. \tag{2.9'}$$

A trajectory corresponding to a soliton passes through the singular point of the hyperbolic type (point O on Fig. 1).

b) **Wave breaking**. Another type of singularity, connected with the loss of analyticity of the solution, is best considered with the problem of nonlinear plasma oscillations as an example:

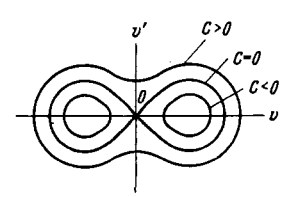

FIG. 1

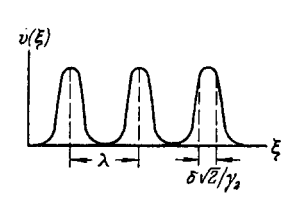

FIG. 2

$$\left.\begin{aligned} \rho_t + (\rho v)_x &= 0, \\ v_t + v v_x &= \frac{e}{m}\varphi_x, \\ \varphi_{xx} &= 4\pi e(\rho - \rho_0), \end{aligned}\right\} \tag{2.10}$$

where ρ and v are the density and velocity of the electrons, and ρ_0 is the constant velocity of the stationary ions. As before, we obtain for traveling waves[17]

$$\left.\begin{aligned} &\rho(v-u) = \text{const} = -u, \quad \sqrt{\Phi} \equiv u - v, \\ &\frac{1}{2}\Phi'^2 - 2\omega_0^2\left(2u\sqrt{\Phi} - \Phi\right) = C, \quad \omega_0^2 = \frac{4\pi e^2\rho_0}{m}, \\ &\frac{u - \sqrt{\Phi}}{\varepsilon u} \equiv \psi = -\cos\left(\pm k\xi - \varepsilon\sqrt{1-\psi^2}\right), \quad \varepsilon = \sqrt{1 - \frac{|C|}{2\omega_0^2 u^2}}. \end{aligned}\right\} \tag{2.11}$$

The phase trajectories (at $C < 0$) and the solution are shown in Figs. 3 and 4 respectively. At $C = 0$ the trajectory passes through the point $\Phi = 0$, i.e., the amplitude of the velocity v reaches the value u, and the density ρ becomes infinite. This phenomenon is called wave breaking. The solution becomes multiply valued, and the system (2.10) becomes meaningless. On approaching the wave breaking, the peaks on Fig. 4 become sharper.

There is no solution in the form of a solitary wave in this example, and the dispersion equation takes the simple form[17]

$$k = \omega_0/u. \tag{2.12}$$

Unlike (2.9), the connection between k and u does not depend on C (i.e., on the amplitude). This leads to the following interesting feature of the plasma oscillations: the nonlinearity of the oscillations becomes manifest only in their anharmonicity, whereas their frequency ($\omega = ku = \omega_0$) does not depend on the amplitude.

c) **Critical velocity.** We turn finally to a case that contains simultaneously both types of singularities that lead to the existence of a solution in the soliton type and to the possibility of wave breaking. It is convenient to consider as a model ion-acoustic oscillations of a plasma[1,2]

$$\left.\begin{aligned} \rho_t &= -(\rho v)_x, \\ v_t + v v_x &= -\frac{e}{M}\varphi_x, \\ \varphi_{xx} &= -4\pi e\left[\rho - \rho_0 \exp\left(\frac{e\varphi}{T}\right)\right], \end{aligned}\right\} \tag{2.13}$$

where M, ρ, and v are the mass, density, and velocity of the ions, and T is the electron temperature. The energy integral of the system (2.13), for solutions of the type (2.1), is equal to

$$v'^2(u-v)^2 = u^2\left(1 - \frac{v}{u}\right) + \exp\left(uv - \frac{v^2}{2}\right) - (1+u^2) - 2C, \tag{2.14}$$

where for convenience the Debye radius $r_d = \sqrt{T/4\pi e^2\rho_0}$ and the ion-sound velocity $c = \sqrt{T/M}$ are set equal to unity. Figures 5 and 6 show two families of phase trajectories: 1) at constant u and at variable C, and 2) at constant C and variable u. In the former case we see that a solution is possible in the form of a soliton at $C = 0$ and any admissible u. From the second family it follows that all C that admit of periodic solutions, wave breaking occurs when the wave velocity reaches the critical value u_C[1,2]. u_C depends on C, and at small C we have $u_C \approx 1.6$.[3)]

d) **The Korteweg-de Vries equation.** An important role in various applications is played by the Korteweg-de Vries equation

$$v_t + c v_x + v v_x + v_{xxx} = 0, \tag{2.15}$$

which describes waves that travel only in one direction. Different problems (waves on "shallow water," Eqs. (2.5), (2.13), etc.) can be reduced to (2.15) in a sufficiently unified manner (see, e.g.,[20]). Usually Eq. (2.15) is derived in the approximation of sufficiently small nonlinearity and sufficiently small dispersion. It must be emphasized, however, that a general requirement in the derivation of (2.15) is

$$\alpha = (u/c) - 1 \ll 1, \tag{2.16}$$

where c is the characteristic velocity of the sound for the given problem. Since nonlinear traveling waves appear in the examples considered here at $u > c$ and, on the other hand, $u < u_c \lesssim c$, the inequality (2.16) denotes that the problem is considered far from the wave-breaking region.

e) **Spectrum** of nonlinear periodic waves. Periodic solutions of the type (2.1) can be expanded in a Fourier series:

$$v(x - ut) = \sum_{n=-\infty}^{\infty} a_n e^{in\hbar(x-ut)}. \tag{2.17}$$

The dependence of a_n on n will be called the spectrum of the wave. In the case close to linear, a_n decreases rapidly with increasing n. This allows us to confine ourselves to the first few terms in the expansion (2.17).

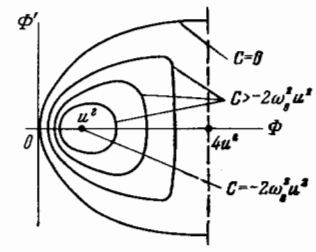

FIG. 3

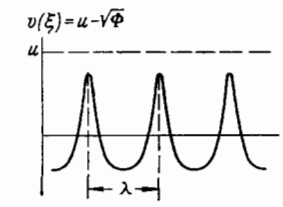

FIG. 4

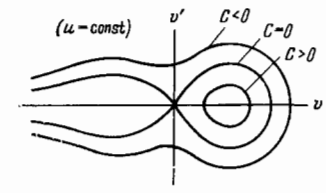

FIG. 5

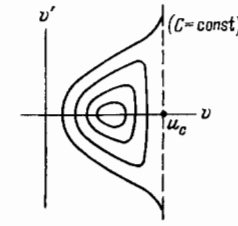

FIG. 6

The next property of nonlinear periodic waves is very important for the subsequent analysis: there exists a number

$$N = N(k, C), \tag{2.18}$$

which determines the characteristic number of modes in the spectrum (i.e., the degree of anharmonicity), and at $n > N$ the sum in (2.17) is effectively cut off. Thus, for example, in the case of a nonlinear string it follows from the series expansion of (2.7) that

$$N = F(\pi/2, \varkappa), \tag{2.19}$$

and it is seen from (2.8) that N is of the order of the ratio of the wavelength to the width of the crest (see Fig. 2). We emphasize that from (2.19) it follows in the limit (2.9′) that $N \sim 1$, and as $C \to 0$ we have $N \gg 1$ and the spectrum has the following structure (Fig. 7):

$$a_n \approx \text{const} \sim \gamma_2/N, \quad n < N, \qquad a_n \sim \exp(-n/N), \quad n > N. \tag{2.20}$$

A similar shape is possessed by the spectrum in the case (2.15)[21], with

$$N = \sqrt{\alpha}/k. \tag{2.21}$$

A more complicated structure of the spectrum appears in the case of plasma oscillations[22] (Fig. 8): at $n < N$ the amplitudes a_n have a power-law dependence on n, and at $n > N$ they decrease exponentially. The value of N is[22]

$$N = (1 - \varepsilon)^{-3/2}, \tag{2.22}$$

where ϵ determines, in accordance with (1.15), the degree of proximity to wave breaking. Near the latter we have $C \to 0$, $\epsilon \to 1$, and $N \to \infty$.

Finally, let us consider, with (2.15) as an example, one more feature of the number N. In analogy with the expression for the Reynolds number, we write down its analog for the case (2.15) at $N \gg 1$[23]:

$$R \sim vv_x/v_{xxx} \sim \alpha/k^2 = N^2 \gg 1. \tag{2.23}$$

Thus, $N = R^{1/2}$ determines the constant of the strong coupling between the harmonics in a nonlinear periodic wave.

3. PERTURBATION THEORY METHODS

The existing approximate analysis methods are based on series that take into account both the smallness of the perturbations and the slowness of the variation of definite variables. Many features of these methods are, to one degree or another, developments of the Krylov-Bogolyubov method[24] in the theory of nonlinear oscillations[4)]. All these methods are asymptotic. This is easiest to illustrate using as an example a solitary wave $v(x - ut)$ perturbed by a small force $\epsilon F(x, t)$ (Fig. 9). Since the edges of a soliton decrease exponentially with respect to the coordinate, there always exists a region in which perturbation theory is not valid at arbitrarily small ϵ. The same conclusion follows from an examination of the curves of the spectrum (see Figs. 7 and 8). The exponential decrease of the latter with increasing n means the existence (starting with a certain $n_0 > N$) of modes for which $a_n \ll \epsilon$. We can therefore state that the existing approximate methods are based on a certain cutoff or averaging over the spectrum. This circumstance is particularly important when one considers strongly-nonlinear waves (i.e., broad wave packets with $N \gg 1$).

Before we proceed to the exposition of the perturbation theory, let us discuss first the variational principles employed in it.

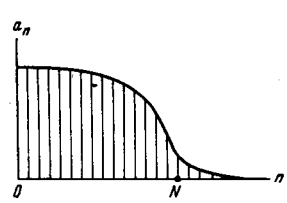

FIG. 7

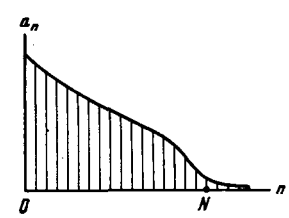

FIG. 8

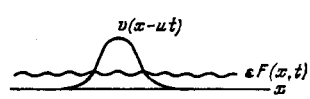

FIG. 9

a) **Variational principles.** The fundamental principle is that of Lagrange, although it is not always possible to find the Lagrangian (for details see[4,26]. It can be written in the form

$$\mathcal{L} = \int dx \int dt\, L, \quad \delta\mathcal{L} = 0, \tag{3.1}$$

where L is the Lagrangian density. In the case of a continuous spectrum, the integral with respect to x is taken between infinite limits. If the spectrum is discrete, then

$$\mathcal{L} = \lim_{l\to\infty}\left(\frac{1}{2l}\int_{-l}^{l} dx \int dtL\right), \quad \delta\mathcal{L} = 0. \tag{3.2}$$

A similar remark applies also to integration with respect to t. The difference between the principles (3.1) and (3.2) can be illustrated by the respective cases of a soliton (the limits with respect to x are infinite and $\int L dx$ is finite) and a nonlinear periodic wave (only the average value of $\int L dt$ per unit length is finite). In the last example, we can also write

$$\mathcal{L} = \frac{1}{2\pi}\int_0^{2\pi} d\vartheta \int dtL, \tag{3.3}$$

where ϑ is the phase of the wave.

In particular, for (2.2) we have

$$L = \frac{1}{2}y_t^2 - \frac{1}{2}c^2 y_x^2 + V(y), \tag{3.4}$$

or for the equation of a nonlinear string (2.5) we have

$$L = \frac{1}{2}y_t^2 - \frac{1}{2}c^2 y_x^2 - \frac{1}{4}\varepsilon c^2 y_x^4 + \frac{1}{2}\delta^2 y_{xx}^2. \tag{3.4'}$$

Unlike the Lagrangian principle, the Hamiltonian principle is not so unique and depends, in particular, on the choice of the canonically-conjugate variables. We turn for simplicity again to the Klein-Gordon equation (2.2) with $F(y) = y^2$. Putting

$$y(x, t) = \sum_n y_n(t)\, e^{inkx}, \tag{3.5}$$

we rewrite (2.2) in the form

$$\frac{d^2 y_n}{dt^2} + n^2k^2c^2 y_n - \sum_{n_1, n_2} y_{n_1} y_{n_2} \delta(n - n_1 - n_2) = 0. \tag{3.6}$$

The Hamiltonian for (3.6) is

$$\mathcal{H} = \frac{1}{2}\sum_n \left(\frac{dy_n}{dt}\frac{dy_{-n}}{dt} + n^2k^2c^2 y_n y_{-n}\right) - \frac{1}{3}\sum_{n_1, n_2, n_3} y_{n_1} y_{n_2} y_{n_3} \delta(n_1 + n_2 + n_3), \quad \frac{d\mathcal{H}}{dt} = 0, \tag{3.7}$$

and the canonical equations of motion, which are equivalent to (3.6) are

$$\frac{d}{dt}\frac{dy_n}{dt} = -\frac{\partial\mathcal{H}}{\partial y_{-n}}, \quad \frac{d}{dt}y_n = \frac{\partial\mathcal{H}}{\partial(dy_{-n}/dt)}. \tag{3.8}$$

The transition to the case of a continuous spectrum is carried out in accordance with (3.1). In analogy with (3.5)–(3.8), we can write

$$y(x, t) = \sum_m y_m(x)\, e^{im\omega t},$$

$$c^2\frac{d^2 y_m}{dx^2} + m^2\omega^2 y_m + \sum_{m_1, m_2} y_{m_1} y_{m_2} \delta(m - m_1 - m_2) = 0, \tag{3.9}$$

and the canonical equations are

$$\tilde{\mathcal{H}} = \frac{1}{2}\left(c^2\frac{dy_m}{dx}\frac{dy_{-m}}{dx} + m^2\omega^2 y_m y_{-m}\right) + \frac{1}{3}\sum_{m_1, m_2, m_3} y_{m_1} y_{m_2} y_{m_3} \delta(m_1 + m_2 + m_3), \quad \frac{d\tilde{H}}{dx} = 0, \tag{3.10}$$

$$\frac{d}{dx}\frac{dy_m}{dx} = -\frac{\partial\tilde{\mathcal{H}}}{\partial y_{-m}}, \quad \frac{d}{dx}y_m = \frac{\partial\tilde{\mathcal{H}}}{\partial(dy_{-m}/dx)}.$$

Although the variation of $\mathscr{H}$ and $\tilde{\mathscr{H}}$ leads to the same equations of motion (2.2), the quantities $\mathscr{H}$ and $\tilde{\mathscr{H}}$ (and consequently the integrals of motions) are different.

The foregoing examples explain the following possible classification of problems, depending on the form of the boundary conditions[5)]:

1) Finding solutions that are periodic in the coordinate, i.e.,

$$y(x, t) = y(x + 2\pi/k, t). \tag{3.11}$$

In this case it is natural to use the principle (3.7), (3.8). The character of the spectrum with respect to x remains unchanged, and the quantities y_n and $\mathscr{H}$ vary in time as a result of the perturbation. Conditions of the type (3.11) arise, for example, in different annular physical systems (cyclic accelerators, toroidal magnetic bottles, etc.).

2) Finding solutions that are periodic in time:

$$y(x, t) = y(x, t + 2\pi/\omega). \tag{3.12}$$

The variational principle takes in this case the form (3.10). What is invariant to the perturbation is the character of the spectrum with respect to t, and what evolves in space is the profile of the wave. This problem, in analogy with quantum mechanics can be naturally called the problem of determining the stationary states. We encounter a similar situation, for example, in scattering problems and in process-control problems.

3) This case is characterized by the absence of hindrances of the type (3.11) and (3.12). A similar situation can be realized also in an unbounded medium, for example when c-wave surfaces are perturbed by a wind. There is no known Hamiltonian principle in this case for the considered examples. A Lagrangian principle in the form (3.1), however, can be used.

We note in conclusion that certain difficulties are encountered when attempts are made to consider the interaction of wave processes one of which has a continuous spectrum and the other a discrete spectrum (for example, the interaction of a soliton with a periodic wave). This is already seen in part from the physical and formal differences between expressions (3.1) and (3.2).

b) Whitham's method[27,28]. The main features of Whitham's method can be presented in the following manner: Let, for example, $y(x - ut)$ be the exact solution of the unperturbed problem in the form of a nonlinear wave. This solution can also be written in the form

$$y = y(\vartheta, A), \tag{3.13}$$

where the phase ϑ satisfies the relations

$$\vartheta_t = -\omega, \quad \vartheta_x = k, \tag{3.14}$$

and the quantity A determines, for example, the amplitude of the wave. Let now some parameters of the problem depend, as a result of the perturbation, adiabatically on x and t (i.e., slowly in comparison with k^{-1} and ω^{-1}). Then the Lagrangian (3.3) averaged over the phase can be calculated approximately without paying attention to the variation of the slow parameters. This yields

$$\bar{\mathscr{L}} = \bar{\mathscr{L}}(k, \omega, A), \tag{3.15}$$

and here k, ω, and A are already functions that vary slowly in time and in space. The next step consists of varying

$$\delta \int dx\, dt\, \bar{\mathscr{L}} = 0$$

with (3.14) and (3.15) taken into account. This leads immediately to the equations

$$\partial\bar{\mathscr{L}}_\omega/\partial t - \partial\bar{\mathscr{L}}_k/\partial x = 0, \quad \bar{\mathscr{L}}_A = 0,$$
$$\bar{\mathscr{L}}_\omega \equiv -\partial\bar{\mathscr{L}}/\partial\vartheta_t, \quad \bar{\mathscr{L}}_k \equiv \partial\bar{\mathscr{L}}/\partial\vartheta_x. \tag{3.16}$$

Equations (3.16) describe the slow evolution of a wave packet in space and in time. They were used to consider a large number of different physical problems[5a] 6)]. The main shortcoming of the described method, as noted by Whitham himself[4], is that it cannot be used to construct the next higher approximations.

c) The Luke-Moser method[8,9]. This method was developed in[8] and subsequently improved and verified by Moser[9] and is apparently the most rigorous one at present. Following Moser, the solution is sought in the form of the series

$$y = y^{(0)}(\vartheta, A) + \varepsilon y^{(1)}(\vartheta, \ldots) + \varepsilon^2 y^{(2)}(\vartheta, \ldots) + \ldots \tag{3.17}$$

Here A is the action, which is defined in the usual manner for Eq. (2.3); ϵ is a small parameter characterizing the "slow" time and coordinate; the points in the brackets denote dependence on all the remaining "slow" variables (A, ϵt, ϵx, ϑ_x, ϑ_t, etc.)[7)].

Substitution of (3.17) in (2.5) yields the zeroth order

$$\vartheta_t^2 - c^2\vartheta_x^2 = -F(y^{(0)})/y^{(0)}_{\vartheta\vartheta}$$

or, taking (3.14) and (2.1) into account

$$\vartheta_t^2 - c^2\vartheta_x^2 = \omega^2 - k^2c^2 = k^2[u^2(A) - c^2] \equiv \tilde{\omega}^2(A). \tag{3.18}$$

We emphasize that in this approximation the solution, as a function of (A, ϑ), has the same form as in the absence of the perturbation. It is precisely because of this circumstance that perturbation theory justifies the qualitative physical considerations on which it is based. A nonlinear wave, being an exact solution, is not a general one. However, small perturbations lead to a weak deformation of the exact solution, and this frees us of the need for using the "complete set" of solutions of the given problem[8)].

In the next higher approximation, the condition that the correction $y^{(1)}$ be orthogonal to $y^{(0)}$ leads to the equation[8]

$$(\vartheta_t A/\tilde{\omega}(A))_t - (\vartheta_x A/\tilde{\omega}(A))_x = 0. \tag{3.19}$$

Moser[9] constructed a general scheme for the successive approximations and for deriving orthogonality conditions of the type (3.19) in all orders in ϵ. It is easy to verify that Eq. (3.19) coincides with (3.16). The first-approximation system (3.18) and (3.19) is analogous to the system of equations in terms of the action and angle variables for ordinary dynamic systems. This statement, which is obvious for the phase equation (3.18), will be verified for (3.19) in a particular case in Sec. e below.

It is convenient to impart to Eqs. (3.16) and (3.19) the following intuitive physical meaning. The zeroth approximation (solution of the type (2.1)) describes a wave packet whose group velocity is equal to the phase velocity. The condition that this property be preserved also in the first-order approximation leads immediately to (3.16). Indeed,

$$\left(\frac{\partial\vartheta_t}{\partial\vartheta_x}\right)_{\bar{\mathscr{H}}=\text{const}} = -\left(\frac{\partial\omega}{\partial k}\right)_{\bar{\mathscr{H}}=\text{const}} = -u(\bar{\mathscr{H}}) = -\left(\frac{\partial x}{\partial t}\right)_{\vartheta=\text{const}}, \tag{3.20}$$

where $\bar{\mathscr{H}}$ is the wave energy averaged over the period. Recognizing that

$$\left(\frac{\partial\omega}{\partial k}\right)_{\bar{\mathscr{H}}=\text{const}} = -\frac{\partial\bar{\mathscr{H}}}{\partial k}\Big/\frac{\partial\bar{\mathscr{H}}}{\partial\omega},$$

we obtain from (3.20)

$$\frac{\partial\bar{\mathscr{H}}}{\partial\omega}dx + \frac{\partial\bar{\mathscr{H}}}{\partial k}dt = 0. \tag{3.21}$$

Expression (3.21) denotes that we can introduce a function Q for which

$$Q_x = \bar{\mathscr{H}}_\omega, \quad Q_t = \bar{\mathscr{H}}_k.$$

This means that

$$\partial\bar{\mathscr{H}}_\omega/\partial t - \partial\bar{\mathscr{H}}_k/\partial x = 0, \tag{3.22}$$

which is equivalent to (3.16), inasmuch as in our case we have $\partial\bar{\mathscr{H}}/\partial\omega = \partial\bar{L}/\partial\omega$ and $\partial\bar{\mathscr{H}}/\partial k = \partial\bar{L}/\partial k$.

In concluding this section, we must call attention to the analogy between the described method and the method proposed by Bogolyubov for obtaining hydrodynamic equations from the kinetic equation[34a].

Kruskal and Miura have recently developed a more detailed method of the WKB-approximation type for the Korteweg-de Vries equation[34b].

d) Generalized Lagrangian formalism. A Lagrangian formulation of the successive approximations was developed in[33,35a]. It is based on the variational principle

$$\int dx \int dt\, \delta(L + W) = 0, \tag{3.23}$$

where W is the power connected with the presence of an external perturbation acting on the wave packet (in particular, W can contain dissipative terms). If we substitute in (3.23) a series of the type (3.17), then variation yields in first order

$$\frac{\partial}{\partial t}\frac{\partial\bar{\mathscr{L}}}{\partial\omega} - \frac{\partial}{\partial x}\frac{\partial\bar{\mathscr{L}}}{\partial k} = \frac{\partial\bar{W}}{\partial\vartheta}, \quad \bar{W} = \frac{1}{2\pi}\int_0^{2\pi} d\vartheta W, \tag{3.24}$$

which coincides with (3.16) at W = 0.

The asymptotic series used in[33,35a] were subsequently generalized and verified in[32,35b]. From the formal point of view, they cover a rather wide circle of applications and include, in particular, averaging over arbitrary nonlinear stationary solutions. Difficulties arise, however, with the determination of the conditions under which the use of the formal series reflects correctly (or fully) the physical aspect of the problem.

e) Hamiltonian formalism[21,23]. For simplicity we consider again Eq. (2.5) with a right-hand side that takes into account the external forces:

$$y_{tt} = c^2 y_{xx} + dV(y)/dy + \varepsilon\Phi(x, t). \tag{3.25}$$

We seek a solution that satisfies the boundary conditions (3.11), and use the Hamiltonian

$$\mathscr{H} = \mathscr{H}_0 + \varepsilon\mathscr{H}_1, \tag{3.26}$$

where $\mathscr{H}_0$ coincides with (3.7), and

$$\mathscr{H}_1 = -\lim_{l\to\infty}\left(\frac{1}{2l}\int_{-l}^{l} y\Phi dx\right) = -\sum_n y_n\Phi_{-n}, \tag{3.27}$$

$$\Phi(x, t) = \sum_n \Phi_n(t)\, e^{inkx}.$$

By direct differentiation we can easily obtain

$$\frac{d\mathscr{H}_0}{dt} = \sum_n\left(\frac{\partial\mathscr{H}_0}{\partial y_n}\dot{y}_n + \frac{\partial\mathscr{H}_0}{\partial\dot{y}_n}\ddot{y}_n\right) = \sum_n \dot{y}_{-n}\Phi_n. \tag{3.28}$$

In the zeroth approximation ($\epsilon = 0$) we have

$$\dot{\mathscr{H}}_0 = 0, \quad \dot{\vartheta} = \omega(\mathscr{H}_0) = ku(\mathscr{H}_0). \tag{3.29}$$

According to (3.14) and (2.1) we have in the zeroth approximation

$$y_n(t) = a_n e^{-in\vartheta} = a_n e^{-inku}, \quad a_n = a_n(\mathscr{H}_0). \tag{3.30}$$

To obtain the next approximation, it suffices to substitute (3.30) in the right-hand side of (3.28):

$$d\mathscr{H}_0/dt = -\varepsilon\omega(\mathscr{H}_0)\,\partial\mathscr{H}_1/\partial\vartheta. \tag{3.30'}$$

We introduce, as in ordinary dynamics, the variable I, which is equal to the action of the wave:

$$d\mathscr{H}/dI = \omega(\mathscr{H}_0). \tag{3.31}$$

This yields immediately in place of (3.30′) the equation

$$\dot{I} = -\varepsilon\,\partial\mathscr{H}_1/\partial\vartheta, \tag{3.32}$$

which has a canonical form. The variables I and ϑ are canonically conjugate. We can write also analogously the equation for ϑ:

$$\dot{\vartheta} = \omega(I) + \varepsilon\,\partial\mathscr{H}_1/\partial I. \tag{3.30}$$

We note the connection between Eqs. (3.32) and (3.24). The term with $\partial\mathscr{L}/\partial k$ in (3.24) vanishes because k does not change under the boundary conditions (3.11). For the canonically conjugate quantities I and ϑ we have the identity

$$I = \partial\mathscr{L}/\partial\dot{\vartheta} = \partial\mathscr{L}/\partial\omega,$$

which leads to equality of (3.24) and (3.32).

The subsequent exposition of different particular problems will be carried out principally with the aid of the Hamiltonian equations.

4. EXTERNAL RESONANT PERTURBATION

a) Isolated resonance. The canonical equations of motion (3.32) and (3.33), obtained in first-order perturbation theory in ϵ, are of general character, since the form of Eq. (3.25) was not used anywhere[9)]. In this approximation the nonlinear wave can be regarded as before as a nonlinear oscillator with variables I and ϑ acted upon by the perturbation $\mathscr{H}_1(I, \vartheta, t)$ of the Hamiltonian[21]. Let us examine the simplest case, when the external force is given by

$$\Phi(x, t) = \Phi_m \cos(mkx - \nu t + \varphi) \tag{4.1}$$

and the resonance

$$m\omega(I) \approx \nu \tag{4.2}$$

is possible between the perturbation and the m-th harmonic of the nonlinear wave. The system (3.32), (3.33) can be approximately rewritten in the form[10)]

$$\dot{I} \approx \psi(I)\cos\theta, \tag{4.3}$$

$$\dot{\theta} \approx m\omega(I) - \nu + O(\varepsilon),$$

where

$$\psi = 2\varepsilon\,|a_m(I)|\,m\Phi_m. \tag{4.4}$$

If the resonance condition (4.2) is satisfied exactly at $I = I_0$, then phase oscillations take place near the resonance; these oscillations are described by the following

approximate integral of motion:

$$m\,[d\omega\,(I)/dI]\,(I-I_0)^2 - 2\psi\,(I_0)\sin\theta = \text{const}. \qquad (4.5)$$

The amplitude of these oscillations is of the order of

$$\delta I \sim [2\psi/m\omega'\,(I)]^{1/2} \propto \sqrt{\varepsilon}. \qquad (4.6)$$

Thus, the amplitude of the wave is modulated in time (see Fig. 10, which shows the time evolution of one crest of the nonlinear wave). In the linear approximation in ϵ, as is well known, the resonance gives rise to an instability, and modulation of the wave-packet profile was observed for gravitational waves on water[37,38] and in a number of other cases[39]. Expression (4.5) yields more complete information, since it describes the saturation of the instability. Simultaneously with modulation of the amplitude, the wave velocity is also modulated with an amplitude

$$\delta u\,(I) = (1/k)\,\delta\omega\,(I) = (1/k)\,(d\omega\,(I)/dI)\,\delta I. \qquad (4.7)$$

It is of interest to note that the width of the resonance (4.6), (4.7) is determined essentially by the spatial dependence of the external perturbation. This is manifest in the dependence of δI on a_m, where the number m is determined by the spatial period of the external force. At $m > N$, the quantities a_m become exponentially small (see Sec. (e) of Chap. 2), and accordingly the width of the resonance decreases sharply. Such perturbations have a smaller wavelength than the width of the crest of the unperturbed wave.

A more rigorous investigation of an isolated resonance, based on constructing and analyzing asymptotic series, was described in[30].

In those cases when the external perturbation contains more than one harmonic which can be at resonance with the modes of the wave, the situation becomes much more complicated. If the widths of individual resonances overlap, the result is a stochastic instability, which will be discussed by us in the next section. However, if there is no overlap of the resonances, then the question of the behavior of the system remains open. The resultant difficulties are easier to understand by turning to a simpler system, namely a nonlinear oscillator acted upon by an external force[36,40]. Such a dynamic system has effectively one-and-a-half degrees of freedom, and it can be described by using the Kolmogorov-Arnold-Moser stability theorem. If the resonances between the oscillator and the harmonics of the external force do not overlap, then the oscillator motion in the vicinity of some resonance is stable. We cannot, however, formulate an analogous result for a system in which the nonlinear wave is perturbed by an external force. The reason is that the number of degrees of freedom of a nonlinear wave is in general infinite.

b) **Stochastic instability.** In the general case of an arbitrary dependence of an external perturbation on x and t, it is impossible to obtain the solution. However, a certain limiting case corresponding to a very strong stochastic instability[36] lends itself to analysis. Different cases of the onset of such an instability of a nonlinear wave were investigated in[21,30]. We confine ourselves here to a qualitative consideration of one simple case.

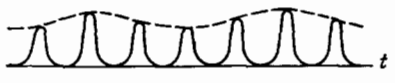

FIG. 10

Let us assume that the spectrum $\Phi(x, t)$ is characterized as before by one frequency ν, but is broad in terms of k, i.e.,

$$\Phi\,(x, t) = \sum_n \Phi_n \cos\,(nkx - \nu t). \qquad (4.8)$$

Assume, as before that a resonance condition of the type (4.2) is satisfied:

$$m\omega\,(I_m) = \nu, \qquad (4.9)$$

where I_m is the action of the nonlinear wave at which exact resonance is obtained between the perturbation and the m-th harmonic of the wave. The resonance closest to the m-th is determined from the condition

$$(m+1)\,\omega\,(I_{m+1}) = \nu. \qquad (4.10)$$

From (4.9) and (4.10) we can determine the distance between the closest resonances:

$$\Omega = \omega\,(I_m) - \omega\,(I_{m+1}) = \nu/m\,(m+1) \approx \nu/m^2 = \omega^2/\nu. \qquad (4.11)$$

On the other hand, each isolated resonance has, in accordance with (4.6) and (4.7), a certain width $\delta\omega$. If

$$K \equiv (\delta\omega/\Omega)^2 \ll 1 \qquad (4.12)$$

stability obtains and the single-resonance approximation is valid. To the contrary, if

$$K = (\delta\omega/\Omega)^2 \gg 1 \qquad (4.13)$$

the wave cannot go out of resonance with the external force for a long time, and the so-called stochastic instability develops[36,40], wherein the phase of the wave ϑ varies randomly with time, and the wave itself behaves like a Brownian particle[21].

For an effective development of the stochastic instability it is obviously necessary to satisfy not only conditions of type (4.13), but also necessary that the harmonics a_{max} and Φ_m in formulas (4.4) and (3.27) be substantially different from zero. This determines the condition for the "saturation" of the stochastic instability. Indeed, let I_{max} be the maximum value of the action which the wave can acquire as a result of the instability. From the condition

$$M\omega\,(I_{max}) = \nu$$

and from (4.4) it follows that $M_{max} = \min\,(N, N_\Phi)$, where N_Φ is the characteristic number of harmonics in the perturbation spectrum.

It is interesting to note that the described instability mechanism is possibly realized when waves are dispersed by wind over the sea surface[41]. Against the background of the fundamental periodic component of the wave profile, there travel randomly modulated ripples, as a result of which the wave becomes accelerated. With increasing wave amplitude, N increases. The limitations on the instability development are connected either with N_Φ or with the critical velocity (see Sec. (c) of Chap. 2), and when this velocity is reached the wave breaks.

5. INTERACTION OF NONLINEAR WAVES

a) **Weak-coupling parameter.** Questions connected with the interaction of nonlinear waves are not only among the most interesting in the theory, but also among the most difficult. As was shown in Chap. 3, under conditions of sufficiently weak perturbation the nonlinear

wave can be regarded as a certain quasiparticle, meaning a sufficiently high stability of the wave packet as a whole relative to spreading out as a result of the perturbation. It will be shown below that a similar concept can be introduced in the theory of the interaction of nonlinear waves. Since the interaction is usually connected with resonances between certain modes, we can point to a simple example of the condition of weakness of the interaction between nonlinear waves:

$$\varepsilon = M/N \ll 1, \tag{5.1}$$

where M is the characteristic number of the packet modes participating simultaneously in the resonant interaction with other packets and N is the characteristic number of the modes in the wave-packet spectrum.

The peculiarity of the introduced small parameter lies in the fact that generally speaking it may not depend on the amplitude of the wave. Moreover, the inequality (5.1) can be realized only at $N \gg 1$, i.e., for strongly nonlinear waves. In accordance with the remarks made in Sec. (e) of Chap. 2, we can state that ϵ is a small parameter proportional to the reciprocal strong-coupling constant.

b) Equations describing the interaction of nonlinear waves. An analysis of wave interaction can be carried out in general form, but it is more convenient to turn to the model of ion-acoustic oscillations (Sec. (b) of Chap. 3) in the non-one-dimensional case. In the approximation in which $kr_d \ll 1$ and the inequality (2.16) is valid, we use the system of equations obtained in[42,43]:

$$\begin{gathered}\rho_t + \nabla(\rho\nabla\Phi) = 0,\\ \Phi_t + \rho + \Delta\rho + \frac{1}{2}(\nabla\Phi)^2 - \frac{1}{2}\rho^2 = 0,\end{gathered} \tag{5.2}$$

where Φ is the potential of the ion velocity, and we put for convenience $\rho_0 = c = r_d = 1$ (the notation here is the same as in Sec. (c) of Chap. 2[11)]). By means of a Fourier expansion

$$\left.\begin{aligned}\rho &= \sum_{\mathbf{q}} \sqrt{\frac{q}{\omega_q}}\,[a(\mathbf{q})\,e^{i\mathbf{q}\cdot} + a^*(\mathbf{q})\,e^{-i\mathbf{q}\mathbf{r}}],\\ \Phi &= -i\sum_{\mathbf{q}} \frac{1}{q}\sqrt{\frac{\omega_q}{q}}\,[a(\mathbf{q})\,e^{i\mathbf{q}\mathbf{r}} - a^*(\mathbf{q})\,e^{-i\mathbf{q}\mathbf{r}}],\\ \omega_q^2 &= q^2(1-q^2), \quad \omega_q \approx q\left(1-\frac{1}{2}q^2\right)\end{aligned}\right\} \tag{5.3}$$

the system (4.2) is reduced to a single equation

$$\begin{gathered}\dot{a}(\mathbf{q}) + i\omega_q a(\mathbf{q}) + \frac{i}{4}q\sum_{\mathbf{q}_1,\mathbf{q}_2} V_{\mathbf{q}\mathbf{q}_1\mathbf{q}_2}[a(\mathbf{q}_1)\,a(\mathbf{q}_2)\,\delta(\mathbf{q}-\mathbf{q}_1-\mathbf{q}_2)\\ + 2a(\mathbf{q}_1)\,a^*(\mathbf{q}_2)\,\delta(\mathbf{q}-\mathbf{q}_1+\mathbf{q}_2) + a^*(\mathbf{q}_1)\,a^*(\mathbf{q}_2)\,\delta(\mathbf{q}+\mathbf{q}_1+\mathbf{q}_2)] = 0,\\ V_{\mathbf{q}\mathbf{q}_1\mathbf{q}_2} = (\mathbf{q}\mathbf{q}_1/qq_1) + (\mathbf{q}\mathbf{q}_2/qq_2) + (\mathbf{q}_1\mathbf{q}_2/q_1q_2) - 1.\end{gathered} \tag{5.4}$$

In the one-dimensional case, for a wave propagating in one direction, we have $V_{\mathbf{q}\mathbf{q}_1\mathbf{q}_2} = 2$, and Eq. (5.4) takes, in spatial variables, the form of the Korteweg-de Vries equation

$$\rho_t + \rho_r + \rho\rho_r + (1/2)\rho_{rrr} = 0, \tag{5.5}$$

where r is the coordinate in the direction of q.

The Hamiltonian for (5.4) is

$$\begin{gathered}\mathscr{H} = \sum_{\mathbf{q}}\left(1-\frac{1}{2}q^2\right)a(\mathbf{q})\,a^*(\mathbf{q}) +\\ + \frac{1}{4}\sum_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3} V_{\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3}\Big[\frac{1}{3}a(\mathbf{q}_1)\,a(\mathbf{q}_2)\,a(\mathbf{q}_3)\,\delta(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3)\\ + a(\mathbf{q}_1)\,a(\mathbf{q}_2)\,a^*(\mathbf{q}_3)\,\delta(\mathbf{q}_1+\mathbf{q}_2-\mathbf{q}_3) + \text{c.c.}\Big],\end{gathered} \tag{5.6}$$

and (4.4) can be written in canonical form

$$\dot{a}(\mathbf{q}) = -iq\,\partial\mathscr{H}/\partial a^*(\mathbf{q}). \tag{5.7}$$

When wave interaction is considered, we start from the fact that the solution in the zeroth approximation is a superposition

$$\rho(\mathbf{r}, t) \approx \sum_s \rho(\mathbf{r}_s - \mathbf{u}_s t), \tag{5.8}$$

where $\mathbf{r}_s \parallel \mathbf{u}_s$. By regrouping the terms in (4.6), we represent H in the form

$$\begin{gathered}\mathscr{H} = \sum_s \mathscr{H}_s + \mathscr{H}_I,\\ \mathscr{H}_s = \sum_{q_s}\left(1-\frac{1}{2}q_s^2\right)a(q_s)\,a^*(q_s) + \frac{1}{2}\sum_{q_s,q_s',q_s''}\Big[\frac{1}{3}a(q_s)\\ \times a(q_s')\,a(q_s'')\,\delta(q_s+q_s'+q_s'') + a(q_s)\,a(q_s')\,a^*(q_s'')\,\delta(q_s+q_s'-q_s'')] + \text{c.c.}\Big],\\ \mathscr{H}_I = \sum_{s_1\neq s_2} a(q_{s_1}=0)\,a(q_{s_2}=0)\\ + \frac{1}{4}\sum_{s_1,s_2,s_3}{}' \sum_{\mathbf{q}_{s_1},\mathbf{q}_{s_2},\mathbf{q}_{s_3}} V_{\mathbf{q}_{s_1}\mathbf{q}_{s_2}\mathbf{q}_{s_3}}\Big[\frac{1}{3}a(\mathbf{q}_{s_1})\,a(\mathbf{q}_{s_2})\,a(\mathbf{q}_{s_3})\,\delta(\mathbf{q}_{s_1}+\mathbf{q}_{s_2}+\mathbf{q}_{s_3})\\ + a(\mathbf{q}_{s_1})\,a(\mathbf{q}_{s_2})\,a^*(\mathbf{q}_{s_3})\,\delta(\mathbf{q}_{s_1}+\mathbf{q}_{s_2}-\mathbf{q}_{s_3}) + \text{c.c.}\Big].\end{gathered} \tag{5.9}$$

The different subscripts s point to different one-dimensional nonlinear waves; q_s, q_s', and q_s'' pertain to one and the same wave; the prime at the summation sign means exclusion of the term with $s_1 = s_2 = s_3$, and the direction of the vector q_s coincides with the velocity direction $\mathbf{u}_s$. The quantity $\mathscr{H}_I$ describes the interaction of the nonlinear waves. Naturally, it should be small enough in order for the representation (5.8), and for the concept of a nonlinear wave to be meaningful in general.

c) Interaction of two waves. Let us see how the small parameter comes into being when two waves whose directions of motion are inclined at not too small an angle interact. We note first that the first term in $\mathscr{H}_I$ does not depend on the time and can always be eliminated by a suitable renormalization of the Hamiltonian. Further, the δ-functions in $\mathscr{H}_I$ select the terms with

$$q_1 = q_1', \quad q_2 = 0 \text{ and } q_1 = 0, \quad q_2 = q_2'. \tag{5.10}$$

According to (4.4) we have

$$V_{q_1,q_1,0} = V_{q_2,q_2,0} = 2\cos(\mathbf{u}_1, \mathbf{u}_2) \equiv 2\cos\gamma,$$

and we obtain for $\mathscr{H}_I$ the expression

$$\begin{gathered}\mathscr{H}_I \approx 2\frac{\cos\gamma}{|\sin\gamma|}\Big(a(q_2=0)\sum_{q_1}|a(q_1)|^2\\ + a(q_1=0)\sum_{q_2}|a(q_2)|^2\Big) \approx 2\frac{\cos\gamma}{|\sin\gamma|}\frac{\alpha_1\alpha_2}{N_1N_2}(\alpha_1+\alpha_2),\end{gathered} \tag{5.11}$$

where relations (2.20) and (2.21) are taken into account. We see already that the selection rules (5.10) decrease the number of terms in $\mathscr{H}_I$ in comparison with the nonlinear term in $\mathscr{H}_s$ by an approximate factor N_s. Indeed, an estimate of the latter yields $\mathscr{H}_{1,2} \sim \alpha_{1,2}^3/N_{1,2}$. Thus, the small parameter of the interaction is

$$\varepsilon = \operatorname{ctg}\gamma/N \ll 1. \tag{5.12}$$

Its physical meaning can be easily understood from Fig. 11, which shows the plane in which the waves move (top view), and the crests of the waves are hatched. At large N, the region of intersection of the crest is relatively small, and this leads to a weakening of the wave interaction. This result was obtained in[44]. With the aid of the Poisson brackets

$$\frac{dP[a, a^*]}{dt} = -i\sum_s q_s\left(\frac{\partial P}{\partial a(q_s)}\frac{\partial\mathscr{H}}{\partial a^*(q_s)} - \frac{\partial P}{\partial a^*(q_s)}\frac{\partial\mathscr{H}}{\partial a(q_s)}\right) \tag{5.13}$$

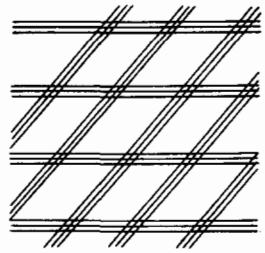

FIG. 11

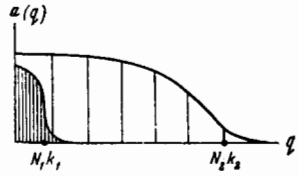

FIG. 12

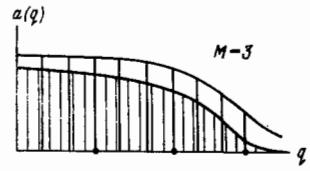

FIG. 13

it is easy to obtain

$$\dot{\mathscr{H}}_1 = \dot{\mathscr{H}}_2 = 0, \tag{5.14}$$

and from the equations of motion

$$\dot{a}(q_1) + iq_1u_1a(q_1) + iq_1\left(\frac{V_{q_1q_10}}{|\sin\gamma|}\right)a(q_2=0)\,a(q_1) = 0 \tag{5.15}$$

it follows that as a result of the interaction a shift takes place in the wave velocity

$$\Delta u_1 = 2\left(\frac{\cos\gamma}{|\sin\gamma|}\right)a(q_2=0) \approx 2\left(\frac{\cos\gamma}{|\sin\gamma|}\right)\frac{\alpha_2}{N_2} \tag{5.16}$$

and analogously

$$\Delta u_2 = 2\left(\frac{\cos\gamma}{|\sin\gamma|}\right)a(q_1=0) \approx 2\left(\frac{\cos\gamma}{|\sin\gamma|}\right)\frac{\alpha_1}{N_1}. \tag{5.17}$$

The condition for the smallness of the velocity shift $\Delta u_i \ll \alpha_i$ is equivalent to (5.12).

The result is also a consequence of the fact that the region of intersection of two periodic waves in Fig. 11, with which the interaction is connected, does not depend on the time, and this leads to a simple reorientation of the wave velocities (in the case of three waves, this, naturally, is not so).

The transition to the one-dimensional case ($\gamma \to 0$) in (5.11) and subsequently is impossible. This is due to the two-dimensional normalization of the energy in (5.9)[12]. Let us examine the interaction of two waves in the one-dimensional case. The trivial case of weak interaction between waves is a weak overlap of the spectra of the nonlinear waves (Fig. 12), which can be written in the form

$$N_1k_1 \lesssim k_2. \tag{5.18}$$

Actually, when the amplitudes of two waves differ little, a weak coupling between the waves is reached already under condition (5.1), where M is determined from the condition of the overlap of the wave spectra

$$N_1k_1 = Mk_2, \quad M \ll N_2. \tag{5.19}$$

We obtain the condition for the smallness of the resonant terms M in the following manner. From the selection rules imposed by the δ-functions in $\mathscr{H}_I$ (5.9) it follows that $n_1k_1 = n_2k_2$. The condition for periodicity of the solution, with period $\lambda_0 = 2\pi/k_0$, means

$$n_1m_1 = n_2m_2, \quad m_{1,2} \equiv \lambda_0/\lambda_{1,2} = k_{1,2}/k_0, \tag{5.20}$$

where $m_{1,2}$ are integers. It follows therefore that M is the number of roots of Eq. (5.20) relative to n_1 and n_2

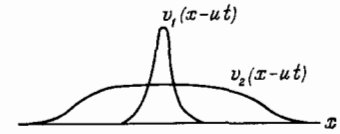

FIG. 14

with integer coefficients subject to the limitation $n_{1,2} \lesssim N_{1,2}$. This means, in particular, that at $N_{1,2} \gg m_{1,2} \gg 1$ the ratio $M/N_{1,2}$ is small (Fig. 13, where the coinciding thick lines correspond to solutions of (5.20)).

Thus, the interaction of two nonlinear periodic waves is connected with two kinds of effects. The first is renormalization of the velocities. It always exists and is determined for each of the waves by the zeroth Fourier harmonic of the other wave (i.e., by the averaged value of the wave profile). The second effect is connected with the resonances between the waves and is determined by a parameter of the type (5.1). Allowance for the resonances leads to modulation oscillations of the harmonics of the spectrum and was considered for nonlinear plasma oscillations in[22,23]. As a result of the interaction of the two nonlinear waves, a bound state is produced, which can be interpreted as a two-stream solution[13)].

Finally, let us point out one more parameter that leads to weak interaction of the wave[45]. It results from weak overlap of the crests of the waves in the one-dimensional case, and its meaning is easily understood from Fig. 14. With the aid of such a parameter, for example, we can construct a two-stream solution in which the wave are opposing[45], namely $u_1 > c$ and $u_2 < -c$.

d) Interaction of three waves. This case differs significantly from the interaction of two waves and it is advantageous to consider it in detail.

We shall assume that conditions are satisfied under which there are no resonant interactions between pairs of waves, but resonance between three waves can be realized. From the general expression (5.9) for the Hamiltonian (s takes on the values 1, 2, 3) and from the expression for the Poisson brackets (5.13) we obtain the following equations:

$$\left.\begin{aligned}
\dot{\mathscr{H}}_1 &= \frac{1}{2}\sum_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3}(-i\mathbf{q}_1u_1)V_{123}a_1^*a_2a_3\delta(\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3) + \text{c.c.},\\
\dot{\mathscr{H}}_2 &= \frac{1}{2}\sum_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3}(-iq_2u_2)V_{123}a_1a_2^*a_3^*\delta(\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3) + \text{c.c.},\\
\dot{\mathscr{H}}_3 &= \frac{1}{2}\sum_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3}(-iq_3u_3)V_{123}a_1a_2^*a_3^*\delta(\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3) + \text{c.c.},
\end{aligned}\right\} \tag{5.21}$$

where $a_i = a(q_i)$. Each of the waves can be expanded in a Fourier series

$$\begin{aligned}
&\rho_s = \rho(\mathbf{r}_s - \mathbf{u}_st) = \sum_{\mathbf{q}} a_se^{i\mathbf{q}_s\mathbf{r}_s}, \quad q_s = n_sk_s,\\
&a_s = |a_s|\exp(-i\vartheta_s - i\vartheta_{s0}),
\end{aligned} \tag{5.22}$$

where ϑ_{s0} is a certain initial phase. Since we shall investigate from now on the resonant phase, we can, just as in Sec. (a) of Chap. IV, neglect in the equation

$$\dot{\vartheta}_s = \omega_s + \partial\mathscr{H}_I/\partial I_s = k_su_s + \partial\mathscr{H}_I/\partial I_s \tag{5.23}$$

the second term in the right-hand side. The action variable I_s is introduced here with the aid of the relation (3.31):

$$d\mathscr{H}_s/dI_s = \omega_s(\mathscr{H}_s). \tag{5.24}$$

It is seen from (5.22) and (5.23) that resonance is possible in the system (5.21) if the following conditions are satisfied simultaneously:

$$\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3 = m_1\mathbf{k}_1 - m_2\mathbf{k}_2 - m_3\mathbf{k}_3 = 0, \tag{5.25}$$

$$q_1u_1 - q_2u_2 - q_3u_3 = m_1k_1u_1 - m_2k_2u_2 - m_3k_3u_3 = 0.$$

In this case the principal role is played by the variation of the wave parameters as a result of the resonance between them, and we can write in place of (5.21), in terms of the canonical variables (5.23) and (5.24)

$$\dot{I}_1 = m_1\Gamma\sin\theta, \quad \dot{I}_2 = -m_2\Gamma\sin\theta, \quad \dot{I}_3 = -m_3\Gamma\sin\theta, \tag{5.26}$$

where

$$\theta = \vartheta_1 - \vartheta_2 - \vartheta_3 + \vartheta_0, \quad \vartheta_0 = \vartheta_{10} - \vartheta_{20} - \vartheta_{30},$$
$$\Gamma = \frac{V_{123}}{|\sin\gamma_{23}|}\,|a_1(I_1)\,a_2(I_2)\,a_3(I_3)| \equiv \Gamma(I_1, I_2, I_3). \tag{5.27}$$

The obtained system (5.26) is analogous to those that occur in nonlinear optics[46] and in plasma theory[47]. It can be integrated with the aid of relations of the Manley-Rowe type, which follow from (5.26):

$$\left.\begin{aligned} I_1m_2 + I_2m_1 &= \text{const},\\ I_1m_3 + I_3m_1 &= \text{const},\\ I_2m_3 - I_3m_2 &= \text{const}. \end{aligned}\right\} \tag{5.28}$$

The integrals (5.28), of which only two are independent, enable us to express the answer in quadratures. The described results were obtained in[22]. We confine ourselves to an analysis of the motion of the waves in the vicinity of the resonance. The resonance conditions (5.25) are satisfied at the action values I_{10}, I_{20}, and I_{30}. It follows from (5.26) that

$$\int_{I_{10}}^{I_1}\omega_1\,dI_1 + \int_{I_{20}}^{I_2}\omega_2\,dI_2 + \int_{I_{30}}^{I_3}\omega_3\,dI_3 + \Gamma_0\cos\theta = \text{const}, \tag{5.29}$$

where $\Gamma_0 = \Gamma(I_{10}, I_{20}, I_{30})$. Expanding $\Gamma_s(I_s)$ in the vicinity of I_{s0} and using relations (5.28), we obtain an integral of motion analogous to (4.5), describing the phase oscillations in a system of three nonlinear waves

$$\sum_{s=1}^{s}\frac{d\omega_s}{dI_s}(I_s - I_{s0})^2 + 2\Gamma_0\cos\theta = \text{const}. \tag{5.30}$$

From this we determine the amplitude of the modulation (the width of the resonance) with respect to the action

$$\Delta I \sim [2\Gamma_0/(d\omega/dI)]^{1/2}, \tag{5.31}$$

and with respect to the frequency

$$\Delta\omega \sim (2\Gamma_0\,d\omega/dI)^{1/2}. \tag{3.32}$$

In particular, for the system (5.2) we can easily estimate

$$\Delta\omega \sim (V_{123}/|\sin\gamma|)^{1/2}k\alpha/N. \tag{5.33}$$

The obtained formula shows that the small parameter of the interaction is of the particular order $\Delta\omega/k \propto N^{-1}$. It has the same cause here as in the case of the interaction of two waves.

In essence, the described process is a nonlinear analysis of decay-type instability. All three waves participating in the resonance are generally speaking strongly nonlinear ($N_s \gg 1$). One could investigate the stability near the resonance of one of the nonlinear waves relative to the excitation of waves of small amplitude in an approximation that is linear with respect to the perturbation. This method of investigation and the analogous conclusions relative to the instability are contained in[37,38] for gravitational waves on the surface of a deep liquid, and for the non-one-dimensional Korteweg-de Vries equation[48,49].

The interaction in question includes resonance between harmonics of nonlinear waves at only one mode. In the more general case, the system (5.25) can admit of several solutions with integer coefficients (m_1, m_2, m_3), subject to the limitation $m_s < N_s$. Perturbation theory can still remain in force, provided the number of such solutions is $M \ll N$. No investigation was carried out in this case. It is not excluded that even in the case of three nonlinear waves one can encounter phenomena of the stochastic-instability type, which lead to breakdown of the integrals (5.28) or (5.29) in certain regions of wave-parameter values.

6. NONLINEAR WAVES IN AN INHOMOGENEOUS MEDIUM

Equations (3.16), which were obtained by Whitham, help consider the motion of a nonlinear wave in a weakly-inhomogeneous medium in the simplest form. This approximation is naturally called adiabatic. Its equivalent is the use of series of the type (3.17). The last method may turn out to be sometimes more preferable than the Whitham method, since it does not presuppose knowledge of the Lagrangian. The adiabatic representations make it possible to obtain, for example, definite results in the problem of the run up of a solitary wave on a smoothly sloping shore[50-52][14]. Naturally, when the adiabatic approximation is used, the principal information is extracted from the conservation laws and from the transport equations for the adiabatic invariance[33,53]. The determination of the nonadiabatic corrections for nonlinear waves is a problem of different type. For nonlinear waves there is at present no method so well developed as that of the theory of dynamic systems. A very important result in this direction is Moser's theorem[9], that the adiabatic invariant of a nonlinear wave is conserved in all orders of perturbation theory. The latter means that the change of the adiabatic invariant should be regarded as exponentially small.

For the case of the boundary-value problem (3.11) and a weakly-inhomogeneous medium, the calculation of the change of the adiabatic invariant can be carried out directly[54]. Assume, for example, that in the Klein-Gordon equation (2.2) the parameter c depends periodically on x:

$$c(x) = c(x + \Lambda), \tag{6.1}$$

and the adiabaticity condition is satisfied

$$k\Lambda \gg 1. \tag{6.2}$$

For a nonlinear string, the function c(x) may be connected either with the inhomogeneity of the elastic forces moving along the string, or with the inhomogeneous distribution of the string mass. Instead of the Hamiltonian in the form (3.7), we now must write

$$\mathscr{H}[c^2] = \frac{1}{2}\sum_n \dot{y}_n\dot{y}_{-n} - \frac{1}{2}\sum_{n_1, n_2, n_3} n_1n_2(c^2)_{n_3}y_{n_1}y_{n_2} \tag{6.3}$$
$$\times\,\delta(n_1 + n_2 + n_3) - \frac{1}{3}\sum_{n_1, n_2, n_3} y_{n_1}y_{n_2}y_{n_3}\delta(n_1 + n_2 + n_3),$$

where

$$(c^2)_n = \frac{1}{\Lambda}\int_{-\Lambda/2}^{\Lambda/2} dx\,e^{-inkx}c^2(x). \tag{6.4}$$

From (6.1), (6.2), and (6.4) it follows that

$$(c^2)_n \propto e^{-nk\Lambda} \tag{6.5}$$

and consequently is exponentially small. Using (6.5), we rewrite $\mathscr{H}$ in the form

$$\mathscr{H} \approx \mathscr{H}_0 + \mathscr{H}_1, \tag{6.6}$$

where $\mathscr{H}_0$ is expression (6.3) in which $(c^2)_{n3}$ is replaced by

$$(c^2)_0 = \frac{1}{\Lambda} \int_{-\Lambda/2}^{\Lambda/2} dx c^2(x) \equiv \overline{c^2}, \tag{6.7}$$

$$\mathscr{H}_0 = \mathscr{H}[(c^2)_0].$$

The quantity

$$\mathscr{H}_1 = -\frac{1}{2} \sum_{n_1, n_2} [(c^2)_1 y_{n_1} y_{n_2} \delta(n_1 + n_2 + 1) + (c^2)_{-1} y_{n_1} y_{n_2} \delta(n_1 + n_2 - 1)] \tag{6.8}$$

is exponentially small in accordance with (6.6) and can be regarded as a perturbation for $\mathscr{H}_0$ by the method described in Sec. (e) of Chap. 3. The qualitative result is already clear, however, since the change of $\mathscr{H}_0$ and the change of the action will be proportional to $\mathscr{H}_1$.

Similar arguments can be advanced also in the general case, when $c = c(x, t)$ and the characteristic scales of the variation of c with respect to x and t are large in comparison with the corresponding scales of the wave. To this end it is necessary to use the Lagrangian and represent it in the form

$$\mathscr{L}[c^2] \approx \mathscr{L}[(c^2)_{00}] + \mathscr{L}_{01}[(c^2)_{01}] + \mathscr{L}_{10}[(c^2)_{10}], \tag{6.9}$$

where the first subscript of c^2 denotes the Fourier harmonic with respect to x, and the second with respect to t.

The calculation of the corrections to the energy and to the action of the wave does not solve completely the problem of propagation of a nonlinear wave in a weakly inhomogeneous medium. What is left aside is the important problem of determining the reflected wave. The formulation of such a problem, of course, is doubtful, since there is no superposition principle in the nonlinear case. In[45] an attempt was made to construct a solution that takes into account the "reflected" waves with the aid of the so-called two-stream solution considered in Sec. (c) of Chap. 5. On the left and on the right of the singular region, from which the reflections come, one considers coupled pairs of nonlinear waves. In each pair there are waves traveling in opposite directions. The conditions for the matching of such two-stream solutions at the boundary of the singular region determine the corresponding transmission and reflection parameters. Scattering by the inhomogeneity changes the wavelength and the phase discontinuity; the values of the changes are calculated in[45].

These results seem to agree with Howe's experimental data[55a].

In analogy with the definition of a reflected wave, one can consider also the problem of transformation of nonlinear waves of different types in a weakly inhomogeneous medium[55b].

Scattering or transformation problems entail basically the determination of the spatial characteristics of the wave for a given periodic time variation with frequency ω (see Sec. (c) of Chap. 1). In the homogeneous case, the dispersion equations (2.9) for nonlinear stationary waves are replaced by the following:

$$\omega = \omega(u, C). \tag{6.10}$$

Let now (6.10) have, for example, two "branches" of solutions:

$$\omega = \omega(u_i, C_i) \qquad (i = 1, 2), \tag{6.11}$$

and let different branches of the oscillations correspond to different (non-overlapping) regions of possible values of u, at which a solution exists in the form of a wave. In the linear case $C \to 0$ and (6.11) leads to two dispersion laws $k_i = k_i(\omega)$.

In the weakly-inhomogeneous case we have in the adiabatic approximation in place of (6.11)

$$\omega = \omega(u_i(x), C_i(x)). \tag{6.12}$$

There can exist a region of values $x \sim x_0$ in which $u_1(x_0) \sim u_2(x_0)$. This causes the wave numbers corresponding to different branches to coincide:

$$k_1(x_0) = \omega/u_1(x_0) \sim k_2(x_0) = \omega/u_2(x_0).$$

This region of the coordinates is singular. The adiabatic approximation is violated in its vicinity. Just as in the linear case, the wave of one of the oscillation modes can "excite" in the vicinity of x_0 a wave of another mode. If the corresponding transformation coefficients (ratio of the amplitude of the scattered or excited wave in the amplitude of the incident wave) are sufficiently small, then they can be determined by using the following circumstance[45,55b]: the n-th harmonic of the new wave is produced with a factor ρ^n, where ρ is an exponentially small quantity. It suffices therefore to confine oneself in the scattered or excited wave to only one first harmonic, i.e., to the linear approximation.

7. KINETICS OF NONLINEAR WAVES

Among the different factors that perturb the motion of nonlinear waves one should separate those representing a certain accidental process. This applies to the propagation of nonlinear waves in random inhomogeneous and nonstationary media, to the interaction of waves representing a statistical ensemble, etc. There exists an approximation in which the nonlinear equations of motion can first be averaged over a certain random process, and the nonlinear interaction between the harmonics is taken into account only after equations of motion are averaged[56]. This approximation is realized in those cases when the connection between the harmonics is sufficiently weak. In this chapter we consider certain cases in which there exists a strong connection between the harmonics, i.e., the waves are essentially nonlinear (the quantity N can be very large).

a) Random external forces. The well known problem of Brownian motion of a particle under the influence of an external random force can be considered to some degree of approximation also for nonlinear waves[57]. Such random forces may be wind perturbations for gravitational waves on the surface of a liquid, a turbulent medium in which a nonlinear wave propagates, a medium with random inhomogeneities, etc.

Let

$$\mathscr{H} = \mathscr{H}_0 + \varepsilon \mathscr{H}_1$$

be the Hamiltonian of the problem, where $\mathscr{H}_0$ corresponds to the unperturbed motion, i.e., the nonlinear wave $y = y(x - ut)$, and $\delta\mathscr{H}_1$ is a perturbation in the form (3.20), in which $\Phi(x, t)$ is a random external force. As to $\Phi(x, t)$, we shall assume that it is a Gaussian random process and

$$\langle \Phi(x, t) \rangle = 0, \tag{7.1}$$

$$\langle \Phi(x, t)\, \Phi(x + \xi, t + \tau) \rangle = R(\xi, \tau).$$

The correlation function introduces into the problem a certain correlation, decrease time τ_c, which we shall assume for simplicity to be shorter than all the times of the problem. As $\tau_c \to 0$, we have $R(\xi, \tau) \to R_0(\xi)\delta(\tau)$, i.e., white noise (with respect to time) is obtained. It is convenient also to introduce the spectral density

$$S(\varkappa, \nu) = \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\tau R(\xi, \tau) e^{i(\varkappa x - \nu t)}. \tag{7.2}$$

If the problem has periodic boundary conditions on x, then the integration with respect to ξ in (7.2) must be carried out in the same manner as in (3.27).

We define the action I in the continuous case in analogy with (3.32)

$$d\mathscr{H}/dI = u(\mathscr{H}), \tag{7.3}$$

and write down the change of the action in the form

$$\Delta I = \int_{-\infty}^{t} dt_1 \frac{dI}{dt_1} = -\varepsilon \int_{-\infty}^{t} dt_1 \int_{-\infty}^{\infty} dx_1 y'(x_1 - ut_1)\Phi(x_1, t_1). \tag{7.4}$$

From this we can calculate the quantity

$$\langle(\Delta I)^2\rangle = \varepsilon^2 \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_2 \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 y'(x_1 - ut_1) \times y'(x_2 - ut_2) R(x_1 - x_2, t_1 - t_2), \tag{7.5}$$

where the right-hand side contains the unperturbed value $y(x - ut)$, which describes the nonlinear wave. It is now necessary to take into account the fact that $R(x, t)$ decreases rapidly at $t > \tau_c$, so that we can integrate in (7.5) with respect to t at $t \gg \tau_c$[57]:

$$D \equiv \frac{\langle(\Delta I)^2\rangle}{t} = 8\pi\varepsilon^2 \int_{-\infty}^{\infty} dq\, q^2 |y_q|^2 S(q, qu). \tag{7.6}$$

As is well known, D is the diffusion coefficient of the corresponding Fokker-Planck equation

$$\frac{\partial f(I, t)}{\partial t} = \frac{1}{2}\frac{\partial}{\partial I} D \frac{\partial f(I, t)}{\partial I}. \tag{7.7}$$

Equation (7.7) determines the Brownian motion of a nonlinear wave with the aid of the distribution function f and the diffusion coefficient (7.6). Equations (7.6) and (7.7) can describe, in particular, the motion of a soliton under the influence of random forces. If we consider a nonlinear periodic wave, then expression (7.6) changes into

$$D = 8\pi\varepsilon^2 \sum_n n^2 |y_n|^2 S(kn, kn\omega), \tag{7.8}$$

and the action I in (7.7) differs, in accordance with the definition (3.32), by a factor k from the preceding case.

Let us estimate the diffusion coefficient of the wave for the particular case of white noise ($\tau_c = 0$):

$$R(\xi, \tau) = \langle\Phi_0^2\rangle \delta(\xi)\delta(\tau),$$

We have

$$D = 4\varepsilon^2 \langle\Phi_0^2\rangle \int_{-\infty}^{\infty} dx (v')^2 = 8\pi\varepsilon^2 \langle\Phi_0^2\rangle I, \tag{7.9}$$

and the solution of (7.7) can be easily written. At $\tau_c \neq 0$ it follows from (7.7) that the wave will be stochastically accelerated and increase its energy.

Thus, the action of random forces can lead to the breaking of a nonlinear wave after the wave reaches a certain critical velocity or critical energy. The characteristic time of acceleration of the wave prior to breaking can be estimated with the aid of the diffusion coefficient D.

b) Nonlinear waves in random media. The motion of nonlinear waves in media with random inhomogeneities is of interest in various branches of physics, and was considered, in particular, for waves on water in[58,59]. If the parameter $\Lambda(x)$, which takes into account the inhomogeneity of the medium, can be represented in the form

$$\Lambda(x) = \Lambda_0 + \Lambda_1(x),$$

where Λ_0 does not depend on x and the random increment Λ_1 is small ($\Lambda_1 \ll \Lambda_0$), then the problem of the propagation of the nonlinear wave can be reduced to the already considered problem. Indeed, we write

$$\mathscr{H}[\Lambda] = \mathscr{H}[\Lambda_0] + \mathscr{H}_1[\Lambda_1],$$

where the perturbation $\mathscr{H}_1$ is proportional, generally speaking to Λ_1, $d\Lambda_1/dx, \ldots$ We can use next the method already described in Sec. a of Chap. 7, recognizing that the correlator (7.1) is determined uniquely in terms of the correlator of the random process $\Lambda_1(x)$:

$$\langle\Lambda_1(x)\rangle = 0, \quad \langle\Lambda_1(x)\Lambda_1(x + \xi)\rangle = R(\xi).$$

It is now necessary to take into consideration the fact that we are considering the problem of the scattering of a nonlinear wave. To this end it is necessary to use the variational principle (3.10) and the condition (3.12). This leads immediately to complications, inasmuch as reflected waves appear as a result of the perturbed part of the Hamiltonian $\mathscr{H}_1[\Lambda_1]$. Thus, for a correct solution of the problem of damping of a nonlinear wave in a medium with random inhomogeneities it is necessary to calculate first the matrix for scattering by the inhomogeneities, i.e., to introduce the reflected waves into consideration (see Sec. (a) of Chap. 7).

There exists one more circumstance that distinguishes in principle the nonlinear case from the linear one. Owing to the random inhomogeneities, the wave velocity fluctuates. At one of such fluctuations the wave velocity can reach the critical value at which the wave breaks. This process competes with the scattering process. Its quantitative analysis reduces to a calculation of the probability of first reaching the critical value of the velocity u_c in a specified segment of length l.

At present there is no rigorous analysis of the problem of propagation of a nonlinear wave in a random medium. The arguments presented above may point to one of the possible realizable ways.

c) Stochastic instability. A kinetic equation analogous to (7.7) can be written in those cases when the external action on the waves leads to its stochastic instability (see Sec. (d) of Chap. 2)[21,30]. The kinetic equation is obtained in this case as a result of averaging over the randomly varying phase, and the time of decrease of the correlation τ_c is estimated directly by starting from an analysis of the stochastic instability.

In all the described cases, the random element of the solution are the small increments due to the perturbation. Therefore the general picture of the form of the solution can be represented as a regular background (unperturbed wave) modulated by a certain random set of ripples. Since the perturbations can increase with time, the presented analysis is naturally limited in

time to the interval in which the distortions of the shape of the nonlinear wave remains small.

d) **Kinetic equation for an ensemble of nonlinear waves.** The notion that the periodic wave changes its parameters little during the course of perturbation makes it possible to consider a large number of nonlinear waves that interact weakly with one another, and to describe their evolution with the aid of a kinetic equation. The parameter describing the weakness of the interaction is determined directly, in a manner used in Secs. (c) and (d) of Chap. 5[15]. From the physical point of view, importance attaches to the following aspect of the investigation of the behavior of a large ensemble of nonlinear waves. When considering problems involving turbulence of a solid medium, groups of strongly correlated harmonics are produced. The limiting case, when all the harmonics are not correlated, is called weak turbulence. Wave packets within which the modes are strongly correlated can be described by the methods of weak-coupling theory if the interaction between the packets is small (this situation is analogous to the possibility of describing weakly interacting systems by individual wave functions in quantum mechanics). In the zeroth approximation, the wave packets do not interact with one another and consequently are exact solutions of the initial equations of motion. We thus arrive at the concept of a "gas" of weakly interacting nonlinear waves. The derivation of the kinetic equation by perturbation theory for an ensemble of weakly interacting nonlinear periodic waves was given in[22,23]. The starting point is the Liouville equation

$$\frac{\partial f}{\partial t}+\sum_{s=1}^{S}\dot{\vartheta}_s\frac{\partial f}{\partial\vartheta_s}+\sum_{s=1}^{S}\dot{I}_s\frac{\partial f}{\partial I_s}=0, \tag{7.10}$$

where $f(t;\vartheta_1, I_1, \ldots, \vartheta_S, I_S)$ is a distribution function that depends on $S \gg 1$ pairs of canonically-conjugate variables (ϑ_i, I_i). These variables satisfy equations of the type (3.25) and (3.26):

$$\begin{aligned}\dot{I}_s&=-\frac{\partial\mathscr{H}}{\partial\vartheta_s}=-\frac{\partial\mathscr{H}_I}{\partial\vartheta_s},\\ \dot{\vartheta}_s&=\frac{\partial\mathscr{H}}{\partial I_s}=\omega_s(I_s)+\frac{\partial\mathscr{H}_I}{\partial I_s},\end{aligned} \tag{7.11}$$

where the total Hamiltonian $\mathscr{H}$ and that part of the Hamiltonian $\mathscr{H}_I$ which takes into account the wave interaction take, e.g., for ion-acoustic oscillations, the form (5.9). If the wave phases ϑ_S are regarded as random, then we can obtain for the distribution function averaged over the phases

$$F(t;\ I_1,\ \ldots,\ I_S)=(2\pi)^{-S}\int_0^{2\pi}d\vartheta_1\ldots d\vartheta_S f(t;\ I_1,\ \vartheta_1,\ \ldots,\ I_S,\ \vartheta_S)$$

the following kinetic equation in the case of ion-acoustic oscillations[16]:

$$\left.\begin{aligned}\frac{\partial F}{\partial t}&=6\pi\sum_{m_1,\,m_2,\,m_3}\ \sum_{n_{m_1},\,n_{m_2},\,n_{m_3}}|V_{m_1m_2m_3}|^2\hat{D}_{m_1m_2m_3}\\ &\times|a(n_{m_1})a(n_{m_2})a^*(n_{m_3})|^2\delta(n_{m_1}\omega_{m_1}+n_{m_2}\omega_{m_2}-n_{m_3}\omega_{m_3})\\ &\qquad\times\delta(n_{m_1}\mathbf{k}_{m_1}+n_{m_2}\mathbf{k}_{m_2}-n_{m_3}\mathbf{k}_{m_3})\hat{D}_{m_1m_2m_3}F,\\ \hat{D}_{m_1m_2m_3}&\equiv n_{m_1}\frac{\partial}{\partial I_{m_1}}+n_{m_2}\frac{\partial}{\partial I_{m_2}}-n_{m_3}\frac{\partial}{\partial I_{m_3}},\end{aligned}\right\} \tag{7.12}$$

where the matrix element V and the amplitudes a(n) have the same meaning as in (5.9).

Equation (7.12) has an equilibrium solution

$$F=F\left(\sum_s(H_s-\mathbf{k}_s\mathbf{w}I_s)\right), \tag{7.13}$$

where the constant w is the macroscopic velocity and can be excluded.

As follows from (5.4), the kernel $V_{m_1m_2m_3}$ is homogeneous with respect to its variables. Since the summation with respect to n_i is carried out effectively from zero to N_i, a similar homogeneity property is possessed (see (3.28)) by a kernel that depends on $a_ia_ka_l$. It can be shown[44] that in this case there exists a distribution function of a certain type

$$F(\mathscr{H}_1,\ \ldots,\ \mathscr{H}_s)=\tilde{F}(\overline{\mathscr{H}}_1,\ \ldots,\ \overline{\mathscr{H}}_s)\frac{(d\overline{\mathscr{H}})}{(d\mathscr{H})}\equiv F(k_1^0\overline{\mathscr{H}}_1,\ \ldots,\ k_s^0\overline{\mathscr{H}}_s)\frac{(d\overline{\mathscr{H}})}{(d\mathscr{H})}, \tag{7.14}$$

which satisfies for all i the equations

$$d\langle\mathscr{H}_i\rangle/dt=0. \tag{7.15}$$

In our case we have $\rho = -2$.[17] The distribution (7.14) has that remarkable property that it does not change the average distribution of the energy over the spectrum of the nonlinear waves, and consequently agrees with the Kolmogorov concept of constancy of the energy flux. Indeed, using the latter property, we can obtain from dimensionality consideration the same value $\rho = -2$. Thus, in this case it is possible to calculate the spectrum of the turbulent motion of the nonlinear waves:

$$\langle\mathscr{H}(k_i)\rangle=\int(d\mathscr{H})F(\mathscr{H})\mathscr{H}(k_i)=k_i^{-2}\cdot\text{const}. \tag{7.16}$$

8. CONCLUSION

The results presented in the review show that in many cases of physical interest the evolution of the nonlinear waves can be regarded from a certain common point of view. The existing methods of such an investigation constitute different generalizations of the Krylov-Bogolyubov method and presuppose smallness of the deviations from the exact solutions. The most convenient situation for the approximate methods constitutes wave packets that are very narrow in the spectrum and spread out slowly, as well as very broad wave packets. This review touched upon mainly the latter case. The results connected with an investigation of narrow wave packets can be found in[3,6]. Although the described approximate methods do seem natural from the physical point of view, they nevertheless require a more rigorous corroboration and a more exact determination of the applicability limits.

The last remark raises problems that call for a more detailed discussion. S. Ulam has carried out a thorough analysis of "unpleasant" points that arise when different approximate models of the solid medium are used to describe the behavior of a large ensemble of particles ([13], Chap. 7, Secs. 2 and 3). Since that time, the situation not only failed to become clearer, but owing to certain examples became apparently even more serious.

It is convenient to start with the classical Fermi-Pasta-Ulam problem[12,13]: under what conditions do stochasticity and relaxation to thermodynamic equilibrium take place in a system consisting of a large number of coupled nonlinear oscillators? These conditions, which were obtained in[61], are connected also with the conditions for the transition to turbulent motion in a medium[62]. In the longwave approximation, the system of oscillators can be approximately described by the equations of motion of the continuum. In this case this is an equation of type (2.5) for a nonlinear string[14]. In the approximation of sufficiently weak nonlinearity, the string equation becomes even simpler and reduces to the Korteweg-de Vries equation (2.15)[14].

In the course of simplifying the description, on going from the system of oscillators to the strong and then to the Korteweg-de Vries equation, it is natural for different limitations to become imposed on the parameters of the problem, in the form of inequalities. Is this sufficient, and to what degree is the description during the last stage of the simplifications equivalent to the description of the processes in the initial system? The following example will demonstrate how far we are still from the understanding of the answer to the question. For a sufficiently arbitrary initial condition, the Korteweg-de Vries can be exactly integrated and has a solution that describes a perfectly regular profile that evolves regularly in time. On the other hand, the same initial condition for an initial system of oscillators leads to the development of stochastic motion (B. V. Chirikov First School on Nonlinear Oscillations and Waves, Gor'kiĭ, 1972), which does not follow from the Korteweg-de Vries equations.

The presented hierarchy of the simplified systems can be continued because of the use of approximate integration methods. Indeed, as was seen from Chaps. 3—5, the expansion of the sought solution into a Fourier series and further study of the behavior of the individual harmonics is equivalent in essence to another transition from the continuum to a certain discrete system. In this case, therefore, the problem consists of determining the extent to which the final approximate discrete system is equivalent to the initial system of particles[18].

Finally, we can formulate a question of rather general form. Following the work by Lax[7b] it has become clear that there exists an infinite class of nonlinear equations of a continuous medium, which can be solved exactly under rather acceptable limitations. On the other hand, the motion of the system of particles generating these equations becomes stochastic under rather arbitrary initial conditions. The resolution of this paradox should contain the answer to the following question: how wide is the range of conditions under which we can use the solutions of the equations of a continuous medium, especially those of them that can be integrated exactly and do not contain as a result any turbulent components?

In conclusion, I am grateful to A. G. Litvak for useful remarks.

APPENDIX

ENERGY-MOMENTUM TENSOR OF NONLINEAR WAVE FIELD

The concept of energy-momentum tensor, which will be introduced here, may be useful for the understanding of a number of general problems that arise in the theory of nonlinear waves[19].

We confine ourselves to a consideration of the simplest example of the nonlinear Klein-Gordon equation (2.2) in order to illustrate this statement. We represent the Lagrangian (3.4) in the form

$$L=\frac{1}{2}\left(\frac{\partial y}{\partial r_i}\right)^2-V(y) \qquad (c=1), \tag{A.1}$$

where $r_i = (x, it)$ and summation over repeated indices is implied. The energy-momentum tensor is defined by the expression[66]

$$T_{ik}=L\delta_{ik}-\frac{\partial y}{\partial r_i}\frac{\partial L}{\partial(\partial y/\partial r_k)}=\delta_{ik}\left[\frac{1}{2}\left(\frac{\partial y}{\partial r_i}\right)^2-V(y)\right]-\frac{\partial u}{\partial r_i}\frac{\partial u}{\partial r_k}. \tag{A.2}$$

We write out the components of T_{ik}:

$$\left.\begin{aligned} T_{xx}&=-\frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2-\frac{1}{2}\left(\frac{\partial y}{\partial t}\right)^2-V(y),\\ T_{xt}&=i\frac{\partial y}{\partial x}\frac{\partial y}{\partial t}=T_{tx},\\ T_{tt}&=\frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial y}{\partial t}\right)^2-V(y).\end{aligned}\right\} \tag{A.3}$$

The conservation law

$$\partial T_{ik}/\partial x_k = 0$$

leads to the following two equations:

$$\frac{\partial T_{xx}}{\partial x}-i\frac{\partial T_{xt}}{\partial t}=0, \tag{A.4}$$

$$\frac{\partial T_{xt}}{\partial x}-i\frac{\partial T_{tt}}{\partial t}=0. \tag{A.5}$$

If, for example, we are considering a class of solutions that are periodic in x, then the integration of (A.5) over the period yields

$$\frac{d}{dt}\left(\frac{k}{2\pi}\oint dx T_{tt}\right)=\frac{d\mathscr{H}}{dt}=0,$$

which coincides with (3.7) if we substitute the expression T_{tt} from (A.3).

In the case when the solution is periodic in time, analogous integration of (A.4) yields

$$\frac{d}{dx}\left(\frac{\omega}{2\pi}\oint dt\, T_{xx}\right)=\frac{d\tilde{\mathscr{H}}}{dx}=0,$$

which coincides with (3.10).

Finally, we consider averaging over the phase ϑ, satisfying the equations (3.14)

$$\partial\vartheta/\partial x = k, \quad \partial\vartheta/\partial t = -\omega. \tag{A.6}$$

From (A.4), (A.5) we have with allowance for (A.3)

$$\begin{aligned}&\frac{1}{2\pi}\frac{\partial}{\partial x}\int_0^{2\pi} d\vartheta T_{xx}+\frac{1}{2\pi}\frac{\partial}{\partial t}\int_0^{2\pi} d\vartheta\frac{\partial y}{\partial x}\frac{\partial y}{\partial t}=0,\\ &\frac{1}{2\pi}\frac{\partial}{\partial t}\int_0^{2\pi} d\vartheta T_{tt}-\frac{1}{2\pi}\frac{\partial}{\partial x}\int_0^{2\pi} d\vartheta\frac{\partial y}{\partial x}\frac{\partial y}{\partial t}=0.\end{aligned} \tag{A.7}$$

If we retain the old notation for $\mathscr{H}$ and $\tilde{\mathscr{H}}$ and introduce the analog of the action

$$\tilde{I}=\frac{1}{2\pi}\oint\frac{\partial y}{\partial\vartheta}dy,$$

then we obtain from (A.7) the following generalization of the equations $d\tilde{\mathscr{H}}/dx = 0$ and $d\tilde{\mathscr{H}}/dt = 0$:

$$\frac{\partial\mathscr{H}}{\partial x}=\frac{\partial}{\partial t}(\omega k\tilde{I}), \quad \frac{\partial\tilde{\mathscr{H}}}{\partial t}=\frac{\partial}{\partial x}(\omega k\tilde{I}),$$

which now must be solved simultaneously with (A.6).

---

1) We note also attempts to employ it in nonlinear quantum field theory [10a] and in the theory of the Josephson effect [10b].

2) Equations of type (2.2) were obtained also in relativistic plasma by A. G. Litvak (Communication at the First School on Nonlinear Oscillations in Waves, Gor'kiĭ, 1972).

3) A similar picture is obtained for nonlinear magnetosonic waves [2, 19].

4) See also [25].

5) See also the Appendix.

6) A closely-related method of averaging without the use of the Lagrangian formalism was developed in [9].

7) Analogous series for different cases and situations were investigated in [30–33].

8) There are at present no rigorous criteria that determine the region of applicability of this approach.

9) A more general expression can be obtained for the right-hand side of (3.28) in those cases when the force Φ depends also on y.

10) See, e.g., [36].

[11]These equations coincide also with the Boussinesq equations for the oscillations of a liquid surface.

[12]In general, the one-dimensional case has a number of unique features, some of which are described below.

[13]The use of the term "two-stream solution" is connected with the fact that a superposition of two nonlinear waves moving with different velocities can be interpreted as a superposition of two streams of particles, each of which executes its own motion. Formally, the coupled pair of waves can be described by two coupled hydrodynamic equations analogous to the well-known equations of multistream hydrodynamics [1].

[14]Some nonadiabatic cases are also considered in [52].

[15]Under certain conditions the small wave-interaction parameter is determined from the condition [22, 23] $\epsilon \propto S/N \ll 1$; $S, N \gg 1$, where N is the characteristic parameter of the nonlinearity for the waves and S is the characteristic number of the effectively interacting waves.

[16]The structure of Eq. (7.12) is sufficiently general, and the difference between the concrete physical cases consists in the form of the kernel $V_{m_1 m_2 m_3}$ and in the relations $a(n, l)$, $\omega = \omega(l)$.

[17]Solutions of this type were first obtained in [60] for the case of weak turbulence and under conditions when the momenta are separable.

[18]Attention should be called in this connection to one surprising model of a chain of atoms with exponential interaction, observed by Toda [63]. In this discrete (!) model there exist not only exact solutions in the form of solitons, but also two-soliton solutions [64] analogous to the corresponding solutions of the Korteweg–de Vries equation.

[19]I am indebted for this remark to G. M. Fraĭman (see also [65]).

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