Duality, dual models, and experiment

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This review is devoted to a systematic exposition of the basic ideas and consequences of duality and the Veneziano model for two-particle and multi-particle processes. It contains an attempt to give an answer to the question as to whether dual models can be regarded as a first approximation in describing the strong interactions. In this connection, emphasis is placed on the discussion of the qualitative consequences of the basic assumptions of duality and their detailed comparison with the experimental data. In addition, the review contains a discussion of the basic theoretical assumptions of duality, various specific variants of dual models and their application to experimentally observed reactions, as well as the difficulties that have been encountered in the consistent development of the idea of duality. On the whole, the review reflects the status of the subject as it existed in June 1971.

CONTENTS

1. INTRODUCTION

Duality—one of the few new and original physical ideas which have arisen in elementary particle physics in recent years—is now a generally accepted language for the discussion of the scattering of strongly interacting particles (hadrons). In the course of three years of intense development (some idea of which is given by the rather lengthy list of reviews, lectures and reports which have appeared during this time[1]), dual models have become a rather complex field of hadron physics that is difficult to study, particularly because of the enormous number of works of a purely technical and formal character. For this reason, the primary purpose of this review is to allow the reader to rapidly acquaint himself with the basic concepts and reasoning of duality to the extent that he is independently enabled to read the original papers on this subject. Furthermore, the desire to compare the simple formulas of dual models with concrete experiment or the elegant and comparatively difficult formalism of these models often causes one (even specialists) to forget the extent to which the ideas of duality are in agreement with reality and are confirmed by the entire set of current experimental data. In this connection, we have allocated considerable space in this review to the comparison of the main predictions of duality with experiment, in order to enable the reader to decide for himself whether it is worth gaining an interest in this subject in general and to decide whether the ideas of duality stand a chance of being correct and not merely attractive. This twofold purpose of the review resulted in its rather great length, and to prevent excessive expansion of the text it was necessary to eliminate the entire history of the ideas which are reviewed (which is elucidated in sufficient detail in [1]) and to practically avoid any discussion of the purely theoretical development of dual models, without which one cannot form an accurate and complete impression of duality (this is remedied to some extent by the references to Appendix II, which provide the shortest means of eliminating ignorance in this respect). To read the review, it is desirable not only to be acquainted with the analyticity properties of the hadronic scattering amplitude, its asymptotic form at high energies, unitary symmetry, quarks and other matters which constitute the usual way of life of the elementary particle physicist, but also to possess a knowledge of the fundamental ideas of duality (e.g., to the extent of [2]).

2. WHAT IS DUALITY?

The physical idea of duality lies in the fact that resonant interactions of elementary particles play a basic role in strong-interaction physics. To be more specific, this means that the hadronic scattering amplitude will be close to the physical amplitude at arbitrary energies if we imagine that the whole of the interaction reduces entirely to the formation and decay of resonances.

Let us assume that this is so; then the scattering amplitude for the reaction $a+b \to a+b$ (Fig. 1) has the form

$$A(s,\ t)=\sum_i\left[B_i(s,\ t)+B_i(u,\ t)\right], \qquad (1)$$

where $B_i(s, t)$ is the contribution of a Breit-Wigner pole to this reaction ($B_i(u, t)$ is the same in the reaction $\bar{a}+b\rightarrow\bar{a}+b$, where $\bar{a}$ is the antiparticle), and s, t and u are the usual Mandelstam variables.

$$B_i(s,\ t)=-\frac{8\pi\sqrt{S}}{p}(2S_i+1)\frac{\Gamma_i^{\mathrm{el}}}{S-m_i^2+i\Gamma_i}P_{S_i}(z), \qquad (2)$$

where m_i, S_i, Γ_i and Γ_i^{el} are the mass, spin, full width and elastic width of the resonance, respectively. $P_l(z)$ is a Legendre polynomial, z is the cosine of the scattering angle, and p is the momentum in the c.m.s. The amplitude has the same form (the sum is over all resonances) for positive t (the physical region for the reaction $a+\bar{a}\rightarrow b+\bar{b}$). However, according to crossing and analyticity arguments, the amplitudes for the reactions $a+b\rightarrow a+b$ and $a+\bar{a}\rightarrow b+\bar{b}$ must be related, i.e., the amplitude (1) must be equal to the analogous amplitude for the reaction $a+\bar{a}\rightarrow b+\bar{b}$ continued into the s-channel physical region ($s>0$, $t<0$). For such a continuation, the set of poles with the same external quantum numbers (charge, isotopic spin, strangeness, parity, etc.) but with different spins (i.e., resonances which lie on a single trajectory $\alpha(t)$) give a contribution for $t<0$

$$R_i(s,\ t)=\beta_i(t)\left(\frac{s}{s_0}\right)^{\alpha_i(t)}\eta_i(t), \qquad (3)$$

where

$$\eta_i(t)=-\frac{\pm e^{-i\pi\alpha_i(t)}+1}{\sin\pi\alpha_i(t)};$$

η is the signature factor of the Regge pole (the + and − signs correspond to positive and negative signatures, respectively), and $\beta_i(t)=g_1^{aa}(t)g_1^{bb}(t)$ (for the reaction $a+b\rightarrow a+b$) is the residue of the reggeon.

Thus,

$$A(s,\ t)=\sum_i\left[B_i(s,\ t)+B_i(u,\ t)\right]=\sum_i B_i(s,\ t), \qquad (4)$$

or

$$\operatorname{Im}A(s,\ t)=\operatorname{Im}\sum_i B_i(s,\ t)=\operatorname{Im}\sum_i R_i(s,\ t), \qquad (5)$$

i.e., the sum of the s-channel resonances of the reaction must be equal to the sum of the reggeon contributions. This can be represented graphically in the form of Fig. 2. The equalities (4) and (5) clearly express a certain interrelationship that yields a bootstrap condition between the resonances of the direct channel ($a+b\rightarrow a+b$) and those of the crossed channel ($a+\bar{a}\rightarrow b+\bar{b}$), and the dual amplitude (i.e., the amplitude constructed entirely as a sum over all the resonances) is a solution of this bootstrap. Such an amplitude obviously does not satisfy the unitarity condition and therefore may be regarded only as a first approximatkon to the actual physical scattering amplitude. Our belief is that this approximation is a good one, i.e., that all the unitarity corrections are small. Thus, it is a question of constructing a perturbation theory for the strong interactions. Any attempts to advance in this direction must undoubtedly arouse, on the one hand, great interest and sympathy and, on the other hand, quite well-founded suspicion and doubt. The latter is especially true of duality, since its ideas are based entirely on agreement with experiment and are not in any sense sufficiently well-tested principles which are indispensible for the general description of current experimental data, all the more so because it is not clear at the present time whether the dual bootstrap is self-consistent and without internal contradictions. However, the ideas of duality have a constructive character, great generality (duality is of course applicable not only to binary reactions; we show in Fig. 3, for example, the dual bootstrap condition for the reaction $a+b\rightarrow c+d+e$) and an intimate relationship with current experiment, so that we would like to hope that this path will not lead to a dead end and that the agreement with experiment is not accidental.

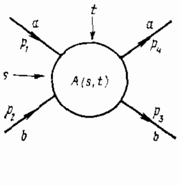


FIG. 1. $s=(p_1+p_2)^2$, $t=(p_1-p_4)^2=-2p^2(1-z)$, $u=(p_2-p_4)^2=-2p^2(1+z)$, $s+t+u=2m_a^2+2m_b^2$, p is the momentum in the c.m.s., $A(s,\ t)=\Sigma_1(21+1)a_l(s)P_l(z)$, $z=\cos\vartheta$, and ϑ is the scattering angle.

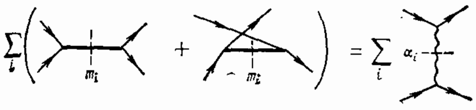


FIG. 2

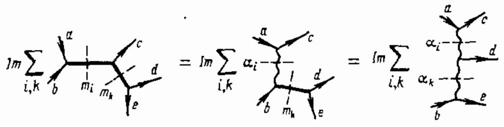


FIG. 3

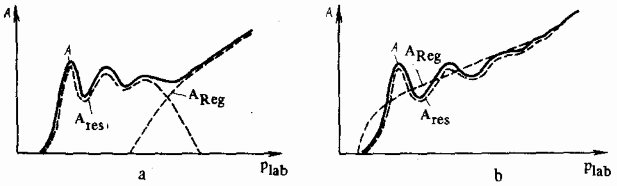


FIG. 4. The scattering amplitude in the interference model (a) and in dual models (b) ($A = A_{res}$, A_{Reg}).

It should be stressed that the usual representation of the scattering amplitude as a sum of a certain number of resonances (A_{res}) and a contribution of Regge poles (A_{Reg}) (i.e., $A=A_{res}+A_{Reg}$) is radically different from the dual description. This difference is clear from Fig. 4.

Unfortunately, the scattering amplitude cannot be completely dual, since the dual bootstrap (see Fig. 2) cannot be imposed on the Pomeranchuk pole (or the vacuum reggeon), which is responsible for the constant values of the total cross sections. The point is that the total cross sections for reactions in which there are no direct-channel resonances (e.g., $K^+p\rightarrow K^+p$ or $pp\rightarrow pp$) do not tend to zero with increasing energy, so that the contribution of the vacuum reggeon must not depend on the presence of direct-channel resonances. Duality arguments can be applied either to processes in which exchange of the vacuum pole does not contribute (e.g., charge-exchange reactions) or to the scattering amplitude from which the contribution of the vacuum reggeon has been subtracted:

$$A_D(s,\ t)=A(s,\ t)-P(s,\ t),$$

where P(s, t) is the contribution of the Pomeranchuk pole, which is of the form (3) with a positive signature (its trajectory $\alpha(t)$ passes through 1 at $t=0$). For further details on the vacuum reggeon in the duality scheme, see Sec. 7.

3. THE RESONANCE SPECTRUM IN DUAL MODELS

As we have already indicated above, duality is essentially an attempt to construct a good approximation to the physical scattering amplitude, confining oneself to the contribution of all the resonances. It is therefore quite obvious that the resonance spectrum required for the solution of the equations (4) plays a fundamental role in dual models. Let us consider what constraints the duality conditions (4) impose on the resonance spectrum independently of the specific dual model.

a) First of all, it follows from (4) that there must exist resonances of arbitrarily large mass (in other words, the trajectories of the Regge poles $\alpha(t)$ must rise indefinitely with increasing t). In fact, if all the resonances have masses less than M (i.e., if $\alpha(t)$ has the form of the dashed curve in Fig. 5), then there are no resonances for $s \gg M^2$ and (5) implies $\operatorname{Im} A(s, t) = 0$, and not $s^{\alpha(t)}$. Thus, $\alpha(t)$ must not "turn around," but must have the form of the solid curve in Fig. 5. Such a behavior of the trajectories is in good agreement with experiment; moreover, all currently known trajectories are straight lines (Fig. 6). Information about a trajectory $\alpha(t)$ can be obtained in the first place from the fact that it contains resonances for $t > 0$ (the particle region); secondly, $\alpha(t)$ determines the behavior of the differential cross section for $t < 0$ (the scattering region). For example, the Δ_δ trajectory determines the $\pi^- p \to \pi^- p$ backward scattering cross section. The data on the trajectories obtained from these two regions are in good mutual agreement (Table I). The longest trajectory at the present time is the Δ_δ trajectory, which contains the heaviest resonance with a mass 3230 MeV. However, it should be noted that the quantum numbers of this resonance and those of the resonances of masses 2850 and 2420 MeV (apart from the isotopic spin) have not been determined [3]. The same is true of the last particles on the Λ and Σ trajectories. The situation is more complicated for the meson trajectories, owing to the obviously poor significance of the quantum numbers of the meson resonances. At the present time, we can speak of only a single meson trajectory in the particle region, namely the ρ trajectory, which contains the mesons $\rho(765)$ and g(1670); its parameters are in good agreement with the data from the scattering region (see Table I).

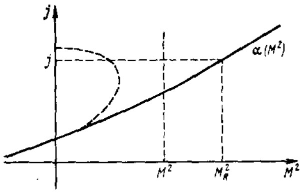


FIG. 5

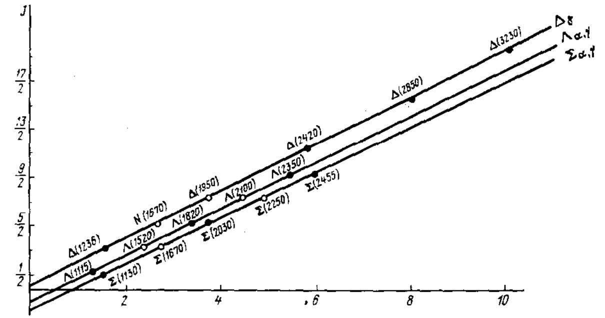


FIG. 6. The Δ_δ, $\Lambda_{\alpha,\gamma}$ and $\Sigma_{\alpha,\gamma}$ trajectories [3].

b) It is well known that, for scattering at high energies, orbital angular momenta $l_0 = pr_0$ are important, where r_0 is the range of the strong interactions, which is asymptotically equal to $r_0 = (\alpha' \ln s)^{1/2}$ (α' being the slope of the trajectory $\alpha(t) = \alpha(0) + \alpha' t$), as the asymptotic behavior (according to (4)) is determined by reggeon exchange. The partial waves $a_l(s)$ (see Fig. 1) are small for $l \gg l_0$ and fall off rapidly (exponentially) with increasing l (see [5]). Let us consider scattering at high energy $s = M_R^2$, where $\alpha(M_R^2) = j$ (see Fig. 5), supposing that $\alpha(M_R^2)$ rises more rapidly than $(\alpha' M_R^2 \ln M_R^2)^{1/2}$ (this condition is satisfied for the known trajectories) and that $j \gg l_0$; then

$$\operatorname{Im} a_j(s) = c(j)\, e^{-aj}, \qquad (6)$$

where c(j) is a power function of j, and $a = \sqrt{t_0}/p$, where t_0 is the square of the mass of the lightest resonance in the crossed channel $(a + \bar{a} \to b + \bar{b})$; but, on the other hand (see (2)),

$$\operatorname{Im} a_j = \frac{8\pi \sqrt{M_R^2}}{p} \frac{\Gamma_i^{el}}{\Gamma_i}. \qquad (7)$$

Comparing (7) with (6), we obtain

TABLE I. Reggeon trajectories (data of [4])

Reggeon	From scattering		From the particle region		
	$\alpha(0)$	$\alpha'(0)$ $(\mathrm{GeV}/c)^{-2}$	$\alpha(0)$	$\alpha'(0)$ $(\mathrm{GeV}/c)^{-2}$	Calculated from the resonances
ρ	0.57±0,02	0,95±0,1	0.50±0.02	0,92±0,08	ρ, ω, A_2, f
ω	0.4±0.1	0,7±0.3			
A_2	0.4±0.1	0.6±0.2			
f	0.5±0.2	0,3±2.2			
ρ	0.57±0,02	0.95±0.1	0,46±0,03	0,98±0,05	$\rho(1^-, 765)$ $g(3^-, 1660)$
π			0.02	0.75±0.04	$\pi(0^-, 140)$ $A_3(2^-. 1640)$
π	0	1.22±0.12	0,02	0.64±0.06	$\pi(0^-, 140)$ $\beta(1^{+-}. 1235)$
$B(1^{+-})$					
π	0.15±0.12	0.62±0,23	0.02	0.85±0.1	$\pi(0^-, 140)$ $A_1(1^{++}, 1070)$
$A_1(1^{++})$					
$K^*(1^-)$			0.33±0.02	0.84±0.03	$K^*(1^-. 890) K^{**}(2^+, 1420)$
$K^{**}(2^+)$					
K			−0.17±0.01	0,69±0.05	$K(0^-. 495)$ $K_A(2^-. 1775)$
K			−0,19±0.02	0.78±0.06	$K(0^-, 495)$ $K_A(1^+. 1230)$
K_A (1230)					
K			−0.16±0.02	0,66±0,06	$K(0^-. 495)$ $K_A(1^+. 1320)$
K_A (1320)					
Δ_δ	0.19±0.05	0,87±0.20	0.15	0.90	(See Fig. 6)
$N_\alpha(1/2^+, 5/2^+)$	−0.38±0.04	0.88±0.09	−0.39	1.01	(See Fig. 39 below)
$N_\gamma(3/2^-. 7/2^-)$			−0.94	0.92	$N(3/2^-, 1520) N(7/2^-. 2190)$
$\Sigma_\alpha(1/2^+. 5/2^+)$	−0.84	1.0	−0.8±0.05	0.90	(See Fig. 6)
$\Sigma_\gamma(3/2^-, 7/2^-)$					
$\Sigma_\delta(3/2^+. 7/2^+)$			−0.25	0.91	$\Sigma(3/2^+. 1385) \Sigma(7/2^+, 2030)$
$\Lambda_\alpha(1/2^+, 5/2^+)$			−0.7	0.97	(See Fig. 6)
$\Lambda_\gamma(3/2^-. 7/2^-)$					

$$\frac{\Gamma_i^{\text{el}}}{\Gamma_i} \sim c(j) \exp\left[-\frac{\alpha(M_R^2)}{M_R} \cdot 2\sqrt{t_0}\right] = c(j) e^{-bM_R}, \tag{8}$$

provided that $\alpha(t) = \alpha(0) + \alpha' t$ ($b = 2\alpha' t_0^{1/2}$).

Thus, the relative probability of decay into each individual channel (in our case, into the channel ab) must fall off rapidly with increasing resonance mass. This assertion is in excellent agreement with the data on baryon resonances (Fig. 7).

c) However, how can $\operatorname{Im} A(s, t)$ behave like $s^{\alpha(t)}$ at high energies when the $\operatorname{Im} a_l(s)$ are small? When $s = M_R^2$, $\operatorname{Im} A(s, t)$ is equal to (see (1))

$$\operatorname{Im} A(M_R^2 t) = \sum_j \operatorname{Im} a_j(M_R^2) P_j(z) = \beta(t) \left(\frac{M_R^2}{s_0}\right)^{\alpha(t)}. \tag{9}$$

The summation is taken in general over all the resonances of mass M_R and different spins. It is obvious that (9) would be violated (see (6)) if only the resonance with spin $j = \alpha(M_R^2)$ is retained in (9). There will be no inconsistency only if the sum (9) has contributions from partial-wave amplitudes with different orbital angular momenta, including those with $j \leq l_0$, for which $\operatorname{Im} a_j \sim 1$. In other words, for each mass there must exist many resonances with different spins, which lie on corresponding trajectories, and there must exist many of these trajectories. Even before any duality, it was shown [6] in considering reactions in which particles with different masses interact that the analyticity of the amplitude requires, in addition to the leading trajectory $\alpha(t)$, the existence of other trajectories (so-called daughter trajectories), which lie below the leading trajectory by an integer at $t = 0$ (Fig. 8). It is therefore natural to expect that the degeneracy of the resonances with respect to the masses is connected with the fact that these daughter trajectories also do not turn around and extend to infinity, and that it is on these trajectories that the resonances lie (Fig. 8). The degeneracy of the resonances with respect to the spins is a most important and interesting consequence of duality. Unfortunately, we have very meager experimental material, and in essence we cannot construct a single daughter trajectory with any degree of confidence from the existing resonances. We shall defer to Sec. 5 what little can be said on this; we merely note here that the existence of a large number of resonances and the increase in their number even in the known mass region (although one may question the authenticity of many of them) make such an assumption possible. Of special interest is the occurrence of a large number of resonances having the same quantum numbers but different masses, which follows from the picture of the trajectories in Fig. 8 (see the dashed line in Fig. 8, on whose intersections with the trajectories there lie particles with the same spin S and different masses). It can be seen from the tables of [3] that there are at the present time three nucleons N(1/2$^+$) with masses 940, 1470 and 1780 MeV, and the pairs of particles N(3/2$^-$) 1520 and 2040, N(1/2$^-$) 1530 and 1700, Λ(1/2$^-$) 1405 and 1670, and Λ(3/2$^-$) 1520 and 1690 MeV. If this phenomenon is not accidental, it strengthens our confidence that there exists a degeneracy of the resonances.

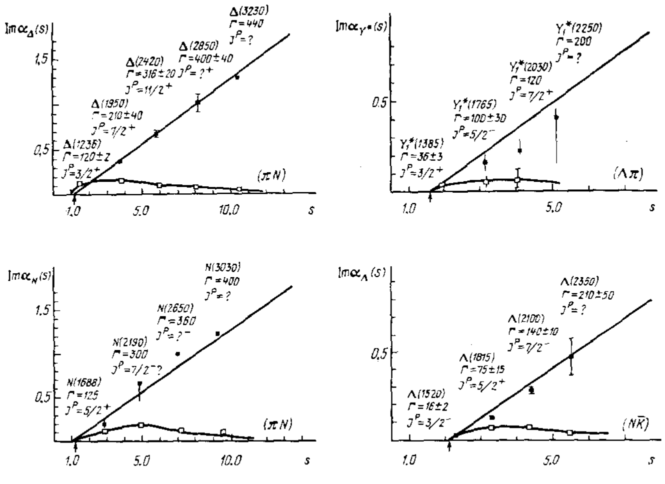


FIG. 7. The full widths and decay widths of baryon resonances into a given channel.

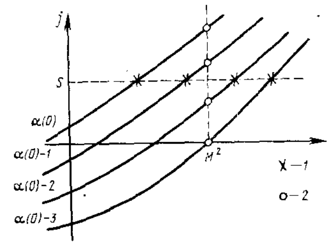


FIG. 8. The leading trajectory and three daughter trajectories: 1—resonances with the same spin S and different masses; 2—resonances with the same mass and different spins.

d) Strictly speaking, allowance in the dual approach for the full widths of resonances, i.e., for the fact that resonances decay into other particles, is inconsistent, since one must then assume that, in addition to the resonances, there is an important contribution from multi-particle states. However, one may hope that this contribution, like all the unitarity corrections, will be small. A natural parameter for the smallness in this case is the quantity

$$\frac{\Gamma}{\Delta M} \ll 1, \tag{10}$$

where Γ is the width of a resonance, and ΔM is the mass difference of the nearest resonances. We see from Fig. 7 that this parameter is not small; moreover, it tends to rise (the straight lines in Fig. 7) and (10) is evidently violated. It is quite difficult to say anything definite about (10), owing to our limited present knowledge of the resonances. In particular, all the heavy resonances that lie on the Δ trajectory, whose widths are given in Fig. 7, are found from phase-shift analyses, the latter having been made without allowing for the possibility that there exist resonances having the same mass and different spins. If they are taken into consideration, the widths may be modified quite appreciably and the status of (10) may become more encouraging. But even if (10) is not satisfied, this does not mean that the unitarity corrections are large. The point is that

$$\Gamma = \frac{2 \operatorname{Im} \alpha(t)}{\alpha'}, \tag{11}$$

and the small magnitude of the unitarity corrections implies that $\operatorname{Im} \alpha(t)$ is small in comparison with $\operatorname{Re} \alpha(t)$, i.e., we have from (11)

$$\Gamma \ll M_R, \tag{12}$$

which is well satisfied. But the detection of resonances in this case (especially those which lie on daughter trajectories) will be difficult (Fig. 9), even with a re-

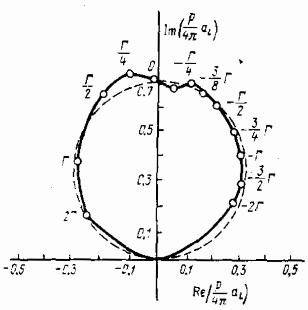


FIG. 9. Argand diagram for $\frac{p}{4\pi}a_l(s)=\frac{1}{2}\left[\frac{\Gamma}{\Delta E+i\Gamma}+\frac{\Gamma}{2(\Delta E+2i\Gamma)}+\frac{\Gamma}{3(\Delta E+3i\Gamma)}\right]$, $\Delta E=\sqrt{s}-\sqrt{s_0}$ (one resonance on the leading trajectory and two on daughter trajectories—solid curve; the dashed curve corresponds to a resonance which can easily be confused with this sum, $pa_1/4\pi=2(5/4)(\Gamma/3)[\Delta E+i(5/4)\Gamma]$; the symbols indicate the values of ΔE).

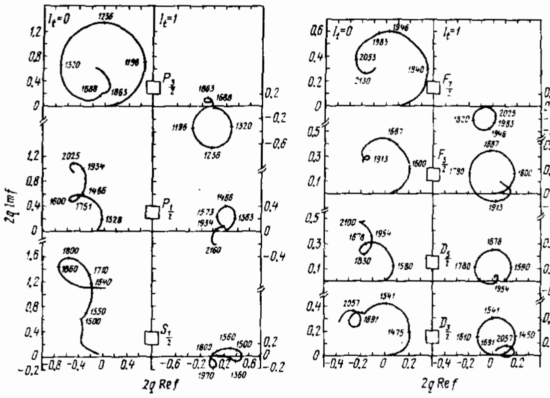


FIG. 10. The Argand diagrams for the combinations of πN scattering amplitudes that have definite isotopic spin in the t-channel [7].

liable phase-shift analysis, since, if (10) is violated in a given partial wave, there must be many resonances (see Fig. 8) separated by distances less than their widths. This last remark may explain why we do not see daughter trajectories, but on the whole the situation becomes very uncertain and the claim that the amplitude can be represented as a sum of only resonances becomes in essence unverifiable beyond certain energies (at any rate, if we confine ourselves to binary reactions).

e) Thus, the spectrum of resonances may be consistent with the idea of duality, but one would like to have a more direct experimental indication that the unitarity corrections to the dual amplitude are small. In this connection, it is of interest that in [7] the phase-shift analysis of πN scattering was presented in a new form, by isolating the combinations of s-channel partial waves $f^T_{j,l}$ which have definite isotopic spin in the t-channel (T is the isotopic spin, j is the total angular momentum, and l is the orbital angular momentum):

$$f^0_{j,l}(s)=\frac{1}{3}(f^{1/2}_{j,l}+2f^{3/2}_{j,l}), \qquad (13)$$

$$f^1_{j,l}(s)=\frac{1}{3}(f^{1/2}_{j,l}-f^{3/2}_{j,l}). \qquad (14)$$

From the point of view of duality, the combinations (13) and (14) should behave differently as a function of s. Only the non-vacuum Regge poles contribute to (14), so that (see (4)) we expect it to have a purely resonant behavior. The vacuum reggeon also contributes to (13), so that the presence of a background in (13) would not surprise us. In Fig. 10 we show the Argand diagrams for (13) and (14). It is clear that (14) describes closed loops with a very small background, whereas the background contribution is large in (13). This qualitative conclusion does not depend strongly on what phase-shift analysis is used and is quite impressive. At any rate, without duality it is not clear why this background,

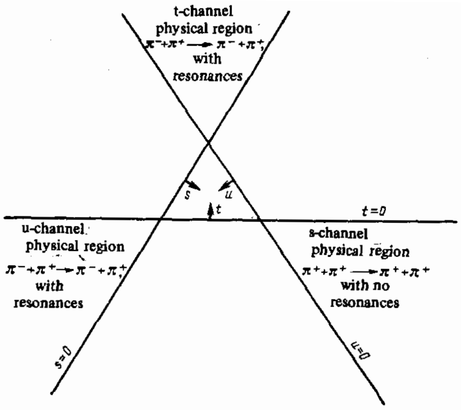


FIG. 11

which is large in each of the $f^{1/2}_{j,l}$ and $f^{3/2}_{j,l}$, has disappeared from (14).

4. DUALITY PLUS THE ABSENCE OF RESONANCES IN EXOTIC CHANNELS

We now turn to the more detailed consequences of duality, first considering those which originate from the fact that resonances are absent in certain reaction channels. Let us first consider, as an example, the reaction $\pi^+\pi^+\to\pi^+\pi^+$ (Fig. 11). There are no resonances in the s-channel of this reaction, so that (5) implies that

$$\mathrm{Im}\sum R_i(s,t)=\mathrm{Im}\,A(s,t)=0. \qquad (15)$$

At high energies, $\pi\pi$ scattering is determined by the exchange of three Regge trajectories: two of them with positive signature (f and f′), containing respectively the particles f (1250) and f′ (1520) with isotopic spin 0 and positive G-parity, and one negative-signature trajectory (ρ), passing through the ρ meson with mass 760 MeV. (Henceforth we shall designate the trajectories by the first particles which lie on them.)

It follows from (15) for this reaction that

$$\beta_f^{\pi\pi}(t)\left(\frac{s}{s_0}\right)^{\alpha_f(t)}+\beta_{f'}^{\pi\pi}\left(\frac{s}{s_0}\right)^{\alpha_{f'}(t)}-\beta_\rho^{\pi\pi}\left(\frac{s}{s_0}\right)^{\alpha_\rho(t)}=0. \qquad (16)$$

It is easy to see that (16) has two solutions:

$$1)\ \alpha_t(t)=\alpha_{f'}(t)=\alpha_\rho(t),\quad \beta_f^{\pi\pi}(t)+\beta_{f'}^{\pi\pi}(t)=\beta_\rho^{\pi\pi}(t); \qquad (17)$$

$$2)\ \alpha_f(t)=\alpha_\rho(t),\quad \beta_f^{\pi\pi}(t)=\beta_\rho^{\pi\pi}(t),\quad \beta_{f'}^{\pi\pi}(t)=0. \qquad (18)$$

The first solution corresponds to the equality of all the trajectories, including $\alpha_f(t)=\alpha_{f'}(t)$, i.e., the masses of the f and f′ mesons must coincide. Therefore this solution is possible only in the limit of exact SU_3 symmetry. If, however, we allow for the mass difference between the f and f′ mesons, then only the solution (18) is possible and this determines the way in which SU_3 is broken.

a) **Exchange Degeneracy.** It is clear from the foregoing example that the absence of resonances in certain reactions leads (in the presence of duality) to the equality of the trajectories and residues of Regge poles of different signatures [8]. This equality has become known as exchange (or signature) degeneracy.

1) Let us first of all consider how well the equality of the trajectories of reggeons with different signatures ("weak exchange degeneracy") is satisfied and what its consequences are. We do not yet assume the equality of the residues. The study[9,10] of a large number of re-

TABLE II

Exchange-degenerate trajectories	Reactions
ρf	$\pi\pi \to \pi\pi$
ρA_2 ωf $\varphi f'$	$K\overline{K} \to K\overline{K}$
ωA_2 $A, (I=0.2^{--})_{1,8}$ $\pi (I=0.1^{+-})_{1,8}$	$\pi\rho \to \pi\rho$
πB $\eta (I=0,\ 1^{+-})_{1,8}$ $A_1 (I=1.2^{-})_1$ $D (I=0.2^{--})_{1,8}$	$\overline{K}K^* \to \overline{K}K^*$
ηB	$\rho\rho \to \rho\rho$
$\eta (I=0.1^{+-})_{1\cdot 8}$ $X_0 (I=0.1^{+-})_{1,8}$ $D (I=0,1^{++})_{1,8}$	$K^*\overline{K^*} \to K^*\overline{K^*}$
K^* (890) K^{**} (1420)	$\pi K \to \pi K$
K (495) K_A (1320) $(I=1/2.2^{--})$ K^* (1240)	$\pi K^* \to \pi K^*$
Δ (3/2+, 7/2+) N (5/2−) N (1/2+, 5/2+) N (3/2−, 7/2−)	$\pi\Delta \to \pi\Delta$
Λ (1/2+, 5/2+) Λ (3/2−, 7/2−) Σ (1/2+, 5/2+) Σ (3/2−, 7/2−) Σ (3/2+, 7/2+) Σ (5/2−)	$\overline{K}N \to N\overline{K}$
Σ (3/2+, 7/2+) Λ (5/2−) Σ (1/2+, 5/2+) Λ (3/2−)	$\pi\Sigma \to \Sigma\pi$
Ξ (1/2+) Ξ (3/2−) Ξ (3/2+) Ξ (5/2−)	$\pi\Xi \to \Xi\pi$
The Ω^- (3/2+) trajectory should not be coupled to the ($\Sigma\overline{K}$) system	$\overline{K}\Sigma \to \Sigma\overline{K}$

actions leads to the equalities shown in Table II (according to [9]). For the meson trajectories, they are in good agreement with the data from the "scattering region" (see Table I). Unfortunately, as we have already noted, these trajectories cannot be reproduced from the "particle region." However, if the equalities of Table II are assumed, then the leading common trajectories (e.g., the trajectory passing through the ρ and f mesons) are in good agreement with the data from the scattering region (see Table I). The strange baryon trajectories coincide very well (see Fig. 6), but degeneracy of the Δ_δ and N_γ trajectories (containing particles $1/2^-$, $5/2^-$, etc.) is not observed. To be sure, the equality of these trajectories followed from the consideration of $\pi\Delta$ scattering (see Table II), so that the π_γ trajectory may not be coupled to the πN system, although the origin of such a selection rule is not clear.

Weak exchange degeneracy implies that the cross sections for two reactions which receive contributions from two non-vacuum poles of different signature (e.g., the reactions $K^-p \to \pi^-\Sigma^+$ and $\pi^+p \to K^+\Sigma^+$, which at high energies receive contributions from the exchange of the K* (890) and K**(1420) reggeons) are equal [11].

In fact, the signature factors (3) are readily rewritten as $\exp[-i\pi\alpha(t)/2][-\cos(\pi\alpha/2)]/\sin\pi\alpha$ for positive signature and $\exp[-i\pi\alpha(t)/2]i[-\sin(\pi\alpha/2)/\sin\pi\alpha$ for negative signature, so that

$$d\sigma\,(K^-p \to \pi^-\Sigma^+) = \frac{s^{(2\alpha(t)-2)}}{\sin^2 \pi\alpha\,(t)}\,[\,|\,\beta_{K^*}(t)\,|^2 + |\,\beta_{K^{**}}(t)\,|^2] = d\sigma\,(\pi^+p \to K^+\Sigma^+) \quad (19)$$

when the trajectories are equal. Relations of type (19) can be tested at the present time for the reactions

$$\left.\begin{array}{l} K^+n \to K^0p\,(K^-p \to \overline{K^0}n),\ K^+p \to K^0\Delta^{++}\,(K^-n \to \overline{K^0}\Delta), \\ K^-N \to \Lambda\pi\,(\pi^-p \to K^0\Lambda),\ K^-p \to \pi^-\Sigma^+\,(K^+p \to \pi^+\Sigma^+), \\ K^-p \to \pi^-\Sigma\,(1385)\,(K^+p \to \pi^+\Sigma^+\,(1385)). \end{array}\right\} \quad (20)$$

The behavior of the cross sections as functions of the momentum transfer is the same for all these reactions, but the cross sections have different values [11]. The cross sections for the reactions $K^-p \to \pi^-\Sigma$, $\Sigma(1385)$ are twice as large as those for $\pi^+p \to K^+\Sigma$, $\Sigma(1385)$, even at relatively high energies (for the last reaction in (20), the cross sections have been compared at incident momenta between 6 and 16 GeV/c). It is not yet clear whether this discrepancy is evidence against exchange degeneracy or whether there is simply a large background contribution in these reactions at current energies; in any case, if the relation (19) continues to be violated with increasing energy, this will be a strong argument against exchange degeneracy.

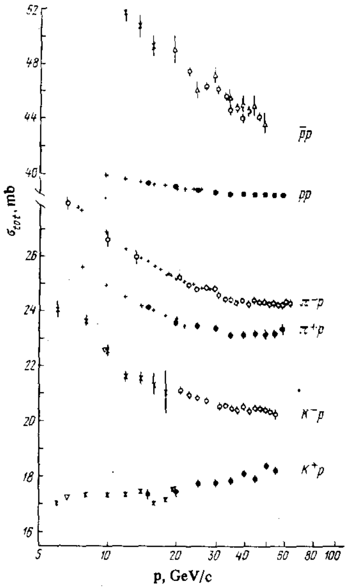


FIG. 12. The total cross sections as functions of the energy of the incident particles [12].

2) It is rather difficult to make a direct test of the equality of the residues, but there exist facts which indicate that this equality is well satisfied.

a) One of these facts is the energy dependence of the total scattering cross sections. The total cross sections for those reactions in which there are no resonances (such as pp, pn, K^+p and K^+n scattering), so that the contributions of all the non-vacuum poles cancel (as was the case for the $\pi^+\pi^+$ reaction, Eq. (16)), must remain unchanged with increasing energy (since these cross sections are due entirely to the contribution of the vacuum pole, which gives a constant cross section). This condition is well satisfied in the case of the indicated reactions (Fig. 12), while the cross sections for other reactions are decreasing at the same energies. Moreover, $\mathrm{Im}\,A(K^+n \to K^0p) = \mathrm{Im}\,A(K^+p) - \mathrm{Im}\,A(K^+n) = s(\sigma(K^+p) - \sigma(K^+n)) = 0$, since there are no resonances in the reaction $K^+n \to K^0p$. The equality of $\sigma(K^+n)$ and $\sigma(K^+p)$ is well satisfied; at any rate, the difference between them is much less than $\sigma(\pi^-p) - \sigma(\pi^+p)$, for example (see Fig. 12).

b) We note further qualitative consequence of relations of type (18). Consider that value of the momentum transfer ($t = t_0$) at which $\alpha f(t_0) = 0$. The signature factor (η_i in (3)) for positive-signature reggeons (in particular, for the f trajectory) becomes infinite at this point, which would correspond to the existence of a particle of negative mass. Since this cannot occur, $\beta_f^{\pi\pi}(t)$ must reduce to zero when $t \to t_0$. But it follows from the equality (18) that $\beta_\rho^{\pi\pi}(t)$ also reduces to zero at the same point [13]. Without duality, such a behavior of $\beta_\rho^{\pi\pi}$ is certainly not necessary, since $\eta(t)$ for the ρ trajectory does not be-

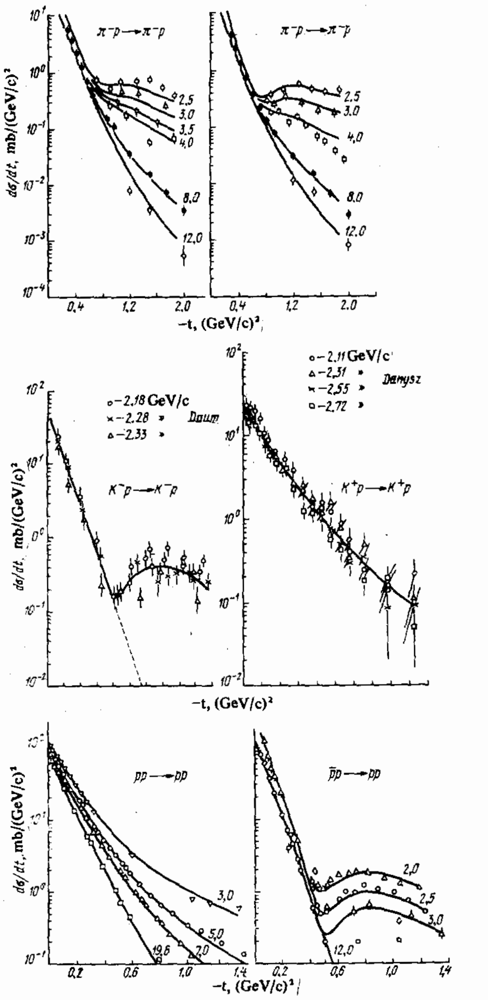


FIG. 13. Behavior of the $\pi^{\pm}p$, $K^{\pm}p$, pp and $\bar{p}p$ differential cross sections as functions of t [14].

come infinite (negative signature). The fact that the residues are forced to be zero leads to interesting consequences, such as minima in the differential cross sections. These minima (or, as they are called, "dips") should be seen in those reactions in which there are direct-channel resonances (for example, in $\pi^{\pm}$p, $\bar{p}$p and K⁻p scattering), but should not occur in reactions in which there are no resonances (K⁺p and pp). In fact,

$$\frac{d\sigma}{dt}=|P(s,t)|^2-2\,\mathrm{Im}\,P(s,t)\,\mathrm{Im}\sum_i R_i+|\sum_i R_i|^2, \tag{21}$$

where the summation is taken over all the non-vacuum reggeons, and P(s, t) is assumed to be purely imaginary. We have $\mathrm{Im}\,\Sigma_i R_i = 0$ for the pp and K⁺p reactions, since there are no resonances for these reactions, Eq. (5); $|\Sigma_i R_i|^2$ is small (of the order 1/s), so that the differential cross section for these reactions is determined by the contribution of the vacuum reggeon. The situation is different for $\bar{p}$p, for example. Here $\mathrm{Im}\,\Sigma_i R_i \neq 0$, and the interference term (which is of order $1/s^{1/2}$) gives a non-zero contribution for all t except the point $t = t_0$, at which it reduces to zero because the residues are forces to vanish. The differential cross sections for these reactions must therefore have characteristic minima for $t \to t_0$.

Such a behavior of the cross sections is actually observed experimentally (Fig. 13). It is of interest to note that duality does not at all require the vanishing of all the residues at $\alpha(t_0)=0$. Thus, for example, there are resonances in all the channels for the reaction $\pi\pi \to \omega\omega$, so that it is not necessary to demand that the residues of reggeons with different signature be equal; consequently, the residue $\beta_\rho^{\pi\omega}$ need not tend to zero as $t \to t_0$. We should therefore not expect minima in all reactions as $t \to t_0$ [15]. In fact if we look at Table III, we find minima only when the reaction contains two mesons with isotopic spin 1 (i.e., when there are exotic channels in the corresponding meson-meson scattering, as for $\pi\pi$), while there are no minima whenever one of the mesons has isotopic spin 0 (for example, for $\pi N \to \omega N$ the residue $\beta_\rho^{\pi\omega} \neq 0$ when $t = t_0$ and there is no minimum). Such a behavior of the cross sections may be somewhat surprising, since the residue $\beta_\rho^{NN}(t=t_0)=0$ and this leads to the vanishing of the contribution of the ρ reggeon in the reaction $\pi N \to \omega N$ (as well as the remaining reactions of the second group in Table III). However, the cross section for these reactions near $t = t_0$ can be written in the form

TABLE III

Reaction	Is there a minimum in dσ/dt?	Contributing Regge poles
$\pi^-p\to\pi^0n$	Yes	ρ
$\pi N\to\rho N_*$	»	ω
$\pi N\to\pi\Delta$	»	ρ
$\gamma\rho\to\pi^0p$	»	ω
$\pi N\to\omega N$	No	ρ
$\gamma N\to\pi^+N$	»	ρ
$\gamma p\to\eta p$	»	ρ
$\pi N\to\omega\Delta$	»	ρ
$\pi^-p\to\eta n$	»	A_2
$\pi^-p\to\eta\Delta$	»	A_2
$K^-p\to\overline{K^0}n$	»	$\rho+A_2$
$K^+p\to K^0\Delta^{++}$	»	$\rho+A_2$
$K^+p\to K^+p$	»	$P+f-\omega$
$pp\to pp$	»	$P+f-\omega$
$\bar{p}p\to\bar{p}p$	Yes	$P+f+\omega$
$K^-p\to K^-p$	»	$P+f+\omega$

$$\frac{d\sigma}{dt}\sim|a|^2+2(\alpha(t))^n\tilde{R}_\rho(s,t)\,a+(\alpha(t))^{2n}|\tilde{R}_\rho(s,t)|^2, \tag{22}$$

where a is the background contribution1), and R(s, t) is the reggeon contribution, in which the power with which the residue is forced to vanish at $t = t_0$ is extracted from this residue. It is clear from (22) that if the background is sufficiently large and slowly varying with t, then there will be a minimum in the differential cross section at $t = t_0$ only in the case n = 2. This behavior can be realized only if transitions to $N\bar{N}$ and the meson-meson system vanish at $t = t_0$ (i.e., only for reactions of the first group in Table III). There is no minimum for the reactions $K^-(K^+)p \to \bar{K}^0(K^0)n(\Delta^{++})$, since, in analogy with (19),

$$\frac{d\sigma}{dt}=s^{2\alpha-2}\left(\frac{\beta_\rho(t)^2}{\sin^2 n\alpha}+a\frac{\beta_\rho(t)}{\sin\pi\alpha}+a^2\right) \tag{23}$$

does not vanish like $\alpha^2(t)$ as $t \to t_0$. We have already discussed the reactions of the last group in Table III. Thus, duality makes it possible to give a qualitative explanation of the characteristic features of the behavior of differential cross sections.

c) We shall now discuss the predictions for the nucleon polarization (P) in the reactions of pseudoscalar meson scattering by nucleons:

$$P=\frac{\mathrm{Im}\,\{f^s_{\Delta\lambda=1}(s,t)\,f^{s*}_{\Delta\lambda=0}(s,t)\}}{d\sigma/dt}, \tag{24}$$

where $f^S_{\Delta\lambda}(s,t)$ is the amplitude for helicity flip $\Delta\lambda$ in the s-channel.

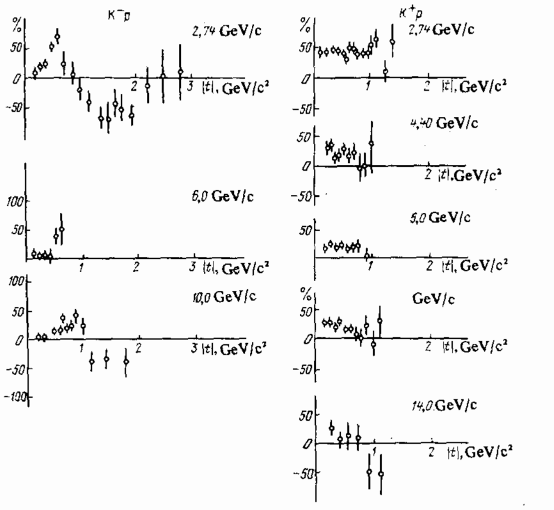


FIG. 14. The nucleon polarization in $K^-p \rightarrow K^-p$ and $K^+p \rightarrow K^+p$ as a function of the momentum transfer t [14].

Let us first consider the reactions $K^-p \rightarrow K^-p$ and $K^+p \rightarrow K^+p$ and examine the relation between the polarizations in these reactions at high energies. Let us assume that the vacuum reggeon gives a purely imaginary contribution for all t (i.e., we neglect the t-dependence of $\alpha_P(t)$); then (24) will be determined by the product of the contribution of the vacuum reggeon and Re $\Sigma_i R_i$, where the R_i are the contributions of the non-vacuum Regge poles (for the Kp reaction, these are the f, A_2, ρ and ω reggeons). For the reactions $K^+p \rightarrow K^+p$ and $K^-p \rightarrow K^-p$,

$$\mathrm{Re}\,\Sigma R_i = S^{\alpha(t)}\,[\beta_f(t)+\beta_{A_2}(t)]\,\mathrm{Re}\,[\eta_f(t)\mp\eta_\rho(t)]. \qquad (25)$$

where the ∓ in the brackets refer to the first and second reaction, respectively. Since

$$\mathrm{Re}\,[\eta_f(t)\mp\eta_\rho(t)]=\begin{cases}-2/\sin\pi\alpha(t) & \text{for the} - \text{sign,}\\ -2\cos\pi\alpha(t)/\sin\pi\alpha(t) & \text{for the} + \text{sign}\end{cases} \qquad (26)$$

(see (3)),

by substituting (26) and (25) in (24) we find

$$P(K^-p)=P(K^+p)\cos\pi\alpha_\rho(t). \qquad (27)$$

Thus, $P(K^-p)$ must tend to zero faster than $\sqrt{t}$ as $t \rightarrow 0(\alpha_\rho(0)\approx 1/2)$, it must be equal to $P(K^+p)$ at $t=t_0$ $(\alpha(t_0)=0)$, it must again reduce to zero at $t=t_0'(\alpha(t_0')=-1/2$, $t_0'\sim -1$ (GeV/c)^2), and it must have the opposite sign to $P(K^+p)$ for $t>t_0'$. In Fig. 14 we see just such a behavior of the polarization [16]. An analogous behavior also follows for the polarizations in pp and $\bar{p}$p scattering, which is also in good agreement with experiment [16].

Let us now consider the behavior of the polarization near $t=t_0(\alpha_\rho(t_0)=0)$. We shall assume, as before, that the vacuum pole gives a purely imaginary contribution and, in addition, that it contributes only to $f^S_{\Delta\lambda=0}$ (conservation of s-channel helicity; for further details, see Sec. VII). Then P is determined mainly by the expression

$$P_s=\frac{P(s,t)\,\mathrm{Re}\,f^s_{\Delta\lambda=1}}{|P(s,t)|^2} \qquad (28)$$

and Re $f^S_{\Delta\lambda=1}$ receives contributions from only the non-vacuum reggeons. If we consider the difference $P(\pi^+p)-P(\pi^-p)$ (or $P(K^+p)-P(K^-p)$), this difference is determined entirely by the negative-signature non-vacuum reggeons (for $P(\pi^+p)-P(\pi^-p)$, entirely by the ρ reggeon), with

$$P(\pi^+p)-P(\pi^-p)=\frac{\beta_\rho(t)\,\mathrm{tg}\,(\pi\alpha(t)/2)}{P(s,t)}. \qquad (29)$$

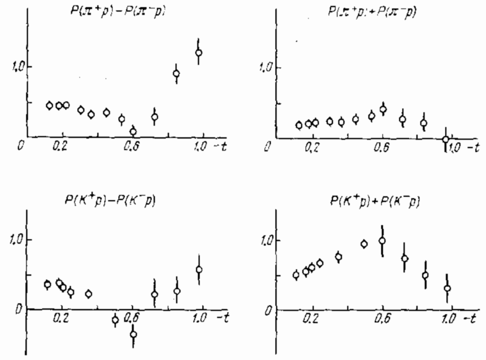
Since $\beta_\rho(t_0)\sim\alpha(t_0)$ and $\tan(\pi\alpha(t_0)/2)\sim\alpha(t_0)$ for $t=t_0$, Eq. (29) must exhibit a double zero at $t=t_0$ (from the point of view of duality [17]), in good agreement with experiment (Fig. 15). For the analogous sum of the polarizations, there remains the contribution of the positive-signature reggeons (the f reggeon for πp scattering), and

$$P(\pi^+p)+P(\pi^-p)=\frac{\beta_f(t)\,\mathrm{ctg}\,(\pi\alpha(t)/2)}{P(s,t)}. \qquad (30)$$

At $t=t_0$, we have $\beta_t\sim\alpha(t)$ and $\cot[\pi\alpha(t)/2]\sim 1/\alpha(t)$, so that (30) should not reduce to zero at $t=t_0$ [17] (see Fig. 15).

Thus, exchange degeneracy leads to a parametrization of the Regge pole residues which is in good (qualitative) agreement with both the behavior of the elastic cross sections and the polarization data.

Of course, this does not mean that duality by itself will ensure satisfactory agreement with experiment with a more quantitative approach. Moreover, it is necessary to allow for the branch cuts due to multiple scattering, i.e., to the exchange of two or more reggeons, one of which must be the vacuum reggeon (Fig. 16), in order to explain phenomena such as the cross-over of $d\sigma(K^-p)/dt$ and $d\sigma(K^+p)/dt$ at $t\approx -0.2$ (GeV/c)^2 [19] and to answer the question as to why the positions of the minima are different in various reactions and even depend on the energy in certain reactions (for example, there is a minimum at $t_0=-0.5$ (GeV/c)^2 for $\bar{p}$p, at $t_0=-0.6$ (GeV/c)^2 for $\pi^\pm$p, and at $t_0=-0.85$ (GeV/c)^2 for K^-p, its position varying from $t_0=-0.7$ to -1.16 (GeV/c)^2 with increasing energy [14]). In this connection, it is curious that the minima in all the reactions of Table III can also be attributed to multiple scattering (see Fig. 16), even without the assumption that the reggeon residues are forced to vanish at $t=t_0$. For this purpose, it is only necessary to consider that, in analyzing all the reactions with allowance for Fig. 16, it was observed that a minimum is obtained at $t=t_0\sim 0.2$ (GeV/c)^2 for a



p_{lab} = 2.74 GeV/c

FIG. 15. $P(K^+p)\pm P(K^-p)$ and $P(\pi^+p)\pm P(\pi^-p)$ as functions of the momentum transfer [18].

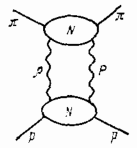


FIG. 16

total s-channel helicity flip ($\Delta\lambda$) equal to 0 (for negative signature), at $t_0 \approx 0.6$ (GeV/c)^2 for $\Delta\lambda = 1$, and at $t_0 \sim -2$ (GeV/c)^2 for $\Delta\lambda = 2$ [20], so that there will be a minimum at $t_0 = -0.6$ (GeV/c)^2 in the differential cross sections for the reactions in which the amplitude with $\Delta\lambda = 1$ dominates, but not in the others. If we accept the duality argument, we obtain in the limit of full SU_3 symmetry an F/D ratio for the meson-nucleon coupling such that the f and ω trajectories do not change $\Delta\lambda$ at the nucleon vertex, while $\Delta\lambda = 1$ dominates for the ρ and A_2. For $\pi N \to \omega N$, the ρ pole contributes, with only $\Delta\lambda = 1$ at the $\rho\omega\pi$ vertex and $\Delta\lambda = 1$ also dominating at the $\rho N\bar{N}$ vertex, so that the total helicity flip in this reaction is equal to 0 or 2 and there is no minimum. For $\pi^+ p \to \rho^+ p$, exchange of the ω reggeon gives the main contribution, so that $\Delta\lambda = 0$ at the $\omega N\bar{N}$ vertex, the total $\Delta\lambda = 1$, and there is a minimum [18]. A reaction which is crucial to these considerations is $\pi^- N \to A_2^0 N$, which is due to exchange of the ρ reggeon, so that the total $\Delta\lambda = 2$ and there should not be a minimum. At the same time, there should be such a minimum according to a consistent dual explanation, since there are exotic channels for the reaction $\pi\pi \to A_2 A_2$.

This possibility of explaining the experimental facts in a different way compels us to have a cautious view of the success of the duality predictions, although their qualitative character and their universality in considering such a large number of facts are sufficiently impressive.

b) **Duality and symmetries.** We shall now consider what the spectrum of resonances should be and what symmetry breaking there should be for the condition (4) to be satisfied, assuming that there exist groups of resonances that are close in mass and correspond to SU_3 multiplets (regarding SU_3, see, e.g., [21]).

1) The mixing angle. In considering $\pi\pi$ and KK scattering, the conditions analogous to (16) yield

$$\beta_{f'}^{\pi\pi}(t) = 0,\ \beta_{A_2}^{KK}(t) = \beta_f^{KK}(t). \tag{31}$$

We recall that there are observed nonets of mesons of approximately the same mass and that two of these mesons have the same quantum numbers ($Y = S = T = 0$); for the vector mesons (1^-) these are the ω and φ mesons, while for the tensor mesons (2^+) they are the f(1260) and the f′(1515). The relation (31) allows us to distinguish these particles by their decays into the $\pi\pi$ system: the f′(1515) does not decay into $\pi\pi$ [22] ($\beta_{f'}^{\pi\pi}(t) = 0$ for all $t < 0$ and hence also for all $t > 0$, in particular at $t = m_{f'}^2$). According to SU_3, the presence of two degenerate states in the nonets leads to a new structure of the wave functions even when SU_3 is broken in zeroth order. This is usually characterized by the mixing angle (ϑ):

$$\begin{aligned} |f'\rangle &= \sin\vartheta\,|f_1\rangle + \cos\vartheta\,|f_8\rangle, \\ |f\rangle &= \cos\vartheta\,|f_1\rangle - \sin\vartheta\,|f_8\rangle, \end{aligned} \tag{32}$$

where f_1 and f_8 are the SU_3 singlet and octet states. Making use of (31) and (32) and the values of the Clebsch-Gordan coefficients for SU_3 [23], we have

$$\begin{aligned} &\frac{g_8}{\sqrt{5}}\cos\vartheta - \frac{g_1}{\sqrt{8}}\sin\vartheta = 0, \\ &\frac{g_1}{\sqrt{8}}\cos\vartheta + \frac{g_8}{\sqrt{20}}\sin\vartheta = -\sqrt{\frac{3}{20}}\,g_8; \end{aligned} \tag{33}$$

g_1 and g_8 are the coupling constants of the SU_3 singlet and octet with the octet of pseudoscalar mesons. From (33), we have $\tan\vartheta = 1/\sqrt{2}$. The same mixing angle is obtained in the quark model (see, e.g., [24]). This is not surprising, since the decay $f' \to 2\pi$ is also forbidden in this model. If we consider the reactions $\pi^+\pi^+ \to \rho^+\rho^+$ and $K^+K^+ \to K^{*+}K^{*+}$, we obtain exactly the same mixing angle for the vector mesons. The identical mixing angles are in good agreement with the mass formulas. Thus, duality yields a result which was previously considered a success of the quark model [24]. In addition, the equalities $\alpha_\rho(t) = \alpha_\omega(t) = \alpha_{A_2}(t) = \alpha_f(t)$ lead to the mass formulas $m_\omega^2 = m_\rho^2$ and $m_{A_2}^2 = m_f^2$, which had also been obtained in the quark model and which are well satisfied.

2) The spectrum of resonances. All the rich experimental data on resonances allow the formulation of quite a simple rule for their quantum numbers: the only resonances that are observed are those predicted by the simplest quark model. In this model, all the mesons consist of a quark-antiquark pair, while the baryons consist of three quarks (for further details, see [24]). It follows from this that there exist no baryon resonances with $Q > 2$ and $S = 0$, $|Q| > 1$ and $S = \pm 1$, or $S > 1$ and $S < -2$, and no meson resonances with $|Q| > 1$ and $|S| > 1$.

We can now formulate the general approach in studying the resonance spectrum according to duality. Let us assume that none of the forbidden (exotic) states have resonances, while all the allowed states have resonances. Then constraints on the residues and trajectories can be derived from the requirements of duality (from the equations analogous to (16), for all the exotic channels). It is more convenient to quote the results in the language of SU_3. For the mesons [25]:

a) the trajectories and residues of Regge poles with the same P_r and G_r must be equal ($P_r = (-1)^j P$ and $G_r = (-1)^j C$, where P is the parity, C is the charge parity, and j is the spin of a resonance);

b) there should exist mesons with $j^{CP} = 0^{-+}$, 1^{--}, 2^{++}, 1^{++} and 0^{++}, i.e., those that are obtained in the quark model when the particles are classified according to the L-S excitations of the quark-antiquark system [24].

For the baryons [26]:

a) for $P_r = +1$ ($P_r = (-1)^{j-1/2}P$) the octets ({8}) with $j^P = 1/2^+, 5/2^+, 9/2^+, \ldots$ (the α trajectories) should be degenerate with the {1 + 8 + 10} with $j^P = 3/2^-, 7/2^-$ (the γ trajectory), all the octets having $F = 1/2$ (where $F = F/(F + D)$) for the coupling with the octet of pseudoscalar mesons;

b) for $P_r = -1$, the {8 + 10} with $j^P = 3/2^+, 7/2^+$ (the δ trajectory) should be degenerate with the {8} with $j^P = 1/2^-, 5/2^-$ (the β trajectory), with $F = -1/2$;

c) duality determines the F/D ratio for the coupling of the nonets of vector and tensor mesons with nucleons [27], giving a pure F coupling for the helicity non-flip ($\Delta\lambda = 0$) amplitude and $F/D = -1/3$ for $\Delta\lambda = 1$.

Let us first of all show by means of a simple example how the need for some particular particle arises from duality. Consider the reaction

$$\pi^+(p_1) + \pi^+(p_2) = \rho_1^+(p_3) + \rho_2^+(p_4). \tag{34}$$

Its amplitude can be written in the form

$$\begin{aligned} A(st) = {} & A_1(\rho_1 p_1)(\rho_2 p_2) + A_2\{(\rho_1 p)(\rho_2 q) + (\rho_2 p_1)(\rho_1 q)\} + A_3(\rho_1 q)(\rho_2 q) \\ & + A_4(\varepsilon_{\mu\lambda\sigma\nu}\rho_{1\mu}p_{2\nu}p_{3\lambda}\rho_{1\sigma})(\varepsilon_{\mu\lambda\sigma\nu}\rho_{1\mu}\rho_{2\lambda}\rho_{3\nu}\rho_{2\sigma}), \end{aligned} \tag{35}$$

where $q = p_1 - p_3$.

Reggeons with $P_r = +1$ (in particular, ω, ρ, A_2 and f) contribute only to A_4 [28], while A_1, A_2 and A_3 receive contributions from only the reggeons with $P_r = -1$ (such as the π meson). However, there are no resonances in (34); hence $\operatorname{Im} A_1 = 0$ from (5), but this cannot be the case if only the exchange of the π reggeon is taken into account. We must therefore require that, in addition to the π reggeon, there is a reggeon with the same P_r but with different signature. The trajectory passing through 1^{+-} is one of this type, and it must be degenerate with the π trajectory.

Let us now consider the spectrum of baryons.

1) The N_α and N_γ should be degenerate with the Δ_γ; in [3] there is a Δ (1670; $3/2^-$) resonance which may lie on the Δ_γ trajectory, but it is much more weakly coupled to the πN system than the N (1520; $3/2^-$).

2) We have a resonance (1670; $5/2^-$) on the N_β trajectory (Fig. 6), but the other particles on this trajectory have not been seen. This is particularly noticeable when compared with the Δ_δ trajectory (the longest one at the present time). The absence of a resonance with $j = 1/2$ on the N_β trajectory (with mass ~700 MeV) can be attributed to the vanishing of the residue at this point, and then the residue of the Δ_δ trajectory must also vanish at this same point with $\alpha_\Delta = 1/2$, owing to exchange degeneracy. From the cross sections for $\pi^\pm p$ scattering and for the charge-exchange process $\pi^+ p \rightarrow \pi^0 n$, we can conclude that the phase difference between the backward scattering amplitudes with isotopic spins 1/2 and 3/2 is equal to 60°, which excludes the appearance of a factor $\alpha_\Delta - 1/2$ in the residue of the Δ reggeon [4a].

3) There should exist a $3/2^+$ octet that is degenerate with the decuplet, and its coupling to the meson-baryon channel cannot be too weak (for example, the Σ of this octet should be suppressed by a factor of two in comparison with the Σ (1385)). Such an octet is not found; to be sure, there exists a $7/2^+$ resonance of mass 1990 MeV that is degenerate with the Δ (1950; $7/2^+$), but it is not clear why the $3/2^+$ octet is absent. (We note that the residue cannot vanish in this case because of the factor $\alpha - 3/2$, in analogy with the $N_\beta(1/2)^-$, since this would imply that the Δ (1236) does not decay into πN because of exchange degeneracy.)

Thus, the situation regarding degeneracy in meson-baryon scattering is not completely satisfactory. It is possible that this is due to the fact that we have assumed the validity of SU_3 in the analysis. In particular, if it is assumed [29] that the $\Lambda_{\alpha\gamma}$ and $\Sigma_{\alpha\gamma}$ are not degenerate (as they should be in exact SU_3), then the solution of the dual equations (4) does not require the existence of a $3/2^+$ octet. If SU_3 is not assumed at all, the N_γ and N_β trajectories can have an arbitrarily weak coupling to the πN channel (although they must have the same decay into the $\pi\Delta$ channel as the N_α and Δ_σ; see Table II).

On the other hand, however, the predicted F/D ratio for the octets is in good agreement with the entire set of experimental data [30,31]. Moreover, it is only for SU_3 invariance that the residue of the N_2 reggeon goes through zero at $u_0 = -0.2$ $(\text{GeV}/c)^2$ $(\alpha_{N_\alpha}(u_0) = -1/2)$ (since there are exotic channels for the K^+p reaction, but not for πN scattering) and guarantees a minimum in backward scattering at $u = u_0$ [4a]. It is therefore quite possible that the trouble here is not with SU_3, but with the violation of duality itself. In particular, if it is assumed that it is not only the resonances that give a large contribution to the channel $B\bar{B} \rightarrow M\bar{M}$ [9], then for meson-baryon scattering one does not require the existence of a $\{8, 3/2^+\}$ trajectory that is degenerate with the $\{10, 3/2^+\}$, the F/D ratios are not fixed in the octets, and the N_α and N_γ are not interrelated, i.e., all the difficulties are also eliminated. Thus, in considering meson-baryon scattering, there appear two alternatives: either SU_3 is significantly broken, or duality is incompatible with the experimental data on the spectrum of baryon resonances.

Unfortunately, this is not the only difficulty of the dual approach. In particular, in considering the reaction $\rho^+\rho^+ \rightarrow B^+B^+$ (the B meson, 1^{+-}) in analogy with (16), we obtain

$$\alpha_\eta(t) = \alpha_\pi(t) \text{ and } \beta^{\rho B}_{x_0} = 0. \tag{36}$$

This implies that $m_\eta^2 = m_\pi^2$ and that the mixing angle in the 0^- nonet is the same as in the nonet of vector mesons, which is of course incompatible with the experimental data. Is there some way out of this difficulty? At the present time, there is reason to believe that there exist a 1^{++} nonet (A_1 (1070), K_α(1243), D (1288), D′(?)) and a 1^{+-} nonet (B (1235), K (1320), H′, H (1000))[3]. If we assume that α_π is degenerate not with α_B and α_H, but with α_{A_1} and α_H (in which case α_η is degenerate with α_B and α_D), the duality condition does not require the equality of m_π and m_η and a definite mixing angle, although in this case none of the resonances lying on the trajectories $\alpha_\pi = \alpha_{A_1} = \alpha_H$ should decay into the $\bar{K}K^*$ (890) system [32]. Thus, the duality condition can be reconciled with the spectrum of observed particles (even for meson-meson scattering) only if there appear new selection rules that are unrelated to the external quantum numbers. This calls for an experimental test.

c) **Difficulties in baryon-baryon scattering.** The foregoing discussion has shown how useful the ideas of duality may be, although it has proved to be impossible to consistently reconcile duality with the absence of exotic states [33]. To elucidate the nature of the inconsistencies which appear, let us consider, for example, the reaction $p + n \rightarrow \Delta^{++} + \Delta^-$. This reaction has no resonances in the direct channel (there are no resonances with baryon number 2) and in the crossed channel (i.e., $\bar{\Delta}^{++} + n \rightarrow \bar{p}\Delta^-$, which should receive contributions from doubly charged meson resonances, which do not exist). The condition (4) therefore implies that the entire amplitude for this reaction, and not merely its imaginary part, should reduce to zero. It follows from this, in particular, that the residues of the f and A_2 reggeons, which contribute to this reaction, should reduce to zero. However, this is incompatible with the experimental data on the reactions $\pi N \rightarrow \pi\Delta$, $KN \rightarrow K\Delta$ and $\pi N \rightarrow \eta\Delta$, which are in good agreement with the exchange of vector and tensor reggeons [4b]. Thus, the requirements of duality are inconsistent with the experimental data. Consequently, in order to rescue duality it is necessary that there exist exotic resonances, and it is simplest to assume that such resonances exist in the $B\bar{B}$ system (for the reaction $pn \rightarrow \Delta^{++}\Delta^-$, it is sufficient to assume that there is a resonance in the $\Delta^{++}\bar{n}$ system $(S = 0, Q = 2)$). We recall that such resonances would help us to avoid the poor spectrum of baryon resonances. Such resonances have not been detected experimentally, and the search for them is of great importance for the ideas of duality.

In this connection, one can ask whether it is not pos-

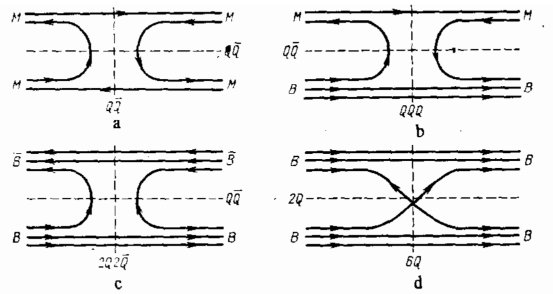


FIG. 17. Quark dual diagrams for MM → MM (a), MB → MB (b), $\bar{B}B \to \bar{B}B$ (c) and BB → BB (d). M–meson, B–baryon.

sible to formulate any rules that would enable one to decide to what processes the requirements of duality can be consistently applied. In [34,35] simple graphs were proposed for the representation of the duality principle and the determination of the channels in which there are exotic resonances. In essence, these rules take into account the fact that the resonances allowed by the simple quark model are observed. These rules are as follows:

1) We represent baryons by three parallel lines directed in the same way (three quarks, QQQ) and mesons by two oppositely directed lines (a quark and antiquark, $Q\bar{Q}$).

2) A scattering process is represented by a redistribution of the lines corresponding to the external particles (Fig. 17). Then, if, for some reaction, it is possible to draw a planar graph (i.e., a graph in which the lines do not intersect) for which one has either $Q\bar{Q}$ or QQQ in the intermediate state, this reaction is dual in the sense of the conditions (4). If one finds a greater number of lines in the intermediate state (even when the graph is planar), the imaginary part of the amplitude is equal to zero. From these rules we find, in particular, that BB and $B\bar{B}$ processes do not obey the duality principle, since any cut for these processes necessarily contains four lines (Fig. 17), whereas duality arguments are applicable to meson-meson and meson-baryon scattering (Fig. 17).

While accepting the fact that one cannot speak about duality for BB scattering, we shall nevertheless try to say something about these processes on the basis of MM and MB scattering. It is clear that we can do this, at least at high energies, by making use of the factorization property of the Regge pole residues. Thus, if we consider the reactions of $\pi^+\pi^+$, K^+K^+, K^+K^0 and K^+p scattering, for example, we fine that

$$g_\rho^{\pi\pi}=g_f^{\pi\pi},\ g_\omega^{KK}=g_{A_2}^{KK}=g_f^{KK},\ \text{while}\ g_\rho^{pp}=g_{A_2}^{pp},\ g_\omega^{pp}=g_f^{pp} \qquad (36')$$

(for the notation, see [3]). Equation (36′) implies that the cross section for pp scattering is constant and that it has no minima with respect to the momentum transfer and still satisfies the remaining predictions of the first part of this section.

A thorough study of such constraints [36] has shown that, if the F/D ratios for the vector and tensor mesons are assumed to be equal, the imaginary part of the amplitude is equal to zero at high energy for BB and BD scattering (B is a member of the octet, and D is a member of the decuplet). In other words, it follows from duality for MB and MM scattering there are no resonances in the BB and BD systems (at least at high energy). If now F/D = 1/2, then there are also no exotic resonances in the $B\bar{B}$ system. For $B\bar{D}$ and $D\bar{D}$ scattering, however, we find a contribution in the exotic channels that cannot be eliminated; consequently, we must assume that resonances exist in these channels if we wish to "rescue duality."

Thus, we see that duality leads to very interesting results that are in agreement with experiment (exchange degeneracy, the F/D ratio for SU_3 multiplets, and the mixing angle); however, in considering baryon-baryon scattering we encounter difficulties which cannot be overcome (without abandoning the requirements of duality), provided that exotic resonances do not exist.

In proceeding to the description of more concrete dual models, we must already content with a lack of consistency of the general principles for BB scattering and the feeling that a strict formulation of duality does not correspond to reality.

5. THE VENEZIANO MODEL

So far we have discussed the constraints that are necessary for the duality conditions (4) to be fulfilled. However, it is not yet clear whether it is possible to construct a scattering amplitude which satisfies all the requirements of duality. Veneziano[37] proposed to construct an amplitude from the sums

$$V(s,\ t)=\sum C_{mk}^{p}\frac{\Gamma(m-\alpha(s))\,\Gamma(k-\alpha(t))}{\Gamma(p-\alpha(s)-\alpha(t))}. \qquad (37)$$

where Γ(x) is the Euler gamma function, and $\alpha(s)$ and $\alpha(t)$ are trajectories containing resonances in the s- and t-channels of the reaction. In particular, the scattering amplitude for scalar particles with Q = S = T = 0 is of the form

$$A(s,\ t)=-[V(s,\ t)+V(u,\ t)+V(s,\ u)]. \qquad (38)$$

If the trajectories are linear, i.e., if

$$\alpha(t)=a+bt, \qquad (39)$$

then the amplitude (38) has only poles in each of the variables and has the correct Regge asymptotic form, i.e., it satisfies the conditions (4). Let us consider the basic properties of (37) and (38) in greater detail.

a) The spectrum of trajectories. With the condition (39) in the Veneziano model (VM), the residue at the pole $s = s_0$ ($\alpha(s_0) = n$), confining ourselves to the first term (p = m = k = 0) in (37), is equal to

$$V(s,\ t)_{s\to s_0}=\frac{\pi}{\sin\pi\alpha(s)}\frac{(-1)^n}{n!}\frac{\Gamma(\alpha(t))}{\Gamma(-n-\alpha(t))}$$
$$\to\frac{\alpha(t)(\alpha(t)+1)\ldots(\alpha(t)+n)}{(\alpha(s)-n)\,n!}=\frac{1}{s-s_0}Q_n(t), \qquad (40)$$

where $Q_n(t)$ is an n-th degree polynomial in t. Since $t=-2p^2(1-z)$, Q_n is also an n-th degree polynomial in z, so that

$$Q_n(t)=\sum_{l=0}^{n}C_lP_l(z). \qquad (41)$$

Equations (40) and (41) imply that there exist resonances of mass s_0 with all the values of the spin from 0 to n in Eq. (38), i.e., that there exists a family of trajectories in the VM that are mutually parallel and displaced by unity (Fig. 18; the circles indicate the particles with spins from 0 to n, for $\alpha(s_0) = n$ and n = 3). Thus, the spectrum of daughter trajectories in the VM which we discussed earlier (in Sec. 3) has a completely concrete form (Fig. 18), which is of the greatest importance in the VM. Unfortunately, almost nothing can be said about these daughter trajectories at the present time. It is of

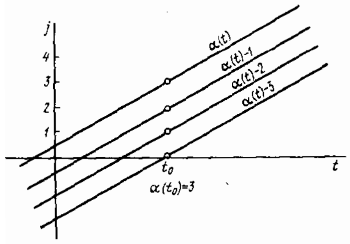


FIG. 18

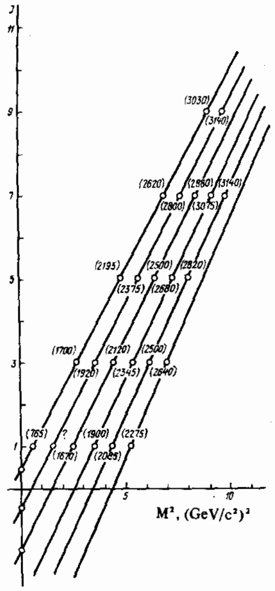


FIG. 19

FIG. 18. The spectrum of trajectories in the Veneziano model (the leading trajectory and three daughter trajectories).

FIG. 19. The ρ trajectory and its daughter trajectories [38] passing through the resonances which entered [3] in some way.

interest to attempt to classify [38] all the currently known boson resonances (i.e., those which have entered the tables of [3] in one way or another) according to trajectories. In Fig. 19 we show the ρ meson trajectory and its daughter trajectories. The picture is obviously quite impressive, as the resonances lie on almost parallel straight lines, whose intercepts on the J-axis differ by 1. Unfortunately, however, of this set of resonances only the ρ meson and perhaps the g(1700) are firmly established, and even in Fig. 19, there is one vacancy: there is no daughter resonance to the f(1250) with quantum numbers 1^- and mass 1300 MeV. Incidentally, it is just this meson which is missing in the description of the data on the electromagnetic form factor of the nucleon [39].

b) If the VM spectrum of trajectories is assumed, it can be shown that the slopes (the parameter in (39)) of all the trajectories must be equal. In fact, let us consider, as an example, the reactions

$$\pi^+\pi^0 \to \pi^+\pi^0,\ \pi^+\pi^0 \to \bar{K}^0K^+,\ K^+\bar{K}^0 \to K^+\bar{K}^0. \tag{42}$$

The reaction (42) receives direct-channel contributions from the resonances lying on the ρ trajectory (as well as the A_2 trajectory for the last reaction), although other trajectories are present in the t-channel (the f reggeon for the first and third reaction, and the K^* reggeon for the second). Consider some resonance on the ρ trajectory in the s-channel; then, according to the factorization condition (Fig. 20), the residues of this pole for the reactions (42) are related by the condition

$$r\,(\pi^+\pi^0 \to \pi^+\pi^0)\,r\,(K^+\bar{K}^0 \to \bar{K}^0K^+) = r^2\,(\pi^+\pi^0 \to \bar{K}^0K^+). \tag{43}$$

However, it is readily found from (40) that the coefficient C_n in the expansion (41) for the reaction $a+b \to c+d$ is equal to $(2b_tp_{ab}p_{cd})^n$, where $\alpha_t = a_t + b_tt$. Thus, for (42) we have

$$\left.\begin{aligned} r\,(\pi^+\pi^0 \to \pi^+\pi^0) &= (2b_fp_{\pi\pi}p_{\pi\pi})^n,\\ r\,(K^+\bar{K}^0 \to K^+\bar{K}^0) &= (2b_fp_{KK}p_{KK})^n,\\ r\,(K^+\bar{K}^0 \to \pi^+\pi^0) &= (2b_{K*}\,p_{KK}p_{\pi\pi})^n. \end{aligned}\right\} \tag{44}$$

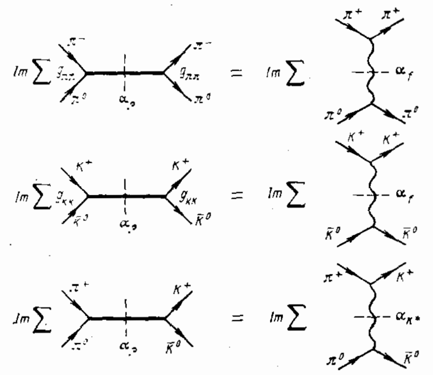


FIG. 20

Consequently, the condition (43) can be satisfied only if $b_f = b_{K^*}$. It is clear that, if we consider the reactions $\pi\pi \to N\bar{N}$ and $N\bar{N} \to N\bar{N}$, we find that the slopes of the meson and baryon trajectories are equal. Of course, the foregoing argument depends on the specific form of Eq. (37). Actually, the equality of the slopes follows entirely from duality and the spectrum of daughter trajectories in the VM. The point is that these two conditions are sufficient for the scattering amplitude to be represented in the form of an infinite sum of the type (37)[40], and it is quite unnecessary to confine oneself to a single term in (37) in discussing Eq. (43). The equality of the slopes of all trajectories is in good agreement with experiment. At any rate, all the existing Regge pole trajectories can be described by straight lines with slopes between $0.8/m_N^2$ and $0.9/m_N^2$ (where m_N is the nucleon mass) (see Table I).

c) The equality of the slopes is a result of very great interest; it is therefore desirable to ascertain more carefully what hypotheses are required for its derivation, and whether it is actually a consequence of the general principles of duality and not merely the VM. To do this, let us consider the reaction (42) at sufficiently high energy (s). We then find a contribution of a resonance with spin $j^2 \sim sb_f \ln s \sim sb_{K^*} \ln s$ to the imaginary part of the partial wave (of course, this resonance will lie on a daughter trajectory if $\alpha(t) > \sqrt{t}$ for large t), and (see Fig. 20 and [5b])

$$\left.\begin{aligned} \operatorname{Im} a_j\,(\pi\pi \to \pi\pi) &= r^j\,(\pi\pi \to \pi\pi) = G_1j^{a_f} \exp\left(-\frac{j^2}{sb_f \ln s}\right),\\ \operatorname{Im} a_j\,(K\bar{K} \to K\bar{K}) &= r^j\,(K\bar{K} \to K\bar{K}) = G_2j^{a_f} \exp\left(-\frac{j^2}{sb_f \ln s}\right),\\ \operatorname{Im} a_j\,(\pi\pi \to K\bar{K}) &= r^j\,(\bar{\pi}\pi \to K\bar{K}) = C_{3}j^{a_{K^*}} \exp\left(-\frac{j^2}{sb_{K^*} \ln s}\right). \end{aligned}\right\} \tag{45}$$

We see from (45) that (43) is violated, i.e., it seems that we have arrived at a contradiction. The way out of this situation is simple: we must assume that the daughter trajectory contains not only one resonance with a given spin and fixed external quantum numbers, but that there must exist an additional degeneracy. In this case, the left-hand sides of the equalities must be multiplied by the number of these resonances (N), and it follows from (43) that

$$\frac{N\,(\pi\pi \to \pi\pi)\,N\,(\bar{K}K \to K\bar{K})}{N^2\,(\pi\pi \to K\bar{K})} = cj^{2(a_f - a_{K^*})} \exp\left[-2\left(\frac{1}{b_{K*}} - \frac{1}{b_f}\right)\frac{j^2}{s \ln s}\right]. \tag{46}$$

Since $N(\pi\pi \to K\bar{K})$ is the number of resonances which decay into both the $\pi\pi$ and $K\bar{K}$ systems,

$$N\,(\pi\pi \to K\bar{K}) \leqslant N\,(\pi\pi \to \pi\pi) \text{ and } N\,(K\bar{K} \to K\bar{K}) \tag{47}$$

and (46) ≥ 1. Then:

1) We may have $b_{K^*} > b_f$, in which case only a very

small fraction (exponentially small for large j) of the total number of resonances decay into both $\pi\pi$ and $K\overline{K}$. Since this number cannot be less than 1, we can say that a very large number of resonances (exponentially large for large j) decay into each of the systems $\pi\pi$ and $K\overline{K}$.

2) Alternatively, if $b_{K^*} = b_f$, then

$$a_{K^*} \leqslant a_f. \tag{48}$$

The degeneracy at each level may behave like a power. Thus, equality of the slopes corresponds to the minimum degeneracy of the daughter states. This minimum degeneracy is guaranteed in the VM.

We note that the condition (48) implies that all the trajectories that contribute to inelastic reactions must lie lower than those that contribute to elastic reactions. In particular, this implies that the strange resonances lying on the trajectory $\alpha_{K^*}(t)$ must be heavier than the non-strange resonances; the baryon resonances corresponding to the trajectories that describe backward scattering of the type $\pi N \rightarrow N\pi$ (an inelastic process) must be heavier than the meson resonances; the resonances of the pion trajectory that contribute to reactions of the type $\pi\pi \rightarrow \rho\rho$ must be heavier (for the same spin) than the resonances lying on the ρ, f, A_2 and ω trajectories that contribute to $\pi\pi$ and $\rho\rho$ scattering, etc. All these results are in good agreement with what we know (see Table I) and, despite the feeling that this statement is obvious, it is interesting and non-trivial, especially when applied to the baryon and pion trajectories. For the strange particles, this result agrees with the usual hypothesis that the strange quark is heavier[24].

d) All of the foregoing referred to reactions of the type $a + b \rightarrow c + d$. At the same time, as already noted in Sec. 2, the conditions of duality may be formulated not only for these simple processes, but also for reactions involving many particles (see Fig. 3). It turns out that one can write down an amplitude for multi-particle processes which has only poles in each of the channels and a Regge asymptotic form in each of the variables, i.e., one which satisfies the dual conditions. For this amplitude, the number of resonances on daughter trajectories with the same mass and spin (for large spin j) is of order

$$\exp\left(c\sqrt{j}\right), \tag{49}$$

provided that the number of particles into which a given resonance decays is assumed to be arbitrarily large[41].

Thus, in the generalized Veneziano model (GVM) for multi-particle processes, the partial waves of order unity for $l \sim (sb_s \ln s)^{1/2}$ (i.e., the waves that are important in the scattering) are built up as a sum of a large number of resonances (49), each of which decays into a given system with a very small probability[8]. At the present time, there are of course no experimental indications that this is actually the case.

e) Let us now consider what experimental indications exist for the specific form (37). Consider once more a particular term of (37). The two Γ functions in the numerator provide poles in both of the variables (s and t), while the Γ function in the denominator is arranged to have a pole at just those values of s and t where there are poles of both of the Γ functions of the numerator, i.e., it ensures that poles do not occur simultaneously in s and t; moreover, it ensures that the residue at the pole in s is a polynomial in t, Eq. (40).

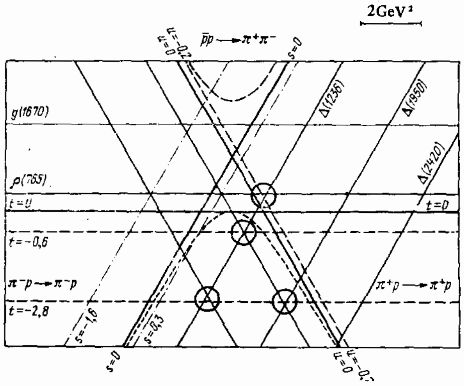


FIG. 21. The Mandelstam plane for πp scattering.

The denominator of each term in (37) becomes infinite (has a pole) at

$$-\alpha(s) - \alpha(t) = -n \text{ (where n is an arbitrary integer)},$$

$n = a_s + a_t + b(t + s)$, and, since $t + s + u = \Sigma_i m_i^2$, it follows from this that the denominator becomes infinite on a line of constant u in the Mandelstam plane. Our amplitude must reduce to zero at those values of s and t at which the numerator does not have poles. Such a behavior of (37) should lead to the occurrence of minima in the differential cross sections at values of the variables which correspond to straight lines passing through the points of intersection of the poles (see Fig. 21), and in studying a given decay we should observe minima ("holes") on the Dalitz plot of this reaction, which should lie on a straight line[42]. The dashed lines in Fig. 21 indicate the values of the variables at which we expect minima in the differential cross section for π^+p scattering ($t = -0.6$ and -2.8 (GeV/c)^2, $u = -0.2$ (GeV/c)^2); for π^-p scattering, there are minima only at $t = -0.6$ and -0.2 (GeV/c)^2. These minima are not associated with high energy, and the fact that they coincide with the positions of the minima in the differential cross section that follow from exchange degeneracy at large s (see Sec. 5) for $\alpha(t_0) = 0$ is simply the result of an accident, namely the small pion mass (for the reaction $\pi N \rightarrow \pi\Delta$, for example, their positions are quite different from $\alpha(t_0) = 0$, namely $t = -0.2$ and -1.95 (GeV/c)^2). This behavior of the differential cross section is confirmed by Fig. 22. A minimum appears practically at the threshold of the reaction at $t = -0.35$ (GeV/c)^2, and then moves towards larger t, settling down at $t = -0.6$ when $P_\pi = 0.8$ GeV/c[44]. The minima on the Dalitz plot for the annihilation $\bar{p}n \rightarrow 3\pi$ seem even more impressive. We show in Fig. 23 the Dalitz plot for this reaction, on which the minima at its center and three minima at a fixed value of $m^2_{\pi^+\pi^+}$ are clearly seen (as is expected from the condition (50)). In this case, we require an explanation of the fact that there do not occur two minima, corresponding to $-\alpha_s - \alpha_t = -4$ (a single minimum corresponds to $-\alpha_s - \alpha_t = -3$, and three minima to $-\alpha_s - \alpha_t = -5$). It may be that this reaction is described by a formula of the type (37) only at rest, while the description is much more complicated if the proton has momentum. All the efforts which have been made in this respect[45], while not yielding complete agreement with experiment, have explained the fact that two minima do not occur (Fig. 24).

However, it would be desirable to have a clearer formulation of which assumptions underlying the VM are supported by the appearance of these minima. In the first place, the latter shows that the "narrow-resonance"

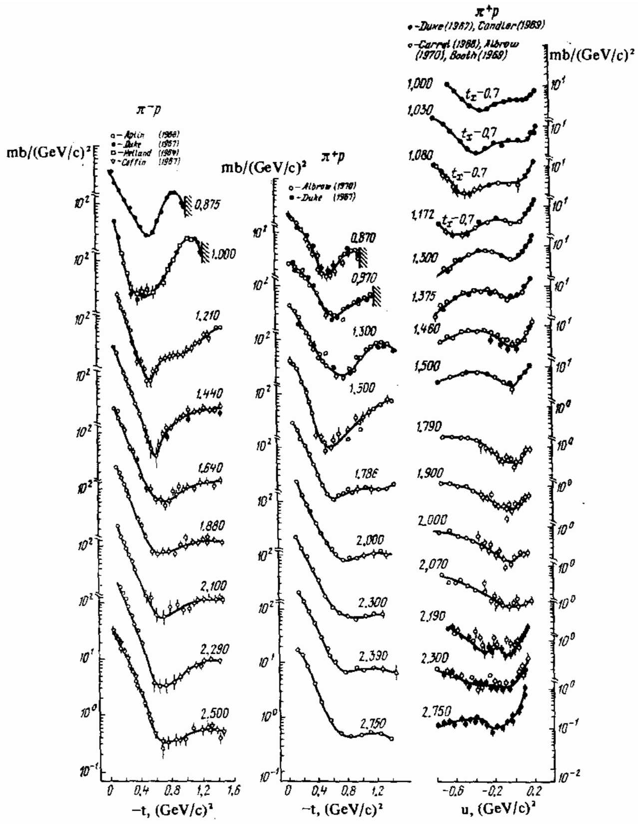


FIG. 22. The $\pi^-p \to \pi^-p$ and $\pi^+p \to \pi^+p$ differential cross sections at low energies ($P_\pi < 3$ GeV/c) [43].

approximation works well (we neglect resonance widths in the VM). In fact, if we have a pole in each of two variables (s and t), there will be a zero of the amplitude between them [46], since A(s, t) near these poles is of the form

$$\frac{g_1}{s-m_1^2}+\frac{g_2}{t-m_2^2}=\frac{g_1(t-m_2^2)+g_2(s-m_1^2)}{(s-m_1^2)(t-m_2^2)}. \tag{50}$$

and there will be a zero at $g_1(t-m_2^2)+g_2(s-m_1^2)=0$. The fact that the zeros on the Dalitz plot lie on a single straight line is evidence for the specific formula (37), in which $g_1=g_2$. However, the absence of two "holes" for $p\bar{n} \to 3\pi$ may be attributed, in particular, to another system of zeros [46]. Consequently, in order to test the VM, we require more experimental data on the reactions $p\bar{n} \to 3\pi$, in particular at higher energies.

f) We shall now study another characteristic property of the VM, namely the mass distribution of some exotic channel (for example, the $(\pi^+\pi^+)$ mass distribution in the reaction $p\bar{n} \to \pi^+\pi^+\pi^-$). There are no resonances in this channel, but the VM nevertheless predicts a distribution that is essentially different from phase space in the region of small masses (see Fig. 27b, in which an almost resonant-like peak is observed in this region). It is quite simple to understand why this occurs. Let us describe the reaction $p\bar{n} \to \pi^+\pi^+\pi^-$ by a sum of the type (37), where $s=m^2_{\pi_1^+\pi^-}$, $t=m^2_{\pi_2^+\pi^-}$, $u=m^2_{\pi^+\pi^+}$ and $s+u+t=3m_\pi^2+(2m_N)^2$. The Γ function in the denominator will depend only on u, rising sharply as a function of u (in the reaction in question, it approaches a pole at the maximum u), which in fact leads to a peak in the region of small u. If we consider the reactions $p\bar{n} \to 5\pi, 7\pi, \ldots$, for which the amplitude is integrated with respect to a large number of variables in obtaining the mass distribution, this peak will naturally be smeared out (which is also observed experimentally [47]). Such a behavior of the spectra is, in itself, also greater evidence for the general dual approach and the presence of "narrow" resonances than for the VM in particular, although it is, on the other hand, more characteristic of the assumed form of the amplitude than of the distribution of zeros. In particular, such a characteristic maximum for all the reactions can hardly be expected for the system of zeros of [46]. Consequently, this characteristic distribution in the $\pi^+\pi^+$ system and the appearance of the minima on the Dalitz plot together constitute a strong (but far from conclusive) argument in favor of the specific Veneziano formula.

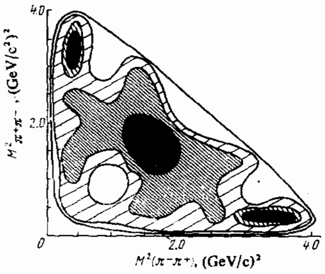


FIG. 23

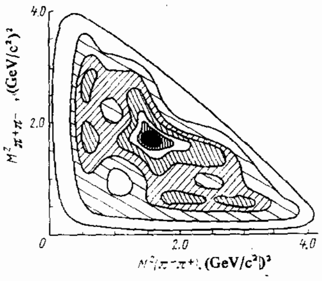


FIG. 24

FIG. 23. Dalitz plot for the reaction $p\bar{n} \to 3\pi$ at $P_p = 1.2$ GeV/c [45].
FIG. 24. Theoretical Dalitz plot for the reaction $p\bar{n} \to 3\pi$ [45].

6. SOME SPECIFIC REACTIONS IN THE GENERALIZED VENEZIANO MODEL

a) **Meson-meson scattering.** In this section we turn to the description of certain specific processes in the GVM, considering first of all meson-meson scattering, an application in which the ideas of duality have so far not encountered any difficulties. We shall confine ourselves to the interaction of pseudoscalar mesons [48] and take ππ scattering as an example. The simplest amplitude for this process in the GVM has the form [49] ($A^T_{s(t)}$ is the amplitude with isotopic spin T in the s-channel (t-channel)

$$\left.\begin{aligned}A_s^0&=-\frac{1}{2}g^2[3(V(s,t)+V(s,u))-V(t,u)],\\A_s^1&=-g^2[V(s,t)-V(s,u)],\\A_s^2&=-g^2[V(t,u)],\end{aligned}\right\} \tag{51}$$

$$\left.\begin{aligned}A_t^0&=-\frac{1}{2}g^2[3(V(s,t)+V(u,t))-V(s,u)],\\A_t^1&=-g^2[V(s,t)-V(u,t)],\\A_t^2&=-g^2[V(s,u)],\end{aligned}\right\} \tag{52}$$

where

$$V(x,y)=\frac{\Gamma(1-\alpha(x))\,\Gamma(1-\alpha(y))}{\Gamma(1-\alpha(x)-\alpha(y))}.$$

Equations (51) and (52) have certain simple properties.

1) The scattering amplitude has only poles in each of the channels (the trajectories are linear), their residues being real and having the form of polynomials of degree n in $a(\alpha(s)=n)$ (see (40) and (41)).

2) There are no resonances in the exotic channel (T = 2) (V(t, u) has poles in u and t, but not in s).

3) Crossing symmetry and Bose statistics for the pions are taken into account in (51) and (52). In fact, if we consider the amplitude A_s^0, it should receive contributions from only the poles with even spin (since the

pions are bosons), while the situation is opposite for A_s^1 (only the odd-spin poles contribute). Since the pole contribution is

$$A_s^0 = -\frac{3g^2}{2}\cdot\frac{1}{s-s_0}\{Q_n(t)+Q_n(u)\} \qquad (53)$$

(see (40)) and since the variable u for this reaction is obtained from t by the substitution $z \to -z$ (see Fig. 1), only the even powers of z contribute to (53), so that for a given mass there will be only even-spin resonances in (41).

4) Eq. (52) has the correct asymptotic form for $s \to \infty$ ($u \to -\infty$, $t < 0$ fixed). From the equations of Appendix I, it is easy to show that

$$A_t^0(st) = -\frac{3}{2}g^2\Gamma(1-\alpha_t)\left[\frac{\sin\pi(\alpha_s+\alpha_t)}{\sin\pi\alpha_s}(\alpha_s)^{\alpha_t}+(-\alpha_u)^{\alpha_t}\right] - g^2\frac{\pi}{2}\frac{1}{\sin\pi\alpha_s}(\alpha_s)^{1-D+\alpha_t}\frac{1}{\Gamma(1+D+\alpha_t)}, \qquad (54)$$

where $D = \alpha_s + \alpha_t + \alpha_u = 3a + 4bm_\pi^2$.

If it is assumed that $\alpha(s)$ has an imaginary part that grows with s, but that $\operatorname{Im}\alpha(s)/\operatorname{Re}\alpha(s) \to 0$ as $s \to \infty$ (or if we seek the asymptotic form not for real s, but at an angle in the complex plane), the contribution of V(s, u) will become small ($\sim e^{-\operatorname{Im}\alpha(s)}$) and

$$A_t^0(s,\ t) = -\beta(t)\frac{e^{-i\pi\alpha_t}+1}{\sin\pi\alpha(t)}(bs)^{\alpha(t)}, \qquad (55)$$

with

$$\beta(t) = g^2\frac{\pi}{\Gamma(\alpha(t))}.$$

Equation (55) corresponds to the exchange of a Regge pole with positive signature (see (3)), in this case the f reggeon. Similarly, it can be shown that a reggeon with negative signature (the ρ reggeon) will contribute to A_t^1 for $s \to \infty$. As was to be expected, these two reggeons are degenerate, and their residues $\beta(t)$ in (55) tend to zero as $\alpha(t) \to 0$. To obtain the correct asymptotic form, it was necessary to assume that $\operatorname{Im}\alpha_s$ grows with s, so that the resonances in the VM must have larger and larger widths. Recalling the discussion of Sec. 4, we see that the situation regarding the determination of these resonances becomes very uncertain in this case.

5) It can be shown that the residues of all the resonances in (51) (including the daughters) are positive for[49]

$$a \geqslant 0{,}496. \qquad (56)$$

This means that, with (56), the coefficients in (41) are all positive. The result (56) itself was derived by considering the residues on the first two daughter trajectories [49]. However, it can be shown that, if (56) is satisfied, all the residues will be positive on the other trajectories as well [50]. The question as to whether the residues are positive is of the utmost importance, since it is only in this case that it is meaningful to regard (51) as a good approximation to reality and to assume that the unitarity corrections are small.

Thus, (51) and (52) satisfy the requirements of duality. It is of interest to see how these equations agree with experiment. Fortunately, we can say something about the $\pi\pi$ interaction from experiment [51]. However, a direct comparison of (51) with experiment is not possible, since all the poles in (51) lie on the real axis, i.e., one must allow for the widths of the resonances in order to make a comparison with experiment. Thus, one writes in essense a formula which has the same spectrum of resonances and the same relations between their residues as in (51), but with the full width arbitrarily inserted in each resonance. Since this width is not known for all resonances, some approximate formula is used for it, on whose form the results will naturally depend. In [49] the expression adopted for α_s was

$$\alpha_\rho(s) = 0{,}483 + 0{,}885s + 0{,}28i\sqrt{s-4\mu^2}\,\theta(s-4\mu^2). \qquad (57)$$

Let us turn to the characteristic features of the amplitude (51). (We note that the one parameter g^2 in (51) can be determined, for example, from the width of the ρ meson.)

1) The phase shifts of the $\pi\pi$ interaction with $T_s = 0$ are positive and small at threshold, while for $T_s = 2$ they are small and negative [49]. Their energy dependence is in agreement with one of the phase-shift analyses of the $\pi\pi$ interaction [52].

2) The $\pi\pi$ scattering lengths in the VM are in agreement with the scattering lengths obtained from current algebra [53,54] (when the width of the ρ meson is equal to 100 MeV). With the usual normalization [49] to the width 110 MeV of the ρ meson, the scattering lengths are found to be [48] $a_0 = 0.20/m_\pi$ and $a_2 = -0.05/m_\pi$ (from current algebra [53], $a_0 = 0.15/m_\pi$ and $a_2 = -0.043/m_\pi$). However, it should be noted that the values of a_0 and a_2 depend significantly on $\alpha(0) = a$; for $a = 0.52$ (instead of 0.485), we have $a_0 \sim 1/m_\pi$, but the positivity of a_0 does not depend on this. Since there is an indication that $a_0 < 0$ [55], Eq. (51), if confirmed, will be incompatible with experiment.

3) The spectrum of resonances and the elastic widths of the resonances are shown in Fig. 25. All the widths are quite small and, with the uncertainty in the classification of the boson resonances, it is difficult to compare them with experiment. However, certain qualitative features of this spectrum are evident:

a) There should exist a scalar resonance (0^+) with the mass of the ρ meson (the σ) and a width much greater than that of the ρ:

$$\Gamma_\sigma/\Gamma_\rho \sim \frac{9}{2}. \qquad (58)$$

Such a resonance evidently does exist (the ϵ meson in the tables of [3]). Moreover, the existence of such a resonance and the relation (58) between the widths are required by chiral symmetries [56].

b) There should be a $1^{(-)}$ meson with the mass of the f meson (~1300 MeV) and a width of the order of the width of the ρ. Such a meson has not been detected, and

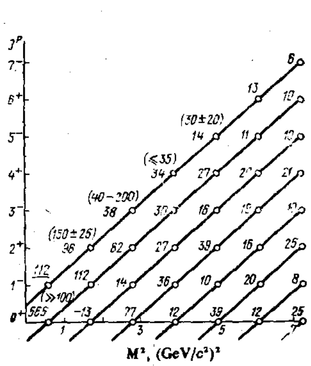


FIG. 25. The spectrum of resonances and their elastic widths in the $\pi\pi$ amplitude in the Veneziano model [49]. The full widths of the meson resonances according to [3] are indicated in parentheses; $\alpha_\rho(t) = 0.48 + 0.90t$.

its absence constitutes a significant contradiction of (51) with experiment.

c) The elastic widths of the resonances fall off rapidly with increasing mass, which is in good agreement with the data on the boson resonances in the regions of the Q, S, T and U [3], provided, of course, that there is no change in the experimental situation in these poorly studied regions.

We note a remarkable feature of the amplitude (50): it satisfies the Adler self-consistency condition [54,57] for

$$\alpha_\rho(m_\pi^2)=\frac{1}{2}. \qquad (59)$$

The Adler condition is a consequence of the conservation of the axial current in the limit $m_\pi \to 0$ [54] and requires that the $\pi\pi$ amplitude vanishes when the 4-momentum of one of the pions is equal to zero. We note that, if $p_\mu = 0$ for one of the pions, then $s = t = u = m_\pi^2$ for $\pi\pi$ scattering. We see that, with the condition (59), the argument of the Γ function in the denominator reduces to zero, and this function itself becomes infinite, which indicates the vanishing of the amplitude (51) [42]. The condition (59), together with the requirement $\alpha_\rho(m_\rho^2) = 1$, enables us to determine $\alpha_\rho(s)$, which is then equal to (57) (of course, without the imaginary part). The analogous condition for the πK amplitude leads to

$$\alpha_{K^*}(m_\pi^2)=\frac{1}{2}. \qquad (60)$$

From the conditions (59) and (60) and the analogous conditions for other processes [58], there follow mass formulas of the type

$$\left.\begin{array}{ccc} \underset{(0{,}211)}{m_{K^*}^2-m_\rho^2} & = & \underset{(0{,}245)}{m_{K^{**}}^2-m_{A_2}^2}=\underset{(0{,}242)}{m_K^2-m_\pi^2}, \\ \underset{(1{,}14)}{m_{A_1}^2} & = & \underset{(1{,}168)}{2m_\rho^2-m_\pi^2}, \\ \underset{(1{,}53)}{m_\Delta^2} & = & \underset{(1{,}463)}{m_N^2+m_\rho^2-m_\pi^2}. \end{array}\right\} \qquad (61)$$

We give in parentheses the experimental values of the differences in $(\mathrm{GeV})^2$ from the tables of [3]. It can be seen that (61) is in good agreement with experiment. We would like to draw attention to the second formula in (61), which was originally derived from current algebra and aroused great interest [59].

The experimental absence of a 1^- meson of mass 1300 MeV (a ρ') compels us to try a more complicated form of (51) by taking several further terms in (37). It should be noted that the positivity of the residues of the daughter resonances limits this possibility to a single additional term, i.e., (51) can be modified by taking

$$V(s,\ t)=\frac{\Gamma(1-\alpha_s)\,\Gamma(1-\alpha_t)}{\Gamma(1-\alpha_s-\alpha_t)}+\beta\,\frac{\Gamma(1-\alpha_s)\,\Gamma(1-\alpha_t)}{\Gamma(2-\alpha_s-\alpha_t)}. \qquad (62)$$

However, if we require that there is no ρ' resonance, we find from (62) that the σ meson enters with a negative residue. The maximally possible variant (corresponding to $\Gamma_\sigma = 0$) gives $\Gamma_{\sigma'} = \Gamma_\rho/4$. Thus, the existence of the ρ' in the $\pi\pi$ system is necessary in the VM. At the same time, the modification (62) of the amplitude (51), even for comparatively small β, enables us to obtain rather large negative scattering lengths a_0 (with $\Gamma_\sigma = 0$, we have $a_0 = -1.6/m_\pi$). Thus, the VM is capable of describing many characteristic features of the $\pi\pi$ interaction and is in good agreement with current algebra, but requires the existence of a ρ' in the 2π system.

b) **Meson-baryon scattering.** A reaction that is of more direct experimental interest is

$$\pi(q_1)+N(p_1)\to\pi(q_2)+N(p_2). \qquad (63)$$

A description of πN scattering in the VM was given in [60,61]. We shall quote only the simplest variant, which takes into account only the baryon resonances which lie on the N_α and Δ_δ trajectories [60]:

$$\left.\begin{array}{l} M=\bar{u}(p_1)\{A(s,\ t)+(\hat{q}_1+\hat{q}_2)\,B(s,\ t)\}\,u(p_2), \\ A^f=\dfrac{\beta_{fN}}{\pi}\,[c^{fN}_{1,\ 3/2}(t,\ s)+c^{fN}_{1,\ 3/2}(t,\ u)+c^{NN}_{3/2\ 3/2}(s,\ u)]+\dfrac{\beta_{t\Delta}}{\pi}\,[N\to\Delta], \\ B^f=\dfrac{\beta_f}{\pi}\,\{[B^{fN}_{1,\ 1/2}(t,\ s)+B^{f\Delta}_{1,\ 1/2}(t,\ s)+B^{N\Delta}_{1/2\ 1/2}(s,\ u)]+[s\to u,\ u\to s]\}, \\ A^\rho=\dfrac{\beta_\rho}{\pi}\,\{[c^{\rho N}_{1,\ 3/2}(t,\ s)+c^{\rho\Delta}_{1,\ 3/2}(t,\ s)+c^{N\Delta}_{3/2\ 3/2}(s,\ u)]+[s\to u,\ u\to s]\}, \\ B^\rho=\dfrac{\beta_{\rho N}}{\pi}\,[B^{\rho N}_{1,\ 1/2}(t,\ s)+B^{\rho N}_{1,\ 1/2}(t,\ u)+B^{NN}_{1/2\ 1/2}(s,\ u)]+\dfrac{\beta_{\rho\Delta}}{\pi}\,[N\to\Delta], \end{array}\right\} \qquad (64)$$

where

$$B^{a,\ b}_{i,\ k}(xy)=\frac{\Gamma(i-\alpha_a(x))\,\Gamma(k-\alpha_b\{y))}{\Gamma(i+k-\alpha_a(x)-\alpha_b(y))},$$

$$C^{a,\ b}_{i,\ k}(xy)=\frac{\Gamma(i-\alpha_a(x))\,\Gamma(k-\alpha_b(y))}{\Gamma(i+k-1-\alpha_a(x)-\alpha_b(y))};$$

f and ρ denote the amplitudes with isotopic spin 0 and 1 in the t-channel of the reaction; for the isotopic amplitudes in the s-channel, we have

$$\left.\begin{array}{l} M_s^{1/2}=-(M^f+2M^\rho), \\ M_s^{3/2}=(M^f-M^\rho). \end{array}\right\} \qquad (65)$$

The amplitude (64) has only poles and the correct Regge asymptotic form, like the $\pi\pi$ amplitude in the VM, but, unlike the latter, it has two significant defects:

1) From the requirement that the resonances lying on the N_α (Δ_δ) trajectory contribute only to $M_S^{1/2}$ ($M_S^{3/2}$), it is readily found that we must have the equalities

$$\alpha_f=\alpha_\rho,\quad \alpha_{N_\alpha}=\alpha_{\Delta_\delta}, \qquad (66)$$

$$\beta_{t,\,\Delta}=-2\beta_\rho,\quad \beta_{\rho\Delta}=-1/2\beta_f,\quad \beta_{fN}=\beta_\delta,\ \beta_{\rho N}=\beta_f. \qquad (67)$$

The first of the equalities (66) corresponds to ordinary exchange degeneracy, but the second does not follow from the general principles of duality (see Sec. 3) and is in conflict with the experimental data (Table I).

2) There must exist in (64) resonances with the same spin but different parity, at one and the same mass (apart from the mass of the nucleon and the Δ resonance). It is simplest to see this by considering the asymptotic form (for example, of $M_S^{3/2}$) at fixed s and large u (which corresponds to backward scattering). From (64) and (65), its form will be

$$M^3_{su\to\infty}\to\bar{u}(p_1)\,(\hat{q}_1+\hat{q}_2)\,u(p_2)\,\eta_\Delta u^{\alpha_\Delta(t)-1/2}, \qquad (68)$$

instead of

$$\bar{u}(p_1)\,(\hat{q}+q)\,u(p_2)\,\eta_\Delta u^{\alpha_\Delta(t)-1/2}, \qquad (69)$$

where $q = p_1 + q_1$, as must be the case for the exchange of a trajectory with positive parity [62]. Such a degeneracy with respect to the parity is not observed in the spectrum of baryon resonances. Equation (64) can be modified in such a way that a doubling of this kind sets in at large masses [61]. However (at least for the time being), no formula has been written (and there are no ideas about how to write one) which takes into account straight-line trajectories and for which there is no parity doubling.

In spite of these significant defects of (64), the analysis of the experimental data has yielded (two free parameters are determined from the pion-nucleon coupling constant and the width of the Δ resonance)[63,64]:

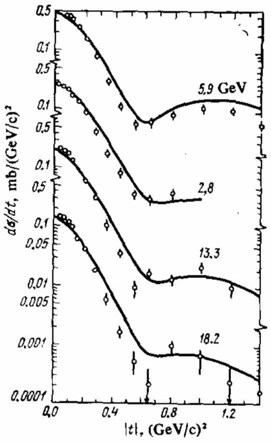


FIG. 26. The $\pi^-p \to \pi^0 n$ differential cross section as a function of the momentum transfer.

1) A good description of πN forward scattering, with regard to both its magnitude and the character of its dependence on the momentum transfer.

2) The correct behavior of the $\pi^-p \to \pi^0 n$ cross section in the vicinity of the minimum at $t \sim -0.6$ (GeV/c)^2 (Fig. 26). This is of interest, since at high energies this reaction receives a contribution from only the exchange of the ρ reggeon, whose residue reduces to zero at $t \sim -0.6$ (GeV/c)^2. The non-zero cross section is usually attributed to the contribution of cuts [4b,16]; however, the satisfactory description of this reaction by means of (64) is evidence that the filling of the minimum at current energies may be due to a non-Regge background which falls off rapidly with the energy (the (su) terms play the role of this background in (64)).

3) A large positive polarization (~0.5) in $\pi^-p \to \pi^0 n$ scattering at $t \sim -0.6$ (GeV/c)^2 (which can also be explained by the interference between the contribution of the ρ reggeon and the (su) term in (64)). Such a polarization is in agreement with recent data [16] and has not yet been obtained using any other description.

4) The cross section for backward scattering is an order of magnitude smaller than the experimental cross section, although Eq. (64) yields a good behavior as a function of the momentum transfer. All of the defects of (64) obviously show up most fully in the description of backward scattering.

Thus, the VM has had very limited success in describing πN scattering, which is in itself interesting. However, there is as yet no possibility of estimating the extent to which this success is due to duality, since no formula has yet been written down which correctly reflects the observed spectrum of baryon resonances.

c) **The reaction $p + \bar{n} \to \pi_1^+ + \pi_2^+ + \pi^-$ at rest.** The $p\bar{n}$ system in this reaction is in a 0^- state with isotopic spin 1, i.e., this state has the same quantum numbers as the pion but a much larger mass ($\alpha_\pi(m_{\bar{n}p}^2) = 3$ instead of $\alpha_\pi(m_\pi^2) = 0$). Therefore this process cannot be described by Eq. (51), and we must revert to Eq. (37), where $m = k$ in (37) because of crossing symmetry for the pions, the role of s is played by $m_{\pi_1^+\pi^-}^2$, and $t = m_{\pi_2^+\pi^-}^2$. The coefficients $C_{m,k}^p$ in (37) are, generally speaking, determined by the form of the $p\bar{n} \to 3\pi$ amplitude at arbitrary energy, but such formulas for multi-particle reactions encounter their own difficulties (see the next section); it is therefore of interest to see how the experimental data can be described by Eq. (37) with arbitrary coefficients. There have now been many attempts to achieve such a description[42,65-67], which differ from one another in the choice of the $C_{m,k}^p$. In [42] C_{11}^2 is non-zero, while in [65] the best result is obtained with

$$C_{11}^1 = 1, \quad C_{11}^2 = 1{,}89, \quad C_{22}^2 = C_{22}^3 = 0, \quad C_{33}^3 = 0{,}57; \tag{70}$$

the trajectory in this case was taken in the form (57). In [67] allowance was also made for the possibility that the full widths of the daughter resonances are greater (for the best fit, by a factor 3.5) than the widths of the resonances on the leading trajectory. In this case, $C_{11}^1 = 1$, $C_{11}^2 = -5.9$, $C_{22}^2 = -5$, $C_{22}^3 = 2.5 \pm 1.8$, $C_{22}^4 = 5.9$ and $C_{33}^3 = -0.35 \pm 0.12$. Despite this wide scatter in the values of the coefficients, good agreement with experiment is obtained in all of the fits. In Fig. 27 we show the results of [65,67]. We have already discussed the characteristic spectrum of the $\pi^+\pi^+$ system (Sec. 5). We note that good agreement was obtained not only for the mass distribution, but also for the angular spectra (which are extremely sensitive to various small corrections). It is therefore all the more interesting to compare the results of the foregoing analysis with the study of the reaction $p\bar{n} \to 3\pi$, as a five-particle reaction that satisfies the conditions of Fig. 3.

Such an approach was developed in [68] and, despite its limitations, it led to very interesting results. In Fig. 28a we show the necessary notation, as well as the trajectories that were taken into account in this approach. The basic assumptions consisted in taking into account only the π trajectory in the $p\bar{n}$ system and only the Δ trajectory in the πN system. The $p\bar{n} \to 3\pi$ amplitude can then be written in terms of the B_5 functions, which give the solution of the conditions of Fig. 3 (see Appendix II), in the following form [68]:

$$A(s, t) = \beta \Big[\alpha_{12}^\rho B_5 \Big(\alpha_{12}^\rho, \alpha_{23}^\rho - 1, \alpha_{34}^\Delta - \frac{1}{2}, \alpha_{45}^\pi, \alpha_{15}^\Delta - \frac{3}{2} \Big)$$
$$+ c \Big(\alpha_{34}^\Delta - \frac{1}{2} \Big) B_5 \Big(\alpha_{12}^\rho - 1, \alpha_{23}^\rho - 1, \alpha_{34}^\Delta - \frac{1}{2}, \alpha_{45}^\pi, \alpha_{15}^\Delta - \frac{1}{2} \Big) \Big] \bar{u}_p \gamma_5 u_n, \tag{71}$$

where $\alpha_{ik}^\rho = \alpha_\rho(s_{ik})$.

Equation (71) has all the necessary poles in each channel and the correct asymptotic form (but the incorrect behavior $s_{12}^{\alpha_\Delta - 1}$ in the energy s_{12} if $c = 0$). With $s_{45} = (2m_N)^2$, the series (37) is readily obtained from (71); we then find that $C_{mm}^p = 0$ for $p > 4$. If, in addition, we require that (71) has a zero at the center of the Dalitz plot (we have already noted the experimental fact that there is a deep minimum in this region), we obtain the following values for the C_{mm}^p: $C_1^{11} = 1$, $C_{11}^2 = 1.8$, $C_{22}^2 = 0.26$ and $C_{22}^3 = 0$; these values are in essence no different from the system (70) and consequently give a good description of experiment. Moreover, since (71) for definite values of the pair energies s_{ik} ($s_{45} = m_\pi^2$, $s_{12} = m_\rho^2$) is completely determined by the graph of Fig. 28b, for which all the constants are known, it is also possible to determine the normalization factor β in (71), i.e., to predict the cross section for the process $p\bar{n} \to 3\pi$ [69]. The predicted value (18 mb) and the experimental value (10 mb) are in reasonable agreement with each other. The result is not significantly altered if we take the N instead of the Δ trajectory in the πN system.

The entire analysis of this reaction provides an excellent example, and one that is typical of the GVM. On the one hand, there is the ambiguity in the formula of type (71), whose concrete form is determined mainly by considerations of simplicity, with respect to both the choice of the trajectories and the representation of the

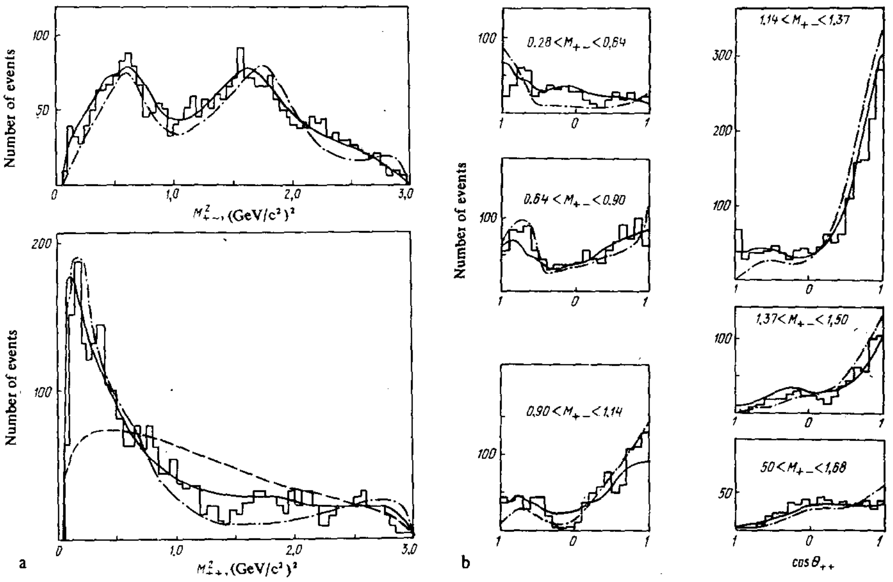


FIG. 27. The mass (a) and angular (b) distributions in the reaction pn → 3π at rest. Solid curves—from [67], dash-dot curves—from [65], dashed curve—phase space.

amplitude in terms of the minimum number of B_5 functions; on the other hand, there is the very good agreement with experiment, which allows us to expect that something significant is expressed in such formulas. In this respect, it is of interest to try to describe this reaction by means of a model which allows for the production of several resonances [70]. With such a model, however, the same good agreement with experiment as in the GVM has been achieved only after allowing for the production of daughter resonances. As we have already noted, the existence of daughter resonances is the most significant feature of the dual approach; consequently, it would be a very strong statement to say that the success of the GVM for this reaction is due to the fact that it correctly reflects the spectrum of observed resonances. In any case, from the study of the reaction $p\bar{n} \to 3\pi$ at rest, we have reason to suppose that this is so.

d) **Multi-particle reactions in the generalized Veneziano model.** We shall consider here in greater detail the reactions

$$a+b \to c+d+e, \tag{72}$$

of which the particular process $p\bar{n} \to 3\pi$ was an example. The description of the reactions (72) is based on the solution of the dual conditions (Fig. 3) given in Appendix II (the B_5 function). An attractive feature of this approach is the fact that the expression for the amplitude for the processes (72) in terms of B_5 functions enables us to write a crossing-symmetric expression of five variables that has the correct poles and asymptotic behavior in each of them.

However, a number of difficulties arise in considering specific reactions.

1) First of all, all the reactions which have been measured experimentally include at least two fermions. But there is no model for fermions at the present time which is free of parity doubling and which correctly allows for the isotopic spin of the nucleons (see part (b) of this section).

2) The general rules for writing dual amplitudes (Appendix II) for multi-particle processes exist only for particles which lie on trajectories with $\alpha(0) < 0$ and with positive G-parity. At the same time, the actual trajec-

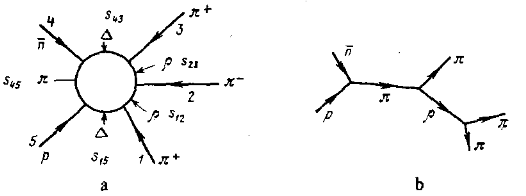


FIG. 28

tories of reggeons (Table I) have either $\alpha(0) > 0$ or negative G-parity (for example, for the π reggeon). The form of the dual amplitude for each process is therefore written on the basis of considerations of simplicity (as was the case in writing (71)).

3) All the formulas of the GVM are written in the zero-width approximation for all the resonances, which is clearly incompatible with experiment. When making a comparison with experiment, one therefore introduces the full widths of the resonances, which are approximated by a rather arbitrary formula (see Fig. 2 and Eq. (57)), and all the resonances of a given mass and different spins have the same width in this approach. This contradicts even the VM itself, since the elastic width of the σ meson for such a parametrization in the VM is found to be greater than its full width (see Fig. 25).

4) For most of the measured processes, a given channel receives a contribution from the vacuum pole, which we cannot take into account in the GVM and whose contribution we must write separately. Therefore, strictly speaking, we should confine ourselves to reactions in which exchange of the Pomeranchuk pole does not contribute (for example, $K^-p \to \pi^+\pi^-\Lambda$). However, there are few such reactions.

5) Two particular particles can usually be the decay products of resonances lying on different trajectories (for example, resonances lying on the N and Δ trajectories decay into πN). Allowance for all the trajectories leads to a considerable complication in the analysis of multi-particle reactions. The analysis is therefore limited to several trajectories (more frequently, a single one), on the basis of some experimental considerations of poor accuracy.

Despite these significant limitations, the application

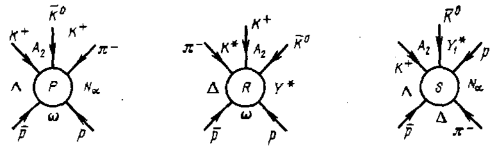


FIG. 29. Dual diagrams contributing to the reactions (73)-(77) and the trajectories which were taken into account in the analysis of [71].

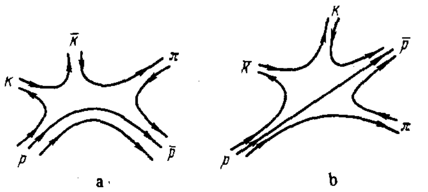


FIG. 30 Quark diagrams for P-(a), and the R(S) diagrams (b) of Fig. 29.

of the GVM to the reactions (72) has yielded surprising results. As an example, we shall consider the processes [71]

$$K^+p \to K^0\pi^+p, \quad (73)$$
$$K^-p \to \bar{K}^0\pi^-p, \quad (74)$$
$$\pi^-p \to K^0K^-p, \quad (75)$$
$$\pi^+p \to \bar{K}^0K^+p, \quad (76)$$
$$p\bar{n} \to K^-K^+\pi^+, \quad (77)$$

which represent different channels of one and the same five-particle amplitude. The contribution of the vacuum pole is small in (73)–(77) (one can apply here the so-called Gribov-Morrison rule [72], according to which such a suppression exists whenever a change in P_r of a particle occurs at one of the vertices ($P_r = (-1)^j P$, where P is the intrinsic parity of the particle and j is its spin)). All the nucleons were regarded as spinless[73]. Of all the various configurations according to the rules of the quark diagrams (part (c) of Sec. 4), only three work for these reactions (Fig. 29; their quark structure is shown in Fig. 30). In Fig. 29 we also show the trajectories whose contributions were taken into account in [71] (where a detailed discussion of the reliability of this choice can be found). Each configuration of Fig. 29 is written in the form (for example, for the P)

$$(P) = \varepsilon_{\mu\nu\delta\sigma} p_\pi{}^\mu p_K^\nu p_{\bar{K}}^\delta p_p^\sigma B_5\left(1-\alpha_{K^*},\ 1-\alpha_{A_2},\ \frac{1}{2}-\alpha_\Lambda,\ 1-\alpha_\omega,\ \frac{1}{2}-\alpha_N\right). \quad (78)$$

Equation (78) has all the poles at the correct masses, and the ϵ factor takes into account the fact that the meson trajectories begin with the p-wave resonances (the baryon spins are neglected).

The relative values of the P, S and R contributions (see Fig. 29) must be R : S = 1 : 1 in order for the exchange of the Δ reggeon to have the correct signature (in analogy with the contribution of V(s, t) + V(u, t) in (51)) and P : S = 1 : 1 in order for the exchange of the N_α reggeon to have the correct signature. Thus, the amplitude for the reactions (73)-(77) in the GVM is written in the form

$$A = \beta\,[(P) + (S) + (R)]. \quad (79)$$

Thus, the five reactions are described by the formula (79) with a single parameter β (which was determined in [71] from the value of the $K^+p \to K^0\pi^+p$ cross section at $p_{K^+} = 5$ GeV/c). This result exhibits the most remarkable feature of the GVM—the fact that it is explicitly crossing-symmetric, which leads to many different consequences. A comparison of (79) with experiment is shown in Figs. 31–33. It is of interest to note that the cross sections for the reactions (73) and (74) are an order of magnitude greater than the cross section for (75), which was also obtained from (79). The discrepancy with experiment by a factor 2 for the cross section for (75) under all the assumptions that were made does not seem surprising, especially if we bear in mind that (79) provides a good description of the way in which this cross section behaves. The other distributions are also in good agreement with the experimental ones (Figs. 32 and 33), not only in describing the contributions of the resonances (their shapes and relative "strengths"), but also in the channels in which there are no resonances (see the Kp mass distribution in Fig. 32). Equally good agreement is found for the other reactions (Table IV).

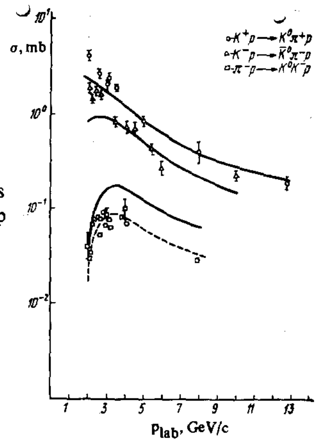


FIG. 31. The total cross sections for the reactions $K^\pm p \to K^0(\bar{K}^0)\pi^\pm p$ and $\pi^-p \to K^0K^-p$ [71]. The dashed curve is the theoretical prediction reduced by a factor of two.

TABLE IV. Mutli-particle reactions which have been analyzed in the GVM

Reaction	Momentum of the incident particle, GeV/c	Reference
$K^+p \to K^0\pi^+p$	2.3—13	71,74
$K^-p \to \bar{K}^0\pi^-p$		
$\pi^-p \to K^0K^-p$		
$K^-p \to \bar{K}^0\pi^-p$	3; 10	75
$K^-p \to K^-\pi^+n$	10	76
$K\pm p \to K\pm\pi^0p$	1—10	77
$K^+p \to K^+p\omega$	4; 6; 9;	78
$K^-p \to \bar{K}^0\pi^-p$	3,3	79
$K^-n \to \pi^-\pi^0\Lambda$	3	80
$\pi^-p \to \pi^-K^+\Lambda$	3—6	
$\pi^-p \to \pi^0K^0\Lambda$	3—6	
$K^-n \to K^{*-}\pi^-p$	4; 5	81
$K^-n \to \bar{K}^0\pi^-n$	3	82
$K^+n \to K^0\pi^+n$	3	
$K^-p \to \pi^+\pi^-\Lambda$	3—10	73
$K^-p \to K^{*-}\pi^+n$		83

Consequently, in spite of the various general assumptions and the artificiality of the approach that has been developed, the GVM has been surprisingly successful in describing multi-particle processes; this is apparently due mainly to the explicit allowance for crossing symmetry in the GVM (which was not previously possible).

e) **Diffraction dissociation reactions.** In reactions of the type

$$NN \to N\,(\pi N), \quad (80)$$
$$\pi N \to \pi\,(\pi N), \quad (81)$$
$$KN \to K\,(\pi N), \quad (82)$$
$$\pi N \to (3\pi)\,N, \quad (83)$$
$$\gamma N \to (2\pi)\,N, \quad (84)$$
$$\gamma N \to (K\bar{K})\,N, \quad (85)$$
$$KN \to (K2\pi)\,N \quad (86)$$

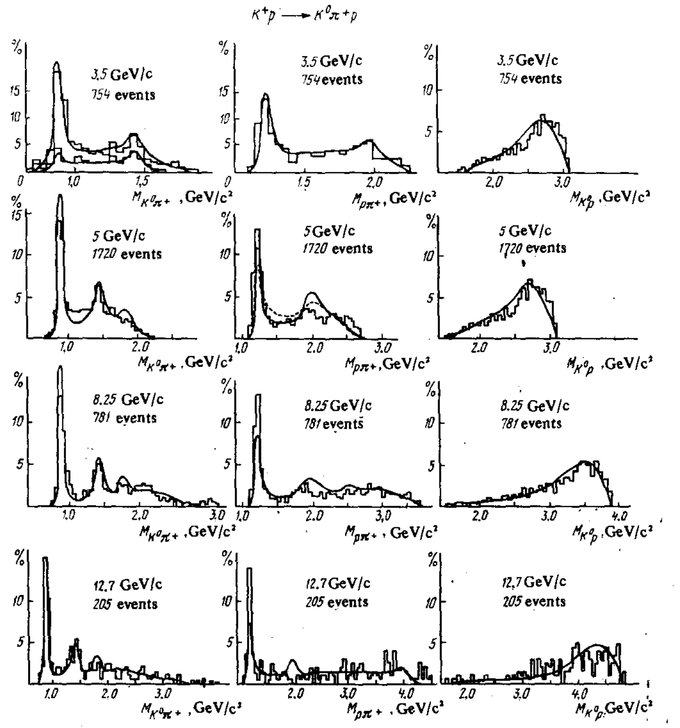


FIG. 32. Mass spectra for the reaction $K^+p \to K^0\pi^+p$ [71].

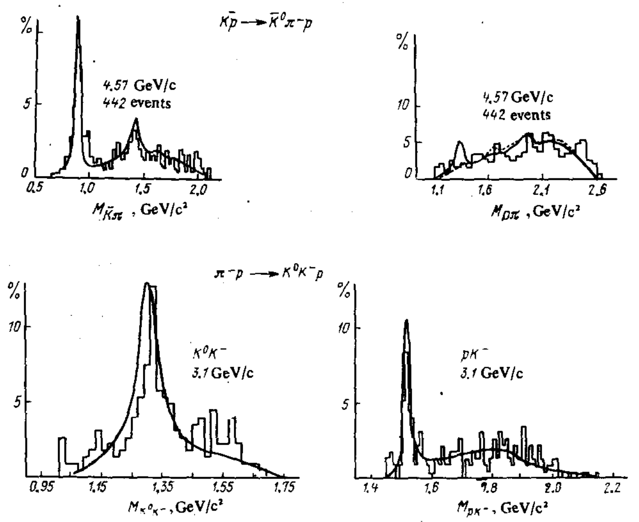


FIG. 33. Mass spectra for the reaction $K^+p \to \bar{K}^0\pi^-p$ and $\pi^-p \to K^0K^-p$ [71].

at high energies, exchange of the vacuum reggeon is important, so that duality arguments are not applicable to these processes. However, in [84] a model was proposed which gives a good description of the characteristic features of these processes; the idea of [84] consists in representing the amplitude for the reaction

$$a + b \to a + (d + c) \tag{87}$$

in the form (Fig. 34)

$$A(s,\ t) = g(t_{aa})\, s^{\alpha_P(t_{aa})}\, V_{Pb\to cd}\,\eta(t_{aa}), \tag{88}$$

where $s = (p_a + p_c)^2$, and $V_{Pb\to cd}$ is given by a Veneziano formula for the amplitude of the hypothetical reaction $P + b \to c + d$ (P is the vacuum reggeon, which is regarded as a scalar particle); for example, for (85)

$$V_{P\gamma\to\pi^+\pi^-} = [B_4(1 - \alpha_\rho^{\bar{s}},\ -\alpha_\pi^t) + (\bar{s} \to u)], \tag{89}$$

where $\bar{s} = (p_d + p_c)^2$ and $t = (p_b - p_d)^2$.

Equation (88) differs from that used in [84-87]. It allows for the fact that the vacuum reggeon moves as t_{aa} varies and, in addition, it has the correct asymptotic form for large s and $\bar{s}$. The form of (88) is not uniquely determined; in particular, we can write $s^{\alpha(t_{aa})}$, where $s = (p_a + p_b)^2$, and make the substitution $\alpha_\pi^t \to \alpha_\pi^t - \alpha_P^t$ in (89). However, these modifications do not show up strongly in the comparison with experiment. Eq. (88) has been applied to the description of the reactions (80)[84], (81)[85], (85)[85], (86)[87] and (84)[88] (in the case of the last reaction, V was replaced by B_5), and extremely encouraging results have been obtained (Figs. 35–38). In this connection, the following points are of interest:

1) The good description of the shape of the ρ peak in (84) (its asymmetry) (Fig. 37).

2) All other resonances (apart from the ρ) are suppressed in the reaction (84), and this may provide an explanation of the fact that one does not see in this reaction the $\rho'(1^-$, 1300 MeV), which is required in the $\pi\pi$ system according to the VM (Fig. 37).

3) The distribution in t_{aa} as a function of the mass of the system (cd) (Fig. 38). In particular, if this dependence is approximated by $d\sigma/dt = e^{bt}$, then b for the reaction (80) is equal to 11 $(\mathrm{GeV}/c)^{-2}$ for small πN masses and 5 $(\mathrm{GeV}/c)^{-2}$ for large masses. Such a dependence is easy to understand, since V (Fig. 34) depends on t_{pp} only because of the condition $\bar{s} + t + \bar{u} = t_{pp} + m_\pi^2 + 2m_N^2$; therefore V is in general independent of t_{pp} at large $\bar{s}$, and the entire dependence is determined by $g(t_{aa})$ (Fig. 34). The fact that the slope for large masses is equal to half the slope for elastic pp scattering ($b \approx 10$ $(\mathrm{GeV}/c)^{-2}$ in pp) is an argument in favor of these considerations. At

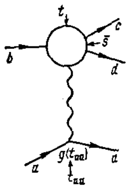


FIG. 34

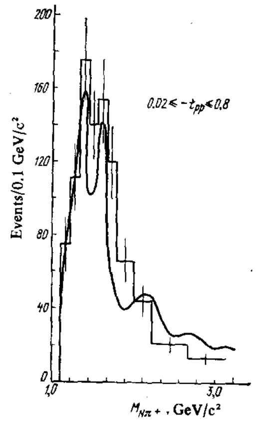


FIG. 35

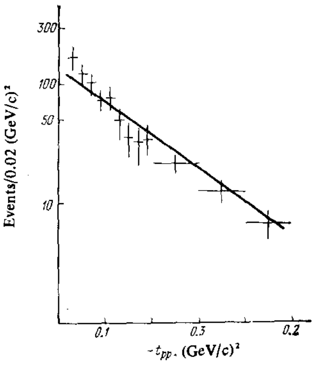


FIG. 36

FIG. 35. Mass distribution of the (π^+N) system in the reaction $pp \to p(N\pi^+)$ [84].

FIG. 36. Distribution in the momentum transfer from the proton to the proton in $pp \to p(N\pi^+)$ [84].

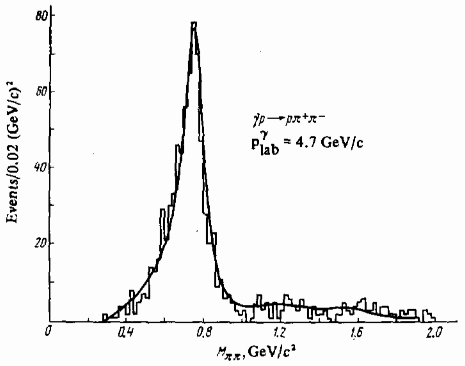


FIG. 37. Distribution in the $(\pi^+\pi^-)$ mass in the reaction $\gamma p \to p(\pi^+ + \pi^-)$ [85].

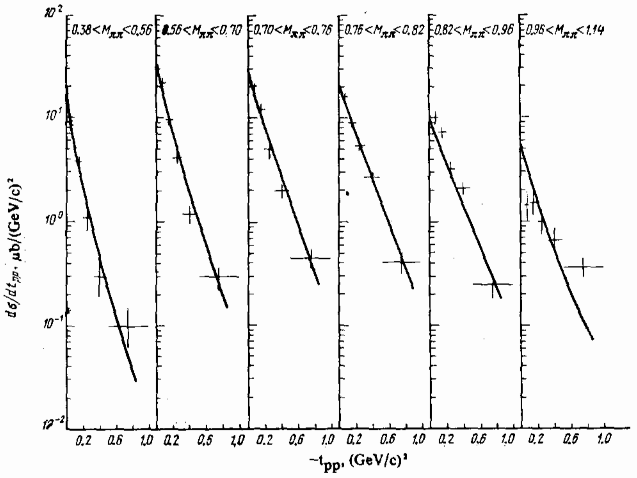


FIG. 38. Distribution in the momentum transfer from the proton to the proton for various ranges of the $\pi^+\pi^-$ mass in the reaction $\gamma p \to p\pi^+\pi^-$.

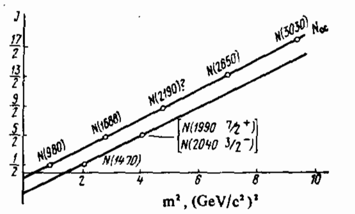


FIG. 39. The N_α trajectory and its daughter trajectories. The resonances are taken from [3]; ? indicates those for which the quantum numbers differ from [3].

small $\bar{s}$, there is a dependence of V on t_{aa}, which yields an increase in the slope.

4) Equation (88) explains the well-known experimental fact that the reaction (80) at small t_{pp} involves mainly the production of the resonance N $(1/2^+, 1470)$, while the contribution of the resonance N $(3/2^-, 1550)$ becomes larger with increasing t_{pp}; this is so because, in the model in question, the production amplitude of the N $(1/2^+, 1470)$ falls off more rapidly with increasing t_{pp} than the production amplitude of the N $(3/2^-, 1550)$, whose mass is larger. We note that the N $(1/2^+, 1470)$ in the VM is a daughter resonance to the N_α trajectory (Fig. 39), i.e., we see in this reaction a certain manifestation of the existence of daughter resonances.

Thus, (88) gives a good description of experiment, and this success gives us reason to hope that the allowance for the vacuum pole in the duality scheme may also be successful; in particular, one can attempt to write an

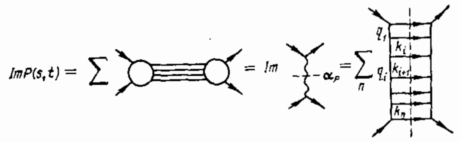


FIG. 40

expression in the GVM for the amplitude for scattering of vacuum reggeons by a particle (N in Fig. 16, where one also takes the P instead of the ρ)[89] and thus determine corrections to the pure pole graph for elastic scattering, provided that further attempts to apply (88) are justified.

7. THE VACUUM POLE IN DUAL MODELS

So far we have subtracted the contribution of the vacuum pole and represented the physical amplitude in the form (see Sec. 2)

$$A(s,t) = A_D(s,t) + P(s,t). \qquad (90)$$

Can we say anything more definite about the contribution of the vacuum reggeon, P(s, t)? As we have already noted (Sec. 2), P(s, t) cannot be associated entirely with the resonances in the direct channel (provided, of course, that resonances are assumed to be absent in exotic channels). In [90] it was conjectured that the contribution of the Pomeranchuk pole is determined by the background in the s-channel of the reaction, i.e., that the duality equations for the vacuum pole have the form (Fig. 40)

$$\operatorname{Im} A_{\text{back}} = \operatorname{Im} P(s,t). \qquad (91)$$

This idea is in good agreement with the currently accepted opinion that the constant cross section at high energies is determined by processes of the so-called "ladder" type, which are kinematically distinguished from other processes by the fact that all $q_i^2 \approx m^2$, while the k_{iL} (the longitudinal momenta of the produced particles) form a geometric progression $k_{(i+1)L} = \lambda k_{iL}$ [91]. Eq. (91) is confirmed experimentally by the behavior of the combination f^0_{jl} of the partial-wave amplitudes for πN scattering (where the t-channel isotopic spin is equal to 0). It is readily seen from Fig. 10 that the non-resonant contribution is very important in this combination and is predominantly imaginary (we recall that the vacuum reggeon gives a purely imaginary contribution at t = 0). This fact is especially conspicuous when compared with f^1_{jl}, which has no contribution from the vacuum reggeon and which describes closed loops on the Argand diagram (see Fig. 10).

This relationship between the non-resonant background and the contribution of the Pomeranchuk pole enables us to make certain predictions about the spin structure of the vacuum reggeon on the basis of a phase-shift analysis at comparatively low energies. In Fig. 41 we show the partial-wave amplitudes with isotopic spin zero in the s-channel, giving both those that conserve the s-channel helicity (F^0_{++}) and those that change it (F^0_{+-}) [92]. It is clear that the background contribution in F^0_{+-} is comparatively small. We can therefore conclude from (91) that the vacuum pole conserves helicity in the s-channel (at high energy, exchange of the vacuum reggeon contributes only to the invariant function B in (64) for πN scattering). We have already successfully made use of this property of the vacuum pole in discussing the behavior of the polarization in Sec. 4. However, it should be noted that this conclusion depends strongly on which phase-shift analysis we employ. In

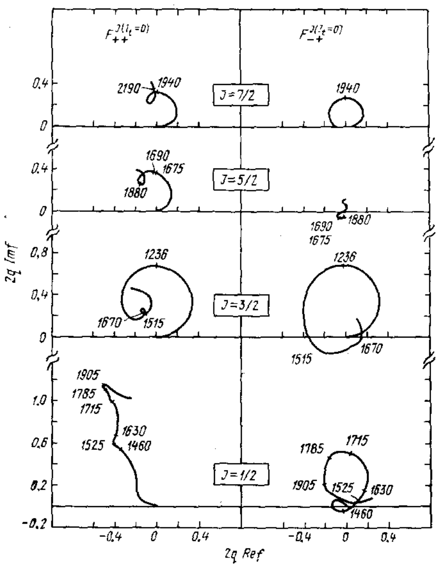


FIG. 41. The Argand diagrams for F^0_{++} and F^0_{+-} [92].

particular, two phase-shift analyses yield completely contradictory results for KN scattering [16]. In addition to the polarization, the spin rotation parameters have now been measured at 6 GeV/c, which makes it possible to determine F^0_{++} and F^0_{+-} at this energy. As we have already noted, the contribution of the nonvacuum poles must reduce to zero in the region $t \sim -0.6\ (\mathrm{GeV/c})^2$; therefore the ratio [93]

$$\frac{F^{(0)}_{++}}{F^{(0)}_{+-}} = -0{,}1 \pm 0{,}1$$

is characteristic of the vacuum pole and is evidence for helicity conservation. Thus, the relationship (91) has proved to be extremely useful. However, (91) in its literal form contradicts crossing symmetry [94]. In fact, in order to correctly take into account all the symmetry properties in $\pi\pi$ scattering, V(s, t) in (51) must be replaced by V(s, t) + P(s, t); we then see at once that the background contributes to the amplitude with isotopic spin 1 in the t-channel ($A^1_t \sim P(s, t) - P(u, t)$), which has no contribution from the vacuum pole. Therefore the background cannot be dual to the vacuum pole alone. In connection with this, there arises the attractive idea that the vacuum pole will appear in the dual theory when the GVM is unitarized. As we have already indicated, we are regarding the GVM as a first approximation to the physical amplitude, supposing that this approximation is a good one in the sense that all the unitarity corrections are small, i.e., that we can imagine the physical amplitude in the form of a series in a small parameter g^2, with the Born term of this series corresponding to the GVM:

$$A(s, t) = A_D(s, t) + g^2 A_1 + g^4 A_2 + \ldots, \qquad (92)$$

where the resonances are contained entirely in A_D (it is obvious that near a resonance the series (92) is meaningless and must be summed with respect to g^2 in order to obtain the correct resonance width). Processes which are not associated with single resonance production (background processes) are already contained in A_1, and the idea of obtaining the vacuum pole consists in the fact that A_1 contains a term that grows like s with the correct vacuum numbers. The importance of this contribution in the comparison with experiment is due to the fact that it is comparable with the first term of (92) at certain energies, i.e., that the series expansion (92) actually goes not in g^2, but rather in $g^2 s^{1/2}$. What grounds do we have for believing that the expansion (92) actually exists, and what is the mechanism by which the vacuum pole is produced?

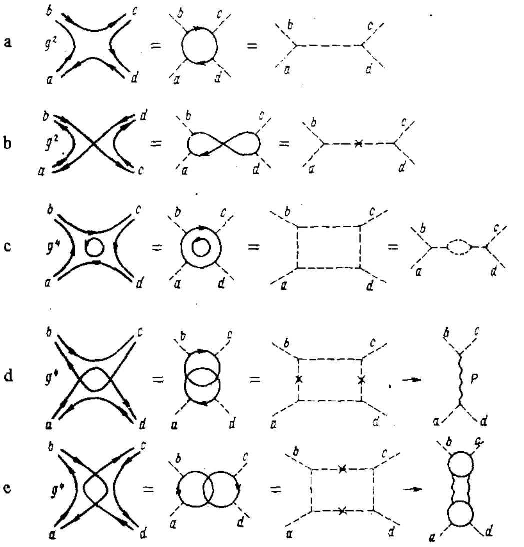


FIG. 42. Quark diagrams for the four-point diagram in the g^2 and g^4 approximations. Solid lines–quarks, dashed lines–mesons.

a) Let us first of all estimate the parameter g^2, assuming that the expansion (92) exists. All estimates must clearly be made at $s \sim 1$ GeV, since the entire dependence on s in dual models enters through $\alpha(s)$ ($\alpha(s) \sim 1$ at $s = 1$ GeV).

1) The Regge trajectories are linear as $g^2 \to 0$, so that $\mathrm{Im}\,\alpha(s)/\mathrm{Re}\,\alpha(s) \sim g^2$ for $s \sim 1$ GeV. This ratio (see (57) and Fig. 2) has the value 0.1–0.3.

2) $\sigma(K^+p)/[\sigma(K^-p) - \sigma(K^+p)] \approx g^2$ (this ratio is the ratio of the vacuum pole contribution to the resonance contribution). Near threshold,

$$\sigma(K^-p) \approx 100\ \mathrm{mb},\quad \sigma(K^+p) \approx 20\ \mathrm{mb},\ \text{i.e.,}\ g^2 \sim 0.25.$$

3) $$\frac{d\sigma(K^-p \to pK^-)}{dt} : \frac{d\sigma(K^+p \to pK^+)}{dt} \approx g^4,$$

since there are no t-channel resonances for the process $K^-p \to pK^-$. The experimental value of this ratio at $p_K = 5$ GeV/c is [12] $\sim 10^{-2}$, i.e., $g^2 \sim 0.1$. An analogous suppression exists for other processes due to exotic exchange (of the type $K^-p \to K^+\Xi^-$ or $K^-p \to K^0\Xi^0$ in the forward direction). Thus, all the estimates yield $g^2 \to 0.2$–0.3, so that the contribution of the vacuum pole can be regarded as a second-order quantity in g^2 (see the second estimate and its agreement with the others).

b) In addition, when the series (92) is constructed (for further details, see [95]) there appear contributions of order g^2 which correspond to the vacuum quantum numbers in the t-channel. In order to see how this happens, let us consider the dual diagrams for the scattering of particles with isotopic spin unity (Fig. 42). The dual amplitude corresponds to the diagrams a and b of Fig. 42 (Fig. 42a corresponds to the (s, t) term, and Fig. 42b to the (s, u) term). The isotopic structure of these graphs is given by a factor $\mathrm{Tr}\,(\tau_a\tau_b\tau_c\tau_d)$ for Fig. 42a and $\mathrm{Tr}\,(\tau_a\tau_b\tau_d\tau_c)$ for Fig. 42b (where the τ are the isotopic matrices) (this form corresponds to the rules of [96]). Corrections to the dual amplitude correspond to graphs containing four quark lines (see c–e of Fig. 42). The diagram of Fig. 42c repeats the isotopic structure of the dual amplitude, and its inclusion only leads to the appearance of resonance widths [97]; Fig. 42d has the isotopic structure $\mathrm{Tr}(\tau_b\tau_c)\mathrm{Tr}(\tau_a\tau_d)$, which corresponds to isotopic spin zero in the t-channel, while Fig. 42e has the structure $\mathrm{Tr}(\tau_a\tau_b)\mathrm{Tr}(\tau_c\tau_d)$, corresponding to isotopic spin zero in the s-channel. Thus, graphs like Fig. 42d correspond to the vacuum quantum numbers of the t-channel and may yield a vacuum pole. Moreover, when one considers concretely such corrections in a theory with single vacuum particles (Appendix II), graphs like Fig. 42d actually exhibit a singularity in the angular momentum plane (a cut), starting from the point $j = (1/3) + (1/2)bt$, where b is the slope of the non-vacuum trajectories (which leads to a behavior $s^{-2/3}/(\ln s)^n$ of the total cross section). However, this singularity does not correspond to a constant total cross section; in addition (and this is the main difficulty in understanding this singularity), its discontinuity is not expressed in terms of real intermediate states, so that its existence violates the unitarity condition. It seems to us that nothing is gained by hopes that allowance for the actual quantum numbers will move this singularity to $j = 1$ and make it a pole, since there is no hope that this will advance our understanding of the relation of this singularity to the real intermediate states. Its appearance is apparently due to the defects of our Born approximation. The point is that the number of resonances with a given mass (M) is enormous ($\sim\exp(C\sqrt{M^2})$) in the GVM, while all the partial-wave amplitudes in a definite multi-particle diagram are of order unity at high energies, since each resonance has a very small probability ($\sim\exp(-a\sqrt{M^2})$) of decaying into a given number of particles. In graphs like c–e of Fig. 42, however, a resonance can decay into an arbitrarily large number of particles (with a significant probability) at a given arbitrarily large energy, so that the contribution of such states to a partial wave is large (proportional to the number of resonances). This leads to a growth of the imaginary part of the amplitude at high energy, which in turn leads to an exponential divergence like that of Fig. 42c and the appearance of a singularity like that of Fig. 42d. While the first can be eliminated by renormalizing the trajectories of the non-vacuum reggeons [97], it is completely unclear what to do with the vacuum singularity, and this is the main difficulty of the theoretical approach.

8. INCLUSIVE REACTIONS

Great interest has recently arisen in so-called "inclusive" reactions, i.e., reactions of the type

$$a + b \rightarrow c + M, \tag{93}$$

where M denotes an arbitrary number of particles that are not detected experimentally. The differential cross section for the reaction (93) can be related to the discontinuity of the amplitude for the process[98]

$$a + b + \bar{c} \rightarrow a + b + \bar{c}. \tag{94}$$

This relation is shown in Fig. 43 for high energies, when the main contribution to the total cross section comes from processes of the ladder type. It is convenient to characterize the reaction (93) by the variable

$$x = \frac{p_{cL}}{p_{cL\,\max}}\ up_{c\perp}, \tag{95}$$

where p_{cL} and $p_{c\perp}$ are the components of the momentum of particle c parallel and perpendicular to the momentum of the incident particle a in the rest system of b.

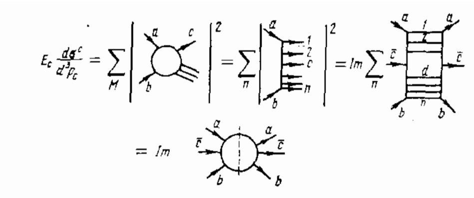


FIG. 43

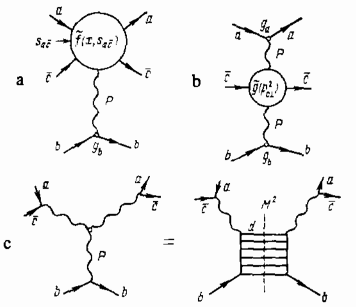


FIG. 44. The single-reggeon (a), double-reggeon (b) and triple-reggeon (c) regimes.

In terms of (95), the invariant variables have the form (for large s)

$$\left.\begin{aligned} s &= (p_a + p_b)^2, \\ s_{a\bar{c}} &= (p_a + p_c)^2 = m_a^2(1 - x) + m_c^2\left(1 - \frac{1}{x}\right) - \frac{p^2_{c\perp}}{x} + O\left(\frac{1}{s}\right), \\ s_{b\bar{s}} &= (p_b - p_c)^2 = xs + O\left(\frac{1}{\sqrt{s}}\right), \\ M^2 &= (p_a + p_b - p_c)^2 = (1 - x)s + O\left(\frac{1}{\sqrt{s}}\right). \end{aligned}\right\} \tag{96}$$

At high energies, it is easy to distinguish three kinematic regions of the reaction (93)[99,100].

a) **Single-reggeon exchange**: $s_{a\bar{c}}$ is fixed (i.e., x is fixed) and M^2/s is fixed.

In this region, the cross section for (93) is described by the diagram of Fig. 44a (which is easy to understand if we bear in mind Figs. 42d and 40) and has the form

$$E_c\frac{d\sigma^c}{dp_{cL}d^2p_{c\perp}} = \tilde{f}(s_{a\bar{c}}, x)\,g_b = f(s_{a\bar{c}}, x). \tag{97}$$

Thus, at high energy the cross section is independent of this energy [99]; such a distribution is in good agreement with the current experimental data [101].

b) **Two-reggeon exchange**:

$$s_{a\bar{c}},\ s_{b\bar{c}} \rightarrow -\infty,$$

and

$$s_{a\bar{c}}s_{b\bar{c}}/s = p^2_{c\perp} + m_c^2$$

is constant.

The diagram of Fig. 44b contributes in this region, and

$$E_c\frac{d\sigma^c}{dp_{cL}d^2p_{c\perp}} = \tilde{g}(p_{c\perp})\,g_a g_b = g(p_{c\perp}). \tag{98}$$

c) **Three-reggeon exchange**: $s/M^2 \rightarrow +\infty$, $M^2 \rightarrow +\infty$. This implies that the energy $s_{\bar{c}d}$ is large, i.e., that a reggeon must pass between $\bar{c}$ and d (see Fig. 44c). It is natural to write the contribution of this region in the form (see Fig. 44c)

$$E_c \frac{d\sigma^c}{dp_{cL}\, d^2p_{c\perp}} = (s-M^2)^{2\alpha(s_{a\bar{c}})} (M^2)^{\alpha(0)-2\alpha(s_{a\bar{c}})} h(s_{a\bar{c}})$$

$$= x^{2\alpha(s_{a\bar{c}})}(1-x)^{1-2\alpha(s_{a\bar{c}})} h(s_{a\bar{c}}) \quad (\text{for}\, \alpha(0)=1). \tag{99}$$

The last two regions may of course be regarded as special limits of the first, and in this sense (99) is an example of the x-dependence of the function $f(x, s_{a\bar{c}})$ in (97).

Dual models attempt to answer the two questions: 1) how rapidly the limiting regime (97) sets in (i.e., what are the energy-dependent corrections to (97)), and 2) what are the unknown functions f, g and h in (97)–(99).

Corrections to (97) which behave like powers in s appear as a result of the exchange of any reggeon, but not of the Pomeranchuk pole in Fig. 44a. For example, they may be due to the exchange of the f reggeon. In this case,

$$E_c \frac{d\sigma^c}{d^3p_c} = f(x, s_{a\bar{c}}) + \varphi(x, s_{a\bar{c}}) s^{\alpha_f(0)-1}. \tag{100}$$

Can one choose reactions for which φ is small and for which the limiting regime is reached at comparatively low energies? To answer this question, let us consider the dual graphs corresponding to the reaction (94) (Fig. 45; the powers of g^2 characterize the lowest orders in which these graphs appear). It is clear from Fig. 45 that the main contribution to f comes from the diagram of Fig. 45b (where we are naturally assuming that the vacuum pole in the dual models is described by the configuration of Fig. 42d, and that this graph guarantees the constancy of the cross sections at high energies). The main corrections are determined by the diagram of Fig. 45a (of course, the other diagrams also contribute to $\varphi(x, s_{a\bar{c}})$ in (100), but they have a small value of order g^2). Consequently, if we consider the reaction (93), in which the channel a + b + c is exotic (for example, $pp \rightarrow \pi + M^{2)}$), Fig. 45a does not contribute, so that φ will be small [102–104]. In Fig. 46 we show the cross section for the reaction at $P_{K^+} = 5$ and 8 GeV/c. It is clear that the regime (97) is a good description at these relatively low energies [106]. This fact constitutes yet another argument for the existence of a small parameter g^2, since the corrections to this reaction like the graph of Fig. 45c contribute to $\varphi(x, s_{a\bar{c}})$. Moreover, on the basis of the diagrams of Fig. 45, it is easy to estimate the parameter g^2 from the inclusive experimental data [104].

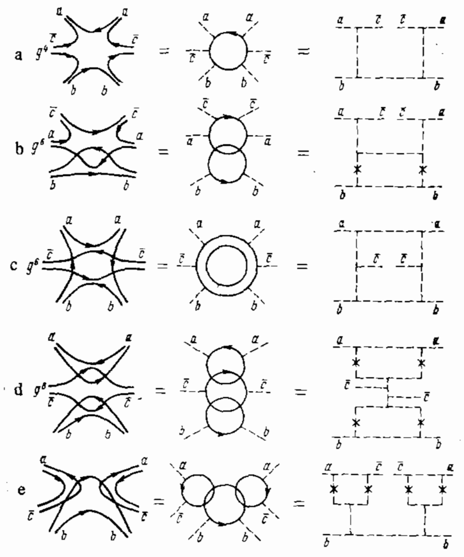


FIG. 45. Quark diagrams for the process a + b + $\bar{c}$ → a + b + $\bar{c}$. The powers of g^2 to the left of the diagrams correspond to the lowest order in which they appear.

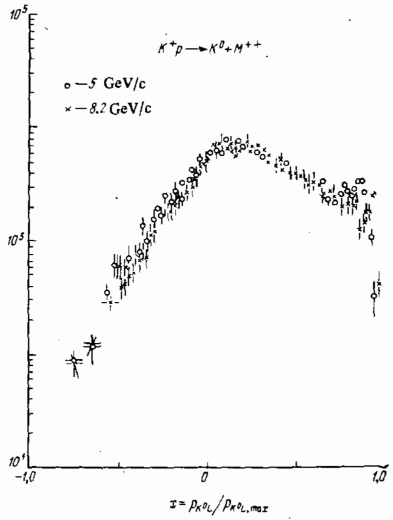


FIG. 46. Cross section for the reaction $K^+p \rightarrow K^0M^{++}$ at $P_{K^+} = 5$ and 8.2 GeV/c as a function of $x = P_{K^0L}/P_{K^0L,max}$ [105].

1) Consider the reactions [106]

$$K^+p \rightarrow \pi^- + M^{+++} \quad \text{at 12.7 GeV/c}, \tag{101}$$

$$K^-p \rightarrow \pi^- + M^+ \quad \text{at 9 GeV/c}, \tag{102}$$

$$\pi^+p \rightarrow \pi^- + M^{++++} \quad \text{at 7 GeV/c}, \tag{103}$$

$$\pi^-p \rightarrow \pi^- + M^+ \quad \text{at 25 GeV/c}. \tag{104}$$

For the quantities

$$f^*_{K\pi} = \frac{E_c\, d\sigma^c/d^3p_c}{[\sigma(K(\pi)p)]_\infty}$$

we have $f^{+}_{\pi^-} - f^*_{\pi^+} \sim g^2$ and $f^*_{K^-} - f^*_{K^+} \sim g^2$, while $f^*_{\pi^+} - f^*_{K^+} \sim g^4$, since the channel $a + b + \bar{c}$ is exotic for the reactions (101) and (103). In fact, the first two differences are experimentally of the same order of magnitude, and an order of magnitude greater than the third[104], i.e., $g^2 \sim 0.1$.

2) Consider the processes

$$p + p \rightarrow K^+ + M^+, \tag{105}$$

$$p + p \rightarrow K^- + M^{+++}. \tag{106}$$

The channel $a + b + \bar{c}$ is exotic in (105) and (106); moreover, the channels $a\bar{c}(pK^+)$ and $b\bar{c}(pK^+)$ are exotic in (106), so that the reaction (105) is determined by the diagram of Fig. 45b (which is of order g^6), while the second reaction is determined only by the diagram of Fig. 45d (g^8). Experimentally, the cross section (for $-0.5 < x < 0.2$ in the c.m.s., with $p_\perp < 0.70$ GeV/c) is larger by a factor 5–300 for the reaction (105) than for (106), i.e., $g^2 \sim 0.1$ [104].

Let us now consider certain characteristic features of the functions f, g and h in dual models. First of all, the main contribution in the two-reggeon region comes from the diagram of Fig. 45d (g^8), so that $d\sigma^c$ in this region is smaller than the cross section in the single-reggeon region by a factor g^2 ($g^2 \sim 0.1$). It is also of interest to note that there is no contribution in the triple-reggeon region from a diagram like that of Fig. 44c with three vacuum reggeons (which does not appear in Fig. 45); if it is assumed that the summation of the graphs in g^2 leads mainly to a change in the trajectories, this contribution is in general absent. This contribution must tend to zero as $s_{a\bar{c}} \to 0$ with increasing energy, in order to ensure the constancy of the total cross section[107]. Therefore $\alpha(s_{a\bar{c}})$ in (99) is a non-vacuum trajectory, which shows up in the character of the behavior of $f(x, s_{a\bar{c}})$ as $x \to 1$. In particular, the cross sections for reactions with an exotic channel $a\bar{c}$ (such as $\pi^+p \to \pi^- + M^{+++}$) must fall off rapidly as $x \to 1$, in good agreement with experiment[106]. The more detailed behavior of the function $f(x, p_\perp^2)$ from the diagram of Fig. 45b was considered in[103,108], and its main properties are as follows (for $pp \to \pi^+ + M^+$):

a) at $p_{cL}^2 = 0$, the distribution in x is close to $\exp(-cx^2)$ with $c = 7.4$;

b) if x is not small, then $f(x, p_\perp^2) = f_1(x)f_2(p_\perp^2)$;

c) the dependence on $p_\perp^2$ is very rapid, particularly for small x (for example, at $x = 0.06$ and $p_\perp^2 = 0.9$ $(\text{GeV}/c)^2$, the cross section becomes two orders of magnitude smaller than the experimental cross section, agreeing with it at $p_\perp^2 = 0$);

d) $g(p_\perp^2)$ at large $p_\perp^2$ is given by $\sim\exp(-4bp_\perp^2)$[108]. This fact is independent of any details of the model, and a comparison with experiment may be of interest. Everything else has a more approximate character, since it is not clear how to deal with the vacuum pole in the duality scheme.

9. CONCLUSIONS

Thus, we see that dual models have led to many results that are in good agreement with experiment. We have exchange degeneracy (part (a) of sec. 4), the connection with symmetries (part (b) of Sec. 4), and simple formulas for multi-particle reactions, which depend explicitly on the Regge trajectories that determine both the spectrum of particles and high-energy scattering, which make it possible to take into account analyticity and crossing symmetry, and which seem to give a correct description of the contributions of the various resonances and even the background in many reactions (Secs. 6 and 8), as well as hopes that there has appeared a small parameter in the theory of strong interactions (Secs. 7 and 8)—all these features constitute the attractive side of dual models and ensure continuing interest in them.

However, to be fair, we must admit the following points:

1. Not a single daughter trajectory has been found experimentally, which means that there has been no confirmation of a fundamental requirement of duality: the degeneracy of resonances with the same mass with respect to the spin. Indirect observations of the spectrum of resonances have merely been compatible with the idea of duality, but have not confirmed it (Secs. 3 and 5).

2. The dual approach is incompatible with the current experimental data on baryon-baryon scattering (part (c) of Sec. 4).

3. There are no ideas which could reconcile straight-line trajectories with the absence of parity degeneracy in the spectrum of baryon resonances (part (b) of Sec. 6).

4. There are no formulas that take into account the actual quantum numbers of the particles and the actual Regge trajectories in multi-particle processes, and attempts to write such formulas encounter serious difficulties.

5. All the agreement with experiment is obtained only after the further introduction of the experimental resonance widths in the formulas (therefore it may be meaningful to take an expression which directly incorporates the resonance widths as the first approximation[109], although such formulas have an unlimited ambiguity).

6. Qualitative confirmations of the formulas of dual models are very meager (Sec. 5).

7. Dual models give a good description of the magnitudes of the contributions of the various resonances and even the background in a rather large number of reactions (see Sec. 6 and Table IV), but in a more concrete approach we find a large number of discrepancies between the simple formulas of dual models and the experimental data. For example, there is the spectrum of baryon resonances in SU_3 (part (b) of Sec. 4), the existence of the ρ' $(1^-, 1300)$ in the scattering of pions (part (a) of Sec. 6), and the impossibility of obtaining the correct value of the $\pi N \to N\pi$ backward scattering cross section (part (b) of Sec. 6). Despite the fact that all these difficulties can be explained in quite a reasonable way and eliminated by making the formulas of the GVM more complicated, there is still the impression that dual models somehow give very much of an average description of experiment, although it is possible that all this is the result of shortcomings in the concrete realizations of the ideas of duality.

8. We encountered difficulties in understanding the significance of the vacuum singularity, even in the purely theoretical discussion.

All this compels us to treat the successes of duality with great caution. In this respect, one would like to decide what experiments are needed to ascertain whether duality exists or not. Unfortunately, it is very difficult to give any unambiguous answer to this question, mainly because the theoretical construction of dual models is at an extremely rudimentary stage. The nature of the corrections to dual models is completely unknown at the present time, and it is therefore not clear how the predictions of duality will be modified when these corrections are taken into account. It is obvious only that "corrections" such as the contribution of the vacuum reggeon and the full resonance widths play a far from minor role in the comparison with experiment. It is therefore significant to enumerate those results of the dual approach which are independent of the concrete formulas of dual models and which will probably be modified only slightly when non-dual corrections are taken into account.

1. In dual models: a) there must exist resonances of arbitrarily large mass; b) there must be many resonances of the same mass but with different spins; and c) the decay widths of resonances in a given channel must fall off rapidly with increasing mass (Sec. 3). Such a spectrum of resonances is the basis of duality, and the detection of resonances lying on daughter trajectories is of the utmost importance for dual models. Since, as we have already discussed in Sec. 3, it is quite possible that the full widths of the resonances grow with increasing mass and become greater than the mass difference of neighboring resonances, it is obvious that efforts in this field should be centered on the search for resonances

of relatively low mass (for example, $m < 2.5$ GeV) and the determination of their quantum numbers. In particular, it would be very interesting to carry out a phase-shift analysis of πN scattering in which all the required resonances are assumed. It would be a real success of dual models to detect a $\rho'(1^-)$ meson of mass between 1.1 and 1.5 GeV, having a full width of up to 250 MeV and an elastic width in the $\pi\pi$ system of 25 to 110 MeV (see part (a) of Sec. 6 and [110]). Since the production of this resonance in various binary reactions may be suppressed (part (e) of Sec. 6), the best reaction in which to seek it is evidently $e^+e^- \rightarrow \rho' \rightarrow (\bar{n}\pi)$.

2. The general requirements of duality imply exchange degeneracy of the non-vacuum Regge poles, which leads to a completely concrete parametrization of the Regge pole residues. At the present time, the complete set of all high-energy scattering data evidently favors just such a parametrization (part (a) of Sec. 4 and [111]). However, the accumulation of data on polarization in various reactions, the study of the minima in the differential cross sections for processes such as $\pi^-p \rightarrow A_2^0 n$, and the measurement of the cross sections for reactions like $K^+n \rightarrow K^0 p$ and $K^-p \rightarrow \bar{K}^0 n$ (see (20)) at higher energies would be of decisive importance in establishing the presence of exchange degeneracy.

3. As already discussed in detail in Sec. 5, the most characteristic requirement of the Veneziano model is the occurrence of minima in the differential cross sections for binary reactions and on the Dalitz plot for reactions of the type $a + b \rightarrow c + d + e$ under the condition (50). In some reactions, such as $\pi N \rightarrow \pi N$ and $K^-p \rightarrow \bar{K}^0 n$, minima are found at the required values of the variables (for $K^-p \rightarrow \bar{K}^0 n$, for example, they should occur at $u = -0.6, -1.6$ and -2.6 $(\text{GeV}/c)^2$ and are observed at $u = -0.1, -0.7$, and -1.7 $(\text{GeV}/c)^2$), while in others, such as $\bar{K}N \rightarrow \pi\Lambda$ and $\pi^-p \rightarrow K^0\Lambda$, there are no minima at fixed t and s, but they are found for some reason at fixed $s-t$ [43,112]. The situation is also not clear regarding the minima on the Dalitz plot for the reaction $p\bar{n} \rightarrow 3\pi$, where for some reason only some of the required minima are present (see Sec. 5 and Figs. 23 and 24). To establish the actual state of affairs, we therefore require measurements of the differential cross sections for other reactions (for example, $\pi N \rightarrow \pi\Delta$) at low energies and the Dalitz plot for $p\bar{n} \rightarrow 3\pi$ (and other similar reactions) at higher energies.

4. For inclusive reactions, dual models predict:

a) Reactions such as $\pi^+p \rightarrow \pi^-M$, $pp \rightarrow \pi(K)M$, $\pi^\pm p \rightarrow K^-M$, $K^+p \rightarrow \pi^\pm M$ and $K^+p \rightarrow K^0 M$ should give the limiting distribution (97) at lower energies than reactions like $\pi^+p \rightarrow \pi^+M$, etc. (see Sec. 7). The experimental data on these reactions are extremely meager and, while the cross sections for $K^+p \rightarrow K^0 M$ agree at 5 and 8 GeV/c (see Fig. 46), for a reaction like $\pi^+p \rightarrow \pi^-M$ they are significantly different for $X > 0.7$ (and in good agreement for smaller X) at 8 and 16 GeV/c (see [112]).

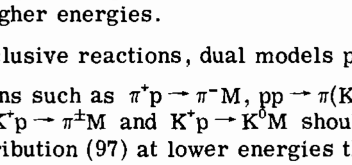


FIG. 47

b) The behavior of $E_c d\sigma_c/d^3p_c$ as a function of the variable $y = (1/2)\ln[(E_c + p_{cL})/(E_c - p_{cL})]$ has the form shown in Fig. 47 (the dashed curve corresponds to phase space for $p_{c\perp} = 0$). Unfortunately, there is no possibility of estimating the value of $y - y_1$ in dual models, so that nothing can be concluded from the fact that we do not observe such a distribution at current energies. To study this point, we require measurements of the cross sections throughout the entire range of p_{cL} at higher energies (cosmic-ray experiments or experiments with the colliding beams at CERN).

5. In parts (d) and (e) of Sec. 6 we gave a rather detailed discussion of the results of specific analyses in the GVM. How should they be regarded, and are they worth continuing? It seems to us that it is meaningful to continue such attempts, but that the main attention should be paid not to the agreement of the formulas with experiment, but to the question of what changes in the model show up weakly in particular characteristics of the reactions. Thus, for example, a change in the specific form of the equations (79) and the method of unitarizing them (by introducing the full widths of the resonances) would have a weak effect on the behavior of the total cross sections for the reactions (73)-(77) and at the same time, strongly modify the mass distributions in these reactions. Therefore the successful description of the behavior and magnitude of the total cross sections for the processes (73)-(77) is evidently connected with the general features of the dual approach (primarily with the explicit allowance for crossing symmetry for multiparticle reactions); at the same time, the mass spectra exploit the specific form of the fitted formula. Moreover, even these crude formulas may be useful in isolating the resonance contribution, since they give a model for the background, which varies rather rapidly with the appropriate energies.

6. Since the ideas of duality can be reconciled with the experimental data on baryon-baryon scattering only if exotic resonances occur in the $B\bar{D}$ and $D\bar{D}$ systems (B is the octet and D is the decuplet of baryons) (part (c) of Sec. 4), the search for them is of great interest.

APPENDIX I

In this appendix we consider the basic properties of $\Gamma(z)$, which plays a major role in the GVM.

1) $\Gamma(z)$ is analytic throughout the complex z-plane, apart from the negative integral points and the point $z = 0$.

2) $\Gamma(z + 1) = z\Gamma(z)$.

3) $\Gamma(n + 1) = n!$

4) All the poles of the gamma function are simple poles, the residue of $\Gamma(z)$ at the pole $z = -n$ being equal to $(-1)^n/n!$ $(n = 0, 1, 2, \ldots)$.

5) $\Gamma(z)$ has no zeros.

6) $\Gamma(z)\Gamma(1 - z) = \pi/\sin\pi z$.

7) For $\operatorname{Re} z > 0$, $\Gamma(z) = \int\limits_0^\infty e^{-t} t^{z-1} dt$.

8) For large positive x, $\Gamma(x + 1) = \sqrt{2\pi x}\,(x/e)^x[1 + O(1/x)]$ (Stirling's formula).

9) $\lim\left[\frac{\Gamma(z + a)}{\Gamma(z)} e^{-a\ln z}\right] = 1$ as $|z| \rightarrow \infty$.

10) $B_4(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx = \Gamma(a)\Gamma(b)/\Gamma(a+b)$.

APPENDIX II

In this appendix we present a general prescription for writing a dual amplitude that has only poles and the correct Regge asymptotic form in each of the channels. These rules are as follows: the multi-particle amplitude (Fig. 48a) is written as a sum of $(N-1)!/2$ terms:

$$A_N=\beta_N \sum_{(P)} B_N(P), \qquad (A.1)$$

where each term in (A.1) corresponds to a definite sequence of momenta in the diagram of Fig. 48b. For a given diagram, we introduce the variables u_{ik} (see Fig. 48b), corresponding to all the diagonals, these variables being not all independent, but satisfying the so-called dual equations

$$u_{i,k}=1-\prod_{\substack{l<i,\\ m>k}} u_{lm}. \qquad (A.2)$$

Geometrically, the u_{lm} appearing in the product correspond to the diagonals of the polygon in Fig. 48b which intersect the diagonal (ik). The significance of the condition (A.2) is that all the $u_{lm} \to 1$ for $u_{ik}=0$. Then

$$B_N=\int_0^1 \prod_{\substack{j,k\\ i\neq k}}{}' u_{ik}^{-\alpha_{ik}-1}\, du_{ik}\, \delta(u_{ik}-\Pi u_{lm}-1), \qquad (A.3)$$

where $\alpha_{ik}=\alpha(s_{ik})$ and $s_{ik}=(p_a+\ldots+p_k)^2$. Equation (A.3) factorizes for each pole for $\alpha(s)\to n$ (Fig. 48c) and $s=k^2$, i.e., it can be rewritten in the form [113]

$$B_N=\frac{1}{\alpha(s)-n}\sum_{\nu}\sum_{p} V^{n,j}_{p\{\mu_l'\}}\, P^{j\{\mu_l\}}_{\{\mu_l'\}}\, \bar{V}^{n,j}_{p\{\mu_{l'}\}}, \qquad (A.4)$$

where j is the spin of a resonance, P^j is the propagator of a spin-j resonance, and p characterizes the additional degeneracy.

V_p^{nj} is the vertex for the decay of a resonance with spin j, mass s_0 ($\alpha(s_0)=n$) and number p into K particles (see Fig. 48c), and $\bar{V}_p^{nj}$ is the same for the decay into M particles.

The sum (A.4) has the following properties [113]:

1) for n = 0, V_0^{00} coincides with the amplitude for K + 1 particles;

2) for j = n (which corresponds to the leading trajectory), there is no additional degeneracy;

3) some of the terms in the sum over the additional degeneracy appear with the negative (antiunitary) sign for K = M (Fig. 48c). Such unphysical states ("ghosts") must disappear after unitarization, but this means that the unitarity corrections cannot be small, so that (A.3) cannot be regarded as a good Born approximation.

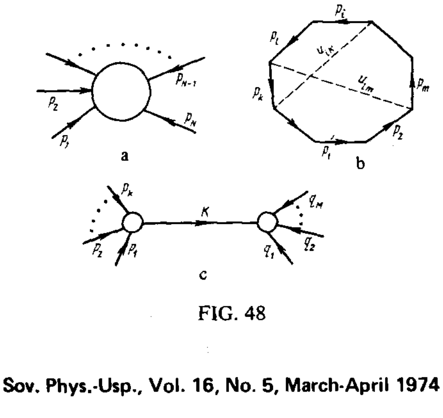


FIG. 48

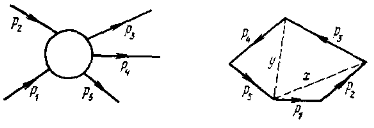


FIG. 49

However, these "ghosts" may simply give no contribution to the sum (A.4) if a certain further symmetry is present. In particular, their present status is as follows. There are no ghosts on the first daughter trajectory, owing to the so-called "twisting symmetry" [113,114] (which takes into account the fact that Eq. (A.3) is invariant under cyclic permutation of the indices i and k); moreover, in the case in which $\alpha(0)=1$, there appears an additional symmetry which completely eliminates the ghosts [115].

The number of states grows with the mass in proportion to $\exp[c\sqrt{\alpha(s)}]$, where $c=2\pi/\sqrt{6}$ [116].

The attempt to use Eq. (A.3) to construct graphs of the next order (as in Fig. 42) has led to the appearance of an exponential divergence in the expressions for Fig. 42c (see [117]) and an obscure singularity in the diagrams of Fig. 42d [118]. If the divergence is removed by renormalizing the trajectories [119], it is not clear what to do about the singularity in the diagram of Fig. 42d. A rather powerful technique has now been developed for the construction of graphs of arbitrary order on the basis of (A.3); this makes use of either the operator formalism [120] and the symmetry properties [121] of Eq. (A.3) or the geometrical interpretation connected with the integration over complex surfaces [122]. However, its actual achievements are limited.

In addition to the attempts to unitarize Eq. (A.3), efforts have been made to incorporate in this equation particles with real quantum numbers contained on real trajectories [123]. In particular, since $\alpha(0)>0$ for the real trajectories, there is a particle with imaginary mass (a "tachyon") in (A.3). Although progress has been made in this respect (especially in excluding "tachyons" from the amplitudes [123]), there is so far no formula that takes into account the physical trajectories.

Thus, the results of the theoretical development can be formulated by the statement that we have a formula which satisfies the requirements of duality for scalar particles with $\alpha(0)=1$ and which has led to difficulties when attempts were made to unitarize it, but that we have no analogous formula for the scattering of real physical particles.

In conclusion, we quote the explicit form of the B_5 function in terms of the independent variables (Fig. 49):

$$B_5=\int_0^1 (1-x)^{-\alpha_{23}-1}x^{-\alpha_{12}-1}\int_0^1 dy\,(1-y)^{-\alpha_{34}-1}y^{-\alpha_{56}-1}(1-xy)^{-\alpha_{24}-\alpha_{23}+\alpha_{34}}.$$

---

1) This background may be associated either with corrections to duality, such as the contribution of two-reggeon branch cuts (in which case it will fall off only logarithmically with energy), or with dual contributions which die out rapidly with increasing energy (such as the (su) term in the Veneziano model; see Sec. 6b below).

2) Here M stands for "missing" and denotes the system "everything else."

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